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Faculty of Science and Bio-engineering Sciences

Units in Integral Group Rings via Fundamental Domains and Hyperbolic Geometry

Doctoral thesis submitted in partial fulfillment of the requirements for the degree of Doctor in Science

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Les charmes enchanteurs de cette sublime science
ne se décèlent dans toute leur beauté
qu'à ceux qui ont le courage de l'approfondir.

Carl Friedrich GAUSS
(letter to Sophie Germain)

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Contents

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Notational Conventions

$\mathbb{B}^n$	Ball model of hyperbolic space of dimension n
$B(a,r)$	Ball of centre a and radius r
$\overline{B(a,r)}$	Closed ball of centre a and radius r
$\mathcal{B}_1(\mathbb{Z}G)$	Group generated by the Bass units in $\mathbb{Z}G$
$\mathcal{B}_2(\mathbb{Z}G)$	Group generated by the bicyclic units in $\mathbb{Z}G$
$\mathrm{Bis}_\gamma(P)$	Bisector of P and $\gamma(P)$
$\mathbb{C}$	Field of complex numbers
$\widehat{\mathbb{C}}$	$\mathbb{C} \cup \{\infty\}$
DAFC	Dirichlet algorithm of finite covolume
$\mathcal{F}$	Fundamental domain
$\mathcal{F}_P$	Fundamental domain of Γ_P
G	Finite group
$\mathrm{GBic}^\iota(\mathcal{O}G)$	Group generated by all the generalized bicyclic units defined by ι
$\mathrm{GL}_n(R)$	Group of invertible matrices in $M_n(R)$
Γ	Discontinuous group
Γ_P	Stabilizer of P in Γ
$h(u)$	Height of a point u in some topological space
$\mathcal{H}\left(\frac{a,b}{K}\right)$	Quaternion algebra with centre K
$\mathcal{H}(K)$	Classical quaternion algebra with centre K
$\mathcal{H}$	Classical Hamilton quaternion algebra with centre $\mathbb{R}$
$\mathbb{H}^n$	Hyperbolic space (upper half-space model) of dimension n
I_γ	Isometric sphere of γ in the upper half-space model
$\mathrm{Iso}(\mathbb{X})$	Group of isometries of $\mathbb{X}$
$\mathrm{Iso}^+(\mathbb{X})$	Group of orientation-preserving isometries of $\mathbb{X}$
K	Number field

$\mathbb{K}^n$	Klein model of hyperbolic space of dimension n
$M_n(R)$	Matrix ring over a ring R
$\mathbb{N}$	Set of natural numbers (including 0)
$\mathbb{N}_0$	Set of natural numbers (without 0)
$N(\ )$	Norm map of a number field K
η_0	Isometry between $\mathbb{H}^3$ and $\mathbb{B}^3$ (defined in Proposition 1.2.28)
$\mathcal{O}$	Order in a ring
$\mathcal{O}_K$	Ring of integers of the number field K
P	Point in some topological space, or convex (fundamental) polyhedron
$\mathrm{PSL}_n(R)$	Projective linear group over a ring R
Ψ	Group isomorphism between $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{SB}_2(\mathcal{H})$ (defined in Proposition 1.2.28)
PID	Principal ideal domain
$\mathbb{Q}$	Field of rational numbers
R	Unital ring
$\mathbb{R}$	Field of real numbers
$\widehat{\mathbb{R}}$	$\mathbb{R} \cup \{\infty\}$
$\mathbb{R}^n$	Euclidean space of dimension n
RG	Group ring of G over R
ρ	Hyperbolic metric
$\mathbb{S}^n$	Unit sphere of dimension n
$S(a,r)$	Sphere of center a and radius r
$\mathrm{SB}_2(\mathcal{H})$	Group of orientation-preserving isometries of $\mathbb{B}^3$
$\mathrm{SL}_n(R)$	Special linear group over a ring R
$\mathrm{SU}_n(R)$	Special unitary group over a ring R
Σ_γ	Bisector of i, resp. j, and $\gamma^{-1}(i)$, resp. $\gamma^{-1}(j)$, in the upper half-space model
$\Sigma_{\Psi(\gamma)}$	Isometric sphere of $\Psi(\gamma)$ (=bisector of 0 and $\Psi(\gamma^{-1})(0)$) in the ball model
$\mathcal{T}$	Tessellation
$\mathcal{U}(R)$	Group of units of the ring R
$\mathbb{X}$	General metric space (of constant curvature in Chapter 2)
x^*	inverse point of x with respect to the reflection in $\mathbb{S}^n$
$\overline{Y}$	Closure of Y
Y°	Interior of Y
∂Y	Boundary of Y
$Z, W, \ldots$	Chapter 2: half-spaces Chapter 4: points of $\mathbb{H}^2 \times \mathbb{H}^2$
$\mathbb{Z}$	Ring of integers
$\mathcal{Z}(R)$	Centre of the ring R

$\| \|$	Euclidean norm
$\|\gamma\|$	Norm of a matrix $\gamma \in M_n(\mathbb{C})$
$\|cZ + d\|$	$|Z_1|^2|Z_2|^2$ for $(Z_1, Z_2) \in \mathbb{H}^2 \times \mathbb{H}^2$

Introduction

"Should you just be an algebraist or a geometer?"
is like saying "Would you rather be deaf or blind?"
Sir Michael Atiyah

The investigation on the structure of the unit group $\mathcal{U}(\mathcal{O})$ of an order $\mathcal{O}$ in a finite-dimensional rational algebra A, or more precisely a $\mathbb{Z}$-order in A, is a question of fundamental interest. Such a unit group is an important example of an arithmetic group. Due to a celebrated result of Borel and Harish-Chandra (see Theorem 1.1.11) it is a finitely presented group.

In this thesis we concentrate on the unit group of a group ring $\mathcal{O}G$ of a finite group G over an order $\mathcal{O}$ in a number field. Group rings were studied at the beginning of the twentieth century in the context of representation theory and since the work of E. Noether and R. Brauer, they have become a subject on their own. They give very interesting algebraic structures and, motivated by fundamental questions such as the Isomorphism Problem or the Zassenhaus conjectures, the interest in unit groups of such group rings became more important. Since the work of Higman in 1940 [Hig40a, Hig40b], the unit group of a group ring $\mathcal{O}G$ of a finite group G over an order $\mathcal{O}$ in a number field has received tremendous attention.

The most natural case is that of integral group rings, i.e. $\mathcal{O} = \mathbb{Z}$. However, only for very few finite non-abelian groups G the unit group $\mathcal{U}(\mathbb{Z}G)$ has been described, and even for fewer groups G a presentation of $\mathcal{U}(\mathbb{Z}G)$ has been obtained. In fact, one important question in the field of units in integral group rings is to obtain finitely many generators, or more general, to get a finite presentation of $\mathcal{U}(\mathbb{Z}G)$ (see for instance Problems 17 and 23 in [Seh93]). A first result in this direction was established in 1965 by Bass and Milnor [Bas65] in the case when G is a finite abelian group (see Theorem 1.1.22). A concrete and generic set of finitely many generators for a subgroup of finite index is described, the so called Bass units. Since 1989, with the work of Ritter and Sehgal [RS89a] and [RS91b], the attention is focussed on finite non-abelian groups. They introduced the first specific constructions of units in integral group rings for finite non-abelian groups; these are called the bicyclic units.

Since then for many finite groups G a specific finite set B of generators of a subgroup of finite index in $\mathcal{U}(\mathbb{Z}G)$ has been given. We shortly describe an approach on how to obtain a finite set of generators. If G is a finite group then $\mathbb{Z}G$ is an order in the rational group algebra $\mathbb{Q}G$. By the Wedderburn-Artin Theorem (see Theorem 1.1.5),

$$\mathbb{Q}G = \prod_{i=1}^{n} M_{n_i}(D_i),$$

where each D_i is a division algebra. If $\mathcal{O}_i$ is an order in D_i, for each i, then $\mathcal{O} = \prod_{i=1}^{n} M_{n_i}(\mathcal{O}_i)$ is an order in $\mathbb{Q}G$ and the group of units $\mathcal{U}(\mathcal{O})$ of $\mathcal{O}$ is commensurable with $\mathcal{U}(\mathbb{Z}G)$, i.e. $\mathcal{U}(\mathcal{O})$ and $\mathcal{U}(\mathbb{Z}G)$ have a common subgroup that is of finite index in both. The group $\mathcal{U}(\mathcal{O})$ is simply $\Pi_{i=1}^{n}\mathrm{GL}_{n_i}(\mathcal{O}_i)$, the direct product of the groups $\mathcal{U}(M_{n_i}(\mathcal{O}_i)) = \mathrm{GL}_{n_i}(\mathcal{O}_i)$. Moreover, $\mathcal{U}(\mathcal{Z}(\mathcal{O}_i)) \times \mathrm{SL}_{n_i}(\mathcal{O}_i)$, the external direct product of the central units in $\mathcal{O}_i$ and the group consisting of the reduced norm one elements in $M_{n_i}(\mathcal{O}_i)$, contains a subgroup of finite index isomorphic to a subgroup of finite index of $\mathrm{GL}_{n_i}(\mathcal{O}_i)$ (see Remark 1.1.14). The structure of the group $\prod_{i=1}^{n}\mathcal{U}(\mathcal{Z}(\mathcal{O}_i))$ may be determined from Dirichlet's Unit Theorem (see Theorem 1.1.6) and it is commensurable with $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$, the group of central units of $\mathbb{Z}G$. For a large class of finite groups G Jespers, Olteanu, del Río and Van Gelder describe generators of a subgroup of finite index in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ [JOdRVG13] (see also [JdRVG14]); these are based on Bass units. Hence, up to commensurability, the problem of finding generators and relations for $\mathcal{U}(\mathbb{Z}G)$ reduces to finding a presentation of $\mathrm{SL}_{n_i}(\mathcal{O}_i)$ for every $1 \leq i \leq n$. The congruence theorems allow to compute generators up to finite index for $\mathrm{SL}_{n_i}(\mathcal{O}_i)$ except if $M_{n_i}(D_i)$ is of so-called exceptional type. More precisely, we call a simple component of the algebra $\mathbb{Q}G$ exceptional if it is of the following type:

(1) a non-commutative division algebra different from a totally definite quaternion algebra,

(2) $M_2(\mathbb{Q})$,

(3) $M_2(K)$ with K a quadratic imaginary extension of $\mathbb{Q}$, or

(4) $M_2(\mathcal{H})$ where $\mathcal{H}$ is a totally definite quaternion algebra with centre $\mathbb{Q}$, i.e. $\mathcal{H} = \mathcal{H}\left(\frac{a,b}{\mathbb{Q}}\right)$ with a and b negative integers.

More details on this will be given in Subsection 1.1.3, in particular in Definition 1.1.19. Combining the results of Bass [Bas64, Theorem 21.1, Corollary 21.4], Vaseršteĭn [Vas73, Theorem 2.4, Lemma 2.6], Liehl [Lie81, Theorem 24], Venkataramana [Ven94] and Kleinert [Kle00, Theorem 2], we get the following Theorem.

