HIGHER LAGRANGIAN MECHANICS ON GRADED BUNDLES

Janusz Grabowski

(Polish Academy of Sciences)

HIGHER GEOMETRY AND FIELD THEORY

Luxembourg, 9-11 December, 2015

Contents

- Graded and double graded bundles
- Tulczyjew triples
- Mechanics on algebroids with vakonomic constraints
- Higher order Lagrangians
- Lagrangian framework for graded bundles
- Higher order Lagrangian mechanics on Lie algebroids
- Geometric mechanics of strings (optionally)

The talk is based on some ideas of W. M. Tulczyjew and my collaboration with A. Bruce, K. Grabowska, M. Rotkiewicz and P. Urbański:

- Grabowski-Rotkiewicz, *J. Geom. Phys.* **62** (2012), 21–36.
- Grabowska-Grabowski-Urbański, *J. Geom. Mech.* **6** (2014), 503–526.
- Bruce-Grabowska-Grabowski, *J. Phys. A* **48** (2015), 205203 (32pp).
- Bruce-Grabowska-Grabowski, *arXiv:1409.0439*.

Contents

- Graded and double graded bundles
- Tulczyjew triples
- Mechanics on algebroids with vakonomic constraints
- Higher order Lagrangians
- Lagrangian framework for graded bundles
- Higher order Lagrangian mechanics on Lie algebroids
- Geometric mechanics of strings (optionally)

The talk is based on some ideas of W. M. Tulczyjew and my collaboration with A. Bruce, K. Grabowska, M. Rotkiewicz and P. Urbański:

- Grabowski-Rotkiewicz, *J. Geom. Phys.* **62** (2012), 21–36.
- Grabowska-Grabowski-Urbański, *J. Geom. Mech.* **6** (2014), 503–526.
- Bruce-Grabowska-Grabowski, *J. Phys. A* **48** (2015), 205203 (32pp).
- Bruce-Grabowska-Grabowski, *arXiv:1409.0439*.

Contents

The talk is based on some ideas of W. M. Tulczyjew and my collaboration with A. Bruce, K. Grabowska, M. Rotkiewicz and P. Urbański:

- Grabowski-Rotkiewicz, *J. Geom. Phys.* **62** (2012), 21–36.
- Grabowska-Grabowski-Urbański, *J. Geom. Mech.* **6** (2014), 503–526.
- Bruce-Grabowska-Grabowski, *J. Phys. A* **48** (2015), 205203 (32pp).
- Bruce-Grabowska-Grabowski, *arXiv:1409.0439*.

Contents

- Graded and double graded bundles
- Tulczyjew triples
- Mechanics on algebroids with vakonomic constraints
- Higher order Lagrangians
- Lagrangian framework for graded bundles
- Higher order Lagrangian mechanics on Lie algebroids
- Geometric mechanics of strings (optionally)

The talk is based on some ideas of W. M. Tulczyjew and my collaboration with A. Bruce, K. Grabowska, M. Rotkiewicz and P. Urbański:

- Grabowski-Rotkiewicz, *J. Geom. Phys.* **62** (2012), 21–36.
- Grabowska-Grabowski-Urbański, *J. Geom. Mech.* **6** (2014), 503–526.
- Bruce-Grabowska-Grabowski, *J. Phys. A* **48** (2015), 205203 (32pp).
- Bruce-Grabowska-Grabowski, *arXiv:1409.0439*.

Contents

- Graded and double graded bundles
- Tulczyjew triples
- Mechanics on algebroids with vakonomic constraints
- Higher order Lagrangians
- Lagrangian framework for graded bundles
- Higher order Lagrangian mechanics on Lie algebroids
- Geometric mechanics of strings (optionally)

The talk is based on some ideas of W. M. Tulczyjew and my collaboration with A. Bruce, K. Grabowska, M. Rotkiewicz and P. Urbański:

- Grabowski-Rotkiewicz, *J. Geom. Phys.* **62** (2012), 21–36.
- Grabowska-Grabowski-Urbański, *J. Geom. Mech.* **6** (2014), 503–526.
- Bruce-Grabowska-Grabowski, *J. Phys. A* **48** (2015), 205203 (32pp).
- Bruce-Grabowska-Grabowski, *arXiv:1409.0439*.

Contents

- Graded and double graded bundles
- Tulczyjew triples
- Mechanics on algebroids with vakonomic constraints
- Higher order Lagrangians
- Lagrangian framework for graded bundles
- Higher order Lagrangian mechanics on Lie algebroids
- Geometric mechanics of strings (optionally)

The talk is based on some ideas of W. M. Tulczyjew and my collaboration with A. Bruce, K. Grabowska, M. Rotkiewicz and P. Urbański:

- Grabowski-Rotkiewicz, *J. Geom. Phys.* **62** (2012), 21–36.
- Grabowska-Grabowski-Urbański, *J. Geom. Mech.* **6** (2014), 503–526.
- Bruce-Grabowska-Grabowski, *J. Phys. A* **48** (2015), 205203 (32pp).
- Bruce-Grabowska-Grabowski, *arXiv:1409.0439*.

Contents

- Graded and double graded bundles
- Tulczyjew triples
- Mechanics on algebroids with vakonomic constraints
- Higher order Lagrangians
- Lagrangian framework for graded bundles
- Higher order Lagrangian mechanics on Lie algebroids
- Geometric mechanics of strings (optionally)

The talk is based on some ideas of W. M. Tulczyjew and my collaboration with A. Bruce, K. Grabowska, M. Rotkiewicz and P. Urbański:

- Grabowski-Rotkiewicz, *J. Geom. Phys.* **62** (2012), 21–36.
- Grabowska-Grabowski-Urbański, *J. Geom. Mech.* **6** (2014), 503–526.
- Bruce-Grabowska-Grabowski, *J. Phys. A* **48** (2015), 205203 (32pp).
- Bruce-Grabowska-Grabowski, *arXiv:1409.0439*.

Contents

- Graded and double graded bundles
- Tulczyjew triples
- Mechanics on algebroids with vakonomic constraints
- Higher order Lagrangians
- Lagrangian framework for graded bundles
- Higher order Lagrangian mechanics on Lie algebroids
- Geometric mechanics of strings (optionally)

The talk is based on some ideas of W. M. Tulczyjew and my collaboration with A. Bruce, K. Grabowska, M. Rotkiewicz and P. Urbański:

- Grabowski-Rotkiewicz, *J. Geom. Phys.* **62** (2012), 21–36.
- Grabowska-Grabowski-Urbański, *J. Geom. Mech.* **6** (2014), 503–526.
- Bruce-Grabowska-Grabowski, *J. Phys. A* **48** (2015), 205203 (32pp).
- Bruce-Grabowska-Grabowski, *arXiv:1409.0439*.

Contents

- Graded and double graded bundles
- Tulczyjew triples
- Mechanics on algebroids with vakonomic constraints
- Higher order Lagrangians
- Lagrangian framework for graded bundles
- Higher order Lagrangian mechanics on Lie algebroids
- Geometric mechanics of strings (optionally)

The talk is based on some ideas of W. M. Tulczyjew and my collaboration with A. Bruce, K. Grabowska, M. Rotkiewicz and P. Urbański:

- Grabowski-Rotkiewicz, *J. Geom. Phys.* **62** (2012), 21–36.
- Grabowska-Grabowski-Urbański, *J. Geom. Mech.* **6** (2014), 503–526.
- Bruce-Grabowska-Grabowski, *J. Phys. A* **48** (2015), 205203 (32pp).
- Bruce-Grabowska-Grabowski, *arXiv:1409.0439*.

Contents

- Graded and double graded bundles
- Tulczyjew triples
- Mechanics on algebroids with vakonomic constraints
- Higher order Lagrangians
- Lagrangian framework for graded bundles
- Higher order Lagrangian mechanics on Lie algebroids
- Geometric mechanics of strings (optionally)

The talk is based on some ideas of W. M. Tulczyjew and my collaboration with A. Bruce, K. Grabowska, M. Rotkiewicz and P. Urbański:

- Grabowski-Rotkiewicz, *J. Geom. Phys.* **62** (2012), 21–36.
- Grabowska-Grabowski-Urbański, *J. Geom. Mech.* **6** (2014), 503–526.
- Bruce-Grabowska-Grabowski, *J. Phys. A* **48** (2015), 205203 (32pp).
- Bruce-Grabowska-Grabowski, *arXiv:1409.0439*.

Contents

- Graded and double graded bundles
- Tulczyjew triples
- Mechanics on algebroids with vakonomic constraints
- Higher order Lagrangians
- Lagrangian framework for graded bundles
- Higher order Lagrangian mechanics on Lie algebroids
- Geometric mechanics of strings (optionally)

The talk is based on some ideas of W. M. Tulczyjew and my collaboration with A. Bruce, K. Grabowska, M. Rotkiewicz and P. Urbański:

- Grabowski-Rotkiewicz, *J. Geom. Phys.* **62** (2012), 21–36.
- Grabowska-Grabowski-Urbański, *J. Geom. Mech.* **6** (2014), 503–526.
- Bruce-Grabowska-Grabowski, *J. Phys. A* **48** (2015), 205203 (32pp).
- Bruce-Grabowska-Grabowski, *arXiv:1409.0439*.

Contents

- Graded and double graded bundles
- Tulczyjew triples
- Mechanics on algebroids with vakonomic constraints
- Higher order Lagrangians
- Lagrangian framework for graded bundles
- Higher order Lagrangian mechanics on Lie algebroids
- Geometric mechanics of strings (optionally)

The talk is based on some ideas of W. M. Tulczyjew and my collaboration with A. Bruce, K. Grabowska, M. Rotkiewicz and P. Urbański:

- Grabowski-Rotkiewicz, *J. Geom. Phys.* **62** (2012), 21–36.
- Grabowska-Grabowski-Urbański, *J. Geom. Mech.* **6** (2014), 503–526.
- Bruce-Grabowska-Grabowski, *J. Phys. A* **48** (2015), 205203 (32pp).
- Bruce-Grabowska-Grabowski, *arXiv:1409.0439*.

Contents

- Graded and double graded bundles
- Tulczyjew triples
- Mechanics on algebroids with vakonomic constraints
- Higher order Lagrangians
- Lagrangian framework for graded bundles
- Higher order Lagrangian mechanics on Lie algebroids
- Geometric mechanics of strings (optionally)

The talk is based on some ideas of W. M. Tulczyjew and my collaboration with A. Bruce, K. Grabowska, M. Rotkiewicz and P. Urbański:

- Grabowski-Rotkiewicz, *J. Geom. Phys.* **62** (2012), 21–36.
- Grabowska-Grabowski-Urbański, *J. Geom. Mech.* **6** (2014), 503–526.
- Bruce-Grabowska-Grabowski, *J. Phys. A* **48** (2015), 205203 (32pp).
- Bruce-Grabowska-Grabowski, *arXiv:1409.0439*.

Vector bundles as graded bundles

- A vector bundle is a locally trivial fibration $\tau : E \to M$ which, locally over $U \subset M$, reads $\tau^{-1}(U) \simeq U \times \mathbb{R}^n$ and admits an atlas in which local trivializations transform linearly in fibers

$$U \cap V \times \mathbb{R}^n \ni (x, y) \mapsto (x, A(x)y) \in U \cap V \times \mathbb{R}^n ,$$

  $A(x) \in \mathrm{GL}(n, \mathbb{R})$.
- The latter property can also be expressed in the terms of the gradation in which base coordinates x have degrees 0 and 'linear coordinates' y have degree 1. Linearity in $y's$ is now equivalent to the fact that changes of coordinates respect the degrees.
- Morphisms in the category of vector bundles are represented by commutative diagram of smooth maps

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} \\ M_1 & \xrightarrow{\varphi} & M_2 \end{array}$$

  being linear in fibres.

Vector bundles as graded bundles

- A vector bundle is a locally trivial fibration $\tau : E \to M$ which, locally over $U \subset M$, reads $\tau^{-1}(U) \simeq U \times \mathbb{R}^n$ and admits an atlas in which local trivializations transform linearly in fibers

$$U \cap V \times \mathbb{R}^n \ni (x, y) \mapsto (x, A(x)y) \in U \cap V \times \mathbb{R}^n \,,$$

  $A(x) \in \mathrm{GL}(n, \mathbb{R})$.
- The latter property can also be expressed in the terms of the gradation in which base coordinates x have degrees 0 and 'linear coordinates' y have degree 1. Linearity in $y's$ is now equivalent to the fact that changes of coordinates respect the degrees.
- Morphisms in the category of vector bundles are represented by commutative diagram of smooth maps

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ \downarrow{\tau_1} & & \downarrow{\tau_2} \\ M_1 & \xrightarrow{\varphi} & M_2 \end{array}$$

  being linear in fibres.

Vector bundles as graded bundles

- A vector bundle is a locally trivial fibration $\tau : E \to M$ which, locally over $U \subset M$, reads $\tau^{-1}(U) \simeq U \times \mathbb{R}^n$ and admits an atlas in which local trivializations transform linearly in fibers

$$U \cap V \times \mathbb{R}^n \ni (x, y) \mapsto (x, A(x)y) \in U \cap V \times \mathbb{R}^n\,,$$

  $A(x) \in \mathrm{GL}(n, \mathbb{R})$.
- The latter property can also be expressed in the terms of the gradation in which base coordinates x have degrees 0 and 'linear coordinates' y have degree 1. Linearity in $y's$ is now equivalent to the fact that changes of coordinates respect the degrees.
- Morphisms in the category of vector bundles are represented by commutative diagram of smooth maps

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} \\ M_1 & \xrightarrow{\varphi} & M_2 \end{array}$$

  being linear in fibres.

Vector bundles as graded bundles

- A vector bundle is a locally trivial fibration $\tau : E \to M$ which, locally over $U \subset M$, reads $\tau^{-1}(U) \simeq U \times \mathbb{R}^n$ and admits an atlas in which local trivializations transform linearly in fibers

$$U \cap V \times \mathbb{R}^n \ni (x, y) \mapsto (x, A(x)y) \in U \cap V \times \mathbb{R}^n ,$$

$A(x) \in \mathrm{GL}(n, \mathbb{R})$.

- The latter property can also be expressed in the terms of the gradation in which base coordinates x have degrees 0 and 'linear coordinates' y have degree 1. Linearity in $y's$ is now equivalent to the fact that changes of coordinates respect the degrees.
- Morphisms in the category of vector bundles are represented by commutative diagram of smooth maps

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} \\ M_1 & \xrightarrow{\varphi} & M_2 \end{array}$$

being linear in fibres.

Vector bundles as graded bundles

- A vector bundle is a locally trivial fibration $\tau : E \to M$ which, locally over $U \subset M$, reads $\tau^{-1}(U) \simeq U \times \mathbb{R}^n$ and admits an atlas in which local trivializations transform linearly in fibers
$$U \cap V \times \mathbb{R}^n \ni (x, y) \mapsto (x, A(x)y) \in U \cap V \times \mathbb{R}^n\,,$$
$A(x) \in \mathrm{GL}(n, \mathbb{R})$.
- The latter property can also be expressed in the terms of the gradation in which base coordinates x have degrees 0 and ‘linear coordinates’ y have degree 1. Linearity in $y's$ is now equivalent to the fact that changes of coordinates respect the degrees.
- Morphisms in the category of vector bundles are represented by commutative diagram of smooth maps
$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} \\ M_1 & \xrightarrow{\varphi} & M_2 \end{array}$$
being linear in fibres.

Vector bundles as graded bundles

- A vector bundle is a locally trivial fibration $\tau : E \to M$ which, locally over $U \subset M$, reads $\tau^{-1}(U) \simeq U \times \mathbb{R}^n$ and admits an atlas in which local trivializations transform linearly in fibers

$$U \cap V \times \mathbb{R}^n \ni (x, y) \mapsto (x, A(x)y) \in U \cap V \times \mathbb{R}^n \,,$$

  $A(x) \in \mathrm{GL}(n, \mathbb{R})$.
- The latter property can also be expressed in the terms of the gradation in which base coordinates x have degrees 0 and 'linear coordinates' y have degree 1. Linearity in $y's$ is now equivalent to the fact that changes of coordinates respect the degrees.
- Morphisms in the category of vector bundles are represented by commutative diagram of smooth maps

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} \\ M_1 & \xrightarrow{\varphi} & M_2 \end{array}$$

  being linear in fibres.

Vector bundles as graded bundles

- A vector bundle is a locally trivial fibration $\tau : E \to M$ which, locally over $U \subset M$, reads $\tau^{-1}(U) \simeq U \times \mathbb{R}^n$ and admits an atlas in which local trivializations transform linearly in fibers
$$U \cap V \times \mathbb{R}^n \ni (x, y) \mapsto (x, A(x)y) \in U \cap V \times \mathbb{R}^n\,,$$
$A(x) \in \mathrm{GL}(n, \mathbb{R})$.
- The latter property can also be expressed in the terms of the gradation in which base coordinates x have degrees 0 and 'linear coordinates' y have degree 1. Linearity in $y's$ is now equivalent to the fact that changes of coordinates respect the degrees.
- Morphisms in the category of vector bundles are represented by commutative diagram of smooth maps
$$\begin{array}{ccc} E_1 & \xrightarrow{\ \Phi\ } & E_2 \\ \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} \\ M_1 & \xrightarrow{\ \varphi\ } & M_2 \end{array}$$
being linear in fibres.

Vector bundles as graded bundles

- A vector bundle is a locally trivial fibration $\tau : E \to M$ which, locally over $U \subset M$, reads $\tau^{-1}(U) \simeq U \times \mathbb{R}^n$ and admits an atlas in which local trivializations transform linearly in fibers

$$U \cap V \times \mathbb{R}^n \ni (x, y) \mapsto (x, A(x)y) \in U \cap V \times \mathbb{R}^n\,,$$

  $A(x) \in \mathrm{GL}(n, \mathbb{R})$.
- The latter property can also be expressed in the terms of the gradation in which base coordinates x have degrees 0 and 'linear coordinates' y have degree 1. Linearity in $y's$ is now equivalent to the fact that changes of coordinates respect the degrees.
- Morphisms in the category of vector bundles are represented by commutative diagram of smooth maps

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} \\ M_1 & \xrightarrow{\varphi} & M_2 \end{array}$$

  being linear in fibres.

Graded bundles

- Canonical examples and constructions: $\mathrm{T}M$, T^*M, $E\otimes_M F$, $\wedge^k E$, etc.
- A straightforward generalization is the concept of a graded bundle $\tau: F\to M$ with a local trivialization by $U\times\mathbb{R}^n$ as before, and with the difference that the local coordinates $(y^1,\dots,y^n)$ in the fibres have now associated positive integer weights $w_1,\dots,w_n$, that are preserved by changes of local trivializations:

$$U\cap V\times\mathbb{R}^n\ni(x,y)\mapsto(x,A(x,y))\in U\cap V\times\mathbb{R}^n\,,$$

- One can show that in this case $A(x,y)$ must be polynomial in fiber coordinates, i.e. any graded bundle is a polynomial bundle.
- As these polynomials need not to be linear, graded bundles do not have, in general, vector space structure in fibers. For instance, if $(y,z)\in\mathbb{R}^2$ are coordinates of degrees $1,2$, respectively, then the map $(y,z)\mapsto(y,z+y^2)$ is a diffeomorphism preserving the degrees, but it is nonlinear.
- If all $w_i\le r$, we say that the graded bundle is of degree r.

Graded bundles

- Canonical examples and constructions: $\mathsf{T}M$, T^*M, $E \otimes_M F$, $\wedge^k E$, etc.
- A straightforward generalization is the concept of a graded bundle $\tau : F \to M$ with a local trivialization by $U \times \mathbb{R}^n$ as before, and with the difference that the local coordinates $(y^1, \ldots, y^n)$ in the fibres have now associated positive integer weights $w_1, \ldots, w_n$, that are preserved by changes of local trivializations:

$$U \cap V \times \mathbb{R}^n \ni (x, y) \mapsto (x, A(x, y)) \in U \cap V \times \mathbb{R}^n ,$$

- One can show that in this case $A(x, y)$ must be polynomial in fiber coordinates, i.e. any graded bundle is a polynomial bundle.
- As these polynomials need not to be linear, graded bundles do not have, in general, vector space structure in fibers. For instance, if $(y, z) \in \mathbb{R}^2$ are coordinates of degrees $1, 2$, respectively, then the map $(y, z) \mapsto (y, z + y^2)$ is a diffeomorphism preserving the degrees, but it is nonlinear.
- If all $w_i \leq r$, we say that the graded bundle is of degree r.

Graded bundles

- Canonical examples and constructions: $\mathsf{T}M$, T^*M, $E \otimes_M F$, $\wedge^k E$, etc.
- A straightforward generalization is the concept of a graded bundle $\tau : F \to M$ with a local trivialization by $U \times \mathbb{R}^n$ as before, and with the difference that the local coordinates $(y^1, \dots, y^n)$ in the fibres have now associated positive integer weights $w_1, \dots, w_n$, that are preserved by changes of local trivializations:

$$U \cap V \times \mathbb{R}^n \ni (x, y) \mapsto (x, A(x, y)) \in U \cap V \times \mathbb{R}^n,$$

- One can show that in this case $A(x, y)$ must be polynomial in fiber coordinates, i.e. any graded bundle is a polynomial bundle.
- As these polynomials need not to be linear, graded bundles do not have, in general, vector space structure in fibers. For instance, if $(y, z) \in \mathbb{R}^2$ are coordinates of degrees $1, 2$, respectively, then the map $(y, z) \mapsto (y, z + y^2)$ is a diffeomorphism preserving the degrees, but it is nonlinear.
- If all $w_i \leq r$, we say that the graded bundle is of degree r.

Graded bundles

- Canonical examples and constructions: $\mathsf{T}M$, T^*M, $E \otimes_M F$, $\wedge^k E$, etc.
- A straightforward generalization is the concept of a graded bundle $\tau : F \to M$ with a local trivialization by $U \times \mathbb{R}^n$ as before, and with the difference that the local coordinates $(y^1, \ldots, y^n)$ in the fibres have now associated positive integer weights $w_1, \ldots, w_n$, that are preserved by changes of local trivializations:

$$U \cap V \times \mathbb{R}^n \ni (x, y) \mapsto (x, A(x, y)) \in U \cap V \times \mathbb{R}^n ,$$

- One can show that in this case $A(x, y)$ must be polynomial in fiber coordinates, i.e. any graded bundle is a polynomial bundle.
- As these polynomials need not to be linear, graded bundles do not have, in general, vector space structure in fibers. For instance, if $(y, z) \in \mathbb{R}^2$ are coordinates of degrees $1, 2$, respectively, then the map $(y, z) \mapsto (y, z + y^2)$ is a diffeomorphism preserving the degrees, but it is nonlinear.
- If all $w_i \leq r$, we say that the graded bundle is of degree r.

Graded bundles

- Canonical examples and constructions: $\mathsf{T}M$, T^*M, $E \otimes_M F$, $\wedge^k E$, etc.
- A straightforward generalization is the concept of a graded bundle $\tau : F \to M$ with a local trivialization by $U \times \mathbb{R}^n$ as before, and with the difference that the local coordinates $(y^1, \ldots, y^n)$ in the fibres have now associated positive integer weights $w_1, \ldots, w_n$, that are preserved by changes of local trivializations:

$$U \cap V \times \mathbb{R}^n \ni (x, y) \mapsto (x, A(x, y)) \in U \cap V \times \mathbb{R}^n \,,$$

- One can show that in this case $A(x, y)$ must be polynomial in fiber coordinates, i.e. any graded bundle is a polynomial bundle.
- As these polynomials need not to be linear, graded bundles do not have, in general, vector space structure in fibers. For instance, if $(y, z) \in \mathbb{R}^2$ are coordinates of degrees $1, 2$, respectively, then the map $(y, z) \mapsto (y, z + y^2)$ is a diffeomorphism preserving the degrees, but it is nonlinear.
- If all $w_i \leq r$, we say that the graded bundle is of degree r.

Graded bundles

- Canonical examples and constructions: $\mathsf{T}M$, T^*M, $E\otimes_M F$, $\wedge^k E$, etc.
- A straightforward generalization is the concept of a graded bundle $\tau: F\to M$ with a local trivialization by $U\times\mathbb{R}^n$ as before, and with the difference that the local coordinates $(y^1,\dots,y^n)$ in the fibres have now associated positive integer weights $w_1,\dots,w_n$, that are preserved by changes of local trivializations:
$$U\cap V\times\mathbb{R}^n\ni(x,y)\mapsto(x,A(x,y))\in U\cap V\times\mathbb{R}^n\,,$$
- One can show that in this case $A(x,y)$ must be polynomial in fiber coordinates, i.e. any graded bundle is a polynomial bundle.
- As these polynomials need not to be linear, graded bundles do not have, in general, vector space structure in fibers. For instance, if $(y,z)\in\mathbb{R}^2$ are coordinates of degrees $1,2$, respectively, then the map $(y,z)\mapsto(y,z+y^2)$ is a diffeomorphism preserving the degrees, but it is nonlinear.
- If all $w_i\le r$, we say that the graded bundle is of degree r.

Graded bundles

- Canonical examples and constructions: $\mathsf{T}M$, T^*M, $E \otimes_M F$, $\wedge^k E$, etc.
- A straightforward generalization is the concept of a graded bundle $\tau : F \to M$ with a local trivialization by $U \times \mathbb{R}^n$ as before, and with the difference that the local coordinates $(y^1, \ldots, y^n)$ in the fibres have now associated positive integer weights $w_1, \ldots, w_n$, that are preserved by changes of local trivializations:
$$U \cap V \times \mathbb{R}^n \ni (x, y) \mapsto (x, A(x, y)) \in U \cap V \times \mathbb{R}^n ,$$
- One can show that in this case $A(x, y)$ must be polynomial in fiber coordinates, i.e. any graded bundle is a polynomial bundle.
- As these polynomials need not to be linear, graded bundles do not have, in general, vector space structure in fibers. For instance, if $(y, z) \in \mathbb{R}^2$ are coordinates of degrees $1, 2$, respectively, then the map $(y, z) \mapsto (y, z + y^2)$ is a diffeomorphism preserving the degrees, but it is nonlinear.
- If all $w_i \leq r$, we say that the graded bundle is of degree r.

Graded bundles

- Canonical examples and constructions: $\mathsf{T}M$, T^*M, $E \otimes_M F$, $\wedge^k E$, etc.
- A straightforward generalization is the concept of a graded bundle $\tau : F \to M$ with a local trivialization by $U \times \mathbb{R}^n$ as before, and with the difference that the local coordinates $(y^1, \ldots, y^n)$ in the fibres have now associated positive integer weights $w_1, \ldots, w_n$, that are preserved by changes of local trivializations:

$$U \cap V \times \mathbb{R}^n \ni (x, y) \mapsto (x, A(x, y)) \in U \cap V \times \mathbb{R}^n ,$$

- One can show that in this case $A(x, y)$ must be polynomial in fiber coordinates, i.e. any graded bundle is a polynomial bundle.
- As these polynomials need not to be linear, graded bundles do not have, in general, vector space structure in fibers. For instance, if $(y, z) \in \mathbb{R}^2$ are coordinates of degrees $1, 2$, respectively, then the map $(y, z) \mapsto (y, z + y^2)$ is a diffeomorphism preserving the degrees, but it is nonlinear.
- If all $w_i \leq r$, we say that the graded bundle is of degree r.

