Abstracts for the School GEOQUANT

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Jorgen ANDERSEN (Aarhus University, CTQM, Denmark),

Quantization of moduli spaces of bundles and the mapping class group

Johannes HÜBSCHMANN (University of Lille, France):

Stratified Kähler spaces

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In the presence of classical phase space singularities the standard methods are insufficient to attack the problem of quantization. In certain situations these difficulties can be overcome by means of *stratified Kähler spaces*. Such a space is a stratified symplectic space together with a complex analytic structure which is compatible with the stratified symplectic structure; in particular each stratum is a Kähler manifold in an obvious fashion.
Examples abound: Symplectic reduction, applied to Kähler manifolds, yields a particular class of examples; this includes adjoint and generalized adjoint quotients of complex semisimple Lie groups which, in turn, underly certain lattice gauge theories. Other examples come from certain moduli spaces of holomorphic vector bundles on a Riemann surface and variants thereof; in physics language, these are spaces of conformal blocks. Still other examples arise from the closure of a holomorphic nilpotent orbit. Symplectic reduction carries a Kähler manifold to a stratified Kähler space in such a way that the sheaf of germs of polarized functions coincides with the ordinary sheaf of germs of holomorphic functions. Projectivization of holomorphic nilpotent orbits yields exotic stratified Kähler structures on complex projective spaces and on certain complex projective varieties including complex projective quadrics. Other physical examples are reduced spaces arising from angular momentum, including our solar system whose correct reduced phase space acquires the structure of an affine stratified Kähler space.
In the presence of singularities, the naive restriction of the quantization problem to a smooth open dense part, the "top stratum", may lead to a loss of information and in fact to inconsistent results. Within the framework of holomorphic quantization, a suitable quantization procedure on stratified Kähler spaces unveils a certain *quantum structure having the classical singularities as its shadow*. The new structure which thus emerges is that of a *costratified Hilbert space*, that is, a Hilbert space together with a system which consists of the subspaces associated with the strata of the reduced phase space and of the corresponding orthoprojectors. The costratified Hilbert space structure reflects the stratification of the reduced phase space. Given a Kähler

manifold, reduction after quantization then coincides with quantization after reduction in the sense that not only the reduced and unreduced quantum phase spaces correspond but the invariant unreduced and reduced quantum observables as well.

We will illustrate the approach with a concrete model: We will present a quantum (lattice) gauge theory which incorporates certain classical singularities. The reduced phase space is a stratified Kähler space, and we make explicit the requisite singular holomorphic quantization procedure and spell out the resulting costratified Hilbert space. In particular, certain tunneling probabilities between the strata emerge, the energy eigenstates can be determined, and corresponding expectation values of the orthoprojectors onto the subspaces associated with the strata in the strong and weak coupling approximations can be explored.

References

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[2] J. Huebschmann: Kähler quantization and reduction. J. reine angew. Math. **591**, 75–109 (2006), `math.SG/0207166`

[3] J. Huebschmann, G. Rudolph and M. Schmidt: A gauge model for quantum mechanics on a stratified space. Commun. Math. Physics **286** (2009), 459–494, `hep-th/0702017`

Lecture Notes can be downloaded from the homepage of the lecturer:
http://math.univ-lille1.fr/~huebschm

Martin SCHLICHENMAIER (University of Luxembourg):

Introduction to Berezin - Toeplitz Quantization

The Berezin-Toeplitz (BT) quantization procedure (either operator quantization or deformation quantization) will appear in many of the lectures of the school and the conference. It can be defined for Kähler manifolds which are *quantizable*. Quantizable means that there exist a certain holomorphic line bundle. It is my goal in this lecture to explain the geometric background and the basic construction of the BT- quantization with the help of the *Toeplitz operators*. In the case that the base manifold is compact one obtains strong approximation results. I.e. it is possible to show that it has the correct semiclassical limit. Moreover it is possible to define an natural deformation quantization (a star product) adapted to the complex structure. It should be mentioned that for certain classes of non-compact manifolds similar approximation results exists.

References

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[2] Martin Schlichenmaier, *Berezin-Toeplitz quantization of compact Kähler manifolds*, in: Quantization, Coherent States and Poisson Structures, Proc. XIV'th Workshop on Geometric Methods in Physics, Białowieza, Poland, 9-15 July 1995, (eds. A. Strasburger, S.T. Ali, J.-P. Antoine, J.-P. Gazeau , A. Odzijewicz), Polish Scientific Publisher PWN 1998, pp.101-115, q-alg/9601016.

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[4] Martin Schlichenmaier, *Berezin-Toeplitz quantization and Berezin transform*, Proc. of the Bologna APTEX International Conference", September 13-17, 1999, (eds. S. Graffi, A. Martinez), World Scientific, 2001, pp. 271–287.

[5] Alexander Karabegov, and Martin Schlichenmaier, *Identification of Berezin-Toeplitz deformation quantization*, J. reine angew. Math. **540**(2001), 49–76.