Theorem. *Let D be a finite dimensional rational division ring and let $\mathcal{O}$ be an order in D. Let n be a positive integer. If $n \geq 3$ or $n = 2$ and the simple algebra $M_2(D)$ is not exceptional of type (2)-(4), then there exists an explicitly computable finite set B which generates a subgroup of finite index in $\mathrm{SL}_n(\mathcal{O})$. If $n = 1$ and D is not exceptional of type (1), then the group of units of norm 1 in $\mathcal{O}$ is finite or commutative. In the latter case it can be described by Dirichlet's Unit Theorem.*

Observe that the theorem does not only prove the theoretical existence of the set B, but this set is explicitly described. Roughly speaking, the set B consists of elementary matrices. More details on this are given in Subsection 1.1.3. This theorem constitutes the starting point for describing a presentation of the group of units of $\mathbb{Z}G$, up to commensurability, provided the group algebra $\mathbb{Q}G$ does not have exceptional simple components. In a series of papers, Ritter and Sehgal [RS91a, RS91b, RS93] showed that for several classes of finite groups, including nilpotent groups of odd order, the group generated by the Bass units and bicyclic units is of finite index in $\mathcal{U}(\mathbb{Z}G)$. Jespers and Leal extended this result in [JL93] to a much wider class of finite groups, only excluding those finite groups G for which the rational group algebra $\mathbb{Q}G$ has exceptional components and groups which have a non-abelian fixed point free epimorphic image. These are remarkable results in the sense that they are based on only two known constructions of units. The proofs of these results ultimately make use of the previous theorem and the result of Bass and Milnor described earlier.

The aim of this thesis is to develop new methods to get finitely many generators and/or a finite presentation of $\mathrm{SL}_n(\mathcal{O})$ with $M_n(\mathcal{O})$ an order in an exceptional component. This then leads to extensions of previously mentioned results on generators of subgroups of finite index in $\mathcal{U}(\mathbb{Z}G)$ by also allowing exceptional components as simple components of the group algebra $\mathbb{Q}G$ in question. Of course, this also yields applications to the unit group of group rings $\mathcal{O}G$ over arbitrary orders $\mathcal{O}$ in a number field. Indeed, by the Wedderburn-Artin theorem, for every order $\mathcal{O}$ in a semisimple finite-dimensional rational algebra A, the investigation on $\mathcal{U}(\mathcal{O})$ is reduced, up to commensurability, to the study of $\mathrm{SL}_{n_i}(\mathcal{O}_i)$ for $n_i \in \mathbb{N}$ and $\mathcal{O}_i$ orders in number fields. By Maschke's theorem (Theorem 1.1.3) this applies in particular to the unit group of an order in KG for K an arbitrary number field and G a finite group. Nevertheless, this thesis is done in the framework of units in integral group rings, for the simple reason that the theory on integral group rings is much more developed than the theory on arbitrary orders in group rings KG of finite groups G over arbitrary number fields K. In this way, we can nicely sketch the theory behind our work and give examples our results can apply to. In this context, we point out that, in a recent result of Eisele, the author and Van Gelder [EKVG14], a classification is given of all the exceptional components of type (3) and (4) that can appear as simple components of $\mathbb{Q}G$ for a finite group G. It is shown that if $\mathbb{Q}G$ has an exceptional component of type (3) or (4), then this component is a 2×2-matrix ring over one of the following rings:

(i) $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$ or $\mathbb{Q}(\sqrt{-3})$;

(ii) $\left(\frac{-1,-1}{\mathbb{Q}}\right)$, $\left(\frac{-1,-3}{\mathbb{Q}}\right)$ or $\left(\frac{-2,-5}{\mathbb{Q}}\right)$.

The fields listed in (i) are all norm Euclidean quadratic fields [Wei63], and the quaternion algebras listed in (ii) all have a maximal order which is norm Euclidean [Fit12]. Furthermore, the finite groups that are faithfully embedded in an exceptional simple component (and generate this component as an algebra) are described. Using this Euclidean norm, one can then obtain a finite set of generators for $\mathrm{SL}_2(\mathcal{O})$ with $\mathcal{O}$ a Euclidean maximal order in one of these exceptional components. The Reidemeister-Schreier method allows one then to obtain generators for a subgroup of finite index.

In this thesis, the main tools for the study of a presentation of $\mathrm{SL}_n(\mathcal{O})$, with $M_n(\mathcal{O})$ an order in an exceptional component, are discontinuous group actions on hyperbolic spaces. This is based on the more general theory of discontinuous group actions on proper metric spaces. If a group Γ acts discontinuously on a proper metric space, it tiles the space into closed parts, whose interiors are disjoint. One "puzzle piece" of this tiling is called a fundamental domain for Γ, or fundamental polyhedron for Γ, in case it is a convex polyhedron. In the three classical metric spaces of constant curvature, i.e. Euclidean, spherical and hyperbolic space, Poincaré's Polyhedron Theorem (see Theorem 2.4.7) describes a presentation of Γ, which is based on its fundamental domain. So the idea is to choose an appropriate space $\mathbb{X}$, on which $\mathrm{SL}_n(\mathcal{O})$ acts discontinuously, construct a fundamental domain for $\mathrm{SL}_n(\mathcal{O})$ in $\mathbb{X}$ and hence deduce a presentation of $\mathrm{SL}_n(\mathcal{O})$. In a first instance, this is done in the hyperbolic 2-space $\mathbb{H}^2$ and hyperbolic 3-space $\mathbb{H}^3$. In fact the groups $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{C})$, which are natural epimorphic images of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$, act via Möbius transformations and Poincaré extensions of Möbius transformations on $\mathbb{H}^2$ and $\mathbb{H}^3$ respectively. One can show that discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$ or $\mathrm{PSL}_2(\mathbb{C})$ have discontinuous actions on $\mathbb{H}^2$ or $\mathbb{H}^3$ (see Theorem 1.2.6). Thus groups of the form $\mathrm{SL}_n(\mathcal{O})$ that are discrete in $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$ may be handled in this way. These groups are linked to exceptional components of type (2), (3) and some of type (1).

There already have been attempts in that direction. In [DJK10], Dooms, Jespers and Konovalov introduced a method (also by computing a fundamental polyhedron for a discrete group of finite covolume) to deal with exceptional simple components of type (2). New generators are introduced, using Farey symbols, which are in one to one correspondence with fundamental polyhedra for congruence subgroups of $\mathrm{PSL}_2(\mathbb{Z})$. In [CJLdR04], Corrales, Jespers, Leal and del Río gave an algorithm to compute the unit group of the order $\mathcal{H}\left(\frac{-1,-1}{\mathcal{O}_K}\right)$ in the quaternion division algebra $\mathcal{H}\left(\frac{-1,-1}{K}\right)$ over an imaginary quadratic field extension K of the rationals ($\mathcal{O}_K$ is the ring of integers in K). The algorithm computes a fundamental polyhedron for the group $\mathcal{U}\left(\mathcal{H}\left(\frac{-1,-1}{\mathcal{O}_K}\right)\right)$, which can be seen as a cocompact discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$ and hence acts discontinuously on the hyperbolic 3-space. In this way one determines a presentation of $\mathcal{U}\left(\mathcal{H}\left(\frac{-1,-1}{\mathcal{O}_K}\right)\right)$ by making use of Poincaré's Polyhedron Theorem. Next, the algorithm is applied to the unit group of $\mathcal{H}\left(\frac{-1,-1}{\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]}\right)$ providing the first example of a description of a finite set of generators of the unit group of a group ring RG with an exceptional component that is a division algebra which is not a totally definite quaternion algebra.

Another approach in this direction is the investigation on groups of Kleinian type. A finite group G is said to be of Kleinian type if each non-commutative simple factor $M_{n_i}(D_i)$ of $\mathbb{Q}G$ is a quaternion algebra over its centre and the natural image of $\mathrm{SL}_{n_i}(\mathcal{O}_i)$ in $\mathrm{PSL}_2(\mathbb{C})$ (obtained by extending some embedding of the centre of D_i in $\mathbb{C}$) is a Kleinian group. A Kleinian group is a subgroup of $\mathrm{PSL}_2(\mathbb{C})$ which is discrete for the natural topology, or equivalently, its action on the 3-dimensional hyperbolic space via the

Poincaré extension of the action by Möbius transformations is discontinuous. Thus, if G is a finite group of Kleinian type then, in theory by Poincaré's Polyhedron Theorem, one can obtain a presentation of a group commensurable with $\mathcal{U}(\mathbb{Z}G)$. Groups of Kleinian type have been classified by Jespers, Pita, del Río and Ruiz in [JPdR$^+$07] and examples on how to find presentations of $\mathcal{U}(\mathbb{Z}G)$ for some groups of Kleinian type of small order have been given by Pita, del Río and Ruiz in [PdRR05] and [PdR06]. Olteanu and del Río, in [OdR07], also obtained a generalization to group rings over commutative orders.

The aim of this PhD thesis is to develop a general approach of the above explained isolated attempts of describing the unit group of an order in an exceptional component via its discontinuous action on some metric space and a fundamental domain for this action. For the applications to units of group rings, we focus on integral group rings $\mathbb{Z}G$ and for convenience and completeness of results we sometimes assume G is nilpotent.

In Chapter 1, we first give a brief overview of the knowledge on generators of the unit group of integral groups rings of finite groups. We describe the results of Bass, Vaseršteĭn, Liehl, Venkataramana and Kleinert, mentioned above, in greater details and explain the origin of the exceptional components. We also give some of the main theorems, which describe generators up to finite index for the unit group of $\mathbb{Z}G$, provided $\mathbb{Q}G$ does not have exceptional components. This overview is principally based on [Seh93, Jes98, JdR]. As already sketched above, the results of this thesis are based on discontinuous group actions, hyperbolic geometry and fundamental domains. In Section 1.2, we first give the necessary background on discontinuous groups and fundamental domains for such groups. We then sketch an overview of hyperbolic geometry and also describe the three main models of this geometry. Details are then given in the two and three dimensional case. This section is mostly based on [Rat06, Bea95, EGM98].

In Chapter 2, we reprove the presentation part of Poincaré's Polyhedron Theorem. As already explained above, our aim is to describe some unit groups via their discontinuous actions on hyperbolic space, using Poincaré's Polyhedron Theorem. A lot on this theorem may be found in the literature, see for instance the books [Bea95, BH99, EGM98, Mas88, Rat06]. There are also various articles on this theorem, such as [EP94, Mas71, dR71, AG11]. However, most of the proofs given in literature are rather unsatisfactory and rely often on the theory of covering spaces. We refer the reader to the paper of Epstein and Petronio [EP94, Section 9] for a long list of references on this topic and problems showing up in each of them. The two original versions written by Poincaré ([Poi82] and [Poi83]) are very complicated to read. Moreover, for the proof of the three-dimensional case one simply refers for a large part to the two-dimensional case, and this without really proving that the results used are still valid. More modern versions often only deal with dimension two, which simplifies the proof a lot and cannot be directly generalized to dimension three. Other modern versions are either very complicated to read or stay hazy on some aspects. The proofs of Maskit in [Mas88] and Epstein and Petronio in [EP94] focus mostly on that part of Poincaré's Theorem which states that if a polyhedron satisfies some conditions then it is a fundamental domain for a discontinuous group. As a consequence, the presentation part of the theorem is obtained somehow indirectly and the intuition on the presentation part is hidden in these proofs. That is why, in Chapter 2, we give a new and self-contained proof of the presentation part of Poincaré's Theorem. This proof just relies on basic concepts from geometry. Moreover, the theorem we obtain, see Theorem 2.4.7, is a bit more general, then the theorems that are often stated in references as it does not use the condition of exactness (for details see Definition 1.2.13). With our new proof, we hope to have contributed in a deeper understanding of this important result. This Chapter is entirely based on the joint work with Jespers and del Río in [JKdR14b].