Graded bundles

- Canonical examples and constructions: $\mathsf{T}M$, T^*M, $E \otimes_M F$, $\wedge^k E$, etc.
- A straightforward generalization is the concept of a graded bundle $\tau : F \to M$ with a local trivialization by $U \times \mathbb{R}^n$ as before, and with the difference that the local coordinates $(y^1, \ldots, y^n)$ in the fibres have now associated positive integer weights $w_1, \ldots, w_n$, that are preserved by changes of local trivializations:

$$U \cap V \times \mathbb{R}^n \ni (x, y) \mapsto (x, A(x, y)) \in U \cap V \times \mathbb{R}^n,$$

- One can show that in this case $A(x, y)$ must be polynomial in fiber coordinates, i.e. any graded bundle is a polynomial bundle.
- As these polynomials need not to be linear, graded bundles do not have, in general, vector space structure in fibers. For instance, if $(y, z) \in \mathbb{R}^2$ are coordinates of degrees $1, 2$, respectively, then the map $(y, z) \mapsto (y, z + y^2)$ is a diffeomorphism preserving the degrees, but it is nonlinear.
- If all $w_i \leq r$, we say that the graded bundle is of degree r.

Graded bundles

- In the above terminology, vector bundles are just graded bundles of degree 1.
- Graded bundles F_k of degree k admit, like many jet bundles, a tower of affine fibrations by their subbundles of lower degrees

$$F_k \xrightarrow{\tau^k} F_{k-1} \xrightarrow{\tau^{k-1}} \cdots \xrightarrow{\tau^3} F_2 \xrightarrow{\tau^2} F_1 \xrightarrow{\tau^1} F_0 = M\,.$$

- Canonical examples: $\mathrm{T}^k M$, with canonical coordinates $(x, \dot{x}, \ddot{x}, \dddot{x}, \dots)$ of degrees, respectively, $0, 1, 2, 3,$ etc.
- Another example. If $\tau : E \to M$ is a vector bundle, then $\wedge^r \mathrm{T}E$ is canonically a graded bundle of degree r with respect to the projection

$$\wedge^r \mathrm{T}\tau : \wedge^r \mathrm{T}E \to \wedge^r \mathrm{T}M\,.$$

- Note that similar objects has been used in supergeometry by Ševera, Voronov, Roytenberg et al. under the name N-manifolds. However, we will work with classical, purely even manifolds during this talk.

Graded bundles

- In the above terminology, vector bundles are just graded bundles of degree 1.
- Graded bundles F_k of degree k admit, like many jet bundles, a tower of affine fibrations by their subbundles of lower degrees

$$F_k \xrightarrow{\tau^k} F_{k-1} \xrightarrow{\tau^{k-1}} \cdots \xrightarrow{\tau^3} F_2 \xrightarrow{\tau^2} F_1 \xrightarrow{\tau^1} F_0 = M\,.$$

- Canonical examples: $\mathrm{T}^k M$, with canonical coordinates $(x, \dot{x}, \ddot{x}, \dddot{x}, \dots)$ of degrees, respectively, $0, 1, 2, 3,$ etc.
- Another example. If $\tau : E \to M$ is a vector bundle, then $\wedge^r \mathrm{T}E$ is canonically a graded bundle of degree r with respect to the projection
$$\wedge^r \mathrm{T}\tau : \wedge^r \mathrm{T}E \to \wedge^r \mathrm{T}M\,.$$
- Note that similar objects has been used in supergeometry by Ševera, Voronov, Roytenberg et al. under the name N-manifolds. However, we will work with classical, purely even manifolds during this talk.

Graded bundles

- In the above terminology, vector bundles are just graded bundles of degree 1.
- Graded bundles F_k of degree k admit, like many jet bundles, a tower of affine fibrations by their subbundles of lower degrees

$$F_k \xrightarrow{\tau^k} F_{k-1} \xrightarrow{\tau^{k-1}} \cdots \xrightarrow{\tau^3} F_2 \xrightarrow{\tau^2} F_1 \xrightarrow{\tau^1} F_0 = M\,.$$

- Canonical examples: $\mathrm{T}^k M$, with canonical coordinates $(x, \dot{x}, \ddot{x}, \dddot{x}, \dots)$ of degrees, respectively, $0, 1, 2, 3$, etc.
- Another example. If $\tau : E \to M$ is a vector bundle, then $\wedge^r \mathrm{T}E$ is canonically a graded bundle of degree r with respect to the projection

$$\wedge^r \mathrm{T}\tau : \wedge^r \mathrm{T}E \to \wedge^r \mathrm{T}M\,.$$

- Note that similar objects has been used in supergeometry by Ševera, Voronov, Roytenberg et al. under the name N-manifolds. However, we will work with classical, purely even manifolds during this talk.

Graded bundles

- In the above terminology, vector bundles are just graded bundles of degree 1.
- Graded bundles F_k of degree k admit, like many jet bundles, a tower of affine fibrations by their subbundles of lower degrees

$$F_k \xrightarrow{\tau^k} F_{k-1} \xrightarrow{\tau^{k-1}} \cdots \xrightarrow{\tau^3} F_2 \xrightarrow{\tau^2} F_1 \xrightarrow{\tau^1} F_0 = M\,.$$

- Canonical examples: $\mathrm{T}^k M$, with canonical coordinates $(x, \dot{x}, \ddot{x}, \dddot{x}, \dots)$ of degrees, respectively, $0, 1, 2, 3,$ etc.
- Another example. If $\tau : E \to M$ is a vector bundle, then $\wedge^r \mathrm{T}E$ is canonically a graded bundle of degree r with respect to the projection

$$\wedge^r \mathrm{T}\tau : \wedge^r \mathrm{T}E \to \wedge^r \mathrm{T}M\,.$$

- Note that similar objects has been used in supergeometry by Ševera, Voronov, Roytenberg et al. under the name N-manifolds. However, we will work with classical, purely even manifolds during this talk.

Graded bundles

- In the above terminology, vector bundles are just graded bundles of degree 1.
- Graded bundles F_k of degree k admit, like many jet bundles, a tower of affine fibrations by their subbundles of lower degrees

$$F_k \xrightarrow{\tau^k} F_{k-1} \xrightarrow{\tau^{k-1}} \cdots \xrightarrow{\tau^3} F_2 \xrightarrow{\tau^2} F_1 \xrightarrow{\tau^1} F_0 = M\,.$$

- Canonical examples: $\mathrm{T}^k M$, with canonical coordinates $(x, \dot{x}, \ddot{x}, \dddot{x}, \dots)$ of degrees, respectively, $0, 1, 2, 3,$ etc.
- Another example. If $\tau : E \to M$ is a vector bundle, then $\wedge^r \mathrm{T}E$ is canonically a graded bundle of degree r with respect to the projection

$$\wedge^r \mathrm{T}\tau : \wedge^r \mathrm{T}E \to \wedge^r \mathrm{T}M\,.$$

- Note that similar objects has been used in supergeometry by Ševera, Voronov, Roytenberg et al. under the name N-manifolds. However, we will work with classical, purely even manifolds during this talk.

Graded bundles

- In the above terminology, vector bundles are just graded bundles of degree 1.
- Graded bundles F_k of degree k admit, like many jet bundles, a tower of affine fibrations by their subbundles of lower degrees

$$F_k \xrightarrow{\tau^k} F_{k-1} \xrightarrow{\tau^{k-1}} \cdots \xrightarrow{\tau^3} F_2 \xrightarrow{\tau^2} F_1 \xrightarrow{\tau^1} F_0 = M\,.$$

- Canonical examples: $\mathrm{T}^k M$, with canonical coordinates $(x, \dot{x}, \ddot{x}, \dddot{x}, \dots)$ of degrees, respectively, $0, 1, 2, 3,$ etc.
- Another example. If $\tau : E \to M$ is a vector bundle, then $\wedge^r \mathrm{T}E$ is canonically a graded bundle of degree r with respect to the projection

$$\wedge^r \mathrm{T}\tau : \wedge^r \mathrm{T}E \to \wedge^r \mathrm{T}M\,.$$

- Note that similar objects has been used in supergeometry by Ševera, Voronov, Roytenberg et al. under the name N-manifolds. However, we will work with classical, purely even manifolds during this talk.

Graded bundles

- In the above terminology, vector bundles are just graded bundles of degree 1.
- Graded bundles F_k of degree k admit, like many jet bundles, a tower of affine fibrations by their subbundles of lower degrees

$$F_k \xrightarrow{\tau^k} F_{k-1} \xrightarrow{\tau^{k-1}} \cdots \xrightarrow{\tau^3} F_2 \xrightarrow{\tau^2} F_1 \xrightarrow{\tau^1} F_0 = M\,.$$

- Canonical examples: $\mathrm{T}^k M$, with canonical coordinates $(x, \dot{x}, \ddot{x}, \dddot{x}, \dots)$ of degrees, respectively, $0, 1, 2, 3,$ etc.
- Another example. If $\tau : E \to M$ is a vector bundle, then $\wedge^r \mathrm{T}E$ is canonically a graded bundle of degree r with respect to the projection

$$\wedge^r \mathrm{T}\tau : \wedge^r \mathrm{T}E \to \wedge^r \mathrm{T}M\,.$$

- Note that similar objects has been used in supergeometry by Ševera, Voronov, Roytenberg et al. under the name N-manifolds. However, we will work with classical, purely even manifolds during this talk.

Graded Bundles

- With the use of coordinates (x^α, y^a) with degrees 0 for basic coordinates x^α, and degrees $w_a > 0$ for the fibre coordinates y^a, we can define on the graded bundle F a globally defined weight vector field (Euler vector field)
$$\nabla_F = \sum_a w_a y^a \partial_{y^a} .$$
- The flow of the weight vector field extends to a smooth action $\mathbb{R} \ni t \mapsto h_t$ of multiplicative reals on F, $h_t(x^\mu, y^a) = (x^\mu, t^{w_a} y^a)$. Such an action $h : \mathbb{R} \times F \to F$, $h_t \circ h_s = h_{ts}$, we will call a homogeneity structure.
- A function $f : F \to \mathbb{R}$ is called homogeneous of degree (weight) k if $f(h_t(x)) = t^k f(x)$; similarly for the homogeneity of tensor fields.
- Morphisms of two homogeneity structures (F_i, h^i), $i = 1, 2$, are defined as smooth maps $\Phi : F_1 \to F_2$ intertwining the $\mathbb{R}$-actions: $\Phi \circ h^1_t = h^2_t \circ \Phi$. Consequently, a homogeneity substructure is a smooth submanifold S invariant with respect to h, $h_t(S) \subset S$.

Graded Bundles

- With the use of coordinates (x^α, y^a) with degrees 0 for basic coordinates x^α, and degrees $w_a > 0$ for the fibre coordinates y^a, we can define on the graded bundle F a globally defined weight vector field (Euler vector field)
$$\nabla_F = \sum_a w_a y^a \partial_{y^a}\,.$$
- The flow of the weight vector field extends to a smooth action $\mathbb{R} \ni t \mapsto h_t$ of multiplicative reals on F, $h_t(x^\mu, y^a) = (x^\mu, t^{w_a} y^a)$. Such an action $h : \mathbb{R} \times F \to F$, $h_t \circ h_s = h_{ts}$, we will call a homogeneity structure.
- A function $f : F \to \mathbb{R}$ is called homogeneous of degree (weight) k if $f(h_t(x)) = t^k f(x)$; similarly for the homogeneity of tensor fields.
- Morphisms of two homogeneity structures (F_i, h^i), $i = 1, 2$, are defined as smooth maps $\Phi : F_1 \to F_2$ intertwining the $\mathbb{R}$-actions: $\Phi \circ h^1_t = h^2_t \circ \Phi$. Consequently, a homogeneity substructure is a smooth submanifold S invariant with respect to h, $h_t(S) \subset S$.

Graded Bundles

- With the use of coordinates (x^α, y^a) with degrees 0 for basic coordinates x^α, and degrees $w_a > 0$ for the fibre coordinates y^a, we can define on the graded bundle F a globally defined weight vector field (Euler vector field)

$$\nabla_F = \sum_a w_a y^a \partial_{y^a}\,.$$

- The flow of the weight vector field extends to a smooth action $\mathbb{R} \ni t \mapsto h_t$ of multiplicative reals on F, $h_t(x^\mu, y^a) = (x^\mu, t^{w_a} y^a)$. Such an action $h : \mathbb{R} \times F \to F$, $h_t \circ h_s = h_{ts}$, we will call a homogeneity structure.
- A function $f : F \to \mathbb{R}$ is called homogeneous of degree (weight) k if $f(h_t(x)) = t^k f(x)$; similarly for the homogeneity of tensor fields.
- Morphisms of two homogeneity structures (F_i, h^i), $i = 1, 2$, are defined as smooth maps $\Phi : F_1 \to F_2$ intertwining the $\mathbb{R}$-actions: $\Phi \circ h^1_t = h^2_t \circ \Phi$. Consequently, a homogeneity substructure is a smooth submanifold S invariant with respect to h, $h_t(S) \subset S$.

Graded Bundles

- With the use of coordinates (x^α, y^a) with degrees 0 for basic coordinates x^α, and degrees $w_a > 0$ for the fibre coordinates y^a, we can define on the graded bundle F a globally defined weight vector field (Euler vector field)

$$\nabla_F = \sum_a w_a y^a \partial_{y^a} .$$

- The flow of the weight vector field extends to a smooth action $\mathbb{R} \ni t \mapsto h_t$ of multiplicative reals on F, $h_t(x^\mu, y^a) = (x^\mu, t^{w_a} y^a)$. Such an action $h : \mathbb{R} \times F \to F$, $h_t \circ h_s = h_{ts}$, we will call a homogeneity structure.
- A function $f : F \to \mathbb{R}$ is called homogeneous of degree (weight) k if $f(h_t(x)) = t^k f(x)$; similarly for the homogeneity of tensor fields.
- Morphisms of two homogeneity structures (F_i, h^i), $i = 1, 2$, are defined as smooth maps $\Phi : F_1 \to F_2$ intertwining the $\mathbb{R}$-actions: $\Phi \circ h^1_t = h^2_t \circ \Phi$. Consequently, a homogeneity substructure is a smooth submanifold S invariant with respect to h, $h_t(S) \subset S$.

Graded Bundles

- With the use of coordinates (x^α, y^a) with degrees 0 for basic coordinates x^α, and degrees $w_a > 0$ for the fibre coordinates y^a, we can define on the graded bundle F a globally defined weight vector field (Euler vector field)

$$\nabla_F = \sum_a w_a y^a \partial_{y^a} .$$

- The flow of the weight vector field extends to a smooth action $\mathbb{R} \ni t \mapsto h_t$ of multiplicative reals on F, $h_t(x^\mu, y^a) = (x^\mu, t^{w_a} y^a)$. Such an action $h : \mathbb{R} \times F \to F$, $h_t \circ h_s = h_{ts}$, we will call a homogeneity structure.
- A function $f : F \to \mathbb{R}$ is called homogeneous of degree (weight) k if $f(h_t(x)) = t^k f(x)$; similarly for the homogeneity of tensor fields.
- Morphisms of two homogeneity structures (F_i, h^i), $i = 1, 2$, are defined as smooth maps $\Phi : F_1 \to F_2$ intertwining the $\mathbb{R}$-actions: $\Phi \circ h^1_t = h^2_t \circ \Phi$. Consequently, a homogeneity substructure is a smooth submanifold S invariant with respect to h, $h_t(S) \subset S$.

Graded Bundles

- With the use of coordinates (x^α, y^a) with degrees 0 for basic coordinates x^α, and degrees $w_a > 0$ for the fibre coordinates y^a, we can define on the graded bundle F a globally defined weight vector field (Euler vector field)
$$\nabla_F = \sum_a w_a y^a \partial_{y^a} \,.$$
- The flow of the weight vector field extends to a smooth action $\mathbb{R} \ni t \mapsto h_t$ of multiplicative reals on F, $h_t(x^\mu, y^a) = (x^\mu, t^{w_a} y^a)$. Such an action $h : \mathbb{R} \times F \to F$, $h_t \circ h_s = h_{ts}$, we will call a homogeneity structure.
- A function $f : F \to \mathbb{R}$ is called homogeneous of degree (weight) k if $f(h_t(x)) = t^k f(x)$; similarly for the homogeneity of tensor fields.
- Morphisms of two homogeneity structures (F_i, h^i), $i = 1, 2$, are defined as smooth maps $\Phi : F_1 \to F_2$ intertwining the $\mathbb{R}$-actions: $\Phi \circ h^1_t = h^2_t \circ \Phi$. Consequently, a homogeneity substructure is a smooth submanifold S invariant with respect to h, $h_t(S) \subset S$.

Graded Bundles

- With the use of coordinates (x^α, y^a) with degrees 0 for basic coordinates x^α, and degrees $w_a > 0$ for the fibre coordinates y^a, we can define on the graded bundle F a globally defined weight vector field (Euler vector field)

$$\nabla_F = \sum_a w_a y^a \partial_{y^a} .$$

- The flow of the weight vector field extends to a smooth action $\mathbb{R} \ni t \mapsto h_t$ of multiplicative reals on F, $h_t(x^\mu, y^a) = (x^\mu, t^{w_a} y^a)$. Such an action $h : \mathbb{R} \times F \to F$, $h_t \circ h_s = h_{ts}$, we will call a homogeneity structure.
- A function $f : F \to \mathbb{R}$ is called homogeneous of degree (weight) k if $f(h_t(x)) = t^k f(x)$; similarly for the homogeneity of tensor fields.
- Morphisms of two homogeneity structures (F_i, h^i), $i = 1, 2$, are defined as smooth maps $\Phi : F_1 \to F_2$ intertwining the $\mathbb{R}$-actions: $\Phi \circ h^1_t = h^2_t \circ \Phi$. Consequently, a homogeneity substructure is a smooth submanifold S invariant with respect to h, $h_t(S) \subset S$.

Double Graded Bundles

The fundamental fact (cf. [Grabowski-Rotkiewicz]) says that graded bundles and homogeneity structures are in fact equivalent concepts.

Theorem

For any homogeneity structure h on a manifold F, there is a smooth submanifold $M = h_0(F) \subset F$, a non-negative integer $k \in \mathbb{N}$, and an $\mathbb{R}$-equivariant map $\Phi_h^k : F \to \mathrm{T}^k F_{|M}$ which identifies F with a graded submanifold of the graded bundle $\mathrm{T}^k F$. In particular, there is an atlas on F consisting of local homogeneous functions.

As two graded bundle structure on the same manifold are just two homogeneity structures, the obvious concept of compatibility leads to the following: A double graded bundle is a manifold equipped with two homogeneity structures h^1, h^2 which are compatible in the sense that

$$h^1_t \circ h^2_s = h^2_s \circ h^1_t \quad \text{for all } s, t \in \mathbb{R}\,.$$

This covers of course the concept of a double vector bundle of Pradines and Mackenzie, and extends to n-tuple graded bundles in the obvious way.

Double Graded Bundles

The fundamental fact (cf. [Grabowski-Rotkiewicz]) says that graded bundles and homogeneity structures are in fact equivalent concepts.

Theorem

For any homogeneity structure h on a manifold F, there is a smooth submanifold $M = h_0(F) \subset F$, a non-negative integer $k \in \mathbb{N}$, and an $\mathbb{R}$-equivariant map $\Phi_h^k : F \to \mathrm{T}^k F_{|M}$ which identifies F with a graded submanifold of the graded bundle $\mathrm{T}^k F$. In particular, there is an atlas on F consisting of local homogeneous functions.

As two graded bundle structure on the same manifold are just two homogeneity structures, the obvious concept of compatibility leads to the following: A double graded bundle is a manifold equipped with two homogeneity structures h^1, h^2 which are compatible in the sense that

$$h_t^1 \circ h_s^2 = h_s^2 \circ h_t^1 \quad \text{for all } s, t \in \mathbb{R}.$$

This covers of course the concept of a double vector bundle of Pradines and Mackenzie, and extends to n-tuple graded bundles in the obvious way.

Double Graded Bundles

The fundamental fact (cf. [Grabowski-Rotkiewicz]) says that graded bundles and homogeneity structures are in fact equivalent concepts.

Theorem

For any homogeneity structure h on a manifold F, there is a smooth submanifold $M = h_0(F) \subset F$, a non-negative integer $k \in \mathbb{N}$, and an $\mathbb{R}$-equivariant map $\Phi_h^k : F \to \mathrm{T}^k F_{|M}$ which identifies F with a graded submanifold of the graded bundle $\mathrm{T}^k F$. In particular, there is an atlas on F consisting of local homogeneous functions.

As two graded bundle structure on the same manifold are just two homogeneity structures, the obvious concept of compatibility leads to the following: A double graded bundle is a manifold equipped with two homogeneity structures h^1, h^2 which are compatible in the sense that

$$h^1_t \circ h^2_s = h^2_s \circ h^1_t \quad \text{for all } s, t \in \mathbb{R}\,.$$

This covers of course the concept of a double vector bundle of Pradines and Mackenzie, and extends to n-tuple graded bundles in the obvious way.

Double Graded Bundles

The fundamental fact (cf. [Grabowski-Rotkiewicz]) says that graded bundles and homogeneity structures are in fact equivalent concepts.

Theorem

For any homogeneity structure h on a manifold F, there is a smooth submanifold $M = h_0(F) \subset F$, a non-negative integer $k \in \mathbb{N}$, and an $\mathbb{R}$-equivariant map $\Phi_h^k : F \to \mathrm{T}^k F_{|M}$ which identifies F with a graded submanifold of the graded bundle $\mathrm{T}^k F$. In particular, there is an atlas on F consisting of local homogeneous functions.

As two graded bundle structure on the same manifold are just two homogeneity structures, the obvious concept of compatibility leads to the following: A double graded bundle is a manifold equipped with two homogeneity structures h^1, h^2 which are compatible in the sense that

$$h_t^1 \circ h_s^2 = h_s^2 \circ h_t^1 \quad \text{for all } s, t \in \mathbb{R}.$$

This covers of course the concept of a double vector bundle of Pradines and Mackenzie, and extends to n-tuple graded bundles in the obvious way.

Double Graded Bundles

The fundamental fact (cf. [Grabowski-Rotkiewicz]) says that graded bundles and homogeneity structures are in fact equivalent concepts.

Theorem

For any homogeneity structure h on a manifold F, there is a smooth submanifold $M = h_0(F) \subset F$, a non-negative integer $k \in \mathbb{N}$, and an $\mathbb{R}$-equivariant map $\Phi_h^k : F \to \mathrm{T}^k F_{|M}$ which identifies F with a graded submanifold of the graded bundle $\mathrm{T}^k F$. In particular, there is an atlas on F consisting of local homogeneous functions.

As two graded bundle structure on the same manifold are just two homogeneity structures, the obvious concept of compatibility leads to the following: A double graded bundle is a manifold equipped with two homogeneity structures h^1, h^2 which are compatible in the sense that

$$h^1_t \circ h^2_s = h^2_s \circ h^1_t \quad \text{for all } s, t \in \mathbb{R}\,.$$

This covers of course the concept of a double vector bundle of Pradines and Mackenzie, and extends to n-tuple graded bundles in the obvious way.

Double Graded Bundles

The fundamental fact (cf. [Grabowski-Rotkiewicz]) says that graded bundles and homogeneity structures are in fact equivalent concepts.

Theorem

For any homogeneity structure h on a manifold F, there is a smooth submanifold $M = h_0(F) \subset F$, a non-negative integer $k \in \mathbb{N}$, and an $\mathbb{R}$-equivariant map $\Phi_h^k : F \to \mathrm{T}^k F_{|M}$ which identifies F with a graded submanifold of the graded bundle $\mathrm{T}^k F$. In particular, there is an atlas on F consisting of local homogeneous functions.

As two graded bundle structure on the same manifold are just two homogeneity structures, the obvious concept of compatibility leads to the following: A double graded bundle is a manifold equipped with two homogeneity structures h^1, h^2 which are compatible in the sense that

$$h_t^1 \circ h_s^2 = h_s^2 \circ h_t^1 \quad \text{for all } s, t \in \mathbb{R} .$$

This covers of course the concept of a double vector bundle of Pradines and Mackenzie, and extends to n-tuple graded bundles in the obvious way.

