ON PAIRS OF COMMUTING INVOLUTIONS IN $\mathrm{Sym}(n)$ AND $\mathrm{Alt}(n)$

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Abstract. The number of pairs of commuting involutions in $\mathrm{Sym}(n)$ and $\mathrm{Alt}(n)$ is determined up to isomorphism. It is also proven that, up to isomorphism and duality, there are exactly two abstract regular polyhedra on which the group $\mathrm{Sym}(6)$ acts as a regular automorphism group.

Keywords: Symmetric groups, Alternating groups, polyhedra.
2000 Math Subj. Class: 20B30, 20D06, 52B10

1. Introduction

In recent years, classifications of all abstract regular polytopes for a given automorphism group G have been made either using the computer [10, 4, 5] or with a theoretical approach: work on polytopes of highest possible rank has been carried out by Leemans for the Suzuki groups $\mathrm{Sz}(q)$ [7], Leemans and Schulte for the $\mathrm{L}_2(q)$ and $\mathrm{PGL}(2,q)$ groups [9, 8], Fernandes and Leemans for the symmetric groups [1], and by Fernandes, Leemans and Mixer [2, 3] for the alternating groups.

In [6], we managed to determine, up to isomorphism and duality, how many abstract regular polytopes have a given Suzuki simple group $\mathrm{Sz}(q)$ as automorphism group. In order to try to get similar results for the symmetric and alternating groups, it is important to first determine, up to isomorphism, how many pairs of commuting involutions these groups have. So a purely group-theoretical question arises here.

Can we determine, up to isomorphism, the number of (unordered) pairs of commuting involutions of a symmetric group $\mathrm{Sym}(n)$ or an alternating group $\mathrm{Alt}(n)$?

We obtain the following two theorems that give an affirmative answer to this question.

Theorem 1.1. *Let $n > 1$ be a positive integer. Define $\lambda(k)$ and $\psi(k,n)$ as follows.*

$$\lambda(k) = \left\lfloor \left(\frac{k}{2}+1\right)^2 \right\rfloor$$

$$\psi(k,n) = \begin{cases} \left[\frac{1}{2}(2k-n)\right]^2 + \frac{1}{2}(2k-n) & \text{if } n \text{ is even,} \\ \left[\frac{1}{2}(2k-n-1)\right]^2 + 2k-n & \text{if } n \text{ is odd.} \end{cases}$$

There are, up to isomorphism,

$$-\frac{3n}{2} + \sum_{k=1}^{n} \lambda(k) \cdot \left(\frac{n-k}{2}+1\right) - \frac{1}{2} \cdot \sum_{k=\lfloor \frac{n}{2} \rfloor + 1}^{n} \psi(k,n)$$

unordered pairs of commuting involutions in $\mathrm{Sym}(2n)$ *and* $\mathrm{Sym}(2n+1)$ *except for* $\mathrm{Sym}(6)$ *in which there are, up to isomorphism, five unordered pairs of commuting involutions.*

Theorem 1.2. *Let $n > 1$ be a positive integer. Define $\phi(k,n)$ and $\mu(n)$ as follows.*

$$\phi(k,n) = \begin{cases} \lambda(k) - 1 & \text{if } k \leq \left\lfloor \frac{n}{2} \right\rfloor, \\ \lambda(k) - \psi(k,n) - 1 & \text{if } k > \left\lfloor \frac{n}{2} \right\rfloor, \end{cases}$$

$$\mu(n) = -2\left\lfloor \frac{n}{2} \right\rfloor + \sum_{\substack{k=1 \\ k \text{ even}}}^{n} \left[\gamma(k) \cdot \left\lceil \frac{1}{2} \cdot \Big(n - k + 1\Big) \right\rceil + \delta(k) \cdot \left\lfloor \frac{1}{2} \cdot \Big(n - k + 1\Big) \right\rfloor \right]$$

where

$$\gamma(k) = \frac{k^2}{8} + \frac{3k}{4} + 1,$$
$$\delta(k) = \frac{k^2}{8} + \frac{k}{4}.$$

There are, up to isomorphism,

$$\frac{1}{2}\Big(\mu(n) + \sum_{\substack{k=1 \\ k \text{ even}}}^{n} \phi(k,n)\Big)$$

unordered pairs of commuting involutions in $\mathrm{Alt}(2n)$ *and* $\mathrm{Alt}(2n+1)$ *except for* $\mathrm{Alt}(6)$ *in which there is, up to isomorphism, a unique unordered pair of commuting involutions.*