In Chapter 3, we show how Theorem 2.4.7 can be applied to approach our problem on determining generators of the group of units of an order in an exceptional component. We consider Kleinian and Fuchsian groups, i.e. discrete subgroups of $\mathrm{PSL}_2(\mathbb{C})$ and $\mathrm{PSL}_2(\mathbb{R})$ respectively. Such a group Γ acts discontinuously on hyperbolic 3-space, or hyperbolic 2-space respectively. In Lemma 3.1.3 and Lemma 3.1.4, we give concrete and simple formulas that describe a Dirichlet fundamental polyhedron for Γ in the upper half-space model and in the ball model of hyperbolic 2- and 3-space. Using this, we then describe an algorithm, the so-called Dirichlet algorithm of finite covolume (DAFC) (Proposition 3.1.8), to compute, in a finite number of steps, a finite set of generators for a subgroup of finite index in Γ. We also describe a refined algorithm (Proposition 3.1.10) that gives a complete set of finitely many generators in case Γ is a cocompact group. In that case, the refined algorithm also gives a fundamental polyhedron for Γ. Similar attempts on describing algorithms to compute fundamental domains for given Kleinian and Fuchsian groups have also been done in [Ril83] and [Pag13]. As we are mostly interested in finding a presentation of a subgroup of finite index in a given group, our algorithm is in some sense more basic and relies on easier formulas. In Section 3.2, we then show how these algorithms can be applied to unit groups of orders in division algebras which are discrete subgroups of $\mathrm{PSL}_2(\mathbb{C})$ as well as to Bianchi groups. The latter constitute in fact unit groups of orders in exceptional components of type (2) and (3). Among the division algebras that our algorithm can handle (so here we consider exceptional components of type (1)) are division algebras with centre a quadratic imaginary extension of $\mathbb{Q}$. We give several examples, including non-cocompact examples. This thus constitutes a generalization of [CJLdR04], where some small cocompact groups are considered. In Section 3.3, we explain how this new information may be concretely applied to units in integral group rings $\mathbb{Z}G$. We give full details in case G is a finite nilpotent group. We finish this chapter with a digression on something different which also shows that our method allows to obtain geometric information. In fact the formulas we established to describe the DAFC may be used to generalize some results on Dirichlet-Ford domains and double Dirichlet domains. Dirichlet-Ford domains are fundamental domains, that are Dirichlet and Ford domains at the same time. Double Dirichlet domains are fundamental domains that have at least two different points as centre. Based on our formulas we improved some of the results of Lakeland in [Lak12]. This chapter is based on a collaboration with Jespers, Juriaans, de Andrade e Silva and Souza Filho in [JJK$^+$13b] and [JJK$^+$13a].

In the investigations on determining presentations of $\mathcal{U}(\mathbb{Z}G)$ it is now natural to deal with finite groups G that are such that $\mathbb{Q}G$ has simple components of which the unit group of some order does not necessarily act discontinuously on one hyperbolic space, but does so on a direct product of hyperbolic 2- or 3-spaces

$$\mathbb{H}_{r,s} = \mathbb{H}^2 \times \overset{(r)}{\cdots} \times \mathbb{H}^2 \times \mathbb{H}^3 \times \overset{(s)}{\cdots} \times \mathbb{H}^3.$$

This is done in Chapter 4. In fact, in Chapter 3, we developed a method to describe, up to commensurability, the unit group of $\mathbb{Z}G$ allowing exceptional components of type (2) and (3) and also some special cases of type (1). However, the class of exceptional components of type (1) contains many algebras that are not of that special type; for example division algebras that are quaternion algebras over a field extension over the rationals of degree larger than 2. So, the next step is tackling the case where $\mathcal{O}$ is an order in a quaternion algebra with centre a number field $\mathbb{Q}(\xi_n)$, where ξ_n is an $n-$th primitive root of unity. This problem seems much harder and thus in first instance one focusses on specific examples. The smallest n of interest is $n = 7$ and hence we consider the specific example $\mathcal{H}(\mathbb{Q}(\xi_7))$, the classical quaternion algebra over $\mathbb{Q}(\xi_7)$. This algebra is, for example, showing up in the Wedderburn-Artin decomposition of the rational group algebra $\mathbb{Q}G$ of the group $G = Q_8 \times C_7$, the direct product of the quaternion group of order 8 and the cyclic group of order 7. Here the study of the unit group is (up to commensurability) reduced to that of $\mathcal{U}(\mathcal{H}(\mathbb{Z}[\xi_7]))$. This group has a discontinuous action on the direct product $\mathbb{H}^3 \times \mathbb{H}^3 \times \mathbb{H}^3$ of three copies of hyperbolic 3-space. More details on this are given in Section 4.1. Thus, the goal is to generalize Theorem 2.4.7, which is established on Riemann varieties of constant curvature, to the direct product $\mathbb{H}^3 \times \mathbb{H}^3 \times \mathbb{H}^3$, which does not have constant curvature any more.

As this problem seems huge, we first handle a test case, which consists in finding generators for the Hilbert Modular group, i.e. $\mathrm{SL}_2(R)$, where R is the ring of integers of K, for $K = \mathbb{Q}(\sqrt{d})$ a quadratic field with d a positive integer. This is mainly based on work of Cohn and Maass in [Coh65a, Coh65b, Maa40]. In fact, the group $\mathrm{SL}_2(R)$ acts discontinuously on the direct product $\mathbb{H}^2 \times \mathbb{H}^2$ of two copies of hyperbolic 2-space, and hence constitutes the “easiest” example of generalization to direct products of hyperbolic spaces. In order to generalize the theory to actions on direct products of (two) hyperbolic spaces several problems have to be overcome. A first problem is the following.

(1) Does Poincaré’s Method remain valid for discontinuous actions on $\mathbb{H}_{r,s}$?

The following problem is one of the first issues to deal with in order to answer the question.

(2) Determine effective methods to construct fundamental domains.

Once this is done, the following problem arises.

(3) The sides of the well-known fundamental polyhedra in $\mathbb{H}^2$ and $\mathbb{H}^3$ are geodesic hyperplanes, i.e. lines and circles or planes and spheres, orthogonal to the border. How should the “sides” of some potential fundamental domain in $\mathbb{H}_{r,s}$ be defined?

We managed to find a method for defining a fundamental domain, whose boundary is given by finitely many real non-singular algebraic varieties, and we are able to generalize

Theorem 2.4.7. So we get a concrete presentation of $\mathrm{SL}_2(R)$, which is finite in case R is a principal ideal domain (PID). Note, however, that the presentation of $\mathrm{SL}_2(R)$ is of theoretical nature. By imposing the condition of PID on R, we get a concrete finite set of generators for $\mathrm{SL}_2(R)$, which is easily computable. But, concerning the relations, we just have a theoretical result and do not know how to effectively compute them.

Chapter 4 is organized as follows. In Section 4.1, we give some background on discontinuous group actions on direct products of copies of $\mathbb{H}^2$ and $\mathbb{H}^3$ and in Section 4.2, we introduce some notation on the Hilbert Modular group and describe its action on $\mathbb{H}^2 \times \mathbb{H}^2$. In Section 4.3, we give a description of a fundamental domain $\mathcal{F}$ for this action, see Theorem 4.3.10. In Section 4.4, we prove some topological lemmas about the fundamental domain which will be used in the two following sections. Finally, in Section 4.5, we describe the sides of the fundamental domain $\mathcal{F}$. In Theorem 4.5.3, we generalize the generating part of Poincaré's Polyhedron Theorem to the case of $\mathrm{PSL}_2(R)$ acting on $\mathbb{H}^2 \times \mathbb{H}^2$ and in Corollary 4.5.8, we give an effective description of the generators of $\mathrm{PSL}_2(R)$. In Section 4.6, we describe the edges of $\mathcal{F}$ and in Theorem 4.6.25, we generalize the presentation part of Poincaré's Polyhedron Theorem. Note that Theorem 4.5.3 and Theorem 4.6.25 could potentially have been deduced from the result of Macbeath in [Mac64, Corollary of Theorem 2]. However, in order to get an effective set of generators and relations as an application of this theorem one needs all the lemmas in Section 4.5 and 4.6 anyway. Moreover, we think that the proofs of Theorem 4.5.3 and Theorem 4.6.25, presented here, are more intuitive from a geometric point of view. Furthermore they are algorithmic in nature. This Chapter is based on joint work with Jespers and del Río in [JKdR14a].

For the convenience of the reader, we include an appendix to this thesis. In fact, some lemmas of Chapter 4 are proved by tedious and brute force calculations. As these are not very pleasant to read, but intuitively clear, we have included them in the appendix. The first section of the appendix was realized with the help of Eisele.

To sum up, in this thesis, we developed new techniques that allow all simple components of the exceptional type (2) and (3) and some components of type (1) in order to determine, up to commensurability, generators for the unit group of the integral group ring $\mathbb{Z}G$ of a finite group G. Although we have not solved the problem in general for exceptional components of type (1), Chapter 4 is a first and important step in the good direction. The fact, that we are able to generalize the group presentation part of Poincaré's Polyhedron Theorem to the Hilbert Modular group acting on $\mathbb{H}^2 \times \mathbb{H}^2$ gives hope for the concrete case of $\mathcal{U}(\mathbb{Z}(Q_8 \times C_7))$.

Inleiding (Nederlands)

Algebra is the offer made by the devil to the mathematician.
The devil says: I will give you this powerful machine,
it will answer any question you like.
All you need to do is give me your soul:
give up geometry and you will have this marvelous machine.
Sir Michael Atiyah

Het onderzoek naar de structuur van de eenhedengroep $\mathcal{U}(\mathcal{O})$ van een order $\mathcal{O}$ in een eindigdimensionale rationale algebra A -of preciezer van een $\mathbb{Z}$-order in A- is van essentieel belang in de algebra. Zulk een eenhedengroep is een belangrijk voorbeeld van een aritmetische groep, waarvan we bovendien weten dat deze eindig voortgebracht is, wegens een beroemd resultaat van Borel en Harish-Chandra (zie Stelling 1.1.11).