Armen SERGEEV (Steklov Institute, Moscow, Russia):

Quantization of universal Teichmller space

Abstract:
In the first part of the course we describe Kähler geometry of the universal Teichmüller space $\mathcal{T}$. This space consists of quasisymmetric homeomorphisms of the unit circle S^1 (i.e. orientation-preserving homeomorphisms of S^1, extending to quasiconformal homeomorphisms of the unit disc Δ), normalized modulo Möbius transformations (i.e. fractional-linear automorphisms of Δ). The space $\mathcal{T}$ is a complex Banach manifold, its complex structure being induced by its embedding into the Banach space of holomorphic quadratic differentials in the unit disc Δ. The classical Teichmüller spaces, associated with compact Riemann surfaces of finite genuses, are embedded into $\mathcal{T}$ as complex submanifolds. The space $\mathcal{S}$ of diffeomorphisms of the unit circle S^1, normalized modulo Möbius transformations, may be considered as a smooth part of $\mathcal{T}$. Quasisymmetric homeomorphisms of S^1 act (by change of variable) on the Sobolev space V of half-differentiable functions on S^1 by bounded symplectic operators. Respectively, diffeomorphisms of S^1 act on V by Hilbert–Schmidt symplectic operators.
The second part of the course is devoted to the quantization of the universal Teichmüller space. We explain first how to quantize the smooth part $\mathcal{S}$ of $\mathcal{T}$. We use for that an embedding of $\mathcal{S}$ into the Hilbert–Schmidt Siegel disc, generated by the action of diffeomorphisms of S^1 as Hilbert–Schmidt symplectic operators on V. We construct a holomorphic Fock bundle over the Siegel disc together with a projective action of the Hilbert–Schmidt symplectic group. The infinitesimal version of this action yields a Dirac quantization of $\mathcal{S}$. Unfortunately, this method does not work for the whole universal Teichmüller space $\mathcal{T}$, since it contains non-smooth homeomorphisms of S^1. For quantization of $\mathcal{T}$ we use another approach, based on the following idea. We still have an action of quasisymmetric homeomorphisms on V but this action has no infinitesimal limit, as in the case of $\mathcal{S}$. However, we can define a quantized infinitesimal version of this action. This quantized action generates the quantization of $\mathcal{T}$.

Program:

I. UNIVERSAL TEICHMÜLLER SPACE

1. Definition of universal Teichmüller space $\mathcal{T}$.
2. Complex and Kähler structures of $\mathcal{T}$.
3. Embedding of classical Teichmüller spaces into $\mathcal{T}$.
4. Grassmann realization of $\mathcal{T}$.

II. QUANTIZATION

5. Dirac quantization.
6. Fock space.
7. Symplectic group action on Fock spaces.
8. Quantum calculus and Connes quantization.
9. Quantization of universal Teichmüller space.

Dmitry TALALAEV (Institute for Theoretical and Experimental Physics, Moscow, Russia)

Quantum integrable systems and Langlands program

1 Classical spectral curve method

1.1 Hitchin construction
1.2 Singular curves
1.3 Gaudin system
1.4 Separated variables

2 Quantum problem

2.1 Deformation quantization paradigm
2.2 Quantum spectral curve
2.3 Bethe ansatz
2.4 Quantum separated variables
2.5 KZ-equation, Drinfeld-Sokolov form

3 Langlands correspondence

3.1 Background
3.2 Center of the affine algebra on the critical level
3.3 Beilinson-Drinfeld scheme

4 Hecke symmetries of Bethe solutions

4.1 Monodromic formulation
4.2 Matrix Bethe ansatz
4.3 Schlesinger transformation

Tatsuya TATE (University of Nagoya, Japan)

Problems on asymptotic analysis over convex polytopes

Content:
Lattice polytopes often appear in many areas in mathematics. In particular, they play essential roles in representation theory of compact Lie groups and the theory of toric varieties. In this series of lectures, we give some account on the asymptotic analysis over polytopes which has some applications to these theories.

In the first lecture, we review some technical tools in asymptotic analysis. In particular, we explain the method of stationary phase with complex phase functions. We then obtain an asymptotic formula for the counting functions of lattice paths attached to certain lattice polytopes as an application of the method of stationary phase with complex phase functions.

In the second lecture, we give some meaning of the asymptotics of the counting functions of the lattice paths from a viewpoint of elementary probability theory. The counting functions also have an application to the representation theory for compact Lie groups. We give a brief account of how it works in the representation theory.

In the last lecture, we explain a result of spectral analysis on toric manifolds, which is closely related to asymptotic analysis over polytopes. The problem is to find asymptotics of distribution functions of monomial sections of line bundles over toric manifolds. We explain this from the background of the problem.

Tilmann WURZBACHER (University of Metz, France):

Geometry of loop spaces

Abstract:
The goal of this set of lectures is to explain the construction of a rotation-equivariant Dirac-type operator on free loop spaces and its “raison d’être”. Given a riemannian manifold (with orientation and spin structure) the equivariant index of such an operator should be the so-called Witten-genus, having values in modular forms. This genus should play in elliptic cohomology a rôle comparable to the Dirac operator in K-theory. The content of the lectures will be roughly as below.

Section 1: Spinors in infinite dimension
We review here the infinite dimensional versions of the Clifford algebra, the vector space of spinors and the Spin^c-group.

Section 2: Topology and geometry underlying the construction of a Dirac operator on loop spaces
We explain here the topological condition for the existence of a “string structure” on a finite dimensional manifold, assuring that its loop space allows for a Spin^c-structure and outline the construction of the Dirac operator acting on smooth sections of the associated spinor bundle.

Section 3: Remarks on index theory of the Dirac operator on loops: some analysis and relations to elliptic genera
We sketch the results in the flat case, where the index is computed to be Dedkind’s eta function, and we comment on the conjectural relations to elliptic genera and cohomology.

Weiping ZHANG (Nankai Institute of Mathematics, Tianjing, China)

Analytic aspects of geometric quantization

Content:
We will introduce the basic idea and techniques of analytic localization in index theory, by starting with Witten's analytic proof of the Morse inequalities. Then we will describe how these idea and techniques can be used to provide an analytic proof of the Guillemin-Sternberg geometric quantization conjecture in symplectic geometry, as well as various generalizations.

References:

1. W. Zhang, Lectures on Chern-Weil Theory and Witten Deformations. World Scientific, Singapore,2001.

2. Y. Tian ad W. Zhang, An analytic proof of the geometric quantization conjecture of Guillemin-Sternberg. In 98,229-259.