Double graded bundles - examples

- Lifts. If $\tau : F \to M$ is a graded bundle of degree k, then $\mathsf{T}F$ and T^*F carry canonical double graded bundle structure: one is the obvious vector bundle, the other is of degree k. A double graded bundle whose one structure is linear we will call a $\mathcal{GL}$-bundle. There are also lifts of graded structures on F to T^rF.
- In particular, if $\tau : E \to M$ is a vector bundle, then $\mathsf{T}E$ and T^*E are double vector bundles. The latter is isomorphic with T^*E^*.
  As a linear Poisson structure on E^* yields a map $\mathsf{T}^*E^* \to \mathsf{T}E^*$, a Lie algebroid structure on E can be encoded as a morphism of double vector bundles, $\varepsilon : \mathsf{T}^*E \to \mathsf{T}E^*$ (!)
- If $\tau : E \to M$ is a vector bundle, then $\wedge^k\mathsf{T}E$ is canonically a $\mathcal{GL}$-bundle:

$$\begin{array}{ccccc} & & \wedge^k\mathsf{T}E & & \\ & \swarrow & & \searrow & \\ E & & & & \wedge^k\mathsf{T}M \\ & \searrow & & \swarrow & \\ & & M & & \end{array}\quad .$$

Double graded bundles - examples

- **Lifts.** If $\tau : F \to M$ is a graded bundle of degree k, then $\mathsf{T}F$ and T^*F carry canonical double graded bundle structure: one is the obvious vector bundle, the other is of degree k. A double graded bundle whose one structure is linear we will call a $\mathcal{GL}$-bundle. There are also lifts of graded structures on F to T^rF.
- In particular, if $\tau : E \to M$ is a vector bundle, then $\mathsf{T}E$ and T^*E are double vector bundles. The latter is isomorphic with T^*E^*. As a linear Poisson structure on E^* yields a map $\mathsf{T}^*E^* \to \mathsf{T}E^*$, a Lie algebroid structure on E can be encoded as a morphism of double vector bundles, $\varepsilon : \mathsf{T}^*E \to \mathsf{T}E^*$ (!)
- If $\tau : E \to M$ is a vector bundle, then $\wedge^k\mathsf{T}E$ is canonically a $\mathcal{GL}$-bundle:

$$\begin{array}{ccccc} & & \wedge^k\mathsf{T}E & & \\ & \swarrow & & \searrow & \\ E & & & & \wedge^k\mathsf{T}M \\ & \searrow & & \swarrow & \\ & & M & & \end{array} .$$

Double graded bundles - examples

- **Lifts.** If $\tau : F \to M$ is a graded bundle of degree k, then $\mathrm{T}F$ and T^*F carry canonical double graded bundle structure: one is the obvious vector bundle, the other is of degree k. A double graded bundle whose one structure is linear we will call a $\mathcal{GL}$-bundle. There are also lifts of graded structures on F to T^rF.
- In particular, if $\tau : E \to M$ is a vector bundle, then $\mathrm{T}E$ and T^*E are double vector bundles. The latter is isomorphic with T^*E^*. As a linear Poisson structure on E^* yields a map $\mathrm{T}^*E^* \to \mathrm{T}E^*$, a Lie algebroid structure on E can be encoded as a morphism of double vector bundles, $\varepsilon : \mathrm{T}^*E \to \mathrm{T}E^*$ (!)
- If $\tau : E \to M$ is a vector bundle, then $\wedge^k\mathrm{T}E$ is canonically a $\mathcal{GL}$-bundle:

$$\begin{array}{ccccc} & & \wedge^k\mathrm{T}E & & \\ & \swarrow & & \searrow & \\ E & & & & \wedge^k\mathrm{T}M \\ & \searrow & & \swarrow & \\ & & M & & \end{array}.$$

Double graded bundles - examples

- Lifts. If $\tau : F \to M$ is a graded bundle of degree k, then $\mathsf{T}F$ and T^*F carry canonical double graded bundle structure: one is the obvious vector bundle, the other is of degree k. A double graded bundle whose one structure is linear we will call a $\mathcal{GL}$-bundle. There are also lifts of graded structures on F to $\mathsf{T}^r F$.
- In particular, if $\tau : E \to M$ is a vector bundle, then $\mathsf{T}E$ and T^*E are double vector bundles. The latter is isomorphic with T^*E^*. As a linear Poisson structure on E^* yields a map $\mathsf{T}^*E^* \to \mathsf{T}E^*$, a Lie algebroid structure on E can be encoded as a morphism of double vector bundles, $\varepsilon : \mathsf{T}^*E \to \mathsf{T}E^*$ (!)
- If $\tau : E \to M$ is a vector bundle, then $\wedge^k\mathsf{T}E$ is canonically a $\mathcal{GL}$-bundle:

$$\begin{array}{ccccc} & & \wedge^k\mathsf{T}E & & \\ & \swarrow & & \searrow & \\ E & & & & \wedge^k\mathsf{T}M \\ & \searrow & & \swarrow & \\ & & M & & \end{array}.$$

Double graded bundles - examples

- Lifts. If $\tau : F \to M$ is a graded bundle of degree k, then $\mathsf{T}F$ and T^*F carry canonical double graded bundle structure: one is the obvious vector bundle, the other is of degree k. A double graded bundle whose one structure is linear we will call a $\mathcal{GL}$-bundle. There are also lifts of graded structures on F to $\mathsf{T}^r F$.
- In particular, if $\tau : E \to M$ is a vector bundle, then $\mathsf{T}E$ and T^*E are double vector bundles. The latter is isomorphic with T^*E^*.
  As a linear Poisson structure on E^* yields a map $\mathsf{T}^*E^* \to \mathsf{T}E^*$, a Lie algebroid structure on E can be encoded as a morphism of double vector bundles, $\varepsilon : \mathsf{T}^*E \to \mathsf{T}E^*$ (!)
- If $\tau : E \to M$ is a vector bundle, then $\wedge^k \mathsf{T}E$ is canonically a $\mathcal{GL}$-bundle:

Double graded bundles - examples

- Lifts. If $\tau : F \to M$ is a graded bundle of degree k, then $\mathsf{T}F$ and T^*F carry canonical double graded bundle structure: one is the obvious vector bundle, the other is of degree k. A double graded bundle whose one structure is linear we will call a $\mathcal{GL}$-bundle. There are also lifts of graded structures on F to $\mathsf{T}^r F$.
- In particular, if $\tau : E \to M$ is a vector bundle, then $\mathsf{T}E$ and T^*E are double vector bundles. The latter is isomorphic with T^*E^*.
  As a linear Poisson structure on E^* yields a map $\mathsf{T}^*E^* \to \mathsf{T}E^*$, a Lie algebroid structure on E can be encoded as a morphism of double vector bundles, $\varepsilon : \mathsf{T}^*E \to \mathsf{T}E^*$ (!)
- If $\tau : E \to M$ is a vector bundle, then $\wedge^k \mathsf{T}E$ is canonically a $\mathcal{GL}$-bundle:

$$
\begin{array}{ccccc}
 & & \wedge^k\mathsf{T}E & & \\
 & \swarrow & & \searrow & \\
E & & & & \wedge^k\mathsf{T}M \\
 & \searrow & & \swarrow & \\
 & & M & &
\end{array}
.
$$

Double graded bundles - examples

- Lifts. If $\tau : F \to M$ is a graded bundle of degree k, then $\mathrm{T}F$ and T^*F carry canonical double graded bundle structure: one is the obvious vector bundle, the other is of degree k. A double graded bundle whose one structure is linear we will call a $\mathcal{GL}$-bundle. There are also lifts of graded structures on F to T^rF.
- In particular, if $\tau : E \to M$ is a vector bundle, then $\mathrm{T}E$ and T^*E are double vector bundles. The latter is isomorphic with T^*E^*.
  As a linear Poisson structure on E^* yields a map $\mathrm{T}^*E^* \to \mathrm{T}E^*$, a Lie algebroid structure on E can be encoded as a morphism of double vector bundles, $\varepsilon : \mathrm{T}^*E \to \mathrm{T}E^*$ (!)
- If $\tau : E \to M$ is a vector bundle, then $\wedge^k\mathrm{T}E$ is canonically a $\mathcal{GL}$-bundle:

$$\begin{array}{ccc} & \wedge^k\mathrm{T}E & \\ \swarrow & & \searrow \\ E & & \wedge^k\mathrm{T}M \\ \searrow & & \swarrow \\ & M & \end{array}.$$

Double graded bundles - examples

- Lifts. If $\tau : F \to M$ is a graded bundle of degree k, then $\mathsf{T}F$ and T^*F carry canonical double graded bundle structure: one is the obvious vector bundle, the other is of degree k. A double graded bundle whose one structure is linear we will call a $\mathcal{GL}$-bundle. There are also lifts of graded structures on F to $\mathsf{T}^r F$.
- In particular, if $\tau : E \to M$ is a vector bundle, then $\mathsf{T}E$ and T^*E are double vector bundles. The latter is isomorphic with T^*E^*.
  As a linear Poisson structure on E^* yields a map $\mathsf{T}^*E^* \to \mathsf{T}E^*$, a Lie algebroid structure on E can be encoded as a morphism of double vector bundles, $\varepsilon : \mathsf{T}^*E \to \mathsf{T}E^*$ (!)
- If $\tau : E \to M$ is a vector bundle, then $\wedge^k \mathsf{T}E$ is canonically a $\mathcal{GL}$-bundle:

Double graded bundles - examples

- Lifts. If $\tau : F \to M$ is a graded bundle of degree k, then $\mathsf{T}F$ and T^*F carry canonical double graded bundle structure: one is the obvious vector bundle, the other is of degree k. A double graded bundle whose one structure is linear we will call a $\mathcal{GL}$-bundle. There are also lifts of graded structures on F to $\mathsf{T}^r F$.
- In particular, if $\tau : E \to M$ is a vector bundle, then $\mathsf{T}E$ and T^*E are double vector bundles. The latter is isomorphic with T^*E^*.
  As a linear Poisson structure on E^* yields a map $\mathsf{T}^*E^* \to \mathsf{T}E^*$, a Lie algebroid structure on E can be encoded as a morphism of double vector bundles, $\varepsilon : \mathsf{T}^*E \to \mathsf{T}E^*$ (!)
- If $\tau : E \to M$ is a vector bundle, then $\wedge^k \mathsf{T}E$ is canonically a $\mathcal{GL}$-bundle:

$$\begin{array}{ccccc} & & \wedge^k \mathsf{T}E & & \\ & \swarrow & & \searrow & \\ E & & & & \wedge^k \mathsf{T}M \\ & \searrow & & \swarrow & \\ & & M & & \end{array} .$$

The Tulczyjew triple - Lagrangian side

M - positions,
$\mathsf{T}M$ - (kinematic) configurations,
$L : \mathsf{T}M \to \mathbb{R}$ - Lagrangian
T^*M - phase space

$$\mathcal{D} = \alpha_M^{-1}(\mathrm{d}L(\mathsf{T}M)) = \mathcal{T}L(\mathsf{T}M)\,,$$

the image of the Tulczyjew differential $\mathcal{T}L$, is the phase dynamics,

$$\mathcal{D} = \left\{ (x, p, \dot{x}, \dot{p}) : \quad p = \frac{\partial L}{\partial \dot{x}}, \quad \dot{p} = \frac{\partial L}{\partial x} \right\},$$

whence the Euler-Lagrange equation: $\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)$.

We have also the Legendre map: $\lambda_L : \mathsf{T}M \to \mathsf{T}^*M, \ \lambda_L(x, \dot{x}) = (x, \frac{\partial L}{\partial \dot{x}})$.

The Tulczyjew triple - Lagrangian side

M - positions,
$\mathsf{T}M$ - (kinematic) configurations,
$L : \mathsf{T}M \to \mathbb{R}$ - Lagrangian
T^*M - phase space

$$\mathcal{D} = \alpha_M^{-1}(\mathrm{d}L(\mathsf{T}M)) = \mathcal{T}L(\mathsf{T}M)\,,$$

the image of the Tulczyjew differential $\mathcal{T}L$, is the phase dynamics,

$$\mathcal{D} = \left\{(x, p, \dot{x}, \dot{p}) : \quad p = \frac{\partial L}{\partial \dot{x}}, \quad \dot{p} = \frac{\partial L}{\partial x}\right\}\,,$$

whence the Euler-Lagrange equation: $\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)$.

We have also the Legendre map: $\lambda_L : \mathsf{T}M \to \mathsf{T}^*M, \quad \lambda_L(x, \dot{x}) = (x, \frac{\partial L}{\partial \dot{x}})$.

The Tulczyjew triple - Lagrangian side

M - positions,
$\mathsf{T}M$ - (kinematic) configurations,
$L : \mathsf{T}M \to \mathbb{R}$ - Lagrangian
T^*M - phase space

$$\mathcal{D} = \alpha_M^{-1}(\mathrm{d}L(\mathsf{T}M)) = \mathcal{T}L(\mathsf{T}M)\,,$$

the image of the Tulczyjew differential $\mathcal{T}L$, is the phase dynamics,

$$\mathcal{D} = \left\{(x, p, \dot{x}, \dot{p}) : \quad p = \frac{\partial L}{\partial \dot{x}}, \quad \dot{p} = \frac{\partial L}{\partial x}\right\}\,,$$

whence the Euler-Lagrange equation: $\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)$.

We have also the Legendre map: $\lambda_L : \mathsf{T}M \to \mathsf{T}^*M, \quad \lambda_L(x, \dot{x}) = \left(x, \frac{\partial L}{\partial \dot{x}}\right)$.

The Tulczyjew triple - Lagrangian side

M - positions,
$\mathsf{T}M$ - (kinematic) configurations,
$L:\mathsf{T}M\to\mathbb{R}$ - Lagrangian
T^*M - phase space

$$\mathcal{D}=\alpha_M^{-1}(\mathrm{d}L(\mathsf{T}M))=\mathcal{T}L(\mathsf{T}M)\,,$$

the image of the Tulczyjew differential $\mathcal{T}L$, is the phase dynamics,

$$\mathcal{D}=\left\{(x,p,\dot{x},\dot{p}):\quad p=\frac{\partial L}{\partial \dot{x}},\quad \dot{p}=\frac{\partial L}{\partial x}\right\}\,,$$

whence the Euler-Lagrange equation: $\frac{\partial L}{\partial x}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)$.

We have also the Legendre map: $\lambda_L:\mathsf{T}M\to\mathsf{T}^*M,\quad \lambda_L(x,\dot{x})=\left(x,\frac{\partial L}{\partial \dot{x}}\right)$.

The Tulczyjew triple - Lagrangian side

M - positions,
$\mathsf{T}M$ - (kinematic) configurations,
$L : \mathsf{T}M \to \mathbb{R}$ - Lagrangian
T^*M - phase space

$$\mathcal{D} = \alpha_M^{-1}(\mathrm{d}L(\mathsf{T}M)) = \mathcal{T}L(\mathsf{T}M)\,,$$

the image of the Tulczyjew differential $\mathcal{T}L$, is the phase dynamics,

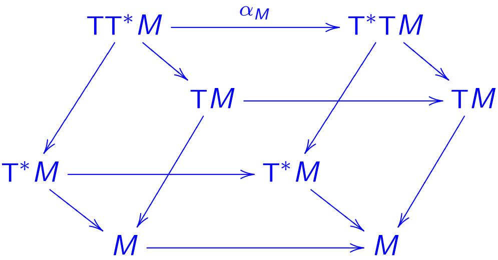
$$\mathcal{D} = \left\{ (x, p, \dot{x}, \dot{p}) : \quad p = \frac{\partial L}{\partial \dot{x}}, \quad \dot{p} = \frac{\partial L}{\partial x} \right\},$$

whence the Euler-Lagrange equation: $\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)$.

We have also the Legendre map: $\lambda_L : \mathsf{T}M \to \mathsf{T}^*M$, $\lambda_L(x, \dot{x}) = \left(x, \frac{\partial L}{\partial \dot{x}}\right)$.

The Tulczyjew triple - Lagrangian side

M - positions,
$\mathsf{T}M$ - (kinematic) configurations,
$L : \mathsf{T}M \to \mathbb{R}$ - Lagrangian
T^*M - phase space



$$\mathcal{D} = \alpha_M^{-1}(\mathrm{d}L(\mathsf{T}M)) = \mathcal{T}L(\mathsf{T}M)\,,$$

the image of the Tulczyjew differential $\mathcal{T}L$, is the phase dynamics,

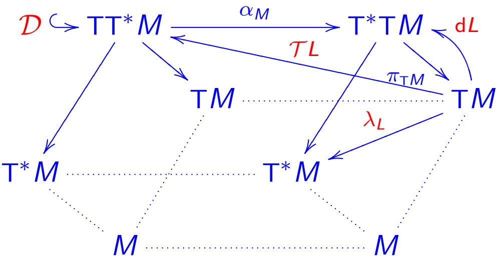
$$\mathcal{D} = \left\{(x,p,\dot{x},\dot{p}):\quad p = \frac{\partial L}{\partial \dot{x}},\quad \dot{p} = \frac{\partial L}{\partial x}\right\}\,,$$

whence the Euler-Lagrange equation: $\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)$.

We have also the Legendre map: $\lambda_L : \mathsf{T}M \to \mathsf{T}^*M,\ \lambda_L(x,\dot{x}) = (x, \frac{\partial L}{\partial \dot{x}})\,.$

The Tulczyjew triple - Lagrangian side

M - positions,
$\mathsf{T}M$ - (kinematic) configurations,
$L : \mathsf{T}M \to \mathbb{R}$ - Lagrangian
T^*M - phase space



$$\mathcal{D} = \alpha_M^{-1}(\mathrm{d}L(\mathsf{T}M)) = \mathcal{T}L(\mathsf{T}M)\,,$$

the image of the Tulczyjew differential $\mathcal{T}L$, is the phase dynamics,

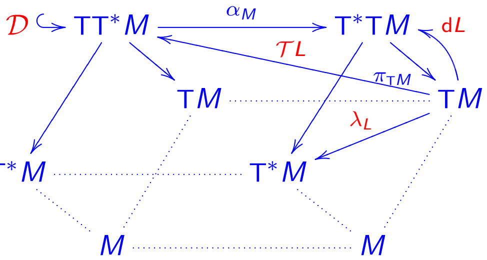
$$\mathcal{D} = \left\{ (x,p,\dot{x},\dot{p}) : \quad p = \frac{\partial L}{\partial \dot{x}}, \quad \dot{p} = \frac{\partial L}{\partial x} \right\},$$

whence the Euler-Lagrange equation: $\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)$.

We have also the Legendre map: $\quad \lambda_L : \mathsf{T}M \to \mathsf{T}^*M, \quad \lambda_L(x,\dot{x}) = (x, \frac{\partial L}{\partial \dot{x}})\,.$

The Tulczyjew triple - Lagrangian side

M - positions,
$\mathsf{T}M$ - (kinematic) configurations,
$L : \mathsf{T}M \to \mathbb{R}$ - Lagrangian
T^*M - phase space



$$\mathcal{D} = \alpha_M^{-1}(\mathrm{d}L(\mathsf{T}M)) = \mathcal{T}L(\mathsf{T}M)\,,$$

the image of the Tulczyjew differential $\mathcal{T}L$, is the phase dynamics,

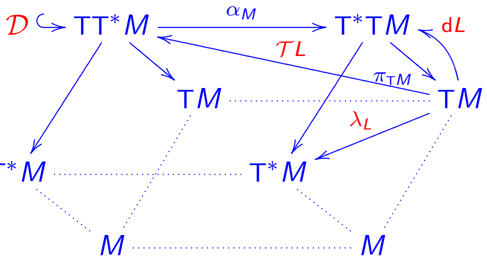
$$\mathcal{D} = \left\{(x,p,\dot{x},\dot{p}) : \quad p = \frac{\partial L}{\partial \dot{x}}, \quad \dot{p} = \frac{\partial L}{\partial x}\right\}\,,$$

whence the Euler-Lagrange equation: $\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)$.

We have also the Legendre map: $\lambda_L : \mathsf{T}M \to \mathsf{T}^*M,\ \ \lambda_L(x,\dot{x}) = \left(x, \frac{\partial L}{\partial \dot{x}}\right)$.

The Tulczyjew triple - Lagrangian side

M - positions,
$\mathsf{T}M$ - (kinematic) configurations,
$L : \mathsf{T}M \to \mathbb{R}$ - Lagrangian
T^*M - phase space



$$\mathcal{D} = \alpha_M^{-1}(\mathrm{d}L(\mathsf{T}M)) = \mathcal{T}L(\mathsf{T}M)\,,$$

the image of the Tulczyjew differential $\mathcal{T}L$, is the phase dynamics,

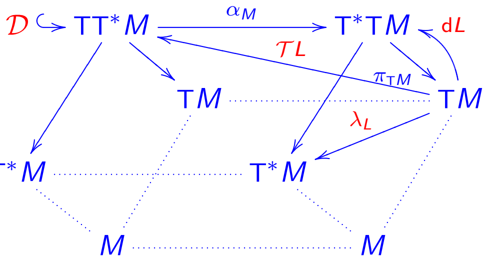
$$\mathcal{D} = \left\{(x, p, \dot{x}, \dot{p}) : \quad p = \frac{\partial L}{\partial \dot{x}}, \quad \dot{p} = \frac{\partial L}{\partial x}\right\}\,,$$

whence the Euler-Lagrange equation: $\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)$.

We have also the Legendre map: $\lambda_L : \mathsf{T}M \to \mathsf{T}^*M, \;\; \lambda_L(x, \dot{x}) = (x, \frac{\partial L}{\partial \dot{x}})\,.$

The Tulczyjew triple - Lagrangian side

M - positions,
$\mathsf{T}M$ - (kinematic) configurations,
$L:\mathsf{T}M\to\mathbb{R}$ - Lagrangian
T^*M - phase space



$$\mathcal{D}=\alpha_M^{-1}(\mathrm{d}L(\mathsf{T}M))=\mathcal{T}L(\mathsf{T}M)\,,$$

the image of the Tulczyjew differential $\mathcal{T}L$, is the phase dynamics,

$$\mathcal{D}=\left\{(x,p,\dot{x},\dot{p}):\ \ p=\frac{\partial L}{\partial\dot{x}},\quad \dot{p}=\frac{\partial L}{\partial x}\right\}\,,$$

whence the Euler-Lagrange equation: $\frac{\partial L}{\partial x}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{x}}\right)$.

We have also the Legendre map: $\lambda_L:\mathsf{T}M\to\mathsf{T}^*M,\ \ \lambda_L(x,\dot{x})=(x,\frac{\partial L}{\partial\dot{x}})\,.$

The Tulczyjew triple - Hamiltonian side

$H : \mathrm{T}^*M \to \mathbb{R}$

$$\mathcal{D} = \beta_M^{-1}(\mathrm{d}H(\mathrm{T}^*M))$$

$$\mathcal{D} = \left\{ (x, p, \dot{x}, \dot{p}) : \quad \dot{p} = -\frac{\partial H}{\partial x}, \quad \dot{x} = \frac{\partial H}{\partial p} \right\},$$

whence the Hamilton equations.

The Tulczyjew triple - Hamiltonian side

$H : \mathsf{T}^*M \to \mathbb{R}$

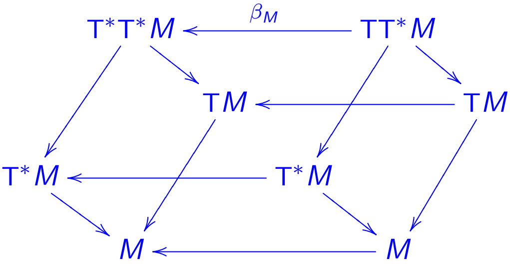
$$\mathcal{D} = \beta_M^{-1}(\mathrm{d}H(\mathsf{T}^*M))$$

$$\mathcal{D} = \left\{(x, p, \dot{x}, \dot{p}) : \quad \dot{p} = -\frac{\partial H}{\partial x}, \quad \dot{x} = \frac{\partial H}{\partial p}\right\},$$

whence the Hamilton equations.

The Tulczyjew triple - Hamiltonian side

$H : \mathsf{T}^*M \to \mathbb{R}$



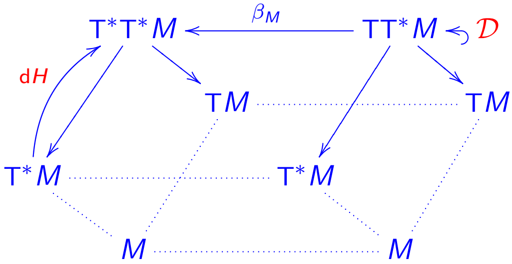
$$\mathcal{D} = \beta_M^{-1}(\mathrm{d}H(\mathsf{T}^*M))$$

$$\mathcal{D} = \left\{(x, p, \dot{x}, \dot{p}) : \quad \dot{p} = -\frac{\partial H}{\partial x}, \quad \dot{x} = \frac{\partial H}{\partial p}\right\},$$

whence the Hamilton equations.

The Tulczyjew triple - Hamiltonian side

$H : \mathsf{T}^*M \to \mathbb{R}$



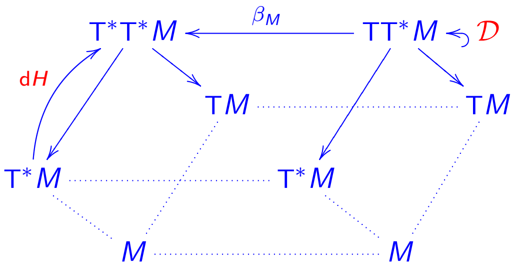
$$\mathcal{D} = \beta_M^{-1}(\mathsf{d}H(\mathsf{T}^*M))$$

$$\mathcal{D} = \left\{(x, p, \dot{x}, \dot{p}) : \quad \dot{p} = -\frac{\partial H}{\partial x}, \quad \dot{x} = \frac{\partial H}{\partial p}\right\},$$

whence the Hamilton equations.

The Tulczyjew triple - Hamiltonian side

$H : \mathsf{T}^*M \to \mathbb{R}$



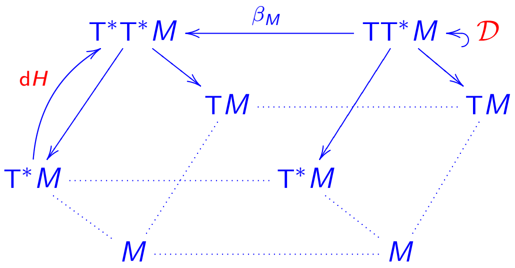
$$\mathcal{D} = \beta_M^{-1}(\mathsf{d}H(\mathsf{T}^*M))$$

$$\mathcal{D} = \left\{ (x, p, \dot{x}, \dot{p}) : \quad \dot{p} = -\frac{\partial H}{\partial x}, \quad \dot{x} = \frac{\partial H}{\partial p} \right\},$$

whence the Hamilton equations.

The Tulczyjew triple - Hamiltonian side

$H : \mathsf{T}^*M \to \mathbb{R}$

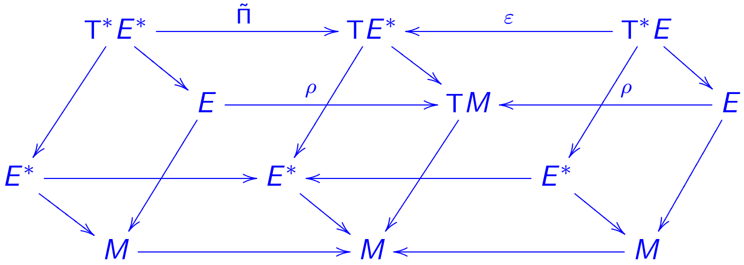


$$\mathcal{D} = \beta_M^{-1}(\mathsf{d}H(\mathsf{T}^*M))$$

$$\mathcal{D} = \left\{(x, p, \dot{x}, \dot{p}) : \quad \dot{p} = -\frac{\partial H}{\partial x}, \quad \dot{x} = \frac{\partial H}{\partial p}\right\},$$

whence the Hamilton equations.

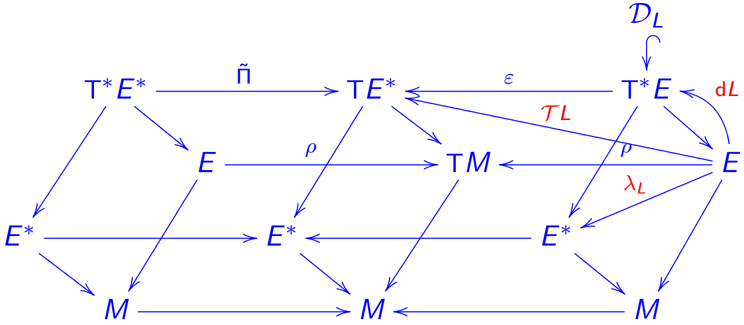
Algebroid setting



$H : E^* \longrightarrow \mathbb{R}$ $\qquad$ $\mathcal{D} = \mathcal{T}L(E)$ $\qquad$ $L : E \longrightarrow \mathbb{R}$

$\mathcal{D}_H \subset \mathsf{T}^*E^*$ $\qquad$ $\mathcal{D} = \tilde{\Pi}(\mathrm{d}H(E^*))$ $\qquad$ $\mathcal{D}_L \subset \mathsf{T}^*E$

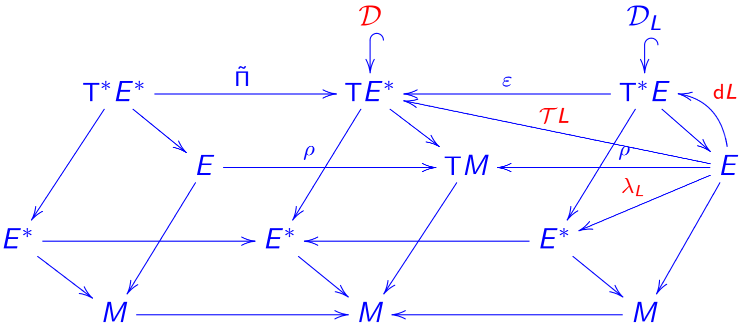
Algebroid setting



$H : E^* \longrightarrow \mathbb{R}$ $\qquad$ $\mathcal{D} = \mathcal{T}L(E)$ $\qquad$ $L : E \longrightarrow \mathbb{R}$

$\mathcal{D}_H \subset \mathsf{T}^*E^*$ $\qquad$ $\mathcal{D} = \tilde{\Pi}(\mathrm{d}H(E^*))$ $\qquad$ $\mathcal{D}_L \subset \mathsf{T}^*E$

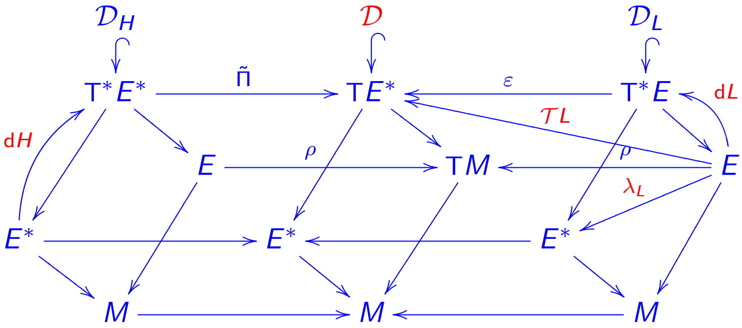
Algebroid setting



$H : E^* \longrightarrow \mathbb{R}$ $\qquad$ $\mathcal{D} = \mathcal{T}L(E)$ $\qquad$ $L : E \longrightarrow \mathbb{R}$

$\mathcal{D}_H \subset \mathsf{T}^*E^*$ $\qquad$ $\mathcal{D} = \tilde{\Pi}(\mathrm{d}H(E^*))$ $\qquad$ $\mathcal{D}_L \subset \mathsf{T}^*E$

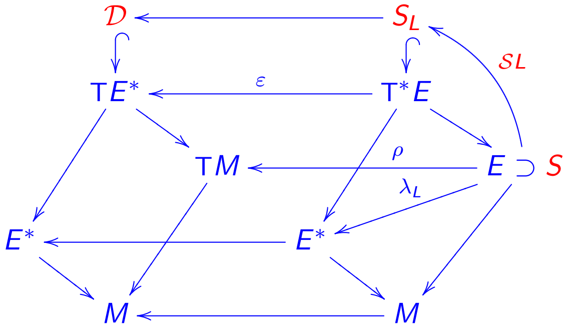
Algebroid setting



$H : E^* \longrightarrow \mathbb{R}$ $\qquad$ $\mathcal{D} = \mathcal{T}L(E)$ $\qquad$ $L : E \longrightarrow \mathbb{R}$

$\mathcal{D}_H \subset \mathsf{T}^*E^*$ $\qquad$ $\mathcal{D} = \tilde{\Pi}(\mathsf{d}H(E^*))$ $\qquad$ $\mathcal{D}_L \subset \mathsf{T}^*E$

Algebroid setting with vakonomic constraints

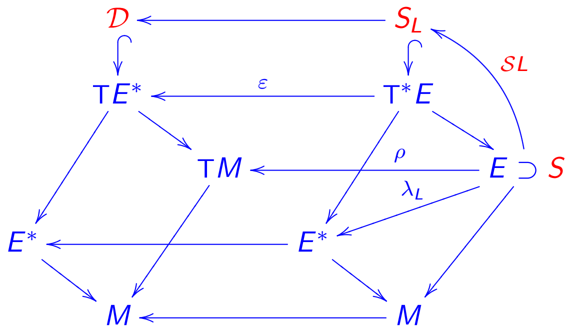


where S_L is the lagrangian submanifold in T^*E induced by the Lagrangian on the constraint S, and $\mathcal{S}L : S \to \mathsf{T}^*E$ is the corresponding relation,

$$S_L = \{\alpha_e \in \mathsf{T}^*_e E : e \in S \text{ and } \langle \alpha_e, v_e \rangle = \mathrm{d}L(v_e) \text{ for every } v_e \in \mathsf{T}_e S\}\,.$$

The vakonomically constrained phase dynamics is just $\mathcal{D} = \varepsilon(S_L) \subset \mathsf{T}E^*$.

