Alternative helix shapes and spirals

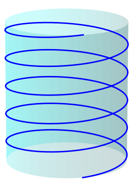
So far, we have focused on **cylindrical helices**: curves that lie on a cylinder and make a constant angle with its axis. However, cylinders are not the only surfaces that can support helical curves. By replacing the cylinder with another surface, we obtain different types of helices.

Alternative Helix shapes:

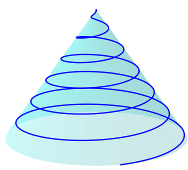
- A **conical helix** lies on the surface of a right circular cone.
  Like a cylindrical helix, it winds around the cone's axis. The main difference is that the radius is no longer constant: as a point traces out the curve, its distance from the axis increases or decreases.

- A **spherical helix** lies on the surface of a sphere.
  Since a sphere is bounded, the curve cannot extend indefinitely. Instead it wraps around the sphere while remaining on its surface.

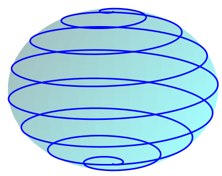
- A **toroidal helix** lies on the surface of a torus, a surface with a doughnut shape.
  It can be viewed as a helix drawn on a closed cylindrical tube. As a point traces out the curve, it winds around the tube while simultaneously travelling around the central hole of the torus.



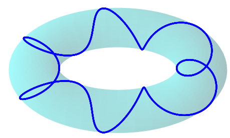
Cylindrical helix



Conical helix



Spherical helix



Toroidal helix

Helix vs. Spiral

In everyday language, the terms "spiral" and "helix" are often used interchangeably, as in the case of a "spiral staircase". However, in mathematics, those concepts are distinct, though deeply connected.

A **spiral** is a **plane (2D) curve** that emanates from a central point, progressively moving farther away from it while rotating around that point.

In $x - y$-coordinates, the curve can be described by parametric equations:

$$x = r(t)\cos(t), \qquad y = r(t)\sin(t)$$

where $r(t)$ is a continuous monotonic function of angle t.

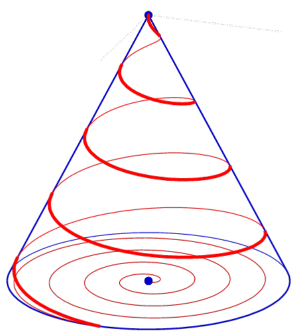
Relation to helices:
While spirals are planar (2D) curves and helices are spatial (3D) curves, several notable spirals can be obtained from helices through suitable projections.
Notable examples:

- **Archimedean spiral:** $r(t) = a \cdot t$, with $a \in \mathbb{R}$.
  This spiral can be obtained as the **orthogonal projection** of a **uniform conical helix** onto the base of the cone.

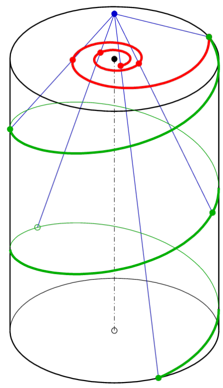
- **Hyperbolic spiral:** $r(t) = \frac{a}{t}$, with $a \in \mathbb{R}$.
  This spiral can be obtained as the **central projection** of a **cylindrical helix** from the apex of a cone onto a plane.

- **Logarithmic spiral**: $r(t) = a \cdot e^{kt}$, with $a \in \mathbb{R}^*_+$ and $k \in \mathbb{R}^*$.
  Unlike the previous examples, the logarithmic spiral does not admit a straightforward geometric relation to helices.

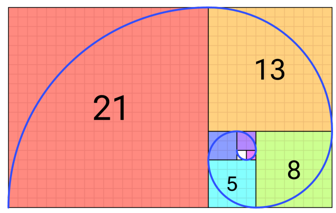
The logarithmic spiral includes the **golden spiral** for $k = \frac{\ln(\Phi)}{\pi/2}$ where $\Phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. A notable approximation of the golden spiral is the **Fibonacci spiral**.



Conical helix with an archimedean spiral as floor projection [5]



Hyperbolic spiral as a central projection of a helix [6]



An approximation of the golden spiral by the Fibonacci spiral [7]

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