In deze thesis concentreren we ons op de eenhedengroep van een groepring $\mathcal{O}G$ van een eindige groep G over een order $\mathcal{O}$ in een getallenlichaam. De studie van groepringen vindt haar oorsprong in het begin van de twintigste eeuw in de context van de representatietheorie en is sinds het werk van E. Noether en R. Brauer tot een onafhankelijk onderzoeksonderwerp uitgegroeid. Groepringen geven in het algemeen heel interessante algebraïsche structuren en de eenhedengroepen van groepringen hebben gaandeweg, gemotiveerd door fundamentele vragen zoals de conjectuur van Zassenhaus en het isomorfismeprobleem, aan bijzondere interesse gewonnen. Sinds het werk van Higman, stammend uit 1940 [Hig40a, Hig40b], heeft de eenhedengroep van een groepring $\mathcal{O}G$ van een eindige groep G over een order in een getallenlichaam geweldige aandacht gekregen.

Het meest natuurlijke geval is het geval van een gehele groepring, i.e. wanneer we $\mathcal{O} = \mathbb{Z}$ nemen. Desondanks is de eenhedengroep $\mathcal{U}(\mathbb{Z}G)$ slechts voor enkele niet-abelse groepen G expliciet beschreven en presentaties van $\mathcal{U}(\mathbb{Z}G)$ zijn nog zeldzamer te noemen. Het is dus inderdaad een van de belangrijke vragen in het domein van de eenheden in gehele groepringen om eindig veel voortbrengers te beschrijven -of algemener nog, om een eindige presentatie te krijgen- voor $\mathcal{U}(\mathbb{Z}G)$ (zie bijvoorbeeld de problemen 17 en 23 in [Seh93]).

Een eerste resultaat in deze richting werd gevestigd in 1965 door Bass en Milnor [Bas65] in het geval dat G een eindige abelse groep is (zie hiervoor Stelling 1.1.22). Ze beschreven een concrete en generieke verzameling van eindig veel voortbrengers voor een deelgroep van eindige index. Deze voortbrengers worden Bass-eenheden genoemd. Sinds 1989 wordt op niet-abelse groepen geconcentreerd, onder impuls van het werk van Ritter en Sehgal [RS89a, RS91b]. Zij gaven de eerste specifieke constructie van eenheden in gehele groepringen voor niet-abelse eindige groepen; deze eenheden worden thans bicyclische eenheden genoemd. Sindsdien is er voor vele eindige groepen G een specifieke verzameling B van voortbrengers van een deelgroep van eindige index in $\mathcal{U}(\mathbb{Z}G)$ beschreven geworden. Laten we kort een methode schetsen om eindig veel zulke generatoren te krijgen.

Als G een eindige groep is, dan is $\mathbb{Z}G$ een order in de rationale groepring $\mathbb{Q}G$. Door de stelling van Wedderburn-Artin (zie Stelling 1.1.5), hebben we dat

$$\mathbb{Q}G = \prod_{i=1}^{n} M_{n_i}(D_i),$$

waarbij elke D_i een delingsalgebra is. Als $\mathcal{O}_i$ een order is in D_i voor elke i, dan is $\mathcal{O} = \prod_{i=1}^{n} M_{n_i}(\mathcal{O}_i)$ een order in $\mathbb{Q}G$ en hebben we dat de groep van eenheden $\mathcal{U}(\mathcal{O})$ van $\mathcal{O}$ commensurabel is met $\mathcal{U}(\mathbb{Z}G)$. We herhalen dat twee deelgroepen van een gegeven groep commensurabel heten als ze een gezamenlijke deelgroep hebben die van eindige index is in beide deze deelgroepen. De groep $\mathcal{U}(\mathcal{O})$ is gewoon $\Pi_{i=1}^{n}\mathrm{GL}_{n_i}(\mathcal{O}_i)$, het direct product van de groepen $\mathcal{U}(M_{n_i}(\mathcal{O}_i)) = \mathrm{GL}_{n_i}(\mathcal{O}_i)$. Bovendien bevat het direct product $\mathcal{U}(\mathcal{Z}(\mathcal{O}_i)) \times \mathrm{SL}_{n_i}(\mathcal{O}_i)$ van de centrale eenheden in $\mathcal{O}_i$ met de groep bestaande uit de elementen van $M_{n_i}(\mathcal{O}_i)$ van gereduceerde norm 1, een deelgroep van eindige index die isomorf is met een deelgroep van eindige index in $\mathrm{GL}_{n_i}(\mathcal{O}_i)$ (zie Opmerking 1.1.14). De eenhedenstelling van Dirichlet geeft ons de structuur van de groep $\prod_{i=1}^{n}\mathcal{U}(\mathcal{Z}(\mathcal{O}_i))$ (zie Stelling 1.1.6). Bovendien is deze groep commensurabel met $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$, de groep der centrale eenheden van $\mathbb{Z}G$. In [JOdRVG13] (zie ook [JdRVG14]) beschreven Jespers, Olteanu, del Río en Van Gelder voortbrengers van een deelgroep van eindige index in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ voor een grote klasse van eindige groepen. Deze beschrijving is gebaseerd op Bass-eenheden. Op commensurabiliteit na, reduceert het probleem van het vinden van eindig veel voortbrengers en relaties voor $\mathcal{U}(\mathbb{Z}G)$ zich dus tot het vraagstuk van het vinden van een presentatie van $\mathrm{SL}_{n_i}(\mathcal{O}_i)$, voor elke $1 \leq i \leq n$. De congruentiestellingen geven de mogelijkheid om voortbrengers (op eindige index na) te berekenen voor $\mathrm{SL}_{n_i}(\mathcal{O}_i)$, behalve als $M_{n_i}(D_i)$ van het zogehete exceptionele type is. Om precies te zijn, noemen we een enkelvoudige algebra exceptioneel als zij van een van de volgende typen is:

(1) een niet-commutatieve delingsalgebra, die verschillend is van een totaal definiete quaternionenalgebra;

(2) $M_2(\mathbb{Q})$,

(3) $M_2(K)$, waarbij K een kwadratische imaginaire uitbreiding van $\mathbb{Q}$ is, of

(4) $M_2(\mathcal{H})$, waarbij $\mathcal{H}$ een totaal definiete quaternionenalgebra is met centrum $\mathbb{Q}$, i.e. $\mathcal{H} = \mathcal{H}\left(\frac{a,b}{\mathbb{Q}}\right)$ met a en b negatieve gehele getallen.

In Paragraaf 1.1.3 (en in het bijzonder in Definitie 1.1.19), geven we meer details hierover. Als we de resultaten [Bas64, Stelling 21.1, Corollarium 21.4], [Vas73, Stelling 2.4, Lemma 2.6], [Lie81, Stelling 24], [Ven94] en [Kle00, Stelling 2] combineren, krijgen we volgende stelling.

Stelling. *Zij D een eindigdimensionale rationale delingsring en zij $\mathcal{O}$ een order in D. Zij n een positief geheel getal. Als $n \geq 3$ of als $n = 2$ en de enkelvoudige algebra $M_2(D)$ niet exceptioneel van type (2)-(4) is, dan bestaat er een concrete eindige verzameling B, die een deelgroep van eindige index in $\mathrm{SL}_n(\mathcal{O})$ voortbrengt. Als $n = 1$ en D is niet exceptioneel van type (1), dan is de groep van eenheden met norm 1 van $\mathcal{O}$ eindig of commutatief. In het laatste geval, kan deze beschreven worden door de eenhedenstelling van Dirichlet.*

We merken op dat de stelling niet alleen het theoretische bestaan van de verzameling B aantoont; B kan er zelfs precies door beschreven worden. Kort gezegd bestaat B uit elementaire matrices. Meer details hierover worden in Paragraaf 1.1.3 gegeven. Onder de voorwaarde dat de groepsalgebra $\mathbb{Q}G$ geen exceptionele componenten heeft, geeft deze stelling ons een startpunt om een presentatie van de eenhedengroep van $\mathbb{Z}G$ te beschrijven, op commensurabiliteit na. In een serie artikelen [RS91a, RS91b, RS93] toonden Ritter en Sehgal aan dat de groep voortgebracht door de Bass- en bicyclische eenheden eindige index heeft in $\mathcal{U}(\mathbb{Z}G)$, en dit voor verschillende klassen van eindige groepen G, waaronder nilpotente groepen van oneven orde. In [JL93] breidden Jespers en Leal dit resultaat naar een grotere klasse van eindige groepen uit. Ze sluiten enkel die groepen uit wiens rationale groepalgebra $\mathbb{Q}G$ exceptionele componenten heeft alsook die groepen die een niet-abels fixpunt-vrij epimorf beeld hebben. Dit zijn opmerkelijke resultaten, als men bedenkt dat ze slechts op twee gekende constructies van eenheden steunen. De bewijzen van al deze resultaten maken gebruik van de hierboven beschreven stelling en het resultaat van Bass en Milnor dat we hogerop vermeld hebben.

Het doel van deze thesis is om nieuwe methodes te ontwikkelen om eindig veel voortbrengers en/of een eindige presentatie voor $\mathrm{SL}_n(\mathcal{O})$ te krijgen, waarbij $M_n(\mathcal{O})$ een order in een exceptionele component is. Dit geeft dan uiteindelijk een uitbreiding van de hierboven beschreven resultaten over voortbrengers van deelgroepen van eindige index in $\mathcal{U}(\mathbb{Z}G)$, omdat we dan ook exceptionele componenten als enkelvoudige componenten in de groepalgebra $\mathbb{Q}G$ kunnen toelaten. Natuurlijk geeft dit ook toepassingen op de eenhedengroep van de groepring $\mathcal{O}G$ over een willekeurige order $\mathcal{O}$ in een getallenlichaam. Inderdaad, dankzij de stelling van Wedderburn-Artin, kan de studie van $\mathcal{U}(\mathcal{O})$, voor elke order $\mathcal{O}$ in een semisimpele eindigdimensionale rationale algebra A, gereduceerd worden tot de studie van $\mathrm{SL}_{n_i}(\mathcal{O}_i)$ voor $n_i \in \mathbb{N}$ en $\mathcal{O}_i$ orders in getallenlichamen, opnieuw op commensurabiliteit na. Bij de stelling van Maschke (zie Stelling 1.1.3) kan dit in het bijzonder toegepast worden op de studie van de eenhedengroep van een order in KG, voor K een willekeurige getallenlichaam en G een eindige groep.

Het kader van deze thesis is desondanks dat van eenheden in gehele groep-ringen om de eenvoudige reden dat deze theorie veel meer ontwikkeld is dan die van willekeurige orders in groepringen KG van eindige groepen G over willekeurige getallenlichamen K. Op deze manier kunnen we een beter beeld geven van de reeds ontwikkelde theorie waarin ons werk kadert en kunnen we mooiere voorbeelden vinden waarop onze resultaten kunnen toegepast worden. In deze context merken we ook op dat Eisele, de auteur en Van Gelder in een recent werk [EKVG14] een classificatie gegeven hebben van alle exceptionele componenten van type (3) en (4), die kunnen voorkomen als enkelvoudige componenten van $\mathbb{Q}G$ voor G een eindige groep. Er wordt aangetoond dat als $\mathbb{Q}G$ een exceptionele component heeft van type (3) of (4), dat deze component dan een 2×2-matrixring over een van de volgende ringen is:

(i) $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$ of $\mathbb{Q}(\sqrt{-3})$;

(ii) $\left(\frac{-1,-1}{\mathbb{Q}}\right)$, $\left(\frac{-1,-3}{\mathbb{Q}}\right)$ of $\left(\frac{-2,-5}{\mathbb{Q}}\right)$.