Table 1 gives the number of unordered pairs of commuting involutions in $\mathrm{Sym}(n)$ and $\mathrm{Alt}(n)$, up to isomorphism, for some small values of n, according to the formulas obtained in our two theorems.

n	$\{\rho_0, \rho_2\}$, with $\rho_0, \rho_2 \in \mathrm{Sym}(n)$	$\{\rho_0, \rho_2\}$, with $\rho_0, \rho_2 \in \mathrm{Alt}(n)$
1,2,3	0	0
4,5	3	1
6	5	1
7	9	2
8,9	21	7
10,11	39	10
12,13	67	21
14,15	105	28
16,17	158	48
18,19	226	61
20	315	93
30	1169	315
40	3105	855
50	6774	1795

TABLE 1. Number of unordered pairs of commuting involutions in $\mathrm{Sym}(n)$ and $\mathrm{Alt}(n)$, up to isomorphism.

Moreover, we classify, without help of a computer, all abstract regular polyhedra for $\mathrm{Sym}(6)$ and $\mathrm{Alt}(6)$.

The paper is organised as follows. In Section 2, we give some preliminaries needed to understand our paper. In Section 3, we prove Theorem 1.1 and we classify all abstract regular polyhedra of $\mathrm{Sym}(6)$. In Section 4, we prove Theorem 1.2.

2. Definitions and notation

In this article we denote by $\mathrm{Sym}(n)$ the symmetric group of degree n, i.e. the group of all permutations of the set $\{1, 2, \ldots, n\}$. We also denote by $\mathrm{Alt}(n)$ the alternating group, i.e. the normal subgroup of $\mathrm{Sym}(n)$ consisting of all even permutations of $\mathrm{Sym}(n)$. For further information on the symmetric and alternating groups, one may refer for instance to [12] and [13].

The classification of pairs of commuting involutions in $\mathrm{Sym}(n)$ and $\mathrm{Alt}(n)$ is useful, for instance, if one wants to classify all abstract regular polyhedra, on which a symmetric or alternating group acts as a regular automorphism group.

For any two elements $g, h \in G$, we define $h^g := ghg^{-1}$. The (ordered) sets S and S' of elements of G are *conjugate* in G provided there exists an element $g \in G$ such that $S^g = S'$ (preserving the order). They are *isomorphic* in G provided there exists an element $g \in Aut(G)$ such that $gS = S'$.

Given an element $g \in Sym(n)$, provided its cycle decomposition contains exactly k_i cycles of length c_i, we define its *cycle type* as the sequence $(c_1^{k_1}, \ldots, c_n^{k_n})$ where we mention only the lengths of cycles that appear in g.

Concerning polytopes, it is well known that there is a bijection between thin regular residually connected geometries with a linear diagram, abstract regular polytopes and string C-groups. The link between these objects, and more information on abstract regular polytopes, may be found for instance in [11]. We take here the viewpoint of string C-groups because it is the easiest and the most efficient one to define abstract regular polytopes.

3. The Symmetric Group Sym(n),

3.1. **Preliminary results.**

We first need the general form of the centralizer $C_{\mathrm{Sym}(n)}(\rho)$ of an involution ρ in $\mathrm{Sym}(n)$. It will be useful to find the pairs of commuting involutions $\{\rho_0, \rho_2\}$ in $\mathrm{Sym}(n)$. This is a special case of [13, Paragraph 2.3.1], hence we leave the proof as an exercise to the interested reader.

Proposition 3.1. *Let ρ be an involution of cycle type $(2^k, 1^{n-2k})$ in $\mathrm{Sym}(n)$, where $1 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor$. The centralizer $C_{\mathrm{Sym}(n)}(\rho) \cong E_{2^k} : \mathrm{Sym}(k) \times \mathrm{Sym}(n-2k)$, where E_{2^k} is an elementary abelian group of order 2^k.*

Observe that, in $\mathrm{Sym}(2k+1)$ (resp. $\mathrm{Alt}(2k+1)$), any pair of commuting involutions will necessarily fix a point. Hence the number of pairs of commuting involutions in $\mathrm{Sym}(2k)$ and $\mathrm{Sym}(2k+1)$ (resp. $\mathrm{Alt}(2k)$ and $\mathrm{Alt}(2k+1)$) is the same up to isomorphism, except for $k = 3$ as in this case, $Aut(\mathrm{Sym}(6)) \neq \mathrm{Sym}(6)$.