Algebroid setting with vakonomic constraints

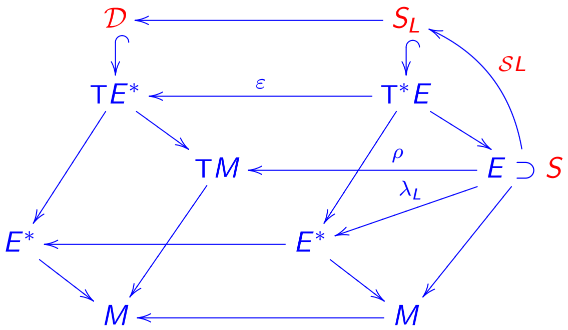


where S_L is the lagrangian submanifold in T^*E induced by the Lagrangian on the constraint S, and $\mathcal{S}L : S \to \mathsf{T}^*E$ is the corresponding relation,

$$S_L = \{\alpha_e \in \mathsf{T}^*_e E : e \in S \text{ and } \langle \alpha_e, v_e \rangle = \mathrm{d}L(v_e) \text{ for every } v_e \in \mathsf{T}_e S\}\,.$$

The vakonomically constrained phase dynamics is just $\mathcal{D} = \varepsilon(S_L) \subset \mathsf{T}E^*$.

Algebroid setting with vakonomic constraints

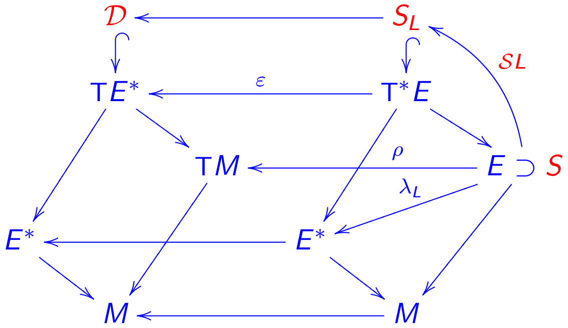


where S_L is the lagrangian submanifold in T^*E induced by the Lagrangian on the constraint S, and $\mathcal{S}L : S \to \mathsf{T}^*E$ is the corresponding relation,

$$S_L = \{\alpha_e \in \mathsf{T}^*_e E : e \in S \text{ and } \langle \alpha_e, v_e \rangle = \mathrm{d}L(v_e) \text{ for every } v_e \in \mathsf{T}_e S\}.$$

The vakonomically constrained phase dynamics is just $\mathcal{D} = \varepsilon(S_L) \subset \mathsf{T}E^*$.

Algebroid setting with vakonomic constraints



where S_L is the lagrangian submanifold in T^*E induced by the Lagrangian on the constraint S, and $\mathcal{S}L : S \to \mathsf{T}^*E$ is the corresponding relation,

$$S_L = \{\alpha_e \in \mathsf{T}^*_e E : e \in S \text{ and } \langle \alpha_e, v_e \rangle = \mathrm{d}L(v_e) \text{ for every } v_e \in \mathsf{T}_e S\}\,.$$

The vakonomically constrained phase dynamics is just $\mathcal{D} = \varepsilon(S_L) \subset \mathsf{T}E^*$.

Higher order Lagrangians

The mechanics with a higher order Lagrangian $L : \mathsf{T}^k Q \to \mathbb{R}$ is traditionally constructed as a vakonomic mechanics, thanks to the canonical embedding of of the higher tangent bundle $\mathsf{T}^k Q$ into the tangent bundle $\mathsf{T}\mathsf{T}^{k-1}Q$ as an affine subbundle of holonomic vectors.

Thus we work with the standard Tulczyjew triple for $\mathsf{T}M$, where $M = \mathsf{T}^{k-1}Q$, with the presence of vakonomic constraint $\mathsf{T}^k Q \subset \mathsf{T}\mathsf{T}^{k-1}Q$:

Higher order Lagrangians

The mechanics with a higher order Lagrangian $L : \mathsf{T}^k Q \to \mathbb{R}$ is traditionally constructed as a vakonomic mechanics, thanks to the canonical embedding of of the higher tangent bundle $\mathsf{T}^k Q$ into the tangent bundle $\mathsf{T}\mathsf{T}^{k-1}Q$ as an affine subbundle of holonomic vectors.

Thus we work with the standard Tulczyjew triple for $\mathsf{T}M$, where $M = \mathsf{T}^{k-1}Q$, with the presence of vakonomic constraint $\mathsf{T}^k Q \subset \mathsf{T}\mathsf{T}^{k-1}Q$:

Higher order Lagrangians

The mechanics with a higher order Lagrangian $L : \mathsf{T}^k Q \to \mathbb{R}$ is traditionally constructed as a vakonomic mechanics, thanks to the canonical embedding of of the higher tangent bundle $\mathsf{T}^k Q$ into the tangent bundle $\mathsf{TT}^{k-1} Q$ as an affine subbundle of holonomic vectors.

Thus we work with the standard Tulczyjew triple for $\mathsf{T}M$, where $M = \mathsf{T}^{k-1}Q$, with the presence of vakonomic constraint $\mathsf{T}^k Q \subset \mathsf{TT}^{k-1}Q$:

Higher order Lagrangians

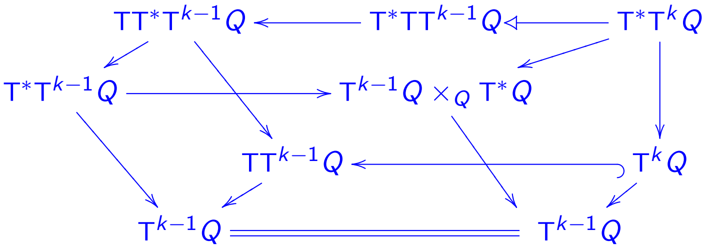
The mechanics with a higher order Lagrangian $L : \mathsf{T}^k Q \to \mathbb{R}$ is traditionally constructed as a vakonomic mechanics, thanks to the canonical embedding of of the higher tangent bundle $\mathsf{T}^k Q$ into the tangent bundle $\mathsf{T}\mathsf{T}^{k-1}Q$ as an affine subbundle of holonomic vectors.

Thus we work with the standard Tulczyjew triple for $\mathsf{T}M$, where $M = \mathsf{T}^{k-1}Q$, with the presence of vakonomic constraint $\mathsf{T}^k Q \subset \mathsf{T}\mathsf{T}^{k-1}Q$:

Higher order Lagrangians

The mechanics with a higher order Lagrangian $L : \mathsf{T}^k Q \to \mathbb{R}$ is traditionally constructed as a vakonomic mechanics, thanks to the canonical embedding of of the higher tangent bundle $\mathsf{T}^k Q$ into the tangent bundle $\mathsf{T}\mathsf{T}^{k-1} Q$ as an affine subbundle of holonomic vectors.

Thus we work with the standard Tulczyjew triple for $\mathsf{T}M$, where $M = \mathsf{T}^{k-1} Q$, with the presence of vakonomic constraint $\mathsf{T}^k Q \subset \mathsf{T}\mathsf{T}^{k-1} Q$:



Higher order Euler-Lagrange equations

The Lagrangian function $L = L(q, \ldots, \overset{(k)}{q})$ generates the phase dynamics

$$\mathcal{D} = \left\{ (v, p, \dot{v}, \dot{p}) : \ \dot{v}_{i-1} = v_i, \ \ \dot{p}_i + p_{i-1} = \frac{\partial L}{\partial \overset{(i)}{q}}, \dot{p}_0 = \frac{\partial L}{\partial q}, p_{k-1} = \frac{\partial L}{\partial \overset{(k)}{q}} \right\}.$$

This leads to the higher Euler-Lagrange equations in the traditional form:

$$\overset{(i)}{q} = \frac{\mathrm{d}^i q}{\mathrm{d}t^i}, \ i = 1, \ldots, k,$$

$$0 = \frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) + \cdots + (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\frac{\partial L}{\partial \overset{(k)}{q}}\right).$$

These equations can be viewed as a system of differential equations of order k on $\mathrm{T}^k Q$ or, which is the standard point of view, as ordinary differential equation of order $2k$ on Q.

Higher order Euler-Lagrange equations

The Lagrangian function $L = L(q, \ldots, \overset{(k)}{q})$ generates the phase dynamics

$$\mathcal{D} = \left\{ (v, p, \dot{v}, \dot{p}) : \ \dot{v}_{i-1} = v_i, \ \ \dot{p}_i + p_{i-1} = \frac{\partial L}{\partial \overset{(i)}{q}}, \dot{p}_0 = \frac{\partial L}{\partial q}, p_{k-1} = \frac{\partial L}{\partial \overset{(k)}{q}} \right\}.$$

This leads to the higher Euler-Lagrange equations in the traditional form:

$$\overset{(i)}{q} = \frac{\mathrm{d}^i q}{\mathrm{d}t^i}, \ i = 1, \ldots, k,$$

$$0 = \frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) + \cdots + (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\frac{\partial L}{\partial \overset{(k)}{q}}\right).$$

These equations can be viewed as a system of differential equations of order k on $\mathsf{T}^k Q$ or, which is the standard point of view, as ordinary differential equation of order $2k$ on Q.

Higher order Euler-Lagrange equations

The Lagrangian function $L = L(q, \ldots, \overset{(k)}{q})$ generates the phase dynamics

$$\mathcal{D} = \left\{ (v, p, \dot v, \dot p) : \ \dot v_{i-1} = v_i, \ \ \dot p_i + p_{i-1} = \frac{\partial L}{\partial \overset{(i)}{q}}, \dot p_0 = \frac{\partial L}{\partial q}, p_{k-1} = \frac{\partial L}{\partial \overset{(k)}{q}} \right\}.$$

This leads to the higher Euler-Lagrange equations in the traditional form:

$$\overset{(i)}{q} = \frac{\mathrm{d}^i q}{\mathrm{d}t^i}, \ i = 1, \ldots, k,$$

$$0 = \frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot q}\right) + \cdots + (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\frac{\partial L}{\partial \overset{(k)}{q}}\right).$$

These equations can be viewed as a system of differential equations of order k on $\mathsf{T}^k Q$ or, which is the standard point of view, as ordinary differential equation of order $2k$ on Q.

Higher order Euler-Lagrange equations

The Lagrangian function $L = L(q, \ldots, \overset{(k)}{q})$ generates the phase dynamics

$$\mathcal{D} = \left\{ (v, p, \dot{v}, \dot{p}) : \ \dot{v}_{i-1} = v_i, \ \ \dot{p}_i + p_{i-1} = \frac{\partial L}{\partial \overset{(i)}{q}}, \dot{p}_0 = \frac{\partial L}{\partial q}, p_{k-1} = \frac{\partial L}{\partial \overset{(k)}{q}} \right\}.$$

This leads to the higher Euler-Lagrange equations in the traditional form:

$$\overset{(i)}{q} = \frac{\mathrm{d}^i q}{\mathrm{d}t^i}, \ i = 1, \ldots, k,$$

$$0 = \frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) + \cdots + (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\frac{\partial L}{\partial \overset{(k)}{q}}\right).$$

These equations can be viewed as a system of differential equations of order k on $\mathsf{T}^k Q$ or, which is the standard point of view, as ordinary differential equation of order $2k$ on Q.

Higher order Euler-Lagrange equations

The Lagrangian function $L = L(q, \dots, \overset{(k)}{q})$ generates the phase dynamics

$$\mathcal{D} = \left\{ (v, p, \dot{v}, \dot{p}) : \ \dot{v}_{i-1} = v_i, \ \ \dot{p}_i + p_{i-1} = \frac{\partial L}{\partial \overset{(i)}{q}}, \dot{p}_0 = \frac{\partial L}{\partial q}, p_{k-1} = \frac{\partial L}{\partial \overset{(k)}{q}} \right\}.$$

This leads to the higher Euler-Lagrange equations in the traditional form:

$$\overset{(i)}{q} = \frac{\mathrm{d}^i q}{\mathrm{d}t^i}, \ \ i = 1, \dots, k,$$

$$0 = \frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) + \cdots + (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\frac{\partial L}{\partial \overset{(k)}{q}}\right).$$

These equations can be viewed as a system of differential equations of order k on $\mathsf{T}^k Q$ or, which is the standard point of view, as ordinary differential equation of order $2k$ on Q.

Higher order Euler-Lagrange equations

The Lagrangian function $L = L(q, \ldots, \overset{(k)}{q})$ generates the phase dynamics

$$\mathcal{D} = \left\{ (v, p, \dot{v}, \dot{p}) : \; \dot{v}_{i-1} = v_i, \;\; \dot{p}_i + p_{i-1} = \frac{\partial L}{\partial \overset{(i)}{q}}, \dot{p}_0 = \frac{\partial L}{\partial q}, p_{k-1} = \frac{\partial L}{\partial \overset{(k)}{q}} \right\}.$$

This leads to the higher Euler-Lagrange equations in the traditional form:

$$\overset{(i)}{q} = \frac{\mathrm{d}^i q}{\mathrm{d}t^i}, \;\; i = 1, \ldots, k,$$

$$0 = \frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) + \cdots + (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\frac{\partial L}{\partial \overset{(k)}{q}}\right).$$

These equations can be viewed as a system of differential equations of order k on $\mathsf{T}^k Q$ or, which is the standard point of view, as ordinary differential equation of order $2k$ on Q.

Higher order Euler-Lagrange equations

The Lagrangian function $L = L(q, \ldots, \overset{(k)}{q})$ generates the phase dynamics

$$\mathcal{D} = \left\{ (v, p, \dot{v}, \dot{p}) : \ \dot{v}_{i-1} = v_i, \ \ \dot{p}_i + p_{i-1} = \frac{\partial L}{\partial \overset{(i)}{q}}, \dot{p}_0 = \frac{\partial L}{\partial q}, p_{k-1} = \frac{\partial L}{\partial \overset{(k)}{q}} \right\}.$$

This leads to the higher Euler-Lagrange equations in the traditional form:

$$\overset{(i)}{q} = \frac{\mathrm{d}^i q}{\mathrm{d}t^i}, \ i = 1, \ldots, k,$$

$$0 = \frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) + \cdots + (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\frac{\partial L}{\partial \overset{(k)}{q}}\right).$$

These equations can be viewed as a system of differential equations of order k on $\mathsf{T}^k Q$ or, which is the standard point of view, as ordinary differential equation of order $2k$ on Q.

Higher order Euler-Lagrange equations

The Lagrangian function $L = L(q, \dots, \overset{(k)}{q})$ generates the phase dynamics

$$\mathcal{D} = \left\{ (v, p, \dot{v}, \dot{p}) : \ \dot{v}_{i-1} = v_i, \ \ \dot{p}_i + p_{i-1} = \frac{\partial L}{\partial \overset{(i)}{q}}, \dot{p}_0 = \frac{\partial L}{\partial q}, p_{k-1} = \frac{\partial L}{\partial \overset{(k)}{q}} \right\}.$$

This leads to the higher Euler-Lagrange equations in the traditional form:

$$\overset{(i)}{q} = \frac{\mathrm{d}^i q}{\mathrm{d}t^i}, \ i = 1, \dots, k,$$

$$0 = \frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) + \cdots + (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\frac{\partial L}{\partial \overset{(k)}{q}}\right).$$

These equations can be viewed as a system of differential equations of order k on $\mathsf{T}^k Q$ or, which is the standard point of view, as ordinary differential equation of order $2k$ on Q.

Linearisation of graded bundles

The possibility of constructing mechanics on graded bundles is based on the following generalization of the embedding $\mathsf{T}^k Q \hookrightarrow \mathsf{T}\mathsf{T}^{k-1} Q$.

Theorem (Bruce-Grabowska-Grabowski)

There is a canonical functor from the category of graded bundles into the category of $\mathcal{GL}$-bundles which assigns, for an arbitrary graded bundle F_k of degree k, a canonical $\mathcal{GL}$-bundle $D(F_k)$ which is linear over F_{k-1}, called the linearisation of F_k, together with a graded embedding $\iota : F_k \hookrightarrow D(F_k)$of F_k as an affine subbundle of the vector bundle $D(F_k)$.

Elements of $F_k \subset D(F_k)$ may be viewed as 'holonomic vectors' in the linear-graded bundle $D(F_k)$. Another geometric part we need is a (Lie) algebroid structure on the vector bundle $D(F_k) \to F_{k-1}$, compatible with the second graded structure (homogeneity). We will call such $\mathcal{GL}$-bundles D weighted (Lie) algebroids and view them as abstract generalizations of the Lie algebroid $\mathsf{T}\mathsf{T}^{k-1} M$. Such D is called a $\mathcal{VB}$-algebroid if it is a double vector bundle.

Linearisation of graded bundles

The possibility of constructing mechanics on graded bundles is based on the following generalization of the embedding $\mathsf{T}^k Q \hookrightarrow \mathsf{T}\mathsf{T}^{k-1} Q$.

Theorem (Bruce-Grabowska-Grabowski)

There is a canonical functor from the category of graded bundles into the category of $\mathcal{GL}$-bundles which assigns, for an arbitrary graded bundle F_k of degree k, a canonical $\mathcal{GL}$-bundle $D(F_k)$ which is linear over F_{k-1}, called the linearisation of F_k, together with a graded embedding $\iota : F_k \hookrightarrow D(F_k)$ of F_k as an affine subbundle of the vector bundle $D(F_k)$.

Elements of $F_k \subset D(F_k)$ may be viewed as 'holonomic vectors' in the linear-graded bundle $D(F_k)$. Another geometric part we need is a (Lie) algebroid structure on the vector bundle $D(F_k) \to F_{k-1}$, compatible with the second graded structure (homogeneity). We will call such $\mathcal{GL}$-bundles D weighted (Lie) algebroids and view them as abstract generalizations of the Lie algebroid $\mathsf{T}\mathsf{T}^{k-1} M$. Such D is called a $\mathcal{VB}$-algebroid if it is a double vector bundle.

Linearisation of graded bundles

The possibility of constructing mechanics on graded bundles is based on the following generalization of the embedding $\mathsf{T}^k Q \hookrightarrow \mathsf{T}\mathsf{T}^{k-1} Q$.

Theorem (Bruce-Grabowska-Grabowski)

There is a canonical functor from the category of graded bundles into the category of $\mathcal{GL}$-bundles which assigns, for an arbitrary graded bundle F_k of degree k, a canonical $\mathcal{GL}$-bundle $D(F_k)$ which is linear over F_{k-1}, called the linearisation of F_k, together with a graded embedding $\iota : F_k \hookrightarrow D(F_k)$of F_k as an affine subbundle of the vector bundle $D(F_k)$.

Elements of $F_k \subset D(F_k)$ may be viewed as 'holonomic vectors' in the linear-graded bundle $D(F_k)$. Another geometric part we need is a (Lie) algebroid structure on the vector bundle $D(F_k) \to F_{k-1}$, compatible with the second graded structure (homogeneity). We will call such $\mathcal{GL}$-bundles D weighted (Lie) algebroids and view them as abstract generalizations of the Lie algebroid $\mathsf{T}\mathsf{T}^{k-1} M$. Such D is called a $\mathcal{VB}$-algebroid if it is a double vector bundle.

Linearisation of graded bundles

The possibility of constructing mechanics on graded bundles is based on the following generalization of the embedding $\mathsf{T}^k Q \hookrightarrow \mathsf{T}\mathsf{T}^{k-1}Q$.

Theorem (Bruce-Grabowska-Grabowski)

There is a canonical functor from the category of graded bundles into the category of $\mathcal{GL}$-bundles which assigns, for an arbitrary graded bundle F_k of degree k, a canonical $\mathcal{GL}$-bundle $D(F_k)$ which is linear over F_{k-1}, called the linearisation of F_k, together with a graded embedding $\iota : F_k \hookrightarrow D(F_k)$ of F_k as an affine subbundle of the vector bundle $D(F_k)$.

Elements of $F_k \subset D(F_k)$ may be viewed as 'holonomic vectors' in the linear-graded bundle $D(F_k)$. Another geometric part we need is a (Lie) algebroid structure on the vector bundle $D(F_k) \to F_{k-1}$, compatible with the second graded structure (homogeneity). We will call such $\mathcal{GL}$-bundles D weighted (Lie) algebroids and view them as abstract generalizations of the Lie algebroid $\mathsf{T}\mathsf{T}^{k-1}M$. Such D is called a $\mathcal{VB}$-algebroid if it is a double vector bundle.

Linearisation of graded bundles

The possibility of constructing mechanics on graded bundles is based on the following generalization of the embedding $\mathsf{T}^k Q \hookrightarrow \mathsf{T}\mathsf{T}^{k-1} Q$.

Theorem (Bruce-Grabowska-Grabowski)

There is a canonical functor from the category of graded bundles into the category of $\mathcal{GL}$-bundles which assigns, for an arbitrary graded bundle F_k of degree k, a canonical $\mathcal{GL}$-bundle $D(F_k)$ which is linear over F_{k-1}, called the linearisation of F_k, together with a graded embedding $\iota : F_k \hookrightarrow D(F_k)$ of F_k as an affine subbundle of the vector bundle $D(F_k)$.

Elements of $F_k \subset D(F_k)$ may be viewed as 'holonomic vectors' in the linear-graded bundle $D(F_k)$. Another geometric part we need is a (Lie) algebroid structure on the vector bundle $D(F_k) \to F_{k-1}$, compatible with the second graded structure (homogeneity). We will call such $\mathcal{GL}$-bundles D weighted (Lie) algebroids and view them as abstract generalizations of the Lie algebroid $\mathsf{T}\mathsf{T}^{k-1} M$. Such D is called a $\mathcal{VB}$-algebroid if it is a double vector bundle.

Linearisation of graded bundles

The possibility of constructing mechanics on graded bundles is based on the following generalization of the embedding $\mathsf{T}^k Q \hookrightarrow \mathsf{T}\mathsf{T}^{k-1} Q$.

Theorem (Bruce-Grabowska-Grabowski)

There is a canonical functor from the category of graded bundles into the category of $\mathcal{GL}$-bundles which assigns, for an arbitrary graded bundle F_k of degree k, a canonical $\mathcal{GL}$-bundle $D(F_k)$ which is linear over F_{k-1}, called the linearisation of F_k, together with a graded embedding $\iota : F_k \hookrightarrow D(F_k)$of F_k as an affine subbundle of the vector bundle $D(F_k)$.

Elements of $F_k \subset D(F_k)$ may be viewed as 'holonomic vectors' in the linear-graded bundle $D(F_k)$. Another geometric part we need is a (Lie) algebroid structure on the vector bundle $D(F_k) \to F_{k-1}$, compatible with the second graded structure (homogeneity). We will call such $\mathcal{GL}$-bundles D weighted (Lie) algebroids and view them as abstract generalizations of the Lie algebroid $\mathsf{T}\mathsf{T}^{k-1} M$. Such D is called a $\mathcal{VB}$-algebroid if it is a double vector bundle.

Linearisation of graded bundles

The possibility of constructing mechanics on graded bundles is based on the following generalization of the embedding $\mathsf{T}^k Q \hookrightarrow \mathsf{T}\mathsf{T}^{k-1} Q$.

Theorem (Bruce-Grabowska-Grabowski)

There is a canonical functor from the category of graded bundles into the category of $\mathcal{GL}$-bundles which assigns, for an arbitrary graded bundle F_k of degree k, a canonical $\mathcal{GL}$-bundle $D(F_k)$ which is linear over F_{k-1}, called the linearisation of F_k, together with a graded embedding $\iota : F_k \hookrightarrow D(F_k)$ of F_k as an affine subbundle of the vector bundle $D(F_k)$.

Elements of $F_k \subset D(F_k)$ may be viewed as 'holonomic vectors' in the linear-graded bundle $D(F_k)$. Another geometric part we need is a (Lie) algebroid structure on the vector bundle $D(F_k) \to F_{k-1}$, compatible with the second graded structure (homogeneity). We will call such $\mathcal{GL}$-bundles D weighted (Lie) algebroids and view them as abstract generalizations of the Lie algebroid $\mathsf{T}\mathsf{T}^{k-1} M$. Such D is called a $\mathcal{VB}$-algebroid if it is a double vector bundle.

Linearisation of graded bundles

The possibility of constructing mechanics on graded bundles is based on the following generalization of the embedding $\mathsf{T}^k Q \hookrightarrow \mathsf{T}\mathsf{T}^{k-1} Q$.

Theorem (Bruce-Grabowska-Grabowski)

There is a canonical functor from the category of graded bundles into the category of $\mathcal{GL}$-bundles which assigns, for an arbitrary graded bundle F_k of degree k, a canonical $\mathcal{GL}$-bundle $D(F_k)$ which is linear over F_{k-1}, called the linearisation of F_k, together with a graded embedding $\iota : F_k \hookrightarrow D(F_k)$ of F_k as an affine subbundle of the vector bundle $D(F_k)$.