Het is zo dat de velden in (1) allen norm-Euclidische velden zijn [Wei63], en de quaternionenalgebras in (2) hebben allen een maximaal order, die norm-Euclidisch is [Fit12]. Bovendien zijn de eindige groepen beschreven die trouw ingebed zijn in een exceptionele enkelvoudige component (en dus deze ook voortbrengen als algebra). Gebruik makend van deze Euclidische norm, is het mogelijk een eindige verzameling van voortbrengers voor $\mathrm{SL}_2(\mathcal{O})$ te krijgen voor $\mathcal{O}$ een maximale order in deze exceptionele component. Aan de hand van de Reidemeister-Schreier methode, kan men dan voortbrengers beschrijven voor een deelgroep van eindige index.

Het hoofdingrediënt voor de studie van een presentatie voor $\mathrm{SL}_n(\mathcal{O})$ waarvoor $M_n(\mathcal{O})$ een order in een exceptionele component is dat we in deze thesis gaan gebruiken, is de theorie van discontinue groepsacties op hyperbolische ruimten. Dit is gebaseerd op de algemenere theorie van discontinue groepsacties op zogehete eindig-compacte metrische ruimten. De term "eindig-compact" is ontleend aan [Rat06, Definitie 5.2] en duidt op een metrische ruimte in welke alle gesloten ballen compact zijn. Als een groep Γ discontinu op een eindig-compacte metrische ruimte ageert, dan betegelt Γ de ruimte in gesloten delen, die disjuncte inwendigen hebben. Een "puzzelstuk" van deze betegeling wordt een fundamenteel domein van Γ genoemd. In de drie klassieke metrische ruimten van constante kromming, i.e. de Euclidische, de bol- en de hyperbolische ruimte, beschrijft de stelling van Poincaré over veelvlakken (zie Stelling 2.4.7) een presentatie van Γ, gebaseerd op zijn fundamenteel domein. De algemene idee is dus een geschikte ruimte $\mathbb{X}$ te vinden waarop $\mathrm{SL}_n(\mathcal{O})$ een discontinue actie uitvoert, vervolgens een fundamenteel domein voor deze actie te construeren in $\mathbb{X}$ om dan uiteindelijk een presentatie via het resultaat van Poincaré te krijgen. Eerst zullen we dit doen in de hyperbolische 2-ruimte $\mathbb{H}^2$ en dito 3-ruimte $\mathbb{H}^3$. De groepen $\mathrm{PSL}_2(\mathbb{R})$ en $\mathrm{PSL}_2(\mathbb{C})$, die natuurlijke epimorfe beelden zijn van $\mathrm{SL}_2(\mathbb{R})$ en $\mathrm{SL}_2(\mathbb{C})$, voeren, via Poincaré extensies van Möbius-transformaties, een actie uit op respectievelijk $\mathbb{H}^2$ en $\mathbb{H}^3$. Men kan aantonen dat discrete deelgroepen van $\mathrm{PSL}_2(\mathbb{R})$ of $\mathrm{PSL}_2(\mathbb{C})$ discontinue acties hebben op $\mathbb{H}^2$ of $\mathbb{H}^3$ (zie Stelling 1.2.6). Groepen van de vorm $\mathrm{SL}_n(\mathcal{O})$, die discreet zijn in $\mathrm{SL}_2(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{C})$ kunnen dus op deze manier bestudeerd worden. Deze groepen zijn geassocieerd met de exceptionele componenten van type (2), (3) en met sommigen van type (1).

In deze richting is al eerder onderzoek uitgevoerd, door verschillende auteurs. In [DJK10] voerden Dooms, Jespers en Konovalov een methode in, die ook gebaseerd is op fundamentele domeinen van discrete groepen van eindig covolume, om exceptionele componenten van type (2) te behandelen. Gebruik makend van Farey-symbolen (die in 1-1 verband staan met fundamentele veelvlakken van congruentiegroepen van $\mathrm{PSL}_2(\mathbb{Z})$), beschreven ze nieuwe voortbrengers. In [CJLdR04] stelden Corrales, Jespers, Leal en del Río een algoritme op om de eenhedengroep van de order $\mathcal{H}\left(\frac{-1,-1}{\mathcal{O}_K}\right)$ in de quaternionendelingsalgebra over een kwadratische imaginaire lichaamsuitbreiding K van de rationale getallen, $\mathcal{H}\left(\frac{-1,-1}{K}\right)$, te berekenen. Het algoritme berekent een fundamenteel veelvlak van de groep $\mathcal{U}\left(\mathcal{H}\left(\frac{-1,-1}{\mathcal{O}_K}\right)\right)$, dat kan gezien worden als een cocompacte discrete deelgroep van $\mathrm{PSL}_2(\mathbb{C})$ (en dat dus een discontinue actie op $\mathbb{H}^3$ heeft). Via de hierboven reeds vermeldde stelling van Poincaré wordt dan een presentatie gegeven. Het algoritme wordt vervolgens toegepast op $\mathcal{H}\left(\frac{-1,-1}{\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]}\right)$ en geeft op deze manier het eerste voorbeeld van een beschrijving van een eindige verzameling van voortbrengers van de eenhedengroep van een groepring RG met een exceptionele component, die een delingsalgebra is verschillend van een totaal definiete quaternionen-algebra.

Een andere benadering in dezelfde richting is de studie van groepen van "Klein-type". Een eindige groep G wordt van Klein-type genoemd als elke niet-commutatieve enkelvoudige component $M_{n_i}(D_i)$ van $\mathbb{Q}G$ een quaternionenalgebra over zijn eigen centrum is en als het beeld van $\mathrm{SL}_{n_i}(\mathcal{O}_i)$ in $\mathrm{PSL}_2(\mathbb{C})$ een Klein-groep is. Een Klein-groep is een deelgroep van $\mathrm{PSL}_2(\mathbb{C})$ die discreet is voor de natuurlijke topologie, of equivalent hiermee, wiens actie op de 3-dimensionale hyperbolische ruimte via de Poincaré-extensie van de Möbius-transformatie actie discontinu is. Dus, als G een eindige groep van Klein-type is, kan men in principe, door gebruik te maken van de stelling van Poincaré over veelvlakken, een presentatie krijgen voor een groep die commensurabel is met $\mathcal{U}(\mathbb{Z}G)$. In [JPdR$^+$07] hebben Jespers, Pita, del Río en Ruiz groepen van Klein-type geclassificeerd en Pita, del Río en Ruiz hebben in [PdRR05] en [PdR06] voorbeelden gegeven van hoe men een presentatie voor $\mathcal{U}(\mathbb{Z}G)$ kan krijgen voor sommige groepen G die van Klein-type zijn en kleine orde hebben. Olteanu en del Río hebben dit veralgemeend voor groepringen over commutatieve orders in [OdR07].

Het uiteindelijke doel van deze thesis is de hierboven geschetste geïsoleerde gevallen te generaliseren en een algemene methode te ontwikkelen om de eenhedengroep van een order in een exceptionele component te beschrijven, en dit via discontinue acties op metrische ruimten en een fundamenteel domein voor deze acties. Onze toepassingen op eenheden in groepringen zijn gericht op gehele groepringen $\mathbb{Z}G$ en voor de eenvoud onderstellen we soms dat G nilpotent is.

In Hoofdstuk 1 geven we eerst een kort overzicht van wat bekend is over voortbrengers van de eenhedengroep in gehele groepringen. We beschrijven de resultaten van Bass [Bas64, Stelling 21.1, Corollarium 21.4], Vaseršteĭn [Vas73, Stelling 2.4, Lemma 2.6], Liehl [Lie81, Stelling 24], Venkataramana [Ven94] en Kleinert [Kle00, Stelling 2] op vrij gedetailleerde wijze en leggen de oorsprong van de exceptionele componenten uit. We presenteren ook enkele hoofdstellingen die voortbrengers beschrijven (op eindige index na) voor de eenhedengroep $\mathbb{Z}G$, onder de voorwaarde dat $\mathbb{Q}G$ geen exceptionele componenten heeft. Zoals hiervoor reeds uitgelegd, baseert ons werk zich op discontinue groepsacties, hyperbolische meetkunde en fundamentele domeinen. In Sectie 1.2 geven we daarom de nodige achtergrond van deze concepten. Dit deel van de thesis is vooral gebaseerd op [Rat06], [Bea95] en [EGM98].

In Hoofdstuk 2 herbewijzen we een deel van de stelling van Poincaré over veelvlakken, namelijk datgene dat handelt over groepsrepresentaties. Zoals al meerdere keren vermeld werd, is het onze bedoeling om eenhedengroepen te beschrijven via hun discontinue actie op hyperbolische ruimten aan de hand van dit resultaat van Poincaré. Er is al veel inkt gevloeid over dit theorema; de geïnteresseerde lezer kan bijvoorbeeld de boeken [Bea95, BH99, EGM98, Mas88, Rat06] raadplegen voor meer informatie. Er zijn ook een aantal artikels over geschreven, zoals bijvoorbeeld [EP94, Mas71, dR71, AG11]. Het probleem is dat de meeste bewijzen die men in de literatuur terug kan vinden, eerder onbevredigend zijn en zich vaak op de theorie van overdekkingsruimten baseren. We verwijzen de lezer op dit punt naar het artikel van Epstein en Petronio [EP94, Section 9], dat een lange lijst van referenties over dit thema bevat en ook alle problemen beschrijft die in deze context optreden. De twee originele versies van de hand van Poincaré ([Poi82] en [Poi83]) zijn heel moeilijk te lezen. Bovendien refereert Poincaré voor het bewijs van het 3-dimensionale geval gewoon naar het 2-dimensionale geval, zonder echter na te gaan dat de resultaten nog altijd geldig blijven. Modernere versies behandelen vaak enkel het geval in dimensie 2, waar het bewijs aanzienlijk eenvoudiger is en zich niet zo maar onmiddellijk leent tot veralgemening naar een dimensie hoger. Voorts blijven zelfs deze moderne versies vaak weinig toegankelijke lectuur, of blijven ze vaag op sommige cruciale plaatsen. De bewijzen van Maskit in [Mas88] en van Epstein en Petronio in [EP94] concentreren zich bijna uitsluitend op het deel van het resultaat van Poincaré waar bewezen wordt dat een veelvlak, dat aan een aantal voorwaarden voldoet, een fundamenteel veelvlak van een discontinue groep is. Het deel over groepspresentaties volgt dan indirect hieruit; dit zorgt ervoor dat er op het intuïtionele vlak ingeboet wordt. Dit is precies waarom we in Hoofdstuk 2 een nieuw, onafhankelijke bewijs van het deel over groepsrepresentaties van deze stelling geven. Hierbij baseren we ons enkel op enkele relatief eenvoudige meetkundige concepten. Het resultaat dat we uiteindelijk krijgen (Stelling 2.4.7), is bovendien iets algemener dan de stelling die meestal geciteerd wordt, omdat we de exactheidsvoorwaarde (zie Definitie 1.2.13) niet gebruiken. We hopen met ons nieuw bewijs bij te dragen aan een beter verstaan van dit belangrijk resultaat. Dit hoofdstuk is volledig gebaseerd op de samenwerking met Jespers en del Río in [JKdR14b].