We now compute the number of pairs of commuting involutions in $\mathrm{Sym}(2n)$. The first step consists of choosing an involution ρ_0, up to isomorphism. If $2n \neq 6$, computing such pairs up to conjugacy or up to isomorphism is the same. When $2n = 6$, the outer automorphisms may fuse some non-conjugated pairs. As two involutions are conjugate in $\mathrm{Sym}(2n)$ if and only if they have same cycle type, the number of different possibilities, up to conjugacy, for ρ_0 in $\mathrm{Sym}(2n)$, equals the number of different cycle types for an involution in $\mathrm{Sym}(2n)$. Clearly an involution in $\mathrm{Sym}(2n)$ has cycle type $(2^k, 1^{2(n-k)})$ with k varying from 1 to n. Thus there are n possibilities for choosing a first involution ρ_0 in $\mathrm{Sym}(2n)$.

In order to count the number of commuting pairs of involutions in $\mathrm{Sym}(2n)$, we fix the involution ρ_0 and look at the number of possibilities to choose ρ_2, up to conjugacy, such that ρ_2 commutes with ρ_0.

We first compute the number of *ordered* pairs of commuting involutions in $\mathrm{Sym}(2n)$. We prove the following lemma.

Lemma 3.2. *Up to conjugacy, there are*

$$-2n + \sum_{k=1}^{n} \lambda(k) \cdot (n-k+1)$$

ordered pairs of commuting involutions ρ_0 *and* ρ_2 *in* $\mathrm{Sym}(2n)$ *(and in* $\mathrm{Sym}(2n+1)$*), where*

$$\lambda(k) = \Big\lfloor \Big(\frac{k}{2}+1\Big)^2 \Big\rfloor.$$

Proof. Suppose ρ_0 is an involution containing k cycles of length 2. Up to conjugacy, there is only one such involution, namely $(1,2)(3,4)\ldots(2k-1,2k)$. As ρ_2 commutes with ρ_0, $\rho_2 \in C_{\mathrm{Sym}(2n)}(\rho_0) \cong E_{2^k} : Sym(k) \times Sym(2(n-k))$.

First we count the possibilities for an involution ρ_2, that fixes every element in $\{2k+1,\ldots,2n\}$, i.e. an involution in $E_{2^k} : Sym(k)$. As we count them up to conjugacy, we have to figure out which of all these involutions are conjugate by an element of $C_{\mathrm{Sym}(2n)}(\rho_0)$. It is obvious that two conjugate involutions have the same cycle type. Moreover it is obvious that there exists an element $g \in \mathrm{Sym}(2n)$ such that $\rho_0^g = \rho_0$ and $\rho_2^g = \rho_2'$ if and only if ρ_2 and ρ_2' have the same number of 2-cycles of the form $(2r-1,2r)$, with $1 \le r \le k$ and the same number of 2-cycles of the form (s,t) with $|s-t| \ge 2$ and $s,t \le 2k$.

Suppose now that ρ_2 is composed of l 2-cycles, where m 2-cycles are of the form (s,t), with $m \le l$ and $l-m$ 2-cycles are of the form $(2r-1,r)$. To count the number of such involutions ρ_2 up to conjugacy, we just have to count the number of possibilities for m. It is trivial to see that m has to be even. So m can take every even value between 0 and l. This gives us $\lfloor \frac{l}{2} \rfloor + 1$ possibilites for m, for a given l. However l can take every value between 0 (in this case ρ_2 is the trivial permutation) and k. If $\lambda(k)$ denotes the number, up to isomorphism, of possible involutions ρ_2 consisting only of elements in $\{1,2,\ldots,2k\}$, that commute with ρ_0, then it is easily proven that

$$\lambda(k) = \sum_{l=0}^{k} \Big(\Big\lfloor \frac{l}{2} \Big\rfloor + 1 \Big) = \Big\lfloor \Big(\frac{k}{2}+1\Big)^2 \Big\rfloor.$$

Now we consider the case where the cycle decomposition of ρ_2 involves also 2-cycles from the group $\mathrm{Sym}(2(n-k))$, i.e. 2-cycles containing the elements of $\{2k+1,\ldots,2n\}$. In fact these 2-cycles are independent of the other ones. So we can simply multiply $\lambda(k)$ by the number of possibilities of forming involutions in $\mathrm{Sym}(2(n-k))$. As ρ_0 fixes all the elements in $\{2k+1,\ldots,2n\}$, all the involutions of same cycle type of $\mathrm{Sym}(2(n-k))$ are conjugate by a permutation fixing ρ_0. To count the number of involutions in $\mathrm{Sym}(2(n-k))$, up to conjugacy, we just have to count the number of different cycle types. An involution of $\mathrm{Sym}(2(n-k))$ can have between 0 and $n-k$ cycles of length 2. So there are $n-k+1$ different cycle types.

