

## Derivation of the arc-length formula for plane function curves

Consider a function  $y = f(x)$  on the interval  $\text{dom} f = [a, b]$ . We assume that the function is *continuously differentiable*, meaning that its derivative  $f'(t)$  exists and is continuous on  $\text{dom} f$ .

Initially we want to approximate the length of the curve  $\mathcal{C}_f$  by dividing the interval  $[a, b]$  into  $n$  subintervals of equal width  $\Delta x := x_{i+1} - x_i$  with  $a = x_0 < x_1 < \dots < x_n = b$ . We call  $P_i$  the point on  $\mathcal{C}_f$  with coordinates  $(x_i; f(x_i))$ .

Then, we can approximate  $\mathcal{C}_f$  by the polygonal chain  $(P_0, P_1, \dots, P_n)$ .

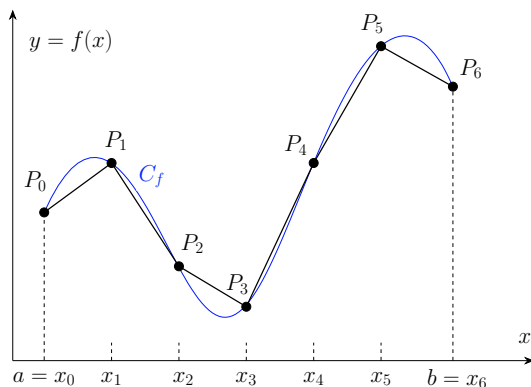
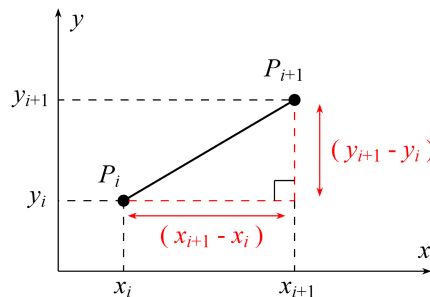


Figure 7: Example of a polygonal chain for  $n = 6$

To find the length of each line segment  $[P_i P_{i+1}]$ , we can use the Pythagorean theorem:

$$\begin{aligned} (\overline{P_i P_{i+1}})^2 &= (x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2 \\ \Rightarrow \overline{P_i P_{i+1}} &= \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2} \\ &= \sqrt{(\Delta x)^2 + (\Delta y)^2} \end{aligned}$$

$$\text{with } \Delta y := y_{i+1} - y_i = f(x_{i+1}) - f(x_i)$$



By the Mean Value Theorem, there exists a  $c_i \in (x_i, x_{i+1})$  such that

$$\begin{aligned} f(x_{i+1}) - f(x_i) &= f'(c_i) \cdot (x_{i+1} - x_i) \\ \Leftrightarrow \Delta y &= f'(c_i) \cdot \Delta x \end{aligned}$$

Then, the length can be written as:

$$\begin{aligned} \overline{P_i P_{i+1}} &= \sqrt{(\Delta x)^2 + (\Delta y)^2} \\ &= \sqrt{(\Delta x)^2 + (f'(c_i) \cdot \Delta x)^2} \\ &= \sqrt{1 + [f'(c_i)]^2} \Delta x \end{aligned}$$

The length  $L$  of  $\mathcal{C}_f$  can thus be approximated by the sum of the lengths of the line segments.

$$L \approx \sum_{i=0}^{n-1} \overline{P_i P_{i+1}} = \sum_{i=0}^{n-1} \left[ \sqrt{1 + [f'(c_i)]^2} \Delta x \right]$$

To increase the accuracy of this approximation, we need to take smaller and smaller subintervals, i.e. an increasing number  $n$  of subintervals. In other words, we need to take the limit  $n \rightarrow \infty$ :

$$L = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \overline{P_i P_{i+1}} = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \left[ \sqrt{1 + [f'(c_i)]^2} \right] \Delta x$$

Since this is the limit of the Riemann sums of a function as the subintervals get smaller, it is the Riemann integral. And we can write:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

### Derivation of the arc-length formula for plane curves:

For a general plane curve, the derivation of the arc-length formula works similarly.

Consider a curve  $C$  parametrised as  $(x(t), y(t))$  for  $t \in [a, b]$ <sup>3</sup>. We assume that the curve is *continuously differentiable*, meaning that the derivatives  $x'(t)$  and  $y'(t)$  exist and are continuous on  $[a, b]$ .

As before, we divide the interval  $[a, b]$  into subintervals of equal width  $\Delta t := t_{i+1} - t_i$  with  $a = t_0 < t_1 < \dots < t_n = b$  and we call  $P_i$  the point on the curve with coordinates  $(x(t_i), y(t_i))$ . Then the length can be approximated by

$$L \approx \sum_{i=0}^{n-1} \overline{P_i P_{i+1}} = \sum_{i=0}^{n-1} \sqrt{[x(t_{i+1}) - x(t_i)]^2 + [y(t_{i+1}) - y(t_i)]^2}$$

Similarly to before, we use the Mean Value Theorem, to write

$$\begin{aligned} x(t_{i+1}) - x(t_i) &= x'(c_i) \cdot \Delta t && \text{for a } c_i \in (t_i, t_{i+1}) \\ y(t_{i+1}) - y(t_i) &= y'(k_i) \cdot \Delta t && \text{for a } k_i \in (t_i, t_{i+1}) \end{aligned}$$

Then:

$$\begin{aligned} L &\approx \sum_{i=0}^{n-1} \sqrt{[x(t_{i+1}) - x(t_i)]^2 + [y(t_{i+1}) - y(t_i)]^2} \\ &= \sum_{i=0}^{n-1} \sqrt{[x'(c_i) \cdot \Delta t]^2 + [y'(k_i) \cdot \Delta t]^2} \\ &= \sum_{i=0}^{n-1} \sqrt{x'(c_i)^2 + y'(k_i)^2} \cdot \Delta t \end{aligned}$$

To increase the accuracy of this approximation, we take the limit as the number of subintervals increases, i.e  $n \rightarrow \infty$  and  $\Delta t \rightarrow 0$ . By continuity of  $x'(t)$  and  $y'(t)$ , we have  $c_i, k_i \rightarrow t_i$ , so the expression becomes a Riemann sum:

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \overline{P_i P_{i+1}} = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \sqrt{x'(c_i)^2 + y'(k_i)^2} \cdot \Delta t \\ &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \sqrt{x'(t_i)^2 + y'(t_i)^2} \cdot \Delta t \end{aligned}$$

Hence,

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt$$

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<sup>3</sup>Note that a function  $y = f(x)$  can be parametrised as  $(x, f(x))$  for  $x \in \text{dom}f$ .