In Hoofdstuk 3 tonen we hoe Stelling 2.4.7 kan toegepast worden op het probleem van de beschrijving van voortbrengers van de eenhedengroep van een order in een exceptionele component. We bekijken Klein- en Fuchs-groepen, i.e. discrete deelgroepen van respectievelijk $\mathrm{PSL}_2(\mathbb{C})$ en $\mathrm{PSL}_2(\mathbb{R})$. Zulk een groep Γ voert een discontinue actie uit op de 3-dimensionale hyperbolische ruimte, respectievelijk op het 2-dimensionale hyperbolische vlak. In Lemma 3.1.3 en Lemma 3.1.4 geven we concrete, eenvoudige formules die het fundamenteel veelvlak van Dirichlet van Γ in het bovenhalfruimtemodel alsook in het bolmodel van de hyperbolische 2- en 3-ruimte beschrijven. Gebruik makend van deze resultaten geven we in Propositie 3.1.8 een algoritme, het zogehete "Dirichlet algorithm of finite covolume (DAFC)", dat -in een eindig aantal stappen- een eindige verzameling van voortbrengers voor een deelgroep van eindige index in Γ berekent. We beschrijven ook een verfijning van dit algoritme (Propositie 3.1.10) die de hele verzameling van eindig veel voortbrengers geeft in het geval dat Γ een cocompacte groep is. In deze situatie geeft het verfijnd algoritme dan ook een fundamenteel veelvlak voor Γ. Gelijkaardige pogingen om een algoritme voor het berekenen van fundamenteel domeinen voor Klein- en Fuchs-groepen te beschrijven, werden ondernomen in [Ril83] en [Pag13]. Omdat we vooral geïnteresseerd zijn in het vinden van een presentatie voor een deelgroep van eindige index in een gegeven groep, is ons algoritme op eenvoudigere formules opgebouwd. In Sectie 3.2 tonen we dan hoe dit algoritme kan toegepast worden op eenhedengroepen van orders in delingsalgebras die discreet zijn in $\mathrm{PSL}_2(\mathbb{C})$, alsook op Bianchi-groepen. Laatstgenoemden zijn inderdaad eenhedengroepen van orders in exceptionele componenten van type (2) en (3). Onder de delingsalgebras, waarop ons algoritme van toepassing is, vallen ondermeer delingsalgebras die kwadratisch imaginaire uitbreidingen van $\mathbb{Q}$ als centrum hebben (dus hier werken we in de exceptionele componenten van type (1)). We geven meerdere voorbeelden, waaronder ook voorbeelden van niet-cocompacte groepen. Het betreft dus een veralgemening van [CJLdR04], waarin enkel cocompacte groepen van kleine orde bekeken worden. In Paragraaf 3.3 leggen we uit hoe deze nieuwe informatie in concreto kan worden toegepast op eenheden in gehele groepringen $\mathbb{Z}G$. We geven alle details in het geval dat G een eindige nilpotente groep is. We eindigen dit hoofdstuk met een kleine uitweiding die mooi illustreert hoe onze resultaten nieuwe meetkundige informatie kunnen geven. De formules, die we opgesteld hebben voor het DAFC, kunnen gebruikt worden om resultaten over Dirichlet-Ford-domeinen en dubbele Dirichlet-domeinen te veralgemenen. Dirichlet-Ford-domeinen zijn fundamentele domeinen die tegelijkertijd Dirichlet- en Ford-domeinen zijn en dubbele Dirichlet-domeinen zijn fundamentele domeinen die ten minste twee punten als centrum hebben. Met behulp van onze formules verbeteren we enkele resultaten van Lakeland in [Lak12]. Dit hoofdstuk is gebaseerd op de samenwerking met Jespers, Juriaans, de Andrade e Silva en Souza Filho in [JJK+13b] en [JJK+13a].

In de zoektocht naar een presentatie van $\mathcal{U}(\mathbb{Z}G)$ worden we op natuurlijke wijze geleid tot het beschouwen van eindige groepen G zodat $\mathbb{Q}G$ enkelvoudige componenten bezit zodat de eenhedengroep van een order van deze componenten niet noodzakelijk een discontinue actie uitvoert op een hyperbolische ruimte, maar wel op een direct product van meerdere hyperbolische 2- en 3-ruimten

$$\mathbb{H}_{r,s} = \mathbb{H}^2 \times \overset{(r)}{\cdots} \times \mathbb{H}^2 \times \mathbb{H}^3 \times \overset{(s)}{\cdots} \times \mathbb{H}^3.$$

Dit is het onderwerp van Hoofdstuk 4. In feite hebben we in Hoofdstuk 3 methodes ontwikkeld om de eenhedengroep van $\mathbb{Z}G$ te beschrijven, op commensurabiliteit na, en daarbij ook exceptionele componenten van type (2) en (3), alsook enige speciale gevallen van type (1), toe te laten. Toch bevat de klasse van exceptionele componenten van type (1) ook quaternionenalgebras die niet van dit speciaal type zijn: we denken bijvoorbeeld aan quaternionenalgebras over een lichaamsuitbreiding van de rationalen van graad groter dan 2. Onze volgende stap is dus het geval bekijken waar $\mathcal{O}$ een order is in een quaternionenalgebra met een getallenlichaam $\mathbb{Q}(\xi_n)$ als centrum, waarbij ξ_n een n-de primitieve eenheidswortel is. Dit probleem blijkt behoorlijk moeilijk; daarom spitsen we onze aandacht in het begin dan ook enkel toe op enkele bijzondere voorbeelden. Het kleinste natuurlijk getal dat ons in dit verband interesseert, is $n = 7$; we bekijken daarom $\mathcal{H}(\mathbb{Q}(\xi_7))$, de klassieke quaternionen algebra over $\mathbb{Q}(\xi_7)$. Deze algebra komt bijvoorbeeld voor in de Wedderburn-Artin ontbinding van de rationale groepalgebra $\mathbb{Q}G$ van de groep $G = Q_8 \times C_7$, het direct product van de quaternionengroep van orde 8 met de cyclische groep van orde 7. In dit geval is de studie van de eenhedengroep (op commensurabiliteit na) gereduceerd tot de studie van $\mathcal{U}(\mathcal{H}(\mathbb{Z}[\xi_7]))$. Deze groep heeft een discontinue actie op het direct product $\mathbb{H}^3 \times \mathbb{H}^3 \times \mathbb{H}^3$ (drie kopieën van de hyperbolische 3-ruimte). Meer details staan beschreven in Paragraaf 4.1. Het doel is dus om Stelling 2.4.7, die geldig is voor Riemann-variëteiten met constante kromming, te veralgemenen naar een versie die geldt voor het direct product $\mathbb{H}^3 \times \mathbb{H}^3 \times \mathbb{H}^3$ (dat geen constante kromming meer heeft).

Omdat dit probleem enorm lijkt, behandelden we eerst een testgeval. De bedoeling is om een presentatie te vinden voor Hilberts modulaire groep, i.e. $\mathrm{SL}_2(R)$, waarbij R de ring van de gehele getallen van K is, voor $K = \mathbb{Q}(\sqrt{d})$ een kwadratische uitbreiding, d een positief geheel getal zijnde. Dit werk is voornamelijk gebaseerd op dat van Cohn en Maass in [Coh65a, Coh65b, Maa40]. Inderdaad voert de groep $\mathrm{SL}_2(R)$ een discontinue actie uit op het direct product $\mathbb{H}^2 \times \mathbb{H}^2$ (twee kopiën van de hyperbolische 2-ruimte). Dit stelt dus het eenvoudigste voorbeeld voor van een veralgemening tot directe producten van hyperbolische ruimten. Om nu de hele theorie naar acties op directe producten van hyperbolische ruimten te veralgemenen, moeten meerdere problemen overwonnen worden. Een eerste is het volgende:

(1) Blijft Poincaré zijn methode geldig voor discontinue acties op $\mathbb{H}_{r,s}$?

Het volgend vraagstuk is een van de obstakels dat overwonnen moet worden om bovenstaande vraag te beantwoorden.

(2) Bepaal -bruikbare- methodes om een fundamenteel domein te construeren.

Eenmaal dit zou lukken, zien we het volgende euvel opduiken: bij de stelling van Poincaré over veelvlakken worden de voortbrengers van een groep aan de hand van de zijvlakken van zijn fundamenteel domein berekend, maar de zijvlakken van het -goed bekende- fundamenteel domein in $\mathbb{H}^2$, resp. $\mathbb{H}^3$, zijn geodetische hypervlakken, i.e. lijnen en cirkels (resp. vlakken en bollen) die orthogonaal met de rand zijn.

(3) Hoe zouden de "zijvlakken" van een potentieel fundamenteel domein in $\mathbb{H}_{r,s}$ moeten gedefinieerd worden?

We zijn erin geslaagd een methode te ontwikkelen voor de constructie van een fundamenteel domein wiens rand bestaat uit eindig veel reële niet-singuliere algebraïsche variëteiten, en we konden ook Stelling 2.4.7 veralgemenen. Op deze manier krijgen we een presentatie voor $\mathrm{SL}_2(R)$, die eindig is in het geval dat R een hoofdideaaldomein is. We merken echter op dat de presentatie van $\mathrm{SL}_2(R)$ van theoretische aard is. Door de voorwaarde van hoofdideealdomein op te leggen aan R, krijgen we concrete voortbrengers voor $\mathrm{SL}_2(R)$, die ook nog gemakkelijk te berekenen zijn. Maar wat de relaties betreft, hebben we op het ogenblik enkel een theoretisch resultaat en weten we nog niet echt hoe we deze effectief kunnen berekenen.