To compute the number of all the involutions ρ_2 that commute with a given ρ_0 of cycle type $(2^k, 1^{2(n-k)})$, we have to multiply $\lambda(k)$ by $n-k+1$. However the trivial permutation and ρ_0 itself are also among these constructions. So we have to subtract 2 from this number.

The number of 2-cycles involved in ρ_0 may vary between 1 and n. We sum up $(\lambda(k)\cdot(n-k+1)-2)$ for every k between 1 and n to finish the proof. □

Some ordered pairs of commuting involutions that are not conjugate under the action of $\mathrm{Sym}(2n)$ are conjugate when seen as unordered pairs. For instance, in $\mathrm{Sym}(6)$, the ordered pairs $[(1,2),(1,2)(3,4)]$ and $[(1,2)(3,4),(1,2)]$ are conjugate when seen as unordered pairs. But $[(1,2),(1,2)(3,4)]$ and $[(3,4)(5,6),(1,2)]$ are obviously not as $[(1,2),(1,2)(3,4)]$ and $[(1,2),(3,4)(5,6)]$ are not conjugate.

Lemma 3.3. *Let $1 \leq k, l \leq n$. Let $[\rho_0, \rho_2]$ be a representative of a conjugacy class of ordered pairs of commuting involutions of* Sym$(2n)$*, with ρ_0 an involution of cycle type $(2^k, 1^{2(n-k)})$ and ρ_2 an involution of cycle type $(2^l, 1^{2(n-l)})$.*
If $k \neq l$ there exists an involution ρ_0' of cycle type $(2^l, 1^{2(n-l)})$ and an involution ρ_2' of cycle type $(2^k, 1^{2(n-k)})$, such that the unordered pairs $\{\rho_0, \rho_2\}$ and $\{\rho_0', \rho_2'\}$ are conjugate.
If $k = l$, two representatives of distinct classes of ordered pairs of commuting involutions are never conjugate, even when seen as unordered pairs.

Proof. The first one to one correspondence is easy to see. Take $[\rho_0, \rho_2]$ as described in the lemma. Take ρ_0' of cycle type $(2^l, 1^{2(n-l)})$. As ρ_2 and ρ_0' have same cycle type, they are conjugate by a permutation α. The involution ρ_0^α is of cycle type $(2^k, 1^{2(n-k)})$ and commutes with $\rho_2^\alpha = \rho_0'$, because ρ_0 commutes with ρ_2. Set $\rho_2' = \rho_0^\alpha$. Hence $[\rho_0', \rho_2']$ is an ordered pair of commuting involutions and $\{\rho_0, \rho_2\}$ and $\{\rho_0', \rho_2'\}$ are conjugate as unordered pairs. As the situation is symmetric in k and l, the one to one correspondence is established.
Suppose now that $k = l$ and suppose, by contradiction, that there exist two ordered pairs $[\rho_0, \rho_2]$ and $[\rho_0', \rho_2']$ from different isomorphism classes and a permutation α such that $\rho_0^\alpha = \rho_2'$ and $\rho_2^\alpha = \rho_0'$. Without loss of generality we may choose two representative pairs with $\rho_0 = \rho_0'$. Then

$$\rho_2' = \rho_0^\alpha = \rho_0'^\alpha = \rho_2^{\alpha^2}.$$

This means that ρ_2 and ρ_2' are conjugate which contradicts the fact that the two ordered pairs $[\rho_0, \rho_2]$ and $[\rho_0', \rho_2']$ are in different conjugacy classes. □

Imagine we have found all the conjugacy classes of ordered pairs of commuting involutions in Sym$(2n)$. We now pick one pair of each class and set up a list. Lemma 3.3 shows that for every ordered pair of commuting involutions in the list, of the form $[\rho_0, \rho_2]$ where ρ_0 and ρ_2 have different cycle types, there exists another ordered pair $[\rho_0', \rho_2']$ in the list such that the two pairs are conjugate as unordered pairs. Hence we have to modify the formula in Lemma 3.2 accordingly.

Remark 3.4. *In terms of polytopes the above reasoning means that two ordered pairs described as above generate two dual polytopes. However an ordered pair in the list, of the form $[\rho_0, \rho_2]$, with ρ_0 and ρ_2 having the same cycle type, has no dual pair on that list.*

In the next proof we apply the above reasoning.

