Decomposing derived categories by decomposing polynomials

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Plan: become superheroes

1. introduce your superpower: the Hodge diamond
2. a brief introduction to mirror symmetry
3. ~~spend 5 hours doing homological algebra~~
4. use Hodge diamonds to decompose really complicated objects

let X be

- a smooth projective algebraic variety, or
- a Kähler manifold

we can probe its topology using singular or de Rham cohomology: C a curve or Riemann surface of genus $g = 3$

$$\dim \mathrm{H}^0(C, \mathbb{C}) = 1$$
$$\dim \mathrm{H}^1(C, \mathbb{C}) = 2g$$
$$\dim \mathrm{H}^2(C, \mathbb{C}) = 1$$

we can decompose it further:

$$\mathrm{H}^i(X, \mathbb{C}) \cong \bigoplus_{p+q=i} \mathrm{H}^q(X, \Omega_X^p)$$

Our superpower: Hodge diamond

$$\mathrm{h}^{p,q}(X) \coloneqq \dim \mathrm{H}^q(X, \Omega_X^p) \qquad p, q = 0, \ldots, \dim X$$

there are symmetries:

$$\begin{aligned} \textbf{Hodge} \quad & \mathrm{h}^{p,q} = \mathrm{h}^{q,p} \\ \textbf{Serre} \quad & \mathrm{h}^{p,q} = \mathrm{h}^{n-q,n-p} \end{aligned}$$

organize this as a diamond:

$$\begin{array}{ccccccc} & & & \mathrm{h}^{n,n} & & & \\ & & \mathrm{h}^{n,n-1} & & \mathrm{h}^{n-1,n} & & \\ & \iddots & & \vdots & & \ddots & \\ \mathrm{h}^{n,0} & & & \mathrm{h}^{n/2,n/2} & & & \mathrm{h}^{0,n} \\ & \ddots & & \vdots & & \iddots & \\ & & \mathrm{h}^{1,0} & & \mathrm{h}^{0,1} & & \\ & & & \mathrm{h}^{0,0} & & & \end{array}$$

Examples

- C a curve of genus g

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}$$

- $S_d \subset \mathbb{P}^3$ a surface of degree d

$$\begin{array}{ccccc} & & d=3 & & \\ & & 1 & & \\ & 0 & & 0 & \\ 0 & & 7 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} & & d=4 & & \\ & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} & & d=4 & & \\ & & 1 & & \\ & 0 & & 0 & \\ 4 & & 45 & & 4 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Exercise remember the curve and $d = 4$ diamonds!

observe that we can also encode a Hodge diamond as a polynomial in $\mathbb{Z}[x, y]$

Mirror symmetry

Hodge diamonds of Calabi–Yau 3-folds

- $\omega_X \cong \mathcal{O}_X$ and $\mathrm{h}^{i,0} = 0$ for $i \neq 0, 3$
- holonomy is equal to $\mathrm{SU}(3)$

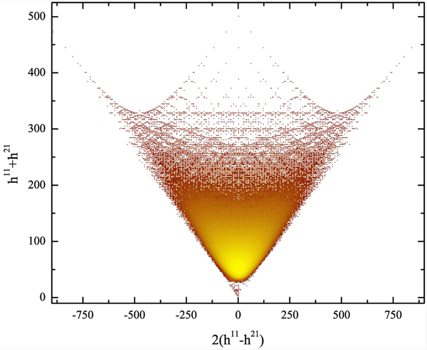
Hodge diamond looks like

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & \mathrm{h}^{1,1} & & 0 & \\ 1 & & \mathrm{h}^{2,1} & & \mathrm{h}^{2,1} & & 1 \\ & 0 & & \mathrm{h}^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Mirror symmetry

the playground of lots of mathematical physics!

plot of Hodge numbers for many known Calabi–Yau 3-folds



curious symmetry: for (almost) every X there is a Y such that

$$\mathrm{h}^{1,1}(X) = \mathrm{h}^{2,1}(Y) \qquad \mathrm{h}^{2,1}(X) = \mathrm{h}^{1,1}(Y)$$

Derived categories and semiorthogonal decompositions

A crash course in homological algebra

Let's skip this!

Decompositions using Hodge diamonds

Decomposing varieties

we can "decompose" a variety in many ways:

- cut-and-paste: $U \subset X$ Zariski-open, Z complement
  anything that behaves like an Euler characteristic is compatible with this
- direct sums of Hodge structures, or Chow motives, …
- semiorthogonal decompositions of derived categories:
  what I do for a living
- …

there are relations between these decompositions,
but there is no decomposition to rule them all

Principle use "weaker" decompositions to guess "stronger" decompositions

Intersection of 2 quadrics in $\mathbb{P}^5$

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
0 & & 2 & & 2 & & 0 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

1972: there is a $g = 2$ curve attached to $Q_1 \cap Q_2 \subset \mathbb{P}^5$

- decomposition of Hodge structures
- 1995: decomposition of derived category

this actually works in arbitrary dimension for $Q_1 \cap Q_2 \subset \mathbb{P}^{2g+1}$ using a hyperelliptic curve of genus g

Cubic fourfolds and K3 surfaces

$X \subset \mathbb{P}^5$

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 1 & & 21 & & 1 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

Hint use that $21 = 20 + 1$

we have that some X have an associated K3 surface (such as a quartic surface), but not all

Strong law of small numbers

> *"There aren't enough small numbers to meet the many demands made of them."*

be careful for overly optimistic conclusions!

- Hodge diamonds as a conjecture generator
- mathematicians as judge of plausibility

Hodge diamond cutter: `https://cutter.ncag.info`

DOI 10.5281/zenodo.3893509 | release v1.2 | license GPL-3.0

Unit tests passing | code style black

Hodge diamond cutter

A collection of Python classes and functions in Sage to deal with Hodge diamonds (and Hochschild homology) of smooth projective varieties, together with many constructions.

You can read its documentation as

- a webpage
- a pdf

Become a superhero using the Hodge diamond cutter

staring at Hodge diamonds has played a sometimes minor, sometimes large role in the following of my papers: 1810.11873, 1909.04321, 2002.04940, 2005.01989, **2009.05568**, 2202.08601, 2311.17004, 2309.06244, **2403.12517**

let's look at an example: **2009.05568**
also known as the BGMN conjecture:

G Galkin

M Mukhopadhyay

N Narasimhan

Numerically verifying the BGMN conjecture

$Q_1 \cap Q_2 \subset \mathbb{P}^5$ generalizes to moduli of rank-two bundles

$$[\mathrm{M}_C(2, \mathcal{L})] = \mathbb{L}^{g-1}[\mathrm{Sym}^{g-1} C] + \sum_{i=0}^{g-2} (\mathbb{L}^i + \mathbb{L}^{3g-3-2i})[\mathrm{Sym}^i C]$$

- semiorthogonal decomposition: theorem by Tevelev–Torres
- Grothendieck ring of varieties: still open

```
L = lefschetz()

assert all(
    L ** (g - 1) * symmetric_power(g - 1, g)
    + sum(
        (L**i + L ** (3 * g - 3 - 2 * i)) * symmetric_power(i, g)
        for i in range(g - 1)
    )
    == moduli_vector_bundles(2, 1, g)
    for g in range(2, 20)
)
```