Hoofdstuk 4 is als volgt georganiseerd. In Paragraaf 4.1 geven we de nodige achtergrondinformatie over discontinue acties op directe producte van $\mathbb{H}^2$ en $\mathbb{H}^3$. In Paragraaf 4.2 leiden we de notatie m.b.t. Hilberts modulaire groep in en beschrijven we zijn actie op $\mathbb{H}^2 \times \mathbb{H}^2$. In Paragraaf 4.3 geven we een beschrijving van een fundamenteel domein $\mathcal{F}$ voor deze actie (zie hiervoor Stelling 4.3.10) en in Paragraaf 4.4 bewijzen we enige topologische lemma's over het fundamenteel domein. Finaal beschrijven we de zijvlakken van het fundamenteel domein $\mathcal{F}$ in Paragraaf 4.5. In Stelling 4.5.3 veralgemenen we het deel over generatoren van de stelling van Poincaré over veelvlakken naar het geval van $\mathrm{PSL}_2(R)$ die op $\mathbb{H}^2 \times \mathbb{H}^2$ ageert en in Gevolg 4.5.8 geven we een effectieve beschrijving van voortbrengers van $\mathrm{PSL}_2(R)$. In Paragraaf 4.6 beschrijven we de ribben van $\mathcal{F}$ en in Stelling 4.6.25 veralgemenen we het deel over presentaties van het theorema van Poincaré. We merken hierbij op dat Stelling 4.5.3 en Stelling 4.6.25 mogelijks ook hadden kunnen afgeleid worden uit het resultaat van Macbeath in [Mac64, Gevolg van Stelling 2]. Maar, om een (bruikbare) verzameling van voortbrengers en relaties te krijgen als toepassing van laatstgenoemde resultaat, heeft men de meeste lemma's uit Paragraaf 4.5 en 4.6 toch nodig. Bovendien vinden we dat onze bewijzen van Stelling 4.5.3 and Stelling 4.6.25 "intuïtiever" zijn en meer inzicht geven vanuit het meetkundige standpunt. En ze zijn ook algoritmisch van aard. Dit hoofdstuk is gebaseerd op het gezamenlijk uitgevoerde werk met Jespers en del Río in [JKdR14a].

Voor het comfort van de lezer, voegen we een appendix aan deze thesis toe. Enkele lemma's uit Hoofdstuk 4 worden namelijk aan de hand van lange "brute kracht" - berekeningen bewezen. Omdat deze berekeningen niet bepaald aangename lectuur zijn, en bovendien meestal intuïtief duidelijk lijken, hebben we ze naar een appendix verhuisd. Het eerste deel van de appendix is geschreven met de hulp van Eisele.

Om het kort samen te vatten, hebben we in deze thesis nieuwe technieken ontwikkeld om ook enkelvoudige componenten van exceptioneel type (2) en (3), en ook sommige componenten van type (1), in resultaten over voortbrengers (op commensurabiliteit na) van de eenhedengroep van een gehele groepring $\mathbb{Z}G$ van een eindige groep G te kunnen bevangen. Ook al hebben we het algemeen probleem van exceptionele componenten van type (1) niet opgelost, toch is Hoofdstuk 4 een eerste en belangrijke stap in de goede richting. Het feit dat we in staat zijn het deel over groepsrepresentaties van de stelling van Poincaré die handelt over veelvlakken, te veralgemenen naar Hilberts modulaire groep (die op $\mathbb{H}^2 \times \mathbb{H}^2$ ageert), geeft ons hoop om het voorbeeld $\mathcal{U}(\mathbb{Z}(Q_8 \times C_7))$ aan te kunnen.

1 Preliminaries

Beauty is the first test:
there is no permanent place in the world
for ugly mathematics.
G.H. Hardy

In this chapter we give the necessary background on unit groups, group algebras and discontinuous actions on hyperbolic spaces. As our results lie on the crossroad of units of orders in group rings and discontinuous group actions on hyperbolic spaces, we include the necessary background on these topics. This chapter is divided into two parts. In the first section, we recall basic definitions on rational group rings, orders and their unit group. We introduce the necessary concepts and give a short state of the art on units in orders in finite dimensional semisimple rational algebras. In the second section, we give a brief background on discontinuous group actions and fundamental domains. We recall the link between these concepts via presentations of a group in terms of generators and relations. As we mainly work with actions on hyperbolic spaces, we also include some background on hyperbolic geometry and Möbius transformations.

1.1 The Group of Units of an Order

In this section we recall basic facts about rational group rings. The concepts mainly are based on [Seh93], [Jes98] and [JdR].

All rings considered are unital. The identity of R is usually denoted by 1_R or simply 1. The identity of a group G is denoted by 1_G or simply 1. Also, when we say that R' is a subring of a a ring R, we assume that they have the same identity.

Definition 1.1.1. *Let R be a unital ring. The unit group of R is the group of invertible elements in R. We denote it by $\mathcal{U}(R)$.*

1.1.1 Group Rings and Orders

We first recall the definition of a group ring.

Definition 1.1.2. *The group ring RG of a group G over a ring R is the set of all formal sums*

$$\sum_{g\in G} a_g g,$$

where $a_g \in R$ and $a_g = 0$ for almost all $g \in G$. By assumption,

$$\sum_{g\in G} a_g g = \sum_{g\in G} b_g g \text{ if and only if } a_g = b_g \text{ for every } g \in G.$$

The set RG is a ring for the following two operations:

$$\left(\sum_{g\in G} a_g g\right) + \left(\sum_{g\in G} b_g g\right) = \sum_{g\in G} (a_g + b_g)\, g,$$

and

$$\left(\sum_{g\in G} a_g g\right)\left(\sum_{g\in G} b_g g\right) = \sum_{g,h\in G} a_g b_h gh.$$

The identity of RG is the element $1_R 1_G$ and we simply denote it by 1. In case R is a field, RG is called the group algebra of G over R. If $R = \mathbb{Z}$, then we call RG the integral group ring of G.

The group G and the ring R can be considered as subsets of the ring RG through the multiplicative embedding $g \mapsto 1_R \cdot g$ and the ring embedding $r \mapsto r \cdot 1_G$. Moreover, according to the definition, the multiplication of elements in RG by elements $r \in R$ is given by

$$r\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} (r a_g)\, g,$$

and hence RG is a free R-module with basis G.

Theorem 1.1.3 (Maschke)**.** *Let R be a ring and G a group. The group ring RG is semisimple if and only if R is a semisimple ring, the group G is finite and $|G| \cdot 1_R$ is invertible in R.*

In the case R is a field, say K, we clearly have that KG is semisimple if and only if $char(K) \nmid |G|$. Hence, we get the following corollary. Recall that a number field is a field that is a finite extension of the rationals $\mathbb{Q}$.

Corollary 1.1.4. *Let G be a finite group and K a number field. Then KG is a semisimple algebra.*

In particular the rational group ring $\mathbb{Q}G$ is semisimple for every finite group G. Semisimple rings have a particular nice structure. This is shown by the following theorem.

Theorem 1.1.5 (Wedderburn-Artin)**.** *Let R be a semisimple ring. Then*

$$R \cong M_{n_1}(D_1) \times \ldots \times M_{n_r}(D_r),$$

for some skew fields $D_1, \ldots, D_r$ and $n_1, \ldots, n_r \in \mathbb{N}_0$. The number r is uniquely determined as well as the pairs $(n_1, D_1), \ldots, (n_r, D_r)$ (up to a permutation).

Another important concept used in this thesis is the concept of orders. Before introducing the notion, we give a basic example. Let K be a number field. This is necessarily of the form $\mathbb{Q}(a)$ for some $a \in \mathbb{C}$. Recall also that an algebraic integer in K is an element that is a root of a monic polynomial with integral coefficients. The algebraic integers $\mathcal{O}_K$ in K form a subring of K that is the most basic example of an order. The unit group of the algebraic integers has been characterized.

Theorem 1.1.6 (Dirichlet's Unit Theorem)**.** *Let K be a number field and assume that K has r real embeddings and s pairs of complex non-real embeddings. If $\mathcal{O}_K$ is the ring of integers of K then*

$$\mathcal{U}(\mathcal{O}_K) \cong F \times C,$$

where F is a free abelian group of rank $r + s - 1$ and C is a finite group formed by the roots of unity in K. The independent generators of F are called fundamental units.

In general, orders are a special class of subrings of $\mathbb{Q}$-algebras.

Definition 1.1.7. *Let A be a finite dimensional $\mathbb{Q}$-algebra with unity. A subring $\mathcal{O}$ is called a $\mathbb{Z}$-order (or, simply, an order) in A if*

(i) $\mathcal{O}$ is a finitely generated $\mathbb{Z}$-submodule,

(ii) $\mathcal{O}$ contains a $\mathbb{Q}$-basis of A, i.e. $\mathbb{Q}\mathcal{O} = A$.

As stated above the most basic example of an order is the ring of integers $\mathcal{O}_K$ of a number field K. In this class, the most trivial and natural example is the ring of integers $\mathbb{Z}$, which is an order in $\mathbb{Q}$. But also the subring $\mathbb{Z}[\alpha]$ is an order in $\mathbb{Q}(\alpha)$, for α an algebraic integer. Clearly, for a finite group G, the integral group ring $\mathbb{Z}G$ is an order in $\mathbb{Q}G$. More generally, $\mathcal{O}_K G$ is an order in KG, for $\mathcal{O}_K$ the ring of integers in a number field K. Also $M_n(\mathcal{O}_K)$ is an order in $M_n(K)$.

The following lemma is very useful. For a proof we refer to [Seh93, Lemma 4.2 and Lemma 4.6].

Lemma 1.1.8. *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be orders in a $\mathbb{Q}$-algebra A. The following properties hold.*

1. *$\mathcal{O}_1 \cap \mathcal{O}_2$ is an order in A.*

2. *If $\mathcal{O}_2 \subseteq \mathcal{O}_1$, then the index $[\mathcal{U}(\mathcal{O}_1) : \mathcal{U}(\mathcal{O}_2)]$ is finite.*

From the previous lemma we get as an immediate consequence the commensurability of the unit groups of two orders in a rational algebra. We first recall the definition of commensurable.

Definition 1.1.9. *Two subgroups G_1 and G_2 of a group G are commensurable if they have a common subgroup which is of finite index in both.*

Corollary 1.1.10. *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be orders in a $\mathbb{Q}$-algebra A. Then $\mathcal{U}(\mathcal{O}_1)$ and $\mathcal{U}(\mathcal{O}_2)$ are commensurable.*

1.1.2 Basic Constructions of Units

The first approach in the study of units is the search for generic constructions of units. However, only a few of such constructions are known. Apart from the trivial units, the most important constructions are the Bass units and the bicyclic units. We recall the definitions in this subsection.

Let $\mathcal{O}$ be an order in a number field and let G be a finite group. Of course, for every $g \in G$ and every unit u of $\mathcal{O}$, the element ug is a unit in $\mathcal{O}G$. These units are called the trivial units of $\mathcal{O}G$. For example, the trivial units of $\mathbb{Z}G$ are $\pm g$ with $g \in G$.