3.2. **Proof of Theorem 1.1.** First we have to compute the number of ordered pairs $[\rho_0, \rho_2]$ where ρ_0 and ρ_2 contain exactly k cycles, for a given $1 \leq k \leq n$. Without any loss of generality, we may again fix $\rho_0 = (1,2)(3,4)\dots(2k-1,2k)$. We have to count the number of possibilities to choose an involution ρ_2 of cycle type $(2^k, 1^{2(n-k)})$, being different from ρ_0 and commuting with ρ_0. As in the proof of Lemma 3.2, we construct involutions of cycle type $(2^l, 1^{2(n-l)})$ and acting only on the set $\{1, 2, \dots, 2k\}$, i.e. fixing all the elements of the set $\{2k+1, \dots, 2n\}$. As we want ρ_2 to be of cycle type $(2^k, 1^{2(n-k)})$, we have to add $k - l$ 2-cycles acting on the set $\{2k+1, \dots, 2n\}$ and fixing the elements of $\{1, \dots, 2k\}$. The integer l may take every value between 0 and k if $n - k \geq k$, that is if $k \leq n/2$. Define $\nu(n) = \lfloor \frac{n}{2} \rfloor + 1$.

If $k < \nu(n)$, the integer l may take every value between 0 and k and 2-cycles involving only elements from the set $\{2k+1, \dots, 2n\}$ are added to get cycle type $(2^k, 1^{2(n-k)})$. To compute the number of possibilities for ρ_2 in this case, we have to compute the number of involutions consisting of l cycles, with $0 \leq l \leq k$, and fixing every element of the set $\{2k+1, \dots, 2n\}$. So we are in exactly the same situation as in the proof of Lemma 3.2. Thus we get exactly $\lambda(k)$ possibilities, with $\lambda(k)$ defined as in Lemma 3.2. However, one of these possibilities is the case where $\rho_2 = \rho_0$. As we do not want this case, we have to substract 1. Finally if $k < \nu(n)$, there are $\lambda(k) - 1$ ordered pairs of commuting involutions $[\rho_0, \rho_2]$ such that both ρ_0 and ρ_2 contain exactly k cycles.

If $k \geq \nu(n)$, we have $2k - n \leq l \leq k$. Let $\lambda'(k)$ be the number of possibilities for ρ_2. We distinguish between two cases.
If $2k - n - 1$ is odd,

$$\lambda'(k) = \sum_{l=2k-n}^{k} \left(\left\lfloor \frac{l}{2} \right\rfloor + 1 \right) = \lambda(k) - \frac{1}{2}\Big(2k - n\Big) - \left[\frac{1}{2}\Big(2k - n\Big)\right]^2.$$

If $2k - n - 1$ is even,

$$\lambda'(k) = \lambda(k) - 2k + n - \left[\frac{1}{2}\Big(2k - n - 1\Big)\right]^2.$$

A simple algebraic manipulation then gives us

$$\lambda'(k) = \begin{cases} \lambda(k) - \frac{1}{2}\Big(2k - n\Big) - \left[\frac{1}{2}\Big(2k - n\Big)\right]^2 & \text{if } n \text{ is even,} \\ \lambda(k) - 2k + n - \left[\frac{1}{2}\Big(2k - n - 1\Big)\right]^2 & \text{if } n \text{ is odd.} \end{cases}$$

Once again we have to subtract 1 from $\lambda(k)'$ as one of these possibilities is the case $\rho_0 = \rho_2$.

Let $\phi(k)$ be the number of pairs of commuting involutions $[\rho_0, \rho_2]$ with ρ_0 and ρ_2 containing both exactly k cycles. By the former reasoning we get

$$\phi(k) = \begin{cases} \lambda(k) - 1 & \text{if } k \leq \lfloor \frac{n}{2} \rfloor, \\ \lambda(k) - \psi(k, n) - 1 & \text{if } k > \lfloor \frac{n}{2} \rfloor, \end{cases} \tag{1}$$

where $\psi(k, n)$ is defined as in Theorem 1.1. Thus the number of pairs of commuting involutions $[\rho_0, \rho_2]$ with ρ_0 and ρ_2 containing the same number of cycles, is the sum of all $\phi(k)$ for $1 \leq k \leq n$.

To compute the general number of unordered pairs of commuting involutions in $\mathrm{Sym}(2n)$, we substract this sum from the formula in Lemma 3.2, divide by 2 and then add again the sum. The number of unordered pairs of commuting involutions in $\mathrm{Sym}(2n)$ is therefore as stated in the theorem, except for $2n = 6$ as $\mathrm{Sym}(6)$ is not its own automorphism group.

We finish the proof of Theorem 1.1 by analysing the case where $2n = 6$.