Elements of $F_k \subset D(F_k)$ may be viewed as 'holonomic vectors' in the linear-graded bundle $D(F_k)$. Another geometric part we need is a (Lie) algebroid structure on the vector bundle $D(F_k) \to F_{k-1}$, compatible with the second graded structure (homogeneity). We will call such $\mathcal{GL}$-bundles D weighted (Lie) algebroids and view them as abstract generalizations of the Lie algebroid $\mathsf{T}\mathsf{T}^{k-1} M$. Such D is called a $\mathcal{VB}$-algebroid if it is a double vector bundle.

Linearisation of graded bundles

The possibility of constructing mechanics on graded bundles is based on the following generalization of the embedding $\mathsf{T}^k Q \hookrightarrow \mathsf{T}\mathsf{T}^{k-1} Q$.

Theorem (Bruce-Grabowska-Grabowski)

There is a canonical functor from the category of graded bundles into the category of $\mathcal{GL}$-bundles which assigns, for an arbitrary graded bundle F_k of degree k, a canonical $\mathcal{GL}$-bundle $D(F_k)$ which is linear over F_{k-1}, called the linearisation of F_k, together with a graded embedding $\iota : F_k \hookrightarrow D(F_k)$of F_k as an affine subbundle of the vector bundle $D(F_k)$.

Elements of $F_k \subset D(F_k)$ may be viewed as 'holonomic vectors' in the linear-graded bundle $D(F_k)$. Another geometric part we need is a (Lie) algebroid structure on the vector bundle $D(F_k) \to F_{k-1}$, compatible with the second graded structure (homogeneity). We will call such $\mathcal{GL}$-bundles D weighted (Lie) algebroids and view them as abstract generalizations of the Lie algebroid $\mathsf{T}\mathsf{T}^{k-1} M$. Such D is called a $\mathcal{VB}$-algebroid if it is a double vector bundle.

Linearisation of graded bundles

The possibility of constructing mechanics on graded bundles is based on the following generalization of the embedding $\mathsf{T}^k Q \hookrightarrow \mathsf{T}\mathsf{T}^{k-1} Q$.

Theorem (Bruce-Grabowska-Grabowski)

There is a canonical functor from the category of graded bundles into the category of $\mathcal{GL}$-bundles which assigns, for an arbitrary graded bundle F_k of degree k, a canonical $\mathcal{GL}$-bundle $D(F_k)$ which is linear over F_{k-1}, called the linearisation of F_k, together with a graded embedding $\iota : F_k \hookrightarrow D(F_k)$ of F_k as an affine subbundle of the vector bundle $D(F_k)$.

Elements of $F_k \subset D(F_k)$ may be viewed as 'holonomic vectors' in the linear-graded bundle $D(F_k)$. Another geometric part we need is a (Lie) algebroid structure on the vector bundle $D(F_k) \to F_{k-1}$, compatible with the second graded structure (homogeneity). We will call such $\mathcal{GL}$-bundles D weighted (Lie) algebroids and view them as abstract generalizations of the Lie algebroid $\mathsf{T}\mathsf{T}^{k-1} M$. Such D is called a $\mathcal{VB}$-algebroid if it is a double vector bundle.

Linearisation in coordinates

If (x^a, y^i_w, z^j_k) are coordinates on a graded bundle F_k such that x^a are of degree 0, y^i_w are of degree w, $0 < w < k$, and z^j_k are of degree k, then the induced coordinate system on $D(F_k)$ is

$$(x^a, y^i_w, \dot{y}^{i'}_{w'}, \dot{z}^j_k)\,,$$

where x^a are of bi-degree $(0,0)$, y^i_w are of bi-degree $(w,1)$, $\dot{y}^i_w$ are of bi-degree $(w,1)$, and z^j_k are of bi-degree $(k,1)$. Thus,

$$(x^a, y^i_w, \dot{y}^{i'}_{w'}, \dot{z}^j_k) \mapsto (x^a, y^i_w)$$

is a linear fibration over F_{k-1}.

The embedding $\iota : F_k \hookrightarrow D(F_k)$ reads

$$\iota(x^a, y^i_w, z^j_k) = (x^a, y^i_w, w' y^{i'}_{w'}, k z^j_k)\,.$$

Linearisation in coordinates

If (x^a, y^i_w, z^j_k) are coordinates on a graded bundle F_k such that x^a are of degree 0, y^i_w are of degree w, $0 < w < k$, and z^j_k are of degree k, then the induced coordinate system on $D(F_k)$ is

$$(x^a, y^i_w, \dot{y}^{i'}_{w'}, \dot{z}^j_k),$$

where x^a are of bi-degree $(0,0)$, y^i_w are of bi-degree $(w,1)$, $\dot{y}^i_w$ are of bi-degree $(w,1)$, and z^j_k are of bi-degree $(k,1)$. Thus,

$$(x^a, y^i_w, \dot{y}^{i'}_{w'}, \dot{z}^j_k) \mapsto (x^a, y^i_w)$$

is a linear fibration over F_{k-1}.

The embedding $\iota : F_k \hookrightarrow D(F_k)$ reads

$$\iota(x^a, y^i_w, z^j_k) = (x^a, y^i_w, w' y^{i'}_{w'}, k z^j_k).$$

Linearisation in coordinates

If (x^a, y^i_w, z^j_k) are coordinates on a graded bundle F_k such that x^a are of degree 0, y^i_w are of degree w, $0 < w < k$, and z^j_k are of degree k, then the induced coordinate system on $D(F_k)$ is

$$(x^a, y^i_w, \dot{y}^{i'}_{w'}, \dot{z}^j_k)\,,$$

where x^a are of bi-degree $(0,0)$, y^i_w are of bi-degree $(w,1)$, $\dot{y}^i_w$ are of bi-degree $(w,1)$, and z^j_k are of bi-degree $(k,1)$. Thus,

$$(x^a, y^i_w, \dot{y}^{i'}_{w'}, \dot{z}^j_k) \mapsto (x^a, y^i_w)$$

is a linear fibration over F_{k-1}.

The embedding $\iota : F_k \hookrightarrow D(F_k)$ reads

$$\iota(x^a, y^i_w, z^j_k) = (x^a, y^i_w, w' y^{i'}_{w'}, k z^j_k)\,.$$

Linearisation in coordinates

If (x^a, y^i_w, z^j_k) are coordinates on a graded bundle F_k such that x^a are of degree 0, y^i_w are of degree w, $0 < w < k$, and z^j_k are of degree k, then the induced coordinate system on $D(F_k)$ is

$$(x^a, y^i_w, \dot{y}^{i'}_{w'}, \dot{z}^j_k),$$

where x^a are of bi-degree $(0,0)$, y^i_w are of bi-degree $(w,1)$, $\dot{y}^i_w$ are of bi-degree $(w,1)$, and z^j_k are of bi-degree $(k,1)$. Thus,

$$(x^a, y^i_w, \dot{y}^{i'}_{w'}, \dot{z}^j_k) \mapsto (x^a, y^i_w)$$

is a linear fibration over F_{k-1}.

The embedding $\iota : F_k \hookrightarrow D(F_k)$ reads

$$\iota(x^a, y^i_w, z^j_k) = (x^a, y^i_w, w' y^{i'}_{w'}, k z^j_k).$$

Linearisation in coordinates

If (x^a, y^i_w, z^j_k) are coordinates on a graded bundle F_k such that x^a are of degree 0, y^i_w are of degree w, $0 < w < k$, and z^j_k are of degree k, then the induced coordinate system on $D(F_k)$ is

$$(x^a, y^i_w, \dot{y}^{i'}_{w'}, \dot{z}^j_k),$$

where x^a are of bi-degree $(0,0)$, y^i_w are of bi-degree $(w,1)$, $\dot{y}^i_w$ are of bi-degree $(w,1)$, and z^j_k are of bi-degree $(k,1)$. Thus,

$$(x^a, y^i_w, \dot{y}^{i'}_{w'}, \dot{z}^j_k) \mapsto (x^a, y^i_w)$$

is a linear fibration over F_{k-1}.

The embedding $\iota : F_k \hookrightarrow D(F_k)$ reads

$$\iota(x^a, y^i_w, z^j_k) = (x^a, y^i_w, w' y^{i'}_{w'}, k z^j_k).$$

Linearisation in coordinates

If (x^a, y^i_w, z^j_k) are coordinates on a graded bundle F_k such that x^a are of degree 0, y^i_w are of degree w, $0 < w < k$, and z^j_k are of degree k, then the induced coordinate system on $D(F_k)$ is

$$(x^a, y^i_w, \dot{y}^{i'}_{w'}, \dot{z}^j_k),$$

where x^a are of bi-degree $(0,0)$, y^i_w are of bi-degree $(w,1)$, $\dot{y}^i_w$ are of bi-degree $(w,1)$, and z^j_k are of bi-degree $(k,1)$. Thus,

$$(x^a, y^i_w, \dot{y}^{i'}_{w'}, \dot{z}^j_k) \mapsto (x^a, y^i_w)$$

is a linear fibration over F_{k-1}.

The embedding $\iota : F_k \hookrightarrow D(F_k)$ reads

$$\iota(x^a, y^i_w, z^j_k) = (x^a, y^i_w, w' y^{i'}_{w'}, k z^j_k).$$

Linearisation in coordinates

If (x^a, y^i_w, z^j_k) are coordinates on a graded bundle F_k such that x^a are of degree 0, y^i_w are of degree w, $0 < w < k$, and z^j_k are of degree k, then the induced coordinate system on $D(F_k)$ is

$$(x^a, y^i_w, \dot{y}^{i'}_{w'}, \dot{z}^j_k)\,,$$

where x^a are of bi-degree $(0,0)$, y^i_w are of bi-degree $(w,1)$, $\dot{y}^i_w$ are of bi-degree $(w,1)$, and z^j_k are of bi-degree $(k,1)$. Thus,

$$(x^a, y^i_w, \dot{y}^{i'}_{w'}, \dot{z}^j_k) \mapsto (x^a, y^i_w)$$

is a linear fibration over F_{k-1}.

The embedding $\iota : F_k \hookrightarrow D(F_k)$ reads

$$\iota(x^a, y^i_w, z^j_k) = (x^a, y^i_w, w' y^{i'}_{w'}, k z^j_k)\,.$$

Linearisation in coordinates

If (x^a, y^i_w, z^j_k) are coordinates on a graded bundle F_k such that x^a are of degree 0, y^i_w are of degree w, $0 < w < k$, and z^j_k are of degree k, then the induced coordinate system on $D(F_k)$ is

$$(x^a, y^i_w, \dot{y}^{i'}_{w'}, \dot{z}^j_k)\,,$$

where x^a are of bi-degree $(0,0)$, y^i_w are of bi-degree $(w,1)$, $\dot{y}^i_w$ are of bi-degree $(w,1)$, and z^j_k are of bi-degree $(k,1)$. Thus,

$$(x^a, y^i_w, \dot{y}^{i'}_{w'}, \dot{z}^j_k) \mapsto (x^a, y^i_w)$$

is a linear fibration over F_{k-1}.

The embedding $\iota : F_k \hookrightarrow D(F_k)$ reads

$$\iota(x^a, y^i_w, z^j_k) = (x^a, y^i_w, w' y^{i'}_{w'}, k z^j_k)\,.$$

Linearisation in coordinates

If (x^a, y^i_w, z^j_k) are coordinates on a graded bundle F_k such that x^a are of degree 0, y^i_w are of degree w, $0 < w < k$, and z^j_k are of degree k, then the induced coordinate system on $D(F_k)$ is

$$(x^a, y^i_w, \dot{y}^{i'}_{w'}, \dot{z}^j_k),$$

where x^a are of bi-degree $(0,0)$, y^i_w are of bi-degree $(w,1)$, $\dot{y}^i_w$ are of bi-degree $(w,1)$, and z^j_k are of bi-degree $(k,1)$. Thus,

$$(x^a, y^i_w, \dot{y}^{i'}_{w'}, \dot{z}^j_k) \mapsto (x^a, y^i_w)$$

is a linear fibration over F_{k-1}.

The embedding $\iota : F_k \hookrightarrow D(F_k)$ reads

$$\iota(x^a, y^i_w, z^j_k) = (x^a, y^i_w, w' y^{i'}_{w'}, k z^j_k).$$

Linearisation in coordinates

If (x^a, y^i_w, z^j_k) are coordinates on a graded bundle F_k such that x^a are of degree 0, y^i_w are of degree w, $0 < w < k$, and z^j_k are of degree k, then the induced coordinate system on $D(F_k)$ is

$$(x^a, y^i_w, \dot{y}^{i'}_{w'}, \dot{z}^j_k),$$

where x^a are of bi-degree $(0,0)$, y^i_w are of bi-degree $(w,1)$, $\dot{y}^i_w$ are of bi-degree $(w,1)$, and z^j_k are of bi-degree $(k,1)$. Thus,

$$(x^a, y^i_w, \dot{y}^{i'}_{w'}, \dot{z}^j_k) \mapsto (x^a, y^i_w)$$

is a linear fibration over F_{k-1}.

The embedding $\iota : F_k \hookrightarrow D(F_k)$ reads

$$\iota(x^a, y^i_w, z^j_k) = (x^a, y^i_w, w' y^{i'}_{w'}, k z^j_k).$$

Linearisation in coordinates

If (x^a, y^i_w, z^j_k) are coordinates on a graded bundle F_k such that x^a are of degree 0, y^i_w are of degree w, $0 < w < k$, and z^j_k are of degree k, then the induced coordinate system on $D(F_k)$ is

$$(x^a, y^i_w, \dot{y}^{i'}_{w'}, \dot{z}^j_k)\,,$$

where x^a are of bi-degree $(0,0)$, y^i_w are of bi-degree $(w,1)$, $\dot{y}^i_w$ are of bi-degree $(w,1)$, and z^j_k are of bi-degree $(k,1)$. Thus,

$$(x^a, y^i_w, \dot{y}^{i'}_{w'}, \dot{z}^j_k) \mapsto (x^a, y^i_w)$$

is a linear fibration over F_{k-1}.

The embedding $\iota : F_k \hookrightarrow D(F_k)$ reads

$$\iota(x^a, y^i_w, z^j_k) = (x^a, y^i_w, w' y^{i'}_{w'}, k z^j_k)\,.$$

Linearisation in coordinates

If (x^a, y^i_w, z^j_k) are coordinates on a graded bundle F_k such that x^a are of degree 0, y^i_w are of degree w, $0 < w < k$, and z^j_k are of degree k, then the induced coordinate system on $D(F_k)$ is

$$(x^a, y^i_w, \dot{y}^{i'}_{w'}, \dot{z}^j_k)\,,$$

where x^a are of bi-degree $(0,0)$, y^i_w are of bi-degree $(w,1)$, $\dot{y}^i_w$ are of bi-degree $(w,1)$, and z^j_k are of bi-degree $(k,1)$. Thus,

$$(x^a, y^i_w, \dot{y}^{i'}_{w'}, \dot{z}^j_k) \mapsto (x^a, y^i_w)$$

is a linear fibration over F_{k-1}.

The embedding $\iota : F_k \hookrightarrow D(F_k)$ reads

$$\iota(x^a, y^i_w, z^j_k) = (x^a, y^i_w, w' y^{i'}_{w'}, k z^j_k)\,.$$

Linearisation in coordinates

If (x^a, y^i_w, z^j_k) are coordinates on a graded bundle F_k such that x^a are of degree 0, y^i_w are of degree w, $0 < w < k$, and z^j_k are of degree k, then the induced coordinate system on $D(F_k)$ is

$$(x^a, y^i_w, \dot{y}^{i'}_{w'}, \dot{z}^j_k)\,,$$

where x^a are of bi-degree $(0,0)$, y^i_w are of bi-degree $(w,1)$, $\dot{y}^i_w$ are of bi-degree $(w,1)$, and z^j_k are of bi-degree $(k,1)$. Thus,

$$(x^a, y^i_w, \dot{y}^{i'}_{w'}, \dot{z}^j_k) \mapsto (x^a, y^i_w)$$

is a linear fibration over F_{k-1}.

The embedding $\iota : F_k \hookrightarrow D(F_k)$ reads

$$\iota(x^a, y^i_w, z^j_k) = (x^a, y^i_w, w' y^{i'}_{w'}, k z^j_k)\,.$$

Weighted Lie algebroids out of reductions

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and consider the subbundle $\mathsf{T}^k\mathcal{G}^{\underline{s}} \subset \mathsf{T}^k\mathcal{G}$ consisting of all higher order velocities tangent to source-leaves. The bundle

$$F_k = A^k(\mathcal{G}) := \mathsf{T}^k\mathcal{G}^{\underline{s}}\Big|_M ,$$

inherits graded bundle structure of degree k as a graded subbundle of $\mathsf{T}^k\mathcal{G}$. Of course, $A = A^1(\mathcal{G})$ can be identified with the Lie algebroid of $\mathcal{G}$.

Theorem

The linearisation of $A^k(\mathcal{G})$ is given as

$$D(A^k(\mathcal{G})) \simeq \{(Y,Z) \in A(\mathcal{G}) \times_M \mathsf{T}A^{k-1}(\mathcal{G}) | \quad \rho(Y) = \mathsf{T}\tau(Z)\},$$

viewed as a vector bundle over $A^{k-1}(\mathcal{G})$ with respect to the obvious projection of part Z onto $A^{k-1}(\mathcal{G})$, where $\rho : A(\mathcal{G}) \to \mathsf{T}M$ is the standard anchor of the Lie algebroid and $\tau : A^{k-1}(\mathcal{G}) \to M$ is the obvious projection. Moreover, the above bundle is canonically a weighted Lie algebroid, a Lie algebroid prolongation in the sense of Popescu and Martínez.

Weighted Lie algebroids out of reductions

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and consider the subbundle $\mathsf{T}^k\mathcal{G}^{\underline{s}} \subset \mathsf{T}^k\mathcal{G}$ consisting of all higher order velocities tangent to source-leaves. The bundle

$$F_k = A^k(\mathcal{G}) := \mathsf{T}^k\mathcal{G}^{\underline{s}}\Big|_M ,$$

inherits graded bundle structure of degree k as a graded subbundle of $\mathsf{T}^k\mathcal{G}$. Of course, $A = A^1(\mathcal{G})$ can be identified with the Lie algebroid of $\mathcal{G}$.

Theorem

The linearisation of $A^k(\mathcal{G})$ is given as

$$D(A^k(\mathcal{G})) \simeq \{(Y, Z) \in A(\mathcal{G}) \times_M \mathsf{T}A^{k-1}(\mathcal{G}) | \quad \rho(Y) = \mathsf{T}\tau(Z)\} ,$$

viewed as a vector bundle over $A^{k-1}(\mathcal{G})$ with respect to the obvious projection of part Z onto $A^{k-1}(\mathcal{G})$, where $\rho : A(\mathcal{G}) \to \mathsf{T}M$ is the standard anchor of the Lie algebroid and $\tau : A^{k-1}(\mathcal{G}) \to M$ is the obvious projection. Moreover, the above bundle is canonically a weighted Lie algebroid, a Lie algebroid prolongation in the sense of Popescu and Martínez.

Weighted Lie algebroids out of reductions

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and consider the subbundle $\mathsf{T}^k\mathcal{G}^{\underline{s}} \subset \mathsf{T}^k\mathcal{G}$ consisting of all higher order velocities tangent to source-leaves. The bundle

$$F_k = A^k(\mathcal{G}) := \mathsf{T}^k\mathcal{G}^{\underline{s}}\Big|_M,$$

inherits graded bundle structure of degree k as a graded subbundle of $\mathsf{T}^k\mathcal{G}$. Of course, $A = A^1(\mathcal{G})$ can be identified with the Lie algebroid of $\mathcal{G}$.

Theorem

The linearisation of $A^k(\mathcal{G})$ is given as

$$D(A^k(\mathcal{G})) \simeq \{(Y, Z) \in A(\mathcal{G}) \times_M \mathsf{T}A^{k-1}(\mathcal{G}) | \quad \rho(Y) = \mathsf{T}\tau(Z)\},$$

viewed as a vector bundle over $A^{k-1}(\mathcal{G})$ with respect to the obvious projection of part Z onto $A^{k-1}(\mathcal{G})$, where $\rho : A(\mathcal{G}) \to \mathsf{T}M$ is the standard anchor of the Lie algebroid and $\tau : A^{k-1}(\mathcal{G}) \to M$ is the obvious projection. Moreover, the above bundle is canonically a weighted Lie algebroid, a Lie algebroid prolongation in the sense of Popescu and Martínez.

Weighted Lie algebroids out of reductions

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and consider the subbundle $\mathsf{T}^k\mathcal{G}^{\underline{s}} \subset \mathsf{T}^k\mathcal{G}$ consisting of all higher order velocities tangent to source-leaves. The bundle

$$F_k = A^k(\mathcal{G}) := \mathsf{T}^k\mathcal{G}^{\underline{s}}\Big|_M ,$$

inherits graded bundle structure of degree k as a graded subbundle of $\mathsf{T}^k\mathcal{G}$. Of course, $A = A^1(\mathcal{G})$ can be identified with the Lie algebroid of $\mathcal{G}$.

Theorem

The linearisation of $A^k(\mathcal{G})$ is given as

$$D(A^k(\mathcal{G})) \simeq \{(Y, Z) \in A(\mathcal{G}) \times_M \mathsf{T}A^{k-1}(\mathcal{G}) | \quad \rho(Y) = \mathsf{T}\tau(Z)\} ,$$

viewed as a vector bundle over $A^{k-1}(\mathcal{G})$ with respect to the obvious projection of part Z onto $A^{k-1}(\mathcal{G})$, where $\rho : A(\mathcal{G}) \to \mathsf{T}M$ is the standard anchor of the Lie algebroid and $\tau : A^{k-1}(\mathcal{G}) \to M$ is the obvious projection. Moreover, the above bundle is canonically a weighted Lie algebroid, a Lie algebroid prolongation in the sense of Popescu and Martínez.

Weighted Lie algebroids out of reductions

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and consider the subbundle $\mathsf{T}^k\mathcal{G}^{\underline{s}} \subset \mathsf{T}^k\mathcal{G}$ consisting of all higher order velocities tangent to source-leaves. The bundle

$$F_k = A^k(\mathcal{G}) := \mathsf{T}^k\mathcal{G}^{\underline{s}}\Big|_M,$$

inherits graded bundle structure of degree k as a graded subbundle of $\mathsf{T}^k\mathcal{G}$. Of course, $A = A^1(\mathcal{G})$ can be identified with the Lie algebroid of $\mathcal{G}$.

Theorem

The linearisation of $A^k(\mathcal{G})$ is given as

$$D(A^k(\mathcal{G})) \simeq \{(Y, Z) \in A(\mathcal{G}) \times_M \mathsf{T}A^{k-1}(\mathcal{G})| \quad \rho(Y) = \mathsf{T}\tau(Z)\},$$

viewed as a vector bundle over $A^{k-1}(\mathcal{G})$ with respect to the obvious projection of part Z onto $A^{k-1}(\mathcal{G})$, where $\rho : A(\mathcal{G}) \to \mathsf{T}M$ is the standard anchor of the Lie algebroid and $\tau : A^{k-1}(\mathcal{G}) \to M$ is the obvious projection. Moreover, the above bundle is canonically a weighted Lie algebroid, a Lie algebroid prolongation in the sense of Popescu and Martínez.

Weighted Lie algebroids out of reductions

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and consider the subbundle $\mathsf{T}^k\mathcal{G}^{\underline{s}} \subset \mathsf{T}^k\mathcal{G}$ consisting of all higher order velocities tangent to source-leaves. The bundle

$$F_k = A^k(\mathcal{G}) := \mathsf{T}^k\mathcal{G}^{\underline{s}}\Big|_M ,$$

inherits graded bundle structure of degree k as a graded subbundle of $\mathsf{T}^k\mathcal{G}$. Of course, $A = A^1(\mathcal{G})$ can be identified with the Lie algebroid of $\mathcal{G}$.

Theorem

The linearisation of $A^k(\mathcal{G})$ is given as

$$D(A^k(\mathcal{G})) \simeq \{(Y,Z) \in A(\mathcal{G}) \times_M \mathsf{T}A^{k-1}(\mathcal{G}) | \quad \rho(Y) = \mathsf{T}\tau(Z)\},$$

viewed as a vector bundle over $A^{k-1}(\mathcal{G})$ with respect to the obvious projection of part Z onto $A^{k-1}(\mathcal{G})$, where $\rho : A(\mathcal{G}) \to \mathsf{T}M$ is the standard anchor of the Lie algebroid and $\tau : A^{k-1}(\mathcal{G}) \to M$ is the obvious projection. Moreover, the above bundle is canonically a weighted Lie algebroid, a Lie algebroid prolongation in the sense of Popescu and Martínez.

Weighted Lie algebroids out of reductions

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and consider the subbundle $\mathsf{T}^k\mathcal{G}^{\underline{s}} \subset \mathsf{T}^k\mathcal{G}$ consisting of all higher order velocities tangent to source-leaves. The bundle

$$F_k = A^k(\mathcal{G}) := \mathsf{T}^k\mathcal{G}^{\underline{s}}\Big|_M ,$$

inherits graded bundle structure of degree k as a graded subbundle of $\mathsf{T}^k\mathcal{G}$. Of course, $A = A^1(\mathcal{G})$ can be identified with the Lie algebroid of $\mathcal{G}$.

Theorem

The linearisation of $A^k(\mathcal{G})$ is given as

$$D(A^k(\mathcal{G})) \simeq \{(Y,Z) \in A(\mathcal{G}) \times_M \mathsf{T}A^{k-1}(\mathcal{G}) | \quad \rho(Y) = \mathsf{T}\tau(Z)\},$$

viewed as a vector bundle over $A^{k-1}(\mathcal{G})$ with respect to the obvious projection of part Z onto $A^{k-1}(\mathcal{G})$, where $\rho : A(\mathcal{G}) \to \mathsf{T}M$ is the standard anchor of the Lie algebroid and $\tau : A^{k-1}(\mathcal{G}) \to M$ is the obvious projection. Moreover, the above bundle is canonically a weighted Lie algebroid, a Lie algebroid prolongation in the sense of Popescu and Martínez.

Weighted Lie algebroids out of reductions

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and consider the subbundle $\mathsf{T}^k\mathcal{G}^{\underline{s}} \subset \mathsf{T}^k\mathcal{G}$ consisting of all higher order velocities tangent to source-leaves. The bundle

$$F_k = A^k(\mathcal{G}) := \mathsf{T}^k\mathcal{G}^{\underline{s}}\Big|_M ,$$

inherits graded bundle structure of degree k as a graded subbundle of $\mathsf{T}^k\mathcal{G}$. Of course, $A = A^1(\mathcal{G})$ can be identified with the Lie algebroid of $\mathcal{G}$.

Theorem

The linearisation of $A^k(\mathcal{G})$ is given as

$$D(A^k(\mathcal{G})) \simeq \{(Y, Z) \in A(\mathcal{G}) \times_M \mathsf{T}A^{k-1}(\mathcal{G}) | \quad \rho(Y) = \mathsf{T}\tau(Z)\} ,$$

viewed as a vector bundle over $A^{k-1}(\mathcal{G})$ with respect to the obvious projection of part Z onto $A^{k-1}(\mathcal{G})$, where $\rho : A(\mathcal{G}) \to \mathsf{T}M$ is the standard anchor of the Lie algebroid and $\tau : A^{k-1}(\mathcal{G}) \to M$ is the obvious projection. Moreover, the above bundle is canonically a weighted Lie algebroid, a Lie algebroid prolongation in the sense of Popescu and Martínez.