3.3. **The case where $2n = 6$.** For $\mathrm{Sym}(6)$, the number of pairs of involutions obtained above does not take into account that there may be pairs that are isomorphic in $Aut(\mathrm{Sym}(6))$. A straightforward analysis of the pairs of commuting involutions of $\mathrm{Sym}(6)$ gives the following representative of each isomorphism class of pairs of commuting involutions in $\mathrm{Sym}(6)$.

$$\rho_0 = (1, 2) \text{ and } \rho_2 = (3, 4) \tag{2a}$$

$$\rho_0 = (1, 2) \text{ and } \rho_2 = (3, 4)(5, 6) \tag{2b}$$

$$\rho_0 = (1, 2) \text{ and } \rho_2 = (1, 2)(3, 4) \tag{2c}$$

$$\rho_0 = (1, 2) \text{ and } \rho_2 = (1, 2)(3, 4)(5, 6) \tag{2d}$$

$$\rho_0 = (1, 2)(3, 4) \text{ and } \rho_2 = (1, 3)(2, 4) \tag{2e}$$

This finishes the proof of Theorem 1.1.

3.4. **Polyhedra with Automorphism Group** Sym(6)**.** We already mentioned that there is a bijection between abstract regular polytopes and string C-groups.

Definition 3.5. *A C-group is a group generated by pairwise distinct involutions* $\rho_0, \rho_1, \rho_{n-1}$ *which satisfy the following property, called the intersection property.*

$$\forall J, K \subseteq \{0, \dots, n-1\}, \langle \rho_j \mid j \in J \rangle \cap \langle \rho_k \mid k \in K \rangle = \langle \rho_j \mid j \in J \cap K \rangle$$

A C-group $(G, \{\rho_0, \dots, \rho_{n-1}\})$ *is a string C-group if its generators satisfy the following relations.*

$$(\rho_j \rho_k)^2 = 1_G \ \forall j, k \in \{0, \dots n-1\} \text{ with } |j-k| \geq 2$$

As seen above, there are, up to isomorphism, 5 pairs of commuting involutions in Sym(6). We want to see which of these 5 pairs can be extended into a string C-group $\langle \rho_0, \rho_1, \rho_2 \rangle$ isomorphic to Sym(6) (see [11]).

By analysing each pair $\{\rho_0, \rho_2\}$ separately, it may be found out that only the two pairs $\rho_0 = (1,2)$ and $\rho_2 = (3,4)(5,6)$ and $\rho_0 = (1,2)$ and $\rho_2 = (1,2)(3,4)(5,6)$ are expandable in a string C-group isomorphic to Sym(6) by adding one extra involution ρ_1. The following theorem summarizes the result we obtain.

Theorem 3.6. *Up to isomorphism and duality,* Sym(6) *is the automorphism group of exactly* 2 *abstract regular polyhedra. These* 2 *polyhedra have Schläfli type* $\{6,5\}$ *and* $\{6,6\}$.

Remark 3.7. *In* [10] *Leemans and Vauthier have come to the same result for* Sym(6)*, with the help of a computer.*

4. The Alternating Group Alt(n)

4.1. **Preliminary results and proof of Theorem 1.2.** In this section we do the same work for the alternating groups. First we give the general form of the centralizer of a given involution in Alt(n). Then we give a formula computing the number of unordered and the number of ordered pairs of commuting involutions in Alt(n).

Proposition 4.1. *Let* ρ *be an involution with of cycle type* $(2^k, 1^{n-2k})$ *in* Alt(n)*. Then* $C_{\mathrm{Alt}(n)}(\rho) = \{\sigma \in E_{2^k} : \mathrm{Sym}(k) \times \mathrm{Sym}(n-2k) \mid \sigma \in \mathrm{Alt}(n)\}$ *and we also get* $|C_{\mathrm{Alt}(n)}(\rho)| = 2^{k-1} \times k! \times (n-2k)!$

Proof. We know that $\mathrm{Alt}(n) \leqslant \mathrm{Sym}(n)$. So $C_{\mathrm{Alt}(n)}(\rho) \leqslant C_{\mathrm{Sym}(n)}(\rho)$. Moreover, as an involution contains at least one cycle of even length, the conjugacy class of an involution in Alt(n) corresponds to the class of all involutions of same cycle type. Thus it is the same as the conjugacy class of that involution in Sym(n). Hence $C_{\mathrm{Alt}(n)}(\rho)$ is not equal to $C_{\mathrm{Sym}(n)}(\rho)$, but is merely a subgroup of index 2 of it. In fact it is the subgroup of all even permutations of $C_{\mathrm{Sym}(n)}(\rho)$. Also the order of $C_{\mathrm{Alt}(n)}(\rho)$ is half of the order of $C_{\mathrm{Sym}(n)}(\rho)$. □