Lagrangian framework for graded bundles

A weighted Lie algebroid on $D(F_k)$ gives the Tulczyjew triple

Here, the diagram consists of relations, $\hat{\varepsilon} : \mathsf{T}^*F_k \longrightarrow\!\!\triangleright \mathsf{T}^*D(F_k) \to \mathsf{T}D^*(F_k)$, and $\mathrm{Mi}(F_k)$ is the so called Mironian of F_k. In the classical case, $\mathrm{Mi}(\mathsf{T}^kM) = \mathsf{T}^{k-1}M \times_M \mathsf{T}^*M$. Forget the Hamiltonian side.

$\mathcal{T}L$ is the Tulczyjew differential and λ_L the Legendre relation.

The fact that we obtain the Euler-Lagrange equations of higher order comes from the vakonomic constraint and the additional gradation.

Lagrangian framework for graded bundles

A weighted Lie algebroid on $D(F_k)$ gives the Tulczyjew triple

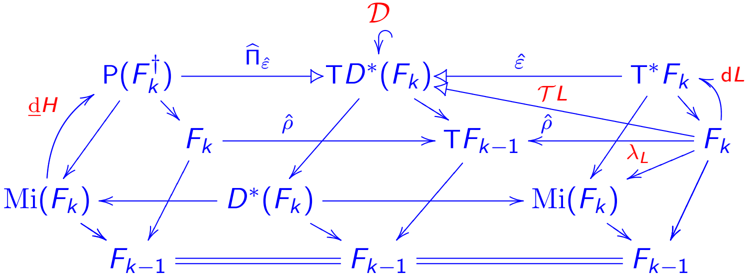
Here, the diagram consists of relations, $\hat\varepsilon : \mathsf{T}^*F_k \longrightarrow \triangleright \mathsf{T}^*D(F_k) \to \mathsf{T}D^*(F_k)$, and $\mathrm{Mi}(F_k)$ is the so called Mironian of F_k. In the classical case, $\mathrm{Mi}(\mathsf{T}^kM) = \mathsf{T}^{k-1}M \times_M \mathsf{T}^*M$. Forget the Hamiltonian side.

$\mathcal{T}L$ is the Tulczyjew differential and λ_L the Legendre relation.

The fact that we obtain the Euler-Lagrange equations of higher order comes from the vakonomic constraint and the additional gradation.

Lagrangian framework for graded bundles

A weighted Lie algebroid on $D(F_k)$ gives the Tulczyjew triple



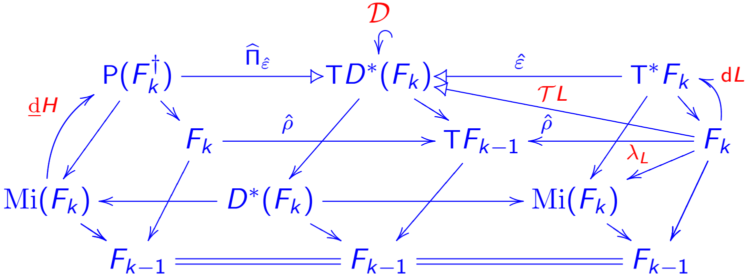
Here, the diagram consists of relations, $\hat{\varepsilon}: \mathrm{T}^*F_k \longrightarrow\!\!\triangleright \mathrm{T}^*D(F_k) \to \mathrm{T}D^*(F_k)$, and $\mathrm{Mi}(F_k)$ is the so called Mironian of F_k. In the classical case, $\mathrm{Mi}(\mathrm{T}^kM) = \mathrm{T}^{k-1}M \times_M \mathrm{T}^*M$. Forget the Hamiltonian side.

$\mathcal{T}L$ is the Tulczyjew differential and λ_L the Legendre relation.

The fact that we obtain the Euler-Lagrange equations of higher order comes from the vakonomic constraint and the additional gradation.

Lagrangian framework for graded bundles

A weighted Lie algebroid on $D(F_k)$ gives the Tulczyjew triple



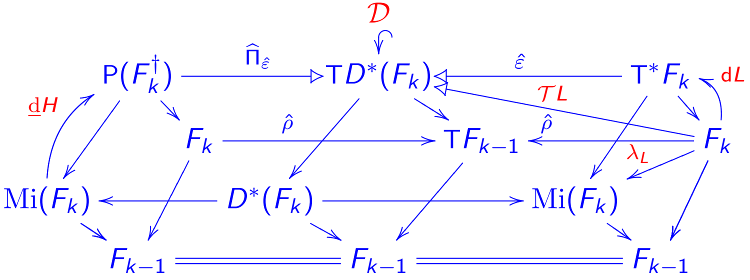
Here, the diagram consists of relations, $\hat\varepsilon : \mathsf{T}^*F_k \longrightarrow\!\triangleright\, \mathsf{T}^*D(F_k) \to \mathsf{T}D^*(F_k)$, and $\mathrm{Mi}(F_k)$ is the so called Mironian of F_k. In the classical case, $\mathrm{Mi}(\mathsf{T}^kM) = \mathsf{T}^{k-1}M \times_M \mathsf{T}^*M$. Forget the Hamiltonian side.

$\mathcal{T}L$ is the Tulczyjew differential and λ_L the Legendre relation.

The fact that we obtain the Euler-Lagrange equations of higher order comes from the vakonomic constraint and the additional gradation.

Lagrangian framework for graded bundles

A weighted Lie algebroid on $D(F_k)$ gives the Tulczyjew triple



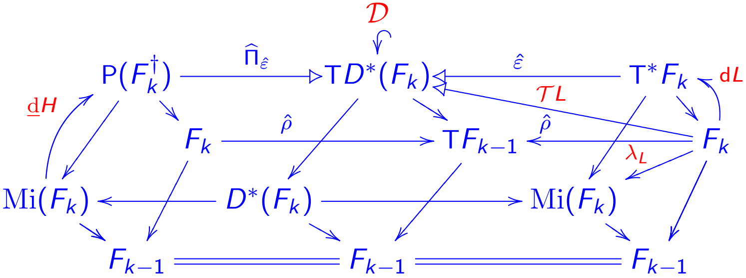
Here, the diagram consists of relations, $\hat{\varepsilon} : \mathsf{T}^*F_k \longrightarrow \mathsf{T}^*D(F_k) \to \mathsf{T}D^*(F_k)$, and $\mathrm{Mi}(F_k)$ is the so called Mironian of F_k. In the classical case, $\mathrm{Mi}(\mathsf{T}^kM) = \mathsf{T}^{k-1}M \times_M \mathsf{T}^*M$. Forget the Hamiltonian side.

$\mathcal{T}L$ is the Tulczyjew differential and λ_L the Legendre relation.

The fact that we obtain the Euler-Lagrange equations of higher order comes from the vakonomic constraint and the additional gradation.

Lagrangian framework for graded bundles

A weighted Lie algebroid on $D(F_k)$ gives the Tulczyjew triple



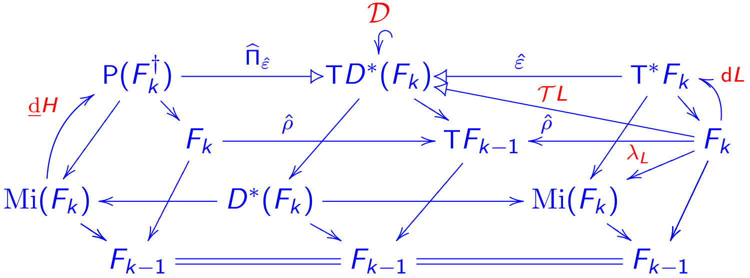
Here, the diagram consists of relations, $\hat\varepsilon : \mathsf{T}^*F_k \longrightarrow\!\!\triangleright\, \mathsf{T}^*D(F_k) \to \mathsf{T}D^*(F_k)$, and $\mathrm{Mi}(F_k)$ is the so called Mironian of F_k. In the classical case, $\mathrm{Mi}(\mathsf{T}^kM) = \mathsf{T}^{k-1}M \times_M \mathsf{T}^*M$. Forget the Hamiltonian side.

$\mathcal{T}L$ is the Tulczyjew differential and λ_L the Legendre relation.

The fact that we obtain the Euler-Lagrange equations of higher order comes from the vakonomic constraint and the additional gradation.

Lagrangian framework for graded bundles

A weighted Lie algebroid on $D(F_k)$ gives the Tulczyjew triple



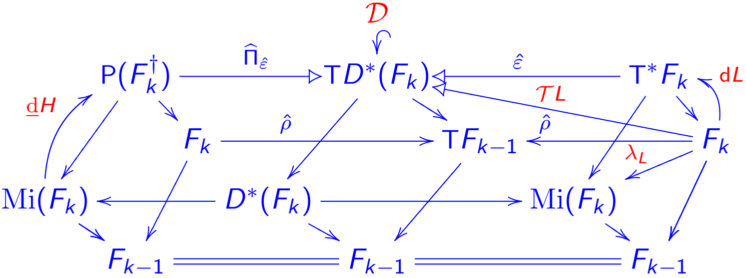
Here, the diagram consists of relations, $\hat\varepsilon : \mathsf{T}^*F_k \longrightarrow\!\triangleright \mathsf{T}^*D(F_k) \to \mathsf{T}D^*(F_k)$, and $\mathrm{Mi}(F_k)$ is the so called Mironian of F_k. In the classical case, $\mathrm{Mi}(\mathsf{T}^kM) = \mathsf{T}^{k-1}M \times_M \mathsf{T}^*M$. Forget the Hamiltonian side.

$\mathcal{T}L$ is the Tulczyjew differential and λ_L the Legendre relation.

The fact that we obtain the Euler-Lagrange equations of higher order comes from the vakonomic constraint and the additional gradation.

Lagrangian framework for graded bundles

A weighted Lie algebroid on $D(F_k)$ gives the Tulczyjew triple



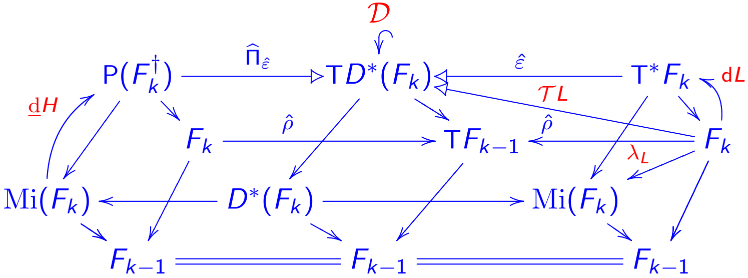
Here, the diagram consists of relations, $\hat\varepsilon : \mathsf{T}^*F_k \longrightarrow\!\!\triangleright \mathsf{T}^*D(F_k) \to \mathsf{T}D^*(F_k)$, and $\mathrm{Mi}(F_k)$ is the so called Mironian of F_k. In the classical case, $\mathrm{Mi}(\mathsf{T}^kM) = \mathsf{T}^{k-1}M \times_M \mathsf{T}^*M$. Forget the Hamiltonian side.

$\mathcal{T}L$ is the Tulczyjew differential and λ_L the Legendre relation.

The fact that we obtain the Euler-Lagrange equations of higher order comes from the vakonomic constraint and the additional gradation.

Lagrangian framework for graded bundles

A weighted Lie algebroid on $D(F_k)$ gives the Tulczyjew triple



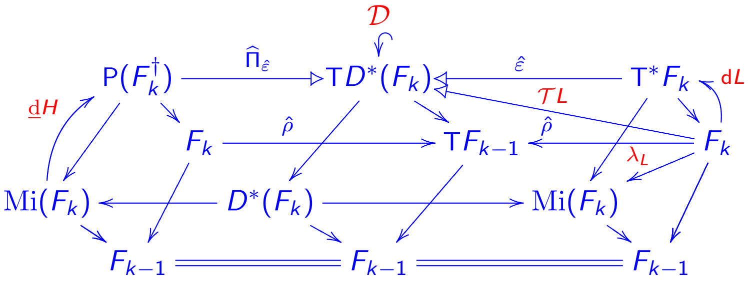
Here, the diagram consists of relations, $\hat{\varepsilon} : \mathsf{T}^*F_k \longrightarrow\!\!\triangleright \mathsf{T}^*D(F_k) \to \mathsf{T}D^*(F_k)$, and $\mathrm{Mi}(F_k)$ is the so called Mironian of F_k. In the classical case, $\mathrm{Mi}(\mathsf{T}^kM) = \mathsf{T}^{k-1}M \times_M \mathsf{T}^*M$. Forget the Hamiltonian side.

$\mathcal{T}L$ is the Tulczyjew differential and λ_L the Legendre relation.

The fact that we obtain the Euler-Lagrange equations of higher order comes from the vakonomic constraint and the additional gradation.

Lagrangian framework for graded bundles

A weighted Lie algebroid on $D(F_k)$ gives the Tulczyjew triple



Here, the diagram consists of relations, $\hat{\varepsilon}: \mathsf{T}^*F_k \longrightarrow\!\triangleright \mathsf{T}^*D(F_k) \to \mathsf{T}D^*(F_k)$, and $\mathrm{Mi}(F_k)$ is the so called Mironian of F_k. In the classical case, $\mathrm{Mi}(\mathsf{T}^kM) = \mathsf{T}^{k-1}M \times_M \mathsf{T}^*M$. Forget the Hamiltonian side.

$\mathcal{T}L$ is the Tulczyjew differential and λ_L the Legendre relation.

The fact that we obtain the Euler-Lagrange equations of higher order comes from the vakonomic constraint and the additional gradation.

Example

Let g be a Lie algebra and put $F_2 = g_2 = g[1] \times g[2]$, with coordinates (x^i, z^j) on g_2 and coordinates (x^i, y^j, z^k) on $D(g_2) = g[1] \times g[1] \times g[2]$.

The embedding $\iota : g_2 \hookrightarrow D(g_2)$ takes the form $\iota(x, z) = (x, x, z)$ and the vector bundle projection is $\tau(x, y, z) = x$.

The Lie algebroid structure $\varepsilon : \mathsf{T}^* D(g_2) \to \mathsf{T} D^*(g_2)$ reads

$$(x, y, z, \alpha, \beta, \gamma) \mapsto (x, \beta, \gamma, z, \mathrm{ad}^*_y \beta, \alpha) .$$

Given a Lagrangian $L : g_2 \to \mathbb{R}$, the Tulczyjew differential relation $\mathcal{T} L : g_2 \to \mathsf{T} D^*(g_2)$ is

$$\mathcal{T} L(x, z) = \left\{ \left(x, \beta, \frac{\partial L}{\partial z}(x, z), z, \mathrm{ad}^*_x \beta, \alpha \right) : \alpha + \beta = \frac{\partial L}{\partial x}(x, z) \right\} .$$

Hence, for the phase dynamics,

$$\beta = \frac{\partial L}{\partial x}(x, z) - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial z}(x, z) \right) .$$

Example

Let g be a Lie algebra and put $F_2 = g_2 = g[1] \times g[2]$, with coordinates (x^i, z^j) on g_2 and coordinates (x^i, y^j, z^k) on $D(g_2) = g[1] \times g[1] \times g[2]$.

The embedding $\iota : g_2 \hookrightarrow D(g_2)$ takes the form $\iota(x, z) = (x, x, z)$ and the vector bundle projection is $\tau(x, y, z) = x$.

The Lie algebroid structure $\varepsilon : \mathsf{T}^* D(g_2) \to \mathsf{T} D^*(g_2)$ reads

$$(x, y, z, \alpha, \beta, \gamma) \mapsto (x, \beta, \gamma, z, \mathrm{ad}^*_y \beta, \alpha).$$

Given a Lagrangian $L : g_2 \to \mathbb{R}$, the Tulczyjew differential relation $\mathcal{T}L : g_2 \to \mathsf{T} D^*(g_2)$ is

$$\mathcal{T}L(x, z) = \left\{ \left(x, \beta, \frac{\partial L}{\partial z}(x, z), z, \mathrm{ad}^*_x \beta, \alpha \right) : \alpha + \beta = \frac{\partial L}{\partial x}(x, z) \right\}.$$

Hence, for the phase dynamics,

$$\beta = \frac{\partial L}{\partial x}(x, z) - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial z}(x, z) \right).$$

Example

Let $\mathfrak{g}$ be a Lie algebra and put $F_2 = \mathfrak{g}_2 = \mathfrak{g}[1] \times \mathfrak{g}[2]$, with coordinates (x^i, z^j) on $\mathfrak{g}_2$ and coordinates (x^i, y^j, z^k) on $D(\mathfrak{g}_2) = \mathfrak{g}[1] \times \mathfrak{g}[1] \times \mathfrak{g}[2]$.

The embedding $\iota : \mathfrak{g}_2 \hookrightarrow D(\mathfrak{g}_2)$ takes the form $\iota(x, z) = (x, x, z)$ and the vector bundle projection is $\tau(x, y, z) = x$.

The Lie algebroid structure $\varepsilon : \mathsf{T}^*D(\mathfrak{g}_2) \to \mathsf{T}D^*(\mathfrak{g}_2)$ reads

$$(x, y, z, \alpha, \beta, \gamma) \mapsto (x, \beta, \gamma, z, \mathrm{ad}^*_y \beta, \alpha) .$$

Given a Lagrangian $L : \mathfrak{g}_2 \to \mathbb{R}$, the Tulczyjew differential relation $\mathcal{T}L : \mathfrak{g}_2 \to \mathsf{T}D^*(\mathfrak{g}_2)$ is

$$\mathcal{T}L(x, z) = \left\{ \left(x, \beta, \frac{\partial L}{\partial z}(x, z), z, \mathrm{ad}^*_x \beta, \alpha \right) : \alpha + \beta = \frac{\partial L}{\partial x}(x, z) \right\} .$$

Hence, for the phase dynamics,

$$\beta = \frac{\partial L}{\partial x}(x, z) - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial z}(x, z) \right) .$$

Example

Let g be a Lie algebra and put $F_2 = g_2 = g[1] \times g[2]$, with coordinates (x^i, z^j) on g_2 and coordinates (x^i, y^j, z^k) on $D(g_2) = g[1] \times g[1] \times g[2]$.

The embedding $\iota : g_2 \hookrightarrow D(g_2)$ takes the form $\iota(x, z) = (x, x, z)$ and the vector bundle projection is $\tau(x, y, z) = x$.

The Lie algebroid structure $\varepsilon : \mathsf{T}^* D(g_2) \to \mathsf{T} D^*(g_2)$ reads

$$(x, y, z, \alpha, \beta, \gamma) \mapsto (x, \beta, \gamma, z, \mathrm{ad}^*_y \beta, \alpha) .$$

Given a Lagrangian $L : g_2 \to \mathbb{R}$, the Tulczyjew differential relation $\mathcal{T} L : g_2 \to \mathsf{T} D^*(g_2)$ is

$$\mathcal{T} L(x, z) = \left\{ \left(x, \beta, \frac{\partial L}{\partial z}(x, z), z, \mathrm{ad}^*_x \beta, \alpha \right) : \alpha + \beta = \frac{\partial L}{\partial x}(x, z) \right\} .$$

Hence, for the phase dynamics,

$$\beta = \frac{\partial L}{\partial x}(x, z) - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial z}(x, z) \right) .$$

Example

Let g be a Lie algebra and put $F_2 = g_2 = g[1] \times g[2]$, with coordinates (x^i, z^j) on g_2 and coordinates (x^i, y^j, z^k) on $D(g_2) = g[1] \times g[1] \times g[2]$.

The embedding $\iota : g_2 \hookrightarrow D(g_2)$ takes the form $\iota(x,z) = (x,x,z)$ and the vector bundle projection is $\tau(x,y,z) = x$.

The Lie algebroid structure $\varepsilon : \mathsf{T}^*D(g_2) \to \mathsf{T}D^*(g_2)$ reads

$$(x,y,z,\alpha,\beta,\gamma) \mapsto (x,\beta,\gamma,z,\mathrm{ad}^*_y\beta,\alpha)\,.$$

Given a Lagrangian $L : g_2 \to \mathbb{R}$, the Tulczyjew differential relation $\mathcal{T}L : g_2 \to \mathsf{T}D^*(g_2)$ is

$$\mathcal{T}L(x,z) = \left\{ \left(x,\beta,\frac{\partial L}{\partial z}(x,z), z, \mathrm{ad}^*_x\beta, \alpha \right) : \alpha + \beta = \frac{\partial L}{\partial x}(x,z) \right\}\,.$$

Hence, for the phase dynamics,

$$\beta = \frac{\partial L}{\partial x}(x,z) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,z)\right)\,.$$

Example

Let g be a Lie algebra and put $F_2 = g_2 = g[1] \times g[2]$, with coordinates (x^i, z^j) on g_2 and coordinates (x^i, y^j, z^k) on $D(g_2) = g[1] \times g[1] \times g[2]$.

The embedding $\iota : g_2 \hookrightarrow D(g_2)$ takes the form $\iota(x, z) = (x, x, z)$ and the vector bundle projection is $\tau(x, y, z) = x$.

The Lie algebroid structure $\varepsilon : \mathsf{T}^* D(g_2) \to \mathsf{T} D^*(g_2)$ reads

$$(x, y, z, \alpha, \beta, \gamma) \mapsto (x, \beta, \gamma, z, \mathrm{ad}^*_y \beta, \alpha)\,.$$

Given a Lagrangian $L : g_2 \to \mathbb{R}$, the Tulczyjew differential relation $\mathcal{T}L : g_2 \to \mathsf{T} D^*(g_2)$ is

$$\mathcal{T}L(x, z) = \left\{ \left(x, \beta, \frac{\partial L}{\partial z}(x, z), z, \mathrm{ad}^*_x \beta, \alpha \right) : \alpha + \beta = \frac{\partial L}{\partial x}(x, z) \right\}\,.$$

Hence, for the phase dynamics,

$$\beta = \frac{\partial L}{\partial x}(x, z) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x, z) \right)\,.$$

Example

This leads to the Euler-Lagrange equations on $\mathfrak{g}_2$:

$$\begin{aligned} \dot{x} &= z\,, \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x}(x,z) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,z)\right)\right) &= \mathrm{ad}^*_x\left(\frac{\partial L}{\partial x}(x,z) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,z)\right)\right). \end{aligned}$$

These equations are second order and induce the Euler-Lagrange equations on $\mathfrak{g}$ which are of order 3:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x}(x,\dot{x}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,\dot{x})\right)\right) = \mathrm{ad}^*_x\left(\frac{\partial L}{\partial x}(x,\dot{x}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,\dot{x})\right)\right).$$

For instance, the 'free' Lagrangian $L(x,z) = \frac{1}{2}\sum_i I_i(z^i)^2$ induces the equations on $\mathfrak{g}$ (c^k_{ij} are structure constants, no summation convention):

$$I_j \dddot{x}^j = \sum_{i,k} c^k_{ij} I_k x^i \ddot{x}^k\,.$$

The latter can be viewed as 'higher Euler equations'.

Example

This leads to the Euler-Lagrange equations on g_2:

$$\dot{x} = z\,,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x}(x,z) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,z)\right)\right) = \mathrm{ad}^*_x\left(\frac{\partial L}{\partial x}(x,z) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,z)\right)\right).$$

These equations are second order and induce the Euler-Lagrange equations on g which are of order 3:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x}(x,\dot{x}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,\dot{x})\right)\right) = \mathrm{ad}^*_x\left(\frac{\partial L}{\partial x}(x,\dot{x}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,\dot{x})\right)\right).$$

For instance, the 'free' Lagrangian $L(x,z) = \frac{1}{2}\sum_i I_i(z^i)^2$ induces the equations on g (c^k_{ij} are structure constants, no summation convention):

$$I_j \dddot{x}^j = \sum_{i,k} c^k_{ij} I_k x^i \ddot{x}^k\,.$$

The latter can be viewed as 'higher Euler equations'.

Example

This leads to the Euler-Lagrange equations on $\mathfrak{g}_2$:

$$\begin{aligned} \dot{x} &= z\,, \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x}(x,z) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,z)\right)\right) &= \mathrm{ad}^*_x\left(\frac{\partial L}{\partial x}(x,z) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,z)\right)\right). \end{aligned}$$

These equations are second order and induce the Euler-Lagrange equations on $\mathfrak{g}$ which are of order 3:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x}(x,\dot{x}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,\dot{x})\right)\right) = \mathrm{ad}^*_x\left(\frac{\partial L}{\partial x}(x,\dot{x}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,\dot{x})\right)\right).$$

For instance, the 'free' Lagrangian $L(x,z) = \frac{1}{2}\sum_i I_i(z^i)^2$ induces the equations on $\mathfrak{g}$ (c^k_{ij} are structure constants, no summation convention):

$$I_j \dddot{x}^j = \sum_{i,k} c^k_{ij} I_k x^i \ddot{x}^k\,.$$

The latter can be viewed as 'higher Euler equations'.

Example

This leads to the Euler-Lagrange equations on g_2:

$$\dot{x} = z\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x}(x,z)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,z)\right)\right) = \mathrm{ad}^*_x\left(\frac{\partial L}{\partial x}(x,z)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,z)\right)\right).$$

These equations are second order and induce the Euler-Lagrange equations on g which are of order 3:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x}(x,\dot{x})-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,\dot{x})\right)\right) = \mathrm{ad}^*_x\left(\frac{\partial L}{\partial x}(x,\dot{x})-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,\dot{x})\right)\right).$$

For instance, the 'free' Lagrangian $L(x,z)=\frac{1}{2}\sum_i I_i(z^i)^2$ induces the equations on g (c^k_{ij} are structure constants, no summation convention):

$$I_j\dddot{x}^j=\sum_{i,k}c^k_{ij}I_k x^i\ddot{x}^k\,.$$

The latter can be viewed as 'higher Euler equations'.

Example

This leads to the Euler-Lagrange equations on g_2:

$$\begin{aligned} \dot{x} &= z\,, \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x}(x,z) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,z)\right)\right) &= \mathrm{ad}^*_x\left(\frac{\partial L}{\partial x}(x,z) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,z)\right)\right). \end{aligned}$$

These equations are second order and induce the Euler-Lagrange equations on g which are of order 3:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x}(x,\dot{x}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,\dot{x})\right)\right) = \mathrm{ad}^*_x\left(\frac{\partial L}{\partial x}(x,\dot{x}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,\dot{x})\right)\right).$$

For instance, the 'free' Lagrangian $L(x,z) = \frac{1}{2}\sum_i I_i(z^i)^2$ induces the equations on g (c^k_{ij} are structure constants, no summation convention):

$$I_j \dddot{x}^j = \sum_{i,k} c^k_{ij} I_k x^i \ddot{x}^k\,.$$

The latter can be viewed as 'higher Euler equations'.

Example

This leads to the Euler-Lagrange equations on g_2:

$$\begin{aligned} \dot{x} &= z\,, \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x}(x,z)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,z)\right)\right) &= \mathrm{ad}^*_x\left(\frac{\partial L}{\partial x}(x,z)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,z)\right)\right). \end{aligned}$$

These equations are second order and induce the Euler-Lagrange equations on g which are of order 3:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x}(x,\dot{x})-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,\dot{x})\right)\right)=\mathrm{ad}^*_x\left(\frac{\partial L}{\partial x}(x,\dot{x})-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,\dot{x})\right)\right).$$

For instance, the 'free' Lagrangian $L(x,z)=\frac{1}{2}\sum_i I_i(z^i)^2$ induces the equations on g (c^k_{ij} are structure constants, no summation convention):

$$I_j\dddot{x}^j=\sum_{i,k}c^k_{ij}I_kx^i\ddot{x}^k\,.$$

The latter can be viewed as 'higher Euler equations'.

Example

This leads to the Euler-Lagrange equations on $\mathfrak{g}_2$:

$$\begin{aligned} \dot{x} &= z\,, \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x}(x,z) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,z)\right)\right) &= \mathrm{ad}^*_x\left(\frac{\partial L}{\partial x}(x,z) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,z)\right)\right). \end{aligned}$$

These equations are second order and induce the Euler-Lagrange equations on $\mathfrak{g}$ which are of order 3:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x}(x,\dot{x}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,\dot{x})\right)\right) = \mathrm{ad}^*_x\left(\frac{\partial L}{\partial x}(x,\dot{x}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,\dot{x})\right)\right).$$

For instance, the 'free' Lagrangian $L(x,z) = \frac{1}{2}\sum_i I_i(z^i)^2$ induces the equations on $\mathfrak{g}$ (c^k_{ij} are structure constants, no summation convention):

$$I_j \dddot{x}^j = \sum_{i,k} c^k_{ij} I_k x^i \ddot{x}^k\,.$$

The latter can be viewed as 'higher Euler equations'.

Higher order Lagrangian mechanics on Lie algebroids

Let us consider a general Lie groupoid $\mathcal{G}$ and a Lagrangian $L : A^k \to \mathbb{R}$ on $A^k = A^k(\mathcal{G})$. We will refer to such systems as a k-th order Lagrangian system on the Lie algebroid $A(\mathcal{G})$. The relevant diagram here is

Here, $D(A^k(\mathcal{G}))$ is the corresponding Lie algebroid prolongation, $\mathcal{D} = \varepsilon \circ r \circ \mathrm{d}L(A^k(\mathcal{G}))$, and λ_L is the Legendre relation.

Note that we deal with reductions: in the case $\mathcal{G}$ is a Lie group,

$$A^k(\mathcal{G}) = \mathsf{T}^k(\mathcal{G})/\mathcal{G} \quad \text{and} \quad D(A^k(\mathcal{G})) = \mathsf{T}\mathsf{T}^{k-1}(\mathcal{G})/\mathcal{G}\,.$$

Higher order Lagrangian mechanics on Lie algebroids

Let us consider a general Lie groupoid $\mathcal{G}$ and a Lagrangian $L : A^k \to \mathbb{R}$ on $A^k = A^k(\mathcal{G})$. We will refer to such systems as a k-th order Lagrangian system on the Lie algebroid $A(\mathcal{G})$. The relevant diagram here is

Here, $D(A^k(\mathcal{G}))$ is the corresponding Lie algebroid prolongation, $\mathcal{D} = \varepsilon \circ r \circ \mathrm{d}L(A^k(\mathcal{G}))$, and λ_L is the Legendre relation.

Note that we deal with reductions: in the case $\mathcal{G}$ is a Lie group,

$$A^k(\mathcal{G}) = \mathsf{T}^k(\mathcal{G})/\mathcal{G} \quad \text{and} \quad D(A^k(\mathcal{G})) = \mathsf{T}\mathsf{T}^{k-1}(\mathcal{G})/\mathcal{G}\,.$$

Higher order Lagrangian mechanics on Lie algebroids

Let us consider a general Lie groupoid $\mathcal{G}$ and a Lagrangian $L: A^k \to \mathbb{R}$ on $A^k = A^k(\mathcal{G})$. We will refer to such systems as a k-th order Lagrangian system on the Lie algebroid $A(\mathcal{G})$. The relevant diagram here is

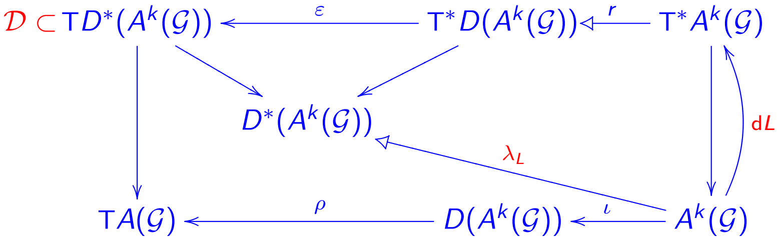
Here, $D(A^k(\mathcal{G}))$ is the corresponding Lie algebroid prolongation, $\mathcal{D} = \varepsilon \circ r \circ \mathrm{d}L(A^k(\mathcal{G}))$, and λ_L is the Legendre relation.

Note that we deal with reductions: in the case $\mathcal{G}$ is a Lie group,

$$A^k(\mathcal{G}) = \mathsf{T}^k(\mathcal{G})/\mathcal{G} \quad \text{and} \quad D(A^k(\mathcal{G})) = \mathsf{T}\mathsf{T}^{k-1}(\mathcal{G})/\mathcal{G}\,.$$

Higher order Lagrangian mechanics on Lie algebroids

Let us consider a general Lie groupoid $\mathcal{G}$ and a Lagrangian $L : A^k \to \mathbb{R}$ on $A^k = A^k(\mathcal{G})$. We will refer to such systems as a k-th order Lagrangian system on the Lie algebroid $A(\mathcal{G})$. The relevant diagram here is



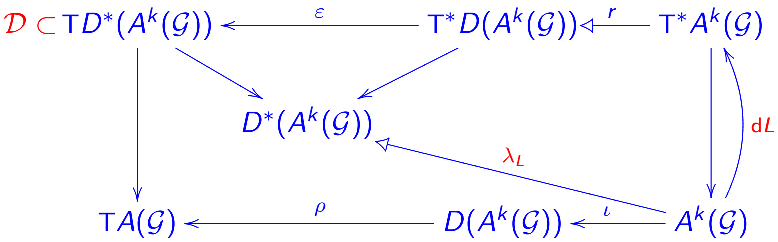
Here, $D(A^k(\mathcal{G}))$ is the corresponding Lie algebroid prolongation, $\mathcal{D} = \varepsilon \circ r \circ \mathrm{d}L(A^k(\mathcal{G}))$, and λ_L is the Legendre relation.

Note that we deal with reductions: in the case $\mathcal{G}$ is a Lie group,

$$A^k(\mathcal{G}) = \mathsf{T}^k(\mathcal{G})/\mathcal{G} \quad \text{and} \quad D(A^k(\mathcal{G})) = \mathsf{T}\mathsf{T}^{k-1}(\mathcal{G})/\mathcal{G}\,.$$

Higher order Lagrangian mechanics on Lie algebroids

Let us consider a general Lie groupoid $\mathcal{G}$ and a Lagrangian $L : A^k \to \mathbb{R}$ on $A^k = A^k(\mathcal{G})$. We will refer to such systems as a k-th order Lagrangian system on the Lie algebroid $A(\mathcal{G})$. The relevant diagram here is



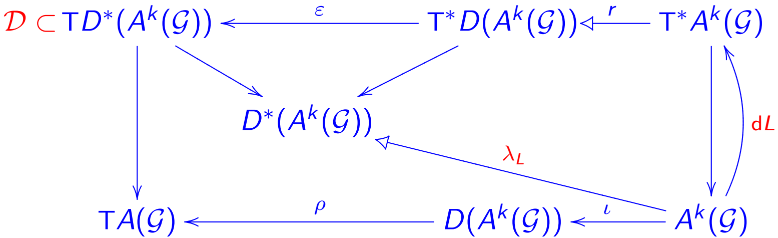
Here, $D(A^k(\mathcal{G}))$ is the corresponding Lie algebroid prolongation, $\mathcal{D} = \varepsilon \circ r \circ \mathrm{d}L(A^k(\mathcal{G}))$, and λ_L is the Legendre relation.

Note that we deal with reductions: in the case $\mathcal{G}$ is a Lie group,

$$A^k(\mathcal{G}) = \mathsf{T}^k(\mathcal{G})/\mathcal{G} \quad \text{and} \quad D(A^k(\mathcal{G})) = \mathsf{T}\mathsf{T}^{k-1}(\mathcal{G})/\mathcal{G}\,.$$

Higher order Lagrangian mechanics on Lie algebroids

Let us consider a general Lie groupoid $\mathcal{G}$ and a Lagrangian $L : A^k \to \mathbb{R}$ on $A^k = A^k(\mathcal{G})$. We will refer to such systems as a k-th order Lagrangian system on the Lie algebroid $A(\mathcal{G})$. The relevant diagram here is



Here, $D(A^k(\mathcal{G}))$ is the corresponding Lie algebroid prolongation, $\mathcal{D} = \varepsilon \circ r \circ \mathrm{d}L(A^k(\mathcal{G}))$, and λ_L is the Legendre relation.

Note that we deal with reductions: in the case $\mathcal{G}$ is a Lie group,

$$A^k(\mathcal{G}) = \mathsf{T}^k(\mathcal{G})/\mathcal{G} \quad \text{and} \quad D(A^k(\mathcal{G})) = \mathsf{T}\mathsf{T}^{k-1}(\mathcal{G})/\mathcal{G}\,.$$

Higher order Lagrangian mechanics on Lie algebroids

For instance, using x^A as base coordinates, and y_i^a as fibre coordinates of degree $i = 1, \ldots, k$ in A^k, extended by the appropriate momenta π_b^j of degree $j = 1, \ldots, k$ in $D^*(A^k)$, we get the equations for the Legendre relation in the form (no Lie algebroid structure appears!):

$$
\begin{aligned}
& k\pi_a^1 = \frac{\partial L}{\partial y_k^a}, \\
& (k-1)\pi_b^2 = \frac{\partial L}{\partial y_{k-1}^b} - \frac{1}{k}\frac{d}{dt}\left(\frac{\partial L}{\partial y_k^b}\right), \\
& \vdots \\
& \pi_d^k = \frac{\partial L}{\partial y_1^d} - \frac{1}{2!}\frac{d}{dt}\left(\frac{\partial L}{\partial y_2^d}\right) + \frac{1}{3!}\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial y_3^d}\right) - \cdots \\
& +(-1)^k\frac{1}{(k-1)!}\frac{d^{k-2}}{dt^{k-2}}\left(\frac{\partial L}{\partial y_{k-1}^d}\right) - (-1)^k\frac{1}{k!}\frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial L}{\partial y_k^d}\right),
\end{aligned}
$$

which we recognise as the Jacobi–Ostrogradski momenta.

Higher order Lagrangian mechanics on Lie algebroids

For instance, using x^A as base coordinates, and y_i^a as fibre coordinates of degree $i = 1, \ldots, k$ in A^k, extended by the appropriate momenta π_b^j of degree $j = 1, \ldots, k$ in $D^*(A^k)$, we get the equations for the Legendre relation in the form (no Lie algebroid structure appears!):

$$
\begin{aligned}
& k\pi_a^1 = \frac{\partial L}{\partial y_k^a}, \\
& (k-1)\pi_b^2 = \frac{\partial L}{\partial y_{k-1}^b} - \frac{1}{k}\frac{d}{dt}\left(\frac{\partial L}{\partial y_k^b}\right), \\
& \vdots \\
& \pi_d^k = \frac{\partial L}{\partial y_1^d} - \frac{1}{2!}\frac{d}{dt}\left(\frac{\partial L}{\partial y_2^d}\right) + \frac{1}{3!}\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial y_3^d}\right) - \cdots \\
& +(-1)^k \frac{1}{(k-1)!}\frac{d^{k-2}}{dt^{k-2}}\left(\frac{\partial L}{\partial y_{k-1}^d}\right) - (-1)^k \frac{1}{k!}\frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial L}{\partial y_k^d}\right),
\end{aligned}
$$

which we recognise as the Jacobi–Ostrogradski momenta.

Higher order Lagrangian mechanics on Lie algebroids

For instance, using x^A as base coordinates, and y^a_i as fibre coordinates of degree $i = 1, \ldots, k$ in A^k, extended by the appropriate momenta π^j_b of degree $j = 1, \ldots, k$ in $D^*(A^k)$, we get the equations for the Legendre relation in the form (no Lie algebroid structure appears!):

$$
\begin{aligned}
&k\pi^1_a = \frac{\partial L}{\partial y^a_k},\\
&(k-1)\pi^2_b = \frac{\partial L}{\partial y^b_{k-1}} - \frac{1}{k}\frac{d}{dt}\left(\frac{\partial L}{\partial y^b_k}\right),\\
&\vdots\\
&\pi^k_d = \frac{\partial L}{\partial y^d_1} - \frac{1}{2!}\frac{d}{dt}\left(\frac{\partial L}{\partial y^d_2}\right) + \frac{1}{3!}\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial y^d_3}\right) - \cdots\\
&+(-1)^k\frac{1}{(k-1)!}\frac{d^{k-2}}{dt^{k-2}}\left(\frac{\partial L}{\partial y^d_{k-1}}\right) - (-1)^k\frac{1}{k!}\frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial L}{\partial y^d_k}\right),
\end{aligned}
$$

which we recognise as the Jacobi–Ostrogradski momenta.

For instance, using x^A as base coordinates, and y^a_i as fibre coordinates of degree $i = 1, \ldots, k$ in A^k, extended by the appropriate momenta π^j_b of degree $j = 1, \ldots, k$ in $D^*(A^k)$, we get the equations for the Legendre relation in the form (no Lie algebroid structure appears!):

$$
\begin{aligned}
k\pi^1_a &= \frac{\partial L}{\partial y^a_k},\\
(k-1)\pi^2_b &= \frac{\partial L}{\partial y^b_{k-1}} - \frac{1}{k}\frac{d}{dt}\left(\frac{\partial L}{\partial y^b_k}\right),\\
&\vdots\\
\pi^k_d &= \frac{\partial L}{\partial y^d_1} - \frac{1}{2!}\frac{d}{dt}\left(\frac{\partial L}{\partial y^d_2}\right) + \frac{1}{3!}\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial y^d_3}\right) - \cdots\\
&+(-1)^k\frac{1}{(k-1)!}\frac{d^{k-2}}{dt^{k-2}}\left(\frac{\partial L}{\partial y^d_{k-1}}\right) - (-1)^k\frac{1}{k!}\frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial L}{\partial y^d_k}\right),
\end{aligned}
$$

which we recognise as the Jacobi–Ostrogradski momenta.

Higher order Lagrangian mechanics on Lie algebroids

For instance, using x^A as base coordinates, and y_i^a as fibre coordinates of degree $i = 1, \ldots, k$ in A^k, extended by the appropriate momenta π_b^j of degree $j = 1, \ldots, k$ in $D^*(A^k)$, we get the equations for the Legendre relation in the form (no Lie algebroid structure appears!):

$$
\begin{aligned}
&k\pi_a^1 = \frac{\partial L}{\partial y_k^a},\\
&(k-1)\pi_b^2 = \frac{\partial L}{\partial y_{k-1}^b} - \frac{1}{k}\frac{d}{dt}\left(\frac{\partial L}{\partial y_k^b}\right),\\
&\vdots\\
&\pi_d^k = \frac{\partial L}{\partial y_1^d} - \frac{1}{2!}\frac{d}{dt}\left(\frac{\partial L}{\partial y_2^d}\right) + \frac{1}{3!}\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial y_3^d}\right) - \cdots\\
&+(-1)^k\frac{1}{(k-1)!}\frac{d^{k-2}}{dt^{k-2}}\left(\frac{\partial L}{\partial y_{k-1}^d}\right) - (-1)^k\frac{1}{k!}\frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial L}{\partial y_k^d}\right),
\end{aligned}
$$

which we recognise as the Jacobi–Ostrogradski momenta.

Higher order Lagrangian mechanics on Lie algebroids

For instance, using x^A as base coordinates, and y_i^a as fibre coordinates of degree $i = 1, \ldots, k$ in A^k, extended by the appropriate momenta π_b^j of degree $j = 1, \ldots, k$ in $D^*(A^k)$, we get the equations for the Legendre relation in the form (no Lie algebroid structure appears!):

$$k\pi_a^1 = \frac{\partial L}{\partial y_k^a},$$

$$(k-1)\pi_b^2 = \frac{\partial L}{\partial y_{k-1}^b} - \frac{1}{k}\frac{d}{dt}\left(\frac{\partial L}{\partial y_k^b}\right),$$

$$\vdots$$

$$\pi_d^k = \frac{\partial L}{\partial y_1^d} - \frac{1}{2!}\frac{d}{dt}\left(\frac{\partial L}{\partial y_2^d}\right) + \frac{1}{3!}\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial y_3^d}\right) - \cdots$$
$$+(-1)^k\frac{1}{(k-1)!}\frac{d^{k-2}}{dt^{k-2}}\left(\frac{\partial L}{\partial y_{k-1}^d}\right) - (-1)^k\frac{1}{k!}\frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial L}{\partial y_k^d}\right),$$

which we recognise as the Jacobi–Ostrogradski momenta.

Higher order Lagrangian mechanics on Lie algebroids

For instance, using x^A as base coordinates, and y_i^a as fibre coordinates of degree $i = 1, \ldots, k$ in A^k, extended by the appropriate momenta π_b^j of degree $j = 1, \ldots, k$ in $D^*(A^k)$, we get the equations for the Legendre relation in the form (no Lie algebroid structure appears!):

$$
\begin{aligned}
&k\pi_a^1 = \frac{\partial L}{\partial y_k^a},\\
&(k-1)\pi_b^2 = \frac{\partial L}{\partial y_{k-1}^b} - \frac{1}{k}\frac{d}{dt}\left(\frac{\partial L}{\partial y_k^b}\right),\\
&\vdots\\
&\pi_d^k = \frac{\partial L}{\partial y_1^d} - \frac{1}{2!}\frac{d}{dt}\left(\frac{\partial L}{\partial y_2^d}\right) + \frac{1}{3!}\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial y_3^d}\right) - \cdots\\
&+(-1)^k\frac{1}{(k-1)!}\frac{d^{k-2}}{dt^{k-2}}\left(\frac{\partial L}{\partial y_{k-1}^d}\right) - (-1)^k\frac{1}{k!}\frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial L}{\partial y_k^d}\right),
\end{aligned}
$$

which we recognise as the Jacobi–Ostrogradski momenta.

Higher order Lagrangian mechanics on Lie algebroids

For instance, using x^A as base coordinates, and y_i^a as fibre coordinates of degree $i = 1, \ldots, k$ in A^k, extended by the appropriate momenta π_b^j of degree $j = 1, \ldots, k$ in $D^*(A^k)$, we get the equations for the Legendre relation in the form (no Lie algebroid structure appears!):

$$
\begin{aligned}
&k\pi_a^1 = \frac{\partial L}{\partial y_k^a},\\
&(k-1)\pi_b^2 = \frac{\partial L}{\partial y_{k-1}^b} - \frac{1}{k}\frac{d}{dt}\left(\frac{\partial L}{\partial y_k^b}\right),\\
&\vdots\\
&\pi_d^k = \frac{\partial L}{\partial y_1^d} - \frac{1}{2!}\frac{d}{dt}\left(\frac{\partial L}{\partial y_2^d}\right) + \frac{1}{3!}\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial y_3^d}\right) - \cdots\\
&+(-1)^k\frac{1}{(k-1)!}\frac{d^{k-2}}{dt^{k-2}}\left(\frac{\partial L}{\partial y_{k-1}^d}\right) - (-1)^k\frac{1}{k!}\frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial L}{\partial y_k^d}\right),
\end{aligned}
$$

which we recognise as the Jacobi–Ostrogradski momenta.

Higher order Lagrangian mechanics on Lie algebroids

The remaining equation for the dynamics is

$$\frac{d}{dt}\pi^k_a = \rho^A_a(x)\frac{\partial L}{\partial x^A} + y^b_1 C^c_{ba}(x)\pi^k_c\,,$$

where ρ^A_a and C^c_{ba} are structure functions of the Lie algebroid $A = A(\mathcal{G})$. The above equation can then be rewritten as

$$\rho^A_a(x)\tfrac{\partial L}{\partial x^A} =$$
$$\left(\delta^c_a \tfrac{d}{dt} - y^b_1 C^c_{ba}(x)\right)\left(\tfrac{\partial L}{\partial y^c_1} - \tfrac{1}{2!}\tfrac{d}{dt}\left(\tfrac{\partial L}{\partial y^c_2}\right)\cdots - (-1)^k\tfrac{1}{k!}\tfrac{d^{k-1}}{dt^{k-1}}\left(\tfrac{\partial L}{\partial y^c_k}\right)\right)$$

which we define to be the k-th order Euler–Lagrange equations on $A(\mathcal{G})$.

The above higher order algebroid Euler-Lagrange equations are in complete agrement with the ones obtained by Jóźwikowski & Rotkiewicz, Colombo & de Diego, as well as Martínez. We clearly recover the standard higher Euler–Lagrange equations on $\mathsf{T}^k M$ as a particular example.

Higher order Lagrangian mechanics on Lie algebroids

The remaining equation for the dynamics is

$$\frac{d}{dt}\pi_a^k = \rho_a^A(x)\frac{\partial L}{\partial x^A} + y_1^b C_{ba}^c(x)\pi_c^k ,$$

where ρ_a^A and C_{ba}^c are structure functions of the Lie algebroid $A = A(\mathcal{G})$. The above equation can then be rewritten as

$$\rho_a^A(x)\frac{\partial L}{\partial x^A} =$$
$$\left(\delta_a^c \frac{d}{dt} - y_1^b C_{ba}^c(x)\right)\left(\frac{\partial L}{\partial y_1^c} - \frac{1}{2!}\frac{d}{dt}\left(\frac{\partial L}{\partial y_2^c}\right) \cdots - (-1)^k \frac{1}{k!}\frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial L}{\partial y_k^c}\right)\right)$$

which we define to be the k-th order Euler–Lagrange equations on $A(\mathcal{G})$.

The above higher order algebroid Euler-Lagrange equations are in complete agrement with the ones obtained by Jóźwikowski & Rotkiewicz, Colombo & de Diego, as well as Martínez. We clearly recover the standard higher Euler–Lagrange equations on $\mathsf{T}^k M$ as a particular example.

Higher order Lagrangian mechanics on Lie algebroids

The remaining equation for the dynamics is

$$\frac{d}{dt}\pi_a^k = \rho_a^A(x)\frac{\partial L}{\partial x^A} + y_1^b C_{ba}^c(x)\pi_c^k\,,$$

where ρ_a^A and C_{ba}^c are structure functions of the Lie algebroid $A = A(\mathcal{G})$.
The above equation can then be rewritten as

$$\rho_a^A(x)\frac{\partial L}{\partial x^A} = \left(\delta_a^c\frac{d}{dt} - y_1^b C_{ba}^c(x)\right)\left(\frac{\partial L}{\partial y_1^c} - \frac{1}{2!}\frac{d}{dt}\left(\frac{\partial L}{\partial y_2^c}\right)\cdots - (-1)^k\frac{1}{k!}\frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial L}{\partial y_k^c}\right)\right)$$

which we define to be the k-th order Euler–Lagrange equations on $A(\mathcal{G})$.

The above higher order algebroid Euler-Lagrange equations are in complete agrement with the ones obtained by Jóźwikowski & Rotkiewicz, Colombo & de Diego, as well as Martínez. We clearly recover the standard higher Euler–Lagrange equations on $\mathrm{T}^k M$ as a particular example.

Higher order Lagrangian mechanics on Lie algebroids

The remaining equation for the dynamics is

$$\frac{d}{dt}\pi^k_a = \rho^A_a(x)\frac{\partial L}{\partial x^A} + y^b_1 C^c_{ba}(x)\pi^k_c\,,$$

where ρ^A_a and C^c_{ba} are structure functions of the Lie algebroid $A = A(\mathcal{G})$. The above equation can then be rewritten as

$$\rho^A_a(x)\frac{\partial L}{\partial x^A} =$$
$$\left(\delta^c_a \frac{d}{dt} - y^b_1 C^c_{ba}(x)\right)\left(\frac{\partial L}{\partial y^c_1} - \frac{1}{2!}\frac{d}{dt}\left(\frac{\partial L}{\partial y^c_2}\right)\cdots - (-1)^k \frac{1}{k!}\frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial L}{\partial y^c_k}\right)\right)$$

which we define to be the k-th order Euler–Lagrange equations on $A(\mathcal{G})$.

The above higher order algebroid Euler-Lagrange equations are in complete agrement with the ones obtained by Jóźwikowski & Rotkiewicz, Colombo & de Diego, as well as Martínez. We clearly recover the standard higher Euler–Lagrange equations on $\mathsf{T}^k M$ as a particular example.

Higher order Lagrangian mechanics on Lie algebroids

The remaining equation for the dynamics is

$$\frac{d}{dt}\pi_a^k = \rho_a^A(x)\frac{\partial L}{\partial x^A} + y_1^b C_{ba}^c(x)\pi_c^k\,,$$

where ρ_a^A and C_{ba}^c are structure functions of the Lie algebroid $A = A(\mathcal{G})$. The above equation can then be rewritten as

$$\rho_a^A(x)\tfrac{\partial L}{\partial x^A} =$$
$$\left(\delta_a^c \tfrac{d}{dt} - y_1^b C_{ba}^c(x)\right)\left(\tfrac{\partial L}{\partial y_1^c} - \tfrac{1}{2!}\tfrac{d}{dt}\left(\tfrac{\partial L}{\partial y_2^c}\right)\cdots - (-1)^k \tfrac{1}{k!}\tfrac{d^{k-1}}{dt^{k-1}}\left(\tfrac{\partial L}{\partial y_k^c}\right)\right)$$

which we define to be the k-th order Euler–Lagrange equations on $A(\mathcal{G})$.

The above higher order algebroid Euler-Lagrange equations are in complete agrement with the ones obtained by Jóźwikowski & Rotkiewicz, Colombo & de Diego, as well as Martínez. We clearly recover the standard higher Euler–Lagrange equations on $\mathrm{T}^k M$ as a particular example.

Higher order Lagrangian mechanics on Lie algebroids

The remaining equation for the dynamics is

$$\frac{d}{dt}\pi^k_a = \rho^A_a(x)\frac{\partial L}{\partial x^A} + y^b_1 C^c_{ba}(x)\pi^k_c\,,$$

where ρ^A_a and C^c_{ba} are structure functions of the Lie algebroid $A = A(\mathcal{G})$. The above equation can then be rewritten as

$$\rho^A_a(x)\tfrac{\partial L}{\partial x^A} =$$
$$\left(\delta^c_a \tfrac{d}{dt} - y^b_1 C^c_{ba}(x)\right)\left(\tfrac{\partial L}{\partial y^c_1} - \tfrac{1}{2!}\tfrac{d}{dt}\left(\tfrac{\partial L}{\partial y^c_2}\right)\cdots - (-1)^k \tfrac{1}{k!}\tfrac{d^{k-1}}{dt^{k-1}}\left(\tfrac{\partial L}{\partial y^c_k}\right)\right)$$

which we define to be the k-th order Euler–Lagrange equations on $A(\mathcal{G})$.

The above higher order algebroid Euler-Lagrange equations are in complete agrement with the ones obtained by Jóźwikowski & Rotkiewicz, Colombo & de Diego, as well as Martínez. We clearly recover the standard higher Euler–Lagrange equations on $\mathsf{T}^k M$ as a particular example.