As in section 3, we concentrate on groups Alt($2n$). We start by choosing a first involution. All involutions of same cycle type in Alt($2n$) are conjugate and thus the number of possibilities to choose a first involution ρ_0, up to conjugacy, equals the number of different cycle types in Alt($2n$). An involution in Alt($2n$) has cycle type $(2^k, 1^{2(n-k)})$ with k an even integer between 1 and n. Hence there are $\lfloor \frac{n}{2} \rfloor$ different possibilities, up to isomorphism, for choosing a first involution ρ_0 in Alt($2n$).

Step 2 consists of choosing a second involution ρ_2, commuting with ρ_0, in Alt($2n$), once ρ_0 is fixed. Hence we choose a pair of commuting involutions in Alt($2n$). We first compute, in the following lemma, the number of orbits of *ordered* pairs of commuting involutions in Alt($2n$) under the action of Sym($2n$).

Lemma 4.2. *Under the action of* $\mathrm{Sym}(2n)$*, there are* $\mu(n)$ *orbits of ordered pairs of commuting involutions* ρ_0 *and* ρ_2 *in* $\mathrm{Alt}(2n)$*, where* $\mu(n)$ *is as stated in Theorem 1.2.*

Proof. We start in the same way as in the proof of Lemma 3.2. We take ρ_0 an involution of cycle type $(2^k, 1^{2(n-k)})$, k even, and $\rho_2 \in C_{\mathrm{Alt}(2n)}(\rho_0) = \{\sigma \in E_{2^k} : \mathrm{Sym}(k) \times \mathrm{Sym}(2(n-k)) \mid \sigma \in \mathrm{Alt}(2n)\}$. We proceed in the same way by counting first the possibilities for an involution ρ_2 acting only on elements of $\{1, 2, \ldots, 2k\}$.

We consider the involutions up to isomorphism and $Aut(\mathrm{Alt}(2n)) = Aut(\mathrm{Sym}(2n))$. If an involution is conjugate to ρ_2 in $\mathrm{Sym}(2n)$, then in $\mathrm{Alt}(2n)$ it is also conjugate to ρ_2. Therefore we can apply the same reasoning as in the proof of Lemma 3.2 and we suppose ρ_2 is composed of l 2-cycles, where m 2-cycles are of the form (s, t), with $|s - t| \geq 2$ and $s, t \leq 2k$, $m \leq l$ and $l - m$ 2-cycles are of the form $(2r - 1, 2r)$, with $1 \leq r \leq k$. We have to count the number of possibilities for m, with m even. So, as before, m can take every even value between 0 and l, which gives us $\lfloor \frac{l}{2} \rfloor + 1$ possibilities for m, for a given l. In the previous proof, l could take every value between 0 and k. In this proof, we have to distinguish between the case when l takes an even value and when l takes an odd value. Let $\gamma(k)$ denote the number of possible involutions ρ_2 with an even number of 2-cycles involving only elements of $\{1, 2, \ldots, 2k\}$. It is easy to see that γ is as claimed. Let $\delta(k)$ denote the number of possible involutions ρ_2 with an odd number of 2-cycles involving elements in $\{1, 2, \ldots, 2k\}$. It is also easy to see that δ is as claimed.

As before we add the 2-cycles composed of elements of the set $\{2k + 1, \ldots, 2n\}$. An involution in $\mathrm{Sym}(2(n - k))$ can have j 2-cycles with j a number between 0 and $n - k$. As the involution ρ_2 is in $\mathrm{Alt}(2n)$, either l and j are even, or l and j are odd. There are exactly $\lceil \frac{1}{2}(n - k + 1) \rceil$ even integers between 0 and $n - k$ and hence there are $\lceil \frac{1}{2}(n - k + 1) \rceil$ possibilities for j taking an even value. These possibilities have to be multiplied by $\gamma(k)$. On the other side there are $\lfloor \frac{1}{2}(n - k + 1) \rfloor$ odd integers between 0 and $n - k$, which gives us $\lfloor \frac{1}{2}(n - k + 1) \rfloor$ possibilities for j taking an odd value. These possibilities have to be multiplied by $\delta(k)$.