Higher order Lagrangian mechanics on Lie algebroids

The remaining equation for the dynamics is

$$\frac{d}{dt}\pi^k_a = \rho^A_a(x)\frac{\partial L}{\partial x^A} + y^b_1 C^c_{ba}(x)\pi^k_c\,,$$

where ρ^A_a and C^c_{ba} are structure functions of the Lie algebroid $A = A(\mathcal{G})$. The above equation can then be rewritten as

$$\rho^A_a(x)\tfrac{\partial L}{\partial x^A} =$$
$$\left(\delta^c_a \tfrac{d}{dt} - y^b_1 C^c_{ba}(x)\right)\left(\tfrac{\partial L}{\partial y^c_1} - \tfrac{1}{2!}\tfrac{d}{dt}\left(\tfrac{\partial L}{\partial y^c_2}\right)\cdots - (-1)^k \tfrac{1}{k!}\tfrac{d^{k-1}}{dt^{k-1}}\left(\tfrac{\partial L}{\partial y^c_k}\right)\right)$$

which we define to be the k-th order Euler–Lagrange equations on $A(\mathcal{G})$.

The above higher order algebroid Euler-Lagrange equations are in complete agrement with the ones obtained by Jóźwikowski & Rotkiewicz, Colombo & de Diego, as well as Martínez. We clearly recover the standard higher Euler–Lagrange equations on $\mathsf{T}^k M$ as a particular example.

The tip of a javelin

For instance, let L be the Lagrangian, governing the motion of the tip of a javelin defined on $\mathsf{T}^2\mathbb{R}^3$,

$$L(x,y,z) = \frac{1}{2}\left(\sum_{i=1}^{3}(y^i)^2 - (z^i)^2\right).$$

We can understand $G = \mathbb{R}^3$ here as a commutative Lie group, and since L is G-invariant, we get immediately the reduction to the graded bundle $\mathbb{R}^3[1] \times \mathbb{R}^3[2]$. The Euler-Lagrange equations on $\mathsf{T}^2\mathbb{R}^3$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^i} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z^i}\right)\right) = 0,$$

give in this case

$$\frac{\mathrm{d}y^i}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}^2 z^i}{\mathrm{d}t^2},$$

so the Euler-Lagrange equation on $\mathbb{R}^3$ reads

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = \frac{1}{2}\frac{\mathrm{d}^4 x^i}{\mathrm{d}t^4}.$$

Jump to end

The tip of a javelin

For instance, let L be the Lagrangian, governing the motion of the tip of a javelin defined on $\mathsf{T}^2\mathbb{R}^3$,

$$L(x,y,z) = \frac{1}{2}\left(\sum_{i=1}^{3}(y^i)^2 - (z^i)^2\right).$$

We can understand $G = \mathbb{R}^3$ here as a commutative Lie group, and since L is G-invariant, we get immediately the reduction to the graded bundle $\mathbb{R}^3[1] \times \mathbb{R}^3[2]$. The Euler-Lagrange equations on $\mathsf{T}^2\mathbb{R}^3$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^i} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z^i}\right)\right) = 0,$$

give in this case

$$\frac{\mathrm{d}y^i}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}^2 z^i}{\mathrm{d}t^2},$$

so the Euler-Lagrange equation on $\mathbb{R}^3$ reads

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = \frac{1}{2}\frac{\mathrm{d}^4 x^i}{\mathrm{d}t^4}.$$

Jump to end

The tip of a javelin

For instance, let L be the Lagrangian, governing the motion of the tip of a javelin defined on $\mathsf{T}^2\mathbb{R}^3$,

$$L(x,y,z)=\frac{1}{2}\left(\sum_{i=1}^{3}(y^i)^2-(z^i)^2\right).$$

We can understand $G=\mathbb{R}^3$ here as a commutative Lie group, and since L is G-invariant, we get immediately the reduction to the graded bundle $\mathbb{R}^3[1]\times\mathbb{R}^3[2]$. The Euler-Lagrange equations on $\mathsf{T}^2\mathbb{R}^3$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^i}-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z^i}\right)\right)=0\,,$$

give in this case

$$\frac{\mathrm{d}y^i}{\mathrm{d}t}=\frac{1}{2}\frac{\mathrm{d}^2z^i}{\mathrm{d}t^2}\,,$$

so the Euler-Lagrange equation on $\mathbb{R}^3$ reads

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}=\frac{1}{2}\frac{\mathrm{d}^4x^i}{\mathrm{d}t^4}\,.$$

Jump to end

The tip of a javelin

For instance, let L be the Lagrangian, governing the motion of the tip of a javelin defined on $\mathsf{T}^2\mathbb{R}^3$,

$$L(x,y,z)=\frac{1}{2}\left(\sum_{i=1}^{3}(y^i)^2-(z^i)^2\right).$$

We can understand $G=\mathbb{R}^3$ here as a commutative Lie group, and since L is G-invariant, we get immediately the reduction to the graded bundle $\mathbb{R}^3[1]\times\mathbb{R}^3[2]$. The Euler-Lagrange equations on $\mathsf{T}^2\mathbb{R}^3$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^i}-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z^i}\right)\right)=0,$$

give in this case

$$\frac{\mathrm{d}y^i}{\mathrm{d}t}=\frac{1}{2}\frac{\mathrm{d}^2z^i}{\mathrm{d}t^2},$$

so the Euler-Lagrange equation on $\mathbb{R}^3$ reads

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}=\frac{1}{2}\frac{\mathrm{d}^4x^i}{\mathrm{d}t^4}.$$

Jump to end

The tip of a javelin

For instance, let L be the Lagrangian, governing the motion of the tip of a javelin defined on $\mathsf{T}^2\mathbb{R}^3$,

$$L(x,y,z) = \frac{1}{2}\left(\sum_{i=1}^{3}(y^i)^2 - (z^i)^2\right).$$

We can understand $G = \mathbb{R}^3$ here as a commutative Lie group, and since L is G-invariant, we get immediately the reduction to the graded bundle $\mathbb{R}^3[1] \times \mathbb{R}^3[2]$. The Euler-Lagrange equations on $\mathsf{T}^2\mathbb{R}^3$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^i} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z^i}\right)\right) = 0\,,$$

give in this case

$$\frac{\mathrm{d}y^i}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}^2 z^i}{\mathrm{d}t^2}\,,$$

so the Euler-Lagrange equation on $\mathbb{R}^3$ reads

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = \frac{1}{2}\frac{\mathrm{d}^4 x^i}{\mathrm{d}t^4}\,.$$

Jump to end

The tip of a javelin

For instance, let L be the Lagrangian, governing the motion of the tip of a javelin defined on $\mathsf{T}^2\mathbb{R}^3$,

$$L(x,y,z) = \frac{1}{2}\left(\sum_{i=1}^{3}(y^i)^2 - (z^i)^2\right) .$$

We can understand $G = \mathbb{R}^3$ here as a commutative Lie group, and since L is G-invariant, we get immediately the reduction to the graded bundle $\mathbb{R}^3[1] \times \mathbb{R}^3[2]$. The Euler-Lagrange equations on $\mathsf{T}^2\mathbb{R}^3$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^i} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z^i}\right)\right) = 0 \,,$$

give in this case

$$\frac{\mathrm{d}y^i}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}^2 z^i}{\mathrm{d}t^2} \,,$$

so the Euler-Lagrange equation on $\mathbb{R}^3$ reads

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = \frac{1}{2}\frac{\mathrm{d}^4 x^i}{\mathrm{d}t^4} .$$

Jump to end

The tip of a javelin

For instance, let L be the Lagrangian, governing the motion of the tip of a javelin defined on $\mathsf{T}^2\mathbb{R}^3$,

$$L(x,y,z)=\frac{1}{2}\left(\sum_{i=1}^{3}(y^i)^2-(z^i)^2\right).$$

We can understand $G=\mathbb{R}^3$ here as a commutative Lie group, and since L is G-invariant, we get immediately the reduction to the graded bundle $\mathbb{R}^3[1]\times\mathbb{R}^3[2]$. The Euler-Lagrange equations on $\mathsf{T}^2\mathbb{R}^3$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^i}-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z^i}\right)\right)=0\,,$$

give in this case

$$\frac{\mathrm{d}y^i}{\mathrm{d}t}=\frac{1}{2}\frac{\mathrm{d}^2z^i}{\mathrm{d}t^2}\,,$$

so the Euler-Lagrange equation on $\mathbb{R}^3$ reads

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}=\frac{1}{2}\frac{\mathrm{d}^4x^i}{\mathrm{d}t^4}\,.$$

The Tulczyjew triple for strings

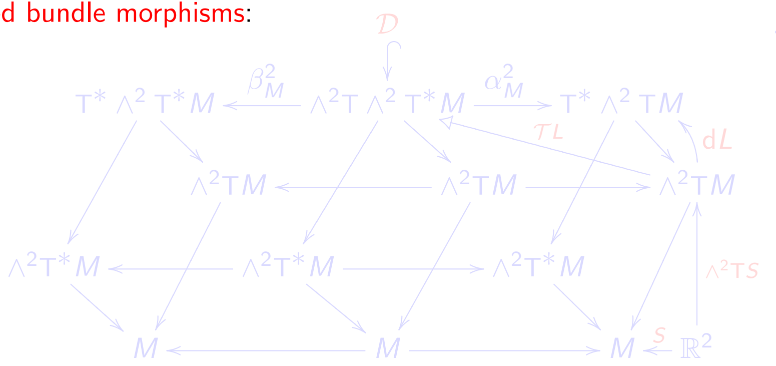
Using the canonical multisymplectic structure on $\wedge^2 \mathrm{T}^* M$, we get the following Tulczyjew triple for multivector bundles, consisting of double graded bundle morphisms:

.

The way of obtaining the implicit phase dynamics $\mathcal{D}$, as a submanifold of $\wedge^2 \mathrm{T} \wedge^2 \mathrm{T}^* M$, from a Lagrangian $L : \wedge^2 \mathrm{T} M \to \mathbb{R}$ (or from a Hamiltonian $H : \wedge^2 \mathrm{T}^* M \to \mathbb{R}$) is now standard: $\mathcal{D} = \mathcal{T}L(\wedge^2 \mathrm{T} M)$.

The Tulczyjew triple for strings

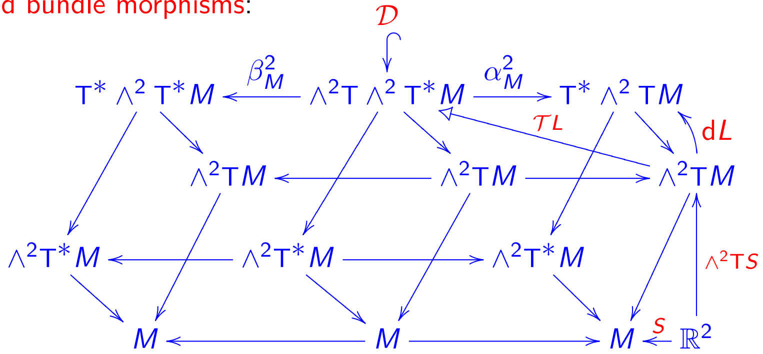
Using the canonical multisymplectic structure on $\wedge^2 \mathsf{T}^* M$, we get the following Tulczyjew triple for multivector bundles, consisting of double graded bundle morphisms:



The way of obtaining the implicit phase dynamics $\mathcal{D}$, as a submanifold of $\wedge^2 \mathsf{T} \wedge^2 \mathsf{T}^* M$, from a Lagrangian $L : \wedge^2 \mathsf{T} M \to \mathbb{R}$ (or from a Hamiltonian $H : \wedge^2 \mathsf{T}^* M \to \mathbb{R}$) is now standard: $\mathcal{D} = \mathcal{T}L(\wedge^2 \mathsf{T} M)$.

The Tulczyjew triple for strings

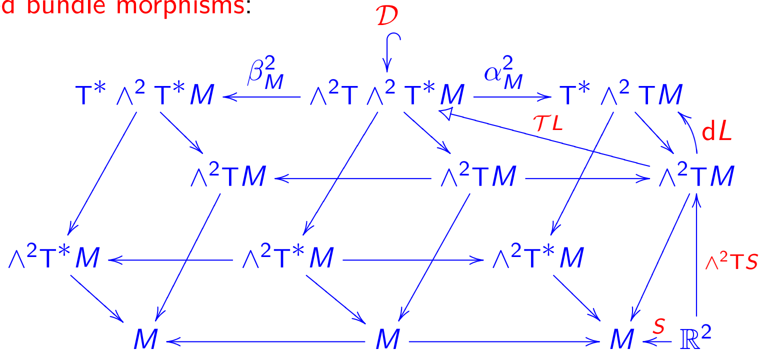
Using the canonical multisymplectic structure on $\wedge^2\mathsf{T}^*M$, we get the following Tulczyjew triple for multivector bundles, consisting of double graded bundle morphisms:



The way of obtaining the implicit phase dynamics $\mathcal{D}$, as a submanifold of $\wedge^2\mathsf{T}\wedge^2\mathsf{T}^*M$, from a Lagrangian $L:\wedge^2\mathsf{T}M\to\mathbb{R}$ (or from a Hamiltonian $H:\wedge^2\mathsf{T}^*M\to\mathbb{R}$) is now standard: $\mathcal{D}=\mathcal{T}L(\wedge^2\mathsf{T}M)$.

The Tulczyjew triple for strings

Using the canonical multisymplectic structure on $\wedge^2 \mathsf{T}^* M$, we get the following Tulczyjew triple for multivector bundles, consisting of double graded bundle morphisms:



The way of obtaining the implicit phase dynamics $\mathcal{D}$, as a submanifold of $\wedge^2 \mathsf{T} \wedge^2 \mathsf{T}^* M$, from a Lagrangian $L : \wedge^2 \mathsf{T} M \to \mathbb{R}$ (or from a Hamiltonian $H : \wedge^2 \mathsf{T}^* M \to \mathbb{R}$) is now standard: $\mathcal{D} = \mathcal{T}L(\wedge^2 \mathsf{T} M)$.

The Euler-Lagrange equations

A surface $S:(t,s)\mapsto(x^\sigma(t,s))$ in M satisfies the Euler-Lagrange equations if the image by $\mathrm{d}L$ of its prolongation to $\wedge^2\mathsf{T}M$,

$$(t,s)\mapsto\left(x^\sigma(t,s),\dot{x}^{\mu\nu}=\frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial s}-\frac{\partial x^\mu}{\partial s}\frac{\partial x^\nu}{\partial t}\right),$$

is α^2_M-related to an admissible surface, i.e. the prolongation of a surface living in the phase space $\wedge^2\mathsf{T}^*M$ to $\wedge^2\mathsf{T}\wedge^2\mathsf{T}^*M$.
In coordinates, the Euler-Lagrange equations read

$$\begin{aligned}\dot{x}^{\mu\nu} &= \frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial s}-\frac{\partial x^\mu}{\partial s}\frac{\partial x^\nu}{\partial t},\\ \frac{\partial L}{\partial x^\sigma} &= \frac{\partial x^\mu}{\partial t}\frac{\partial}{\partial s}\left(\frac{\partial L}{\partial \dot{x}^{\mu\sigma}}(t,s)\right)-\frac{\partial x^\mu}{\partial s}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}^{\mu\sigma}}(t,s)\right).\end{aligned}$$

The Euler-Lagrange equations

A surface $S:(t,s)\mapsto(x^\sigma(t,s))$ in M satisfies the Euler-Lagrange equations if the image by $\mathrm{d}L$ of its prolongation to $\wedge^2\mathrm{T}M$,

$$(t,s)\mapsto\left(x^\sigma(t,s),\dot{x}^{\mu\nu}=\frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial s}-\frac{\partial x^\mu}{\partial s}\frac{\partial x^\nu}{\partial t}\right),$$

is α^2_M-related to an admissible surface, i.e. the prolongation of a surface living in the phase space $\wedge^2\mathrm{T}^*M$ to $\wedge^2\mathrm{T}\wedge^2\mathrm{T}^*M$.
In coordinates, the Euler-Lagrange equations read

$$\begin{aligned}\dot{x}^{\mu\nu} &= \frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial s}-\frac{\partial x^\mu}{\partial s}\frac{\partial x^\nu}{\partial t},\\ \frac{\partial L}{\partial x^\sigma} &= \frac{\partial x^\mu}{\partial t}\frac{\partial}{\partial s}\left(\frac{\partial L}{\partial \dot{x}^{\mu\sigma}}(t,s)\right)-\frac{\partial x^\mu}{\partial s}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}^{\mu\sigma}}(t,s)\right).\end{aligned}$$

The Euler-Lagrange equations

A surface $S : (t, s) \mapsto (x^\sigma(t, s))$ in M satisfies the Euler-Lagrange equations if the image by $\mathrm{d}L$ of its prolongation to $\wedge^2 \mathsf{T} M$,

$$(t, s) \mapsto \left(x^\sigma(t, s), \dot{x}^{\mu\nu} = \frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial s} - \frac{\partial x^\mu}{\partial s}\frac{\partial x^\nu}{\partial t} \right),$$

is α^2_M-related to an admissible surface, i.e. the prolongation of a surface living in the phase space $\wedge^2 \mathsf{T}^* M$ to $\wedge^2 \mathsf{T} \wedge^2 \mathsf{T}^* M$.

In coordinates, the Euler-Lagrange equations read

$$\begin{aligned} \dot{x}^{\mu\nu} &= \frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial s} - \frac{\partial x^\mu}{\partial s}\frac{\partial x^\nu}{\partial t}, \\ \frac{\partial L}{\partial x^\sigma} &= \frac{\partial x^\mu}{\partial t}\frac{\partial}{\partial s}\left(\frac{\partial L}{\partial \dot{x}^{\mu\sigma}}(t, s)\right) - \frac{\partial x^\mu}{\partial s}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}^{\mu\sigma}}(t, s)\right). \end{aligned}$$

The Euler-Lagrange equations

A surface $S:(t,s)\mapsto(x^\sigma(t,s))$ in M satisfies the Euler-Lagrange equations if the image by $\mathrm{d}L$ of its prolongation to $\wedge^2\mathsf{T}M$,

$$(t,s)\mapsto\left(x^\sigma(t,s),\dot{x}^{\mu\nu}=\frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial s}-\frac{\partial x^\mu}{\partial s}\frac{\partial x^\nu}{\partial t}\right),$$

is α^2_M-related to an admissible surface, i.e. the prolongation of a surface living in the phase space $\wedge^2\mathsf{T}^*M$ to $\wedge^2\mathsf{T}\wedge^2\mathsf{T}^*M$.

In coordinates, the Euler-Lagrange equations read

$$\begin{aligned}\dot{x}^{\mu\nu} &= \frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial s}-\frac{\partial x^\mu}{\partial s}\frac{\partial x^\nu}{\partial t},\\ \frac{\partial L}{\partial x^\sigma} &= \frac{\partial x^\mu}{\partial t}\frac{\partial}{\partial s}\left(\frac{\partial L}{\partial \dot{x}^{\mu\sigma}}(t,s)\right)-\frac{\partial x^\mu}{\partial s}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}^{\mu\sigma}}(t,s)\right).\end{aligned}$$

The Euler-Lagrange equations

A surface $S:(t,s)\mapsto(x^\sigma(t,s))$ in M satisfies the Euler-Lagrange equations if the image by $\mathrm{d}L$ of its prolongation to $\wedge^2\mathsf{T}M$,

$$(t,s)\mapsto\left(x^\sigma(t,s),\dot{x}^{\mu\nu}=\frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial s}-\frac{\partial x^\mu}{\partial s}\frac{\partial x^\nu}{\partial t}\right),$$

is α^2_M-related to an admissible surface, i.e. the prolongation of a surface living in the phase space $\wedge^2\mathsf{T}^*M$ to $\wedge^2\mathsf{T}\wedge^2\mathsf{T}^*M$.
In coordinates, the Euler-Lagrange equations read

$$\begin{aligned}\dot{x}^{\mu\nu} &= \frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial s}-\frac{\partial x^\mu}{\partial s}\frac{\partial x^\nu}{\partial t},\\ \frac{\partial L}{\partial x^\sigma} &= \frac{\partial x^\mu}{\partial t}\frac{\partial}{\partial s}\left(\frac{\partial L}{\partial \dot{x}^{\mu\sigma}}(t,s)\right)-\frac{\partial x^\mu}{\partial s}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}^{\mu\sigma}}(t,s)\right).\end{aligned}$$

Plateau problem

In particular, if $M = \mathbb{R}^3 = \{(x^1 = x, x^2 = y, x^3 = z)\}$ with the Euclidean metric, the canonically induced 'free' Lagrangian on $\wedge^2 \mathsf{T}M$ reads

$$L(x^\mu, \dot{x}^{\kappa\lambda}) = \sqrt{\sum_{\kappa,\lambda} (\dot{x}^{\kappa\lambda})^2}\,.$$

The Euler-Lagrange equation for surfaces being graphs $(x, y) \mapsto (x, y, z(x, y))$ provides the well-known equation for minimal surfaces, found already by Lagrange :

$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}}\right) = 0\,.$$

In another form:

$$(1 + z_x^2) z_{yy} - 2 z_x z_y z_{xy} + (1 + z_y^2) z_{xx} = 0\,.$$

Starting with a Lorentz metric, we can obtain analogously the Euler-Lagrange equations for the Nambu-Goto Lagrangian.

Plateau problem

In particular, if $M = \mathbb{R}^3 = \{(x^1 = x, x^2 = y, x^3 = z)\}$ with the Euclidean metric, the canonically induced 'free' Lagrangian on $\wedge^2 TM$ reads

$$L(x^\mu, \dot{x}^{\kappa\lambda}) = \sqrt{\sum_{\kappa,\lambda} (\dot{x}^{\kappa\lambda})^2}\,.$$

The Euler-Lagrange equation for surfaces being graphs $(x, y) \mapsto (x, y, z(x, y))$ provides the well-known equation for minimal surfaces, found already by Lagrange :

$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1+z_x^2+z_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1+z_x^2+z_y^2}}\right) = 0\,.$$

In another form:

$$(1+z_x^2)z_{yy} - 2z_xz_yz_{xy} + (1+z_y^2)z_{xx} = 0\,.$$

Starting with a Lorentz metric, we can obtain analogously the Euler-Lagrange equations for the Nambu-Goto Lagrangian.

Plateau problem

In particular, if $M = \mathbb{R}^3 = \{(x^1 = x, x^2 = y, x^3 = z)\}$ with the Euclidean metric, the canonically induced 'free' Lagrangian on $\wedge^2 \mathsf{T}M$ reads

$$L(x^\mu, \dot{x}^{\kappa\lambda}) = \sqrt{\sum_{\kappa,\lambda} \left(\dot{x}^{\kappa\lambda}\right)^2}\,.$$

The Euler-Lagrange equation for surfaces being graphs $(x, y) \mapsto (x, y, z(x, y))$ provides the well-known equation for minimal surfaces, found already by Lagrange :

$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}}\right) = 0\,.$$

In another form:

$$(1 + z_x^2)z_{yy} - 2z_x z_y z_{xy} + (1 + z_y^2)z_{xx} = 0\,.$$

Starting with a Lorentz metric, we can obtain analogously the Euler-Lagrange equations for the Nambu-Goto Lagrangian.

Plateau problem

In particular, if $M = \mathbb{R}^3 = \{(x^1 = x, x^2 = y, x^3 = z)\}$ with the Euclidean metric, the canonically induced 'free' Lagrangian on $\wedge^2 \mathsf{T}M$ reads

$$L(x^\mu, \dot{x}^{\kappa\lambda}) = \sqrt{\sum_{\kappa,\lambda} \left(\dot{x}^{\kappa\lambda}\right)^2}\,.$$

The Euler-Lagrange equation for surfaces being graphs $(x, y) \mapsto (x, y, z(x, y))$ provides the well-known equation for minimal surfaces, found already by Lagrange :

$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1+z_x^2+z_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1+z_x^2+z_y^2}}\right) = 0\,.$$

In another form:

$$(1+z_x^2)z_{yy} - 2z_x z_y z_{xy} + (1+z_y^2)z_{xx} = 0\,.$$

Starting with a Lorentz metric, we can obtain analogously the Euler-Lagrange equations for the Nambu-Goto Lagrangian.

Plateau problem

In particular, if $M = \mathbb{R}^3 = \{(x^1 = x, x^2 = y, x^3 = z)\}$ with the Euclidean metric, the canonically induced 'free' Lagrangian on $\wedge^2 \mathsf{T}M$ reads

$$L(x^\mu, \dot{x}^{\kappa\lambda}) = \sqrt{\sum_{\kappa,\lambda} \left(\dot{x}^{\kappa\lambda}\right)^2}\,.$$

The Euler-Lagrange equation for surfaces being graphs $(x, y) \mapsto (x, y, z(x, y))$ provides the well-known equation for minimal surfaces, found already by Lagrange :

$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}}\right) = 0\,.$$

In another form:

$$(1 + z_x^2) z_{yy} - 2 z_x z_y z_{xy} + (1 + z_y^2) z_{xx} = 0\,.$$

Starting with a Lorentz metric, we can obtain analogously the Euler-Lagrange equations for the Nambu-Goto Lagrangian.

Plateau problem

In particular, if $M = \mathbb{R}^3 = \{(x^1 = x, x^2 = y, x^3 = z)\}$ with the Euclidean metric, the canonically induced 'free' Lagrangian on $\wedge^2 \mathsf{T}M$ reads

$$L(x^\mu, \dot{x}^{\kappa\lambda}) = \sqrt{\sum_{\kappa,\lambda} (\dot{x}^{\kappa\lambda})^2}\,.$$

The Euler-Lagrange equation for surfaces being graphs $(x, y) \mapsto (x, y, z(x, y))$ provides the well-known equation for minimal surfaces, found already by Lagrange :

$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}}\right) = 0\,.$$

In another form:

$$(1 + z_x^2)z_{yy} - 2z_x z_y z_{xy} + (1 + z_y^2)z_{xx} = 0\,.$$

Starting with a Lorentz metric, we can obtain analogously the Euler-Lagrange equations for the Nambu-Goto Lagrangian.

Plateau problem

In particular, if $M = \mathbb{R}^3 = \{(x^1 = x, x^2 = y, x^3 = z)\}$ with the Euclidean metric, the canonically induced 'free' Lagrangian on $\wedge^2 \mathsf{T}M$ reads

$$L(x^\mu, \dot{x}^{\kappa\lambda}) = \sqrt{\sum_{\kappa,\lambda} (\dot{x}^{\kappa\lambda})^2}\,.$$

The Euler-Lagrange equation for surfaces being graphs $(x, y) \mapsto (x, y, z(x, y))$ provides the well-known equation for minimal surfaces, found already by Lagrange :

$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}}\right) = 0\,.$$

In another form:

$$(1 + z_x^2)z_{yy} - 2z_x z_y z_{xy} + (1 + z_y^2)z_{xx} = 0\,.$$

Starting with a Lorentz metric, we can obtain analogously the Euler-Lagrange equations for the Nambu-Goto Lagrangian.

Plateau problem

In particular, if $M = \mathbb{R}^3 = \{(x^1 = x, x^2 = y, x^3 = z)\}$ with the Euclidean metric, the canonically induced 'free' Lagrangian on $\wedge^2 \mathsf{T}M$ reads

$$L(x^\mu, \dot{x}^{\kappa\lambda}) = \sqrt{\sum_{\kappa,\lambda} (\dot{x}^{\kappa\lambda})^2}\,.$$

The Euler-Lagrange equation for surfaces being graphs $(x, y) \mapsto (x, y, z(x, y))$ provides the well-known equation for minimal surfaces, found already by Lagrange :

$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}}\right) = 0\,.$$

In another form:

$$(1 + z_x^2) z_{yy} - 2 z_x z_y z_{xy} + (1 + z_y^2) z_{xx} = 0\,.$$

Starting with a Lorentz metric, we can obtain analogously the Euler-Lagrange equations for the Nambu-Goto Lagrangian.

THANK YOU FOR YOUR ATTENTION!