As before we have to remove the two possibilities where ρ_2 is the identity and where ρ_2 is equal to ρ_0. In both cases l and j are even. So we have to remove 2 from the number $\gamma(k) \cdot \lceil \frac{1}{2}(n - k + 1) \rceil$ and sum up all these possibilities for every even k between 1 and n. This leads to the desired value of $\mu(n)$. □

By Lemma 4.2, we may compute the number of *unordered* pairs of commuting involutions in $\mathrm{Alt}(2n)$ and obtain a proof of Theorem 1.2 using the same techniques as in the proof of Theorem 1.1. Therefore, we leave the proof to the interested reader.

For the case where $2n = 6$, it remains to look at the action of $Aut(\mathrm{Alt}(6))$ on pairs of commuting involutions.

Lemma 4.3. *In* $\mathrm{Alt}(6)$ *there is a unique pair of commuting involutions, up to isomorphism.*

Proof. In $\mathrm{Sym}(6)$ we are left with 5 pairs of commuting involutions, up to isomorphism. They are shown in (2). Only the pair (2e) corresponds to a pair of commuting involutions in $\mathrm{Alt}(6)$. Hence in $\mathrm{Alt}(6)$ we are left with exactly one pair of commuting involutions, namely $(1, 2)(3, 4)$ and $(1, 3)(2, 4)$. □

4.2. **Polyhedra with Automorphism Group** $\mathrm{Alt}(6)$**.** Lemma 4.3 shows that there is, up to isomorphism, only one pair of commuting involutions in $\mathrm{Alt}(6)$. We choose $\rho_0 = (1, 2)(3, 4)$ and $\rho_2 = (1, 3)(2, 4)$. We are looking for an involution ρ_1 such that $\langle \rho_0, \rho_1, \rho_2 \rangle$ is a string C-group isomorphic to $\mathrm{Alt}(6)$. It can be shown that such a ρ_1 does not exist. In fact all the ρ_1 that satisfy the conditions of a string C-group are such that the group $\langle \rho_0, \rho_1, \rho_2 \rangle$ is not isomorphic to Alt(n). This confirms the following well known theorem.

Theorem 4.4. *No polyhedron has automorphism group isomorphic to* Alt(6).

5. Acknowledgements

The authors thank the anonymous referees for useful comments on a previous version of this paper.

References

[1] M.-E. Fernandes and D. Leemans. Polytopes of high rank for the symmetric groups. *Adv. Math.*, 228:3207–3222, 2011.

[2] M.-E. Fernandes, D. Leemans, and M. Mixer. Polytopes of high rank for the alternating groups. *J. Combin. Theory Ser. A*, 119:42–56, 2012.

[3] M.-E. Fernandes, D. Leemans, and M. Mixer. All alternating groups A_n with $n \geq 12$ have polytopes of rank $\lfloor \frac{n-1}{2} \rfloor$. *SIAM J. Discrete Math.*, 26(2):482-498, 2012.

[4] M. I. Hartley. The atlas of small regular polytopes. *Periodoca Math. Hungar. (Special volume on discrete geometry)*, 2006, http://www.abstract-polytopes.com/atlas.

[5] M. I. Hartley and A. Hulpke. Polytopes Derived from Sporadic Simple Groups. *Contributions to Discrete Mathematics*, 5:106-118 (2010). Full atlas online at http://www.abstract-polytopes.com/sporpolys/

[6] A. Kiefer and D. Leemans. On the number of abstract regular polytopes whose automorphism group is a Suzuki simple group Sz(q). *J. Combin. Theory Ser. A*, 117(8):1248–1257, 2010.

[7] D. Leemans. Almost simple groups of Suzuki type acting on polytopes. *Proc. Amer. Math. Soc.*, 134(12):3649–3651 (electronic), 2006.

[8] D. Leemans and E. Schulte. Groups of type $\mathrm{L}_2(q)$ acting on polytopes. *Adv. Geom.*, 7(4):529–539, 2007.

[9] D. Leemans and E. Schulte. Polytopes with groups of type $PGL_2(q)$. *Ars Math. Contemp.*, 2(2):163–171, 2009.

[10] D. Leemans and L. Vauthier. An atlas of abstract regular polytopes for small groups. *Aeq. Math.*, 72(3):313–320, 2006, http://www.math.auckland.ac.nz/~dleemans/polytopes/.

[11] P. McMullen and E. Schulte. *Abstract regular polytopes*, volume 92 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2002.

[12] M. Suzuki. *Group theory. I*, volume 247 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1982. Translated from the Japanese by the author.

[13] R. A. Wilson. *The finite simple groups*, volume 251 of *Graduate Texts in Mathematics*. Springer-Verlag London Ltd., London, 2009.

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