In a group ring $\mathcal{O}G$, we set for a subgroup H of G

$$\widetilde{H} = \sum_{h \in H} h \in \mathcal{O}G. \tag{1.1}$$

Observe that, for every $h \in H$, $(1-h)\widetilde{H} = 0$ and thus $1 + (1-h)a\widetilde{H}$ and $1 + \widetilde{H}a(1-h)$ are units in $\mathcal{O}G$ for every $a \in \mathcal{O}G$ and $h \in H$. Their inverses are $1 - (1-h)a\widetilde{H}$ and $1 - \widetilde{H}a(1-h)$ respectively. If $g \in G$, then we set

$$\tilde{g} = \widetilde{\langle g \rangle}.$$

The construction of the Bass units is inspired by the cyclotomic units in number theory. Let $g \in G$ be of order n and let k and l be positive integers with $kl \equiv 1 \mod n$. The expression $\frac{g^k - 1}{g-1}$ makes no sense here, but we may consider

$$1 + g + g^2 + \ldots + g^{k-1} \in \mathbb{Z}G.$$

This is not a unit in $\mathbb{Z}G$. However, if m is a positive integer such that $k^m \equiv 1 \mod n$, then the following element

$$u_{k,m}(g) = \left(1 + g + g^2 + \ldots + g^{k-1}\right)^m + \frac{1-k^m}{n}\tilde{g} \tag{1.2}$$

is a unit in $\mathbb{Z}G$ with inverse $u_{l,m}(g)$. These units are called *Bass units* of G and we denote by

$$\mathcal{B}_1(\mathbb{Z}G) \tag{1.3}$$

the group generated by the Bass units in $\mathbb{Z}G$.

Observe that this construction may be given in a more general context of a ring R. For this, the element g appearing in (1.2) may be replaced by an element $g' \in RG$ having finite order. For example, if we take $R = \mathbb{Z}[\xi_n] G$, for ξ_n a n-th primitive root of unity, then the Bass units are given by $u_{k,m}(\xi_n^i g)$ for some $g \in G$ and $0 \leq i < n$. More details on this may be found in [JL93].

Intuitively, the construction of bicyclic units tries to generalize elementary matrices. In general, for a ring R, if $x, y \in R$ are such that $xy = 0$, then for every $a \in R$, $1 + xay$ is a unit in R with inverse $1 - xay$. The units of the form

$$b(g, \tilde{h}) = 1 + (1-h)g\tilde{h} \qquad \text{and} \qquad b(\tilde{h}, g) = 1 + \tilde{h}g(1-h),$$

with $g, h \in G$, are called *bicyclic units* of G. They were first introduced by Ritter and Sehgal in [RS89b]. The group generated by the bicyclic units of $\mathbb{Z}G$ is denoted by

$$\mathcal{B}_2(\mathbb{Z}G). \tag{1.4}$$

The bicyclic units can be generalized as follows. Let A be a finite dimensional rational algebra such that AG is semisimple and let $\mathcal{O}$ be an order in A. This generalization is based on idempotents of AG. The reason being that the construction of bicyclic units is based on the almost idempotents $\tilde{g}$. In fact, it is easily seen that $\frac{1}{|\langle g \rangle|}\tilde{g}$ is an idempotent of $\mathbb{Q}G$. So let $f \in AG$ be an idempotent and denote by n_f the minimal positive integer such that $n_f f \in \mathcal{O}G$. Let $x_1, \ldots x_m$ be generators of $\mathcal{O}$ as a $\mathbb{Z}$-module. Then we define

$$b(x_i g, f) = 1 + n_f^2(1-f)x_i g f \text{ and } b(f, x_i g) = 1 + n_f^2 f x_i g(1-f).$$

As f is an idempotent, $((1-f)x_i g f)^2 = 0 = (f x_i g(1-f))^2$ and hence $b(x_i g, f)$ and $b(f, x_i g)$ are units in $\mathcal{O}G$. They are called *generalized bicyclic units*. They were first introduced in [JL93]. For a given set of idempotents ι, as in [JdR], we denote by

$$\text{GBic}^{\iota}(\mathcal{O}G) = \langle b(x_i g, f), b(f, x_i g) \mid f \in \iota, g \in G, 1 \leq i \leq m \rangle \tag{1.5}$$

the group generated by all the generalized bicyclic units defined by ι. Observe that, if $\iota = \{\frac{1}{|\langle g \rangle|}\tilde{g} \mid g \in G\}$, then

$$\mathcal{B}_2(\mathbb{Z}G) \subseteq \text{GBic}^{\iota}(\mathbb{Z}G).$$

More details on this may be found in [JL93].

1.1.3 The Group of Units of an Order in a Rational Group Ring

A fundamental problem in the investigation on units of orders (in particular of integral group rings) is to determine a presentation of generators and relations. Because of the following result, one knows that a finite set of generators and relations exists.

Theorem 1.1.11. *The group of units of an order in a finite dimensional semisimple rational algebra A is finitely presented. In particular, it is finitely generated.*

In the case of commutative finite dimensional semisimple algebras, this result is due to Dirichlet's Unit Theorem (see 1.1.6). Hurwitz [Hur33] afterwards proved the result for orders of the form $M_n(R)$, with R a commutative order. Later the result was generalized to arithmetic groups by Borel and Harish-Chandra [BHC62] . We do not give a complete proof of this result here, but we will quickly sketch a possible proof of the fact that the group is finitely generated. First, one shows that one only has to consider orders of the type $M_n(R)$, with R an order in a division algebra. Indeed, by Theorem 1.1.5, a semisimple rational algebra is isomorphic to a direct product $\prod_{i=1}^r M_{n_i}(D_i)$, with each D_i a division algebra. Let $\mathcal{O}$ be an arbitrary order in A and let $\mathcal{O}_i$ be orders in D_i for every $1 \leq i \leq r$. Then $\mathcal{O}' = \prod_{i=1}^r M_{n_i}(\mathcal{O}_i)$ is another order in A and, by Corollary 1.1.10, $\mathcal{U}(\mathcal{O}')$ is commensurable with $\mathcal{U}(\mathcal{O})$. Recall that a subgroup N of finite index in a group G is finitely generated if and only if G is. (This may be shown by Schreier's Lemma for example. More details on this may be found in [Rob96, Section 6.1].) Thus, as $\mathcal{U}(\mathcal{O})$ and $\mathcal{U}(\mathcal{O}')$ have a common subgroup of finite index, $\mathcal{U}(\mathcal{O})$ is finitely generated if and only if $\mathcal{U}(\mathcal{O}')$ is finitely generated. Observe that $\mathcal{U}(\mathcal{O}') = \prod_{i=1}^r \mathrm{GL}_{n_i}(\mathcal{O}_i)$ and hence $\mathcal{U}(\mathcal{O}')$ is finitely generated if and only if all $\mathrm{GL}_{n_i}(\mathcal{O}_i)$ are.

So, to prove the result, we only have to prove that $\mathrm{GL}_n(\mathcal{O})$, the group of invertible matrices in $M_n(\mathcal{O})$, is finitely generated for $\mathcal{O}$ an order in a finite dimensional rational division algebra D and n a positive integer. This is done via the following lemma.

Lemma 1.1.12. *Let Γ be a group acting on the right by homeomorphisms on a connected topological space T, $\mathcal{F}$ a subset of T and let*

$$X = \{\gamma \in \Gamma \mid \mathcal{F} \cap \mathcal{F}^\gamma \neq \emptyset\}.$$

If $T = \bigcup_{\gamma\in\Gamma} \mathcal{F}^\gamma$ and $\mathcal{F}$ is included in the interior of $\bigcup_{\gamma\in X} \mathcal{F}^\gamma$, then X is a generating set for Γ.

The proof of this lemma is based on elementary techniques and hence we omit it here. Now to finish the sketch of the proof of Theorem 1.1.11, one constructs an action of $\mathrm{GL}_n(\mathcal{O})$ on a connected topological space which contains a subset $\mathcal{F}$ satisfying the conditions required in the lemma above.

On the way, we have shown that the study of the group of units of an order in a finite dimensional semisimple rational group algebra is up to commensurability reduced to the study of $\mathrm{GL}_n(\mathcal{O})$, for $\mathcal{O}$ an order in a finite dimensional rational division algebra. The following lemma allows to reduce even more. Let

$$\mathcal{O}^1 = \{u \in \mathcal{U}(\mathcal{O}) \mid \mathrm{RNr}_{A/\mathcal{Z}(A)}(u) = 1\},$$

where RNr denotes the reduced norm. Of course $\mathcal{O}^1$ is a subgroup of $\mathcal{U}(\mathcal{O})$. In case of an order of the type $M_n(\mathcal{O})$, we denote $(M_n(\mathcal{O}))^1$ simply as $\mathrm{SL}_n(\mathcal{O})$.

Using standard arguments, one easily shows the following well-known lemma. For a detailed proof, we refer to [JdR].

Lemma 1.1.13. *If $\mathcal{O}$ is an order in a finite dimensional simple rational algebra A then $\langle \mathcal{O}^1, \mathcal{U}(\mathcal{Z}(\mathcal{O}))\rangle$ has finite index in $\mathcal{U}(\mathcal{O})$ and $\mathcal{O}^1 \cap \mathcal{U}(\mathcal{Z}(\mathcal{O}))$ is cyclic and finite.*

Remark 1.1.14. *Up to commensurability, the study of the unit group of an order in a finite dimensional semisimple rational algebra is reduced to the study of* $\mathrm{SL}_n(\mathcal{O})$ *for* $\mathcal{O}$ *an order in a finite dimensional rational division algebra and* n *a positive integer.*

Indeed, we have already seen that, up to commensurability, the study is reduced to the study of $\mathrm{GL}_n(\mathcal{O})$, for $\mathcal{O}$ an order in a finite dimensional rational division algebra D. Observe that $\mathcal{Z}(M_n(\mathcal{O})) \cong \mathcal{Z}(\mathcal{O})$. Hence, by Lemma 1.1.13, $\mathcal{U}(\mathcal{Z}(\mathcal{O})) \times \mathrm{SL}_n(\mathcal{O})$, the direct product of the centre of the unit group of $\mathcal{O}$ with $\mathrm{SL}_n(\mathcal{O})$, contains a subgroup of finite index isomorphic to a subgroup of finite index in $\mathrm{GL}_n(\mathcal{O})$. The group $\mathcal{U}(\mathcal{Z}(\mathcal{O}))$ may be determined by Dirichlet's Unit Theorem and thus up to commensurability we are left with the study of $\mathrm{SL}_n(\mathcal{O})$.

Generators for $\mathrm{SL}_n(\mathcal{O})$ have been obtained in many cases. These are very deep results the proofs of which are beyond the scope of this thesis. In order to state these, we need to introduce some notation. Let I be an ideal of $\mathcal{O}$. An $n \times n$ matrix having 1 on the diagonal and all other entries 0 except possibly one which is an element of I, is called an elementary matrix modulo I. The group generated by all $n \times n$ elementary matrices modulo I is denoted by $E_n(I)$. The following result follows from [Bas64, Theorem 21.1, Corollary 21.4], [Vas73, Theorem 2.4, Lemma 2.6] and [Lie81, Theorem 24].

Theorem 1.1.15 (Bass-Vaseršteĭn-Liehl)**.** *Let* $\mathcal{O}$ *be an order in a finite dimensional rational division algebra* D*. If* $n \geq 3$ *or* $n = 2$ *and* D *is a number field such that* $\mathcal{U}(\mathcal{O})$ *is infinite, then* $E_n(I)$ *has finite index in* $\mathrm{SL}_n(\mathcal{O})$ *for every non-zero ideal* I *in* $\mathcal{O}$*.*

Because of Dirichlet's Unit Theorem, for a number field D, we know that $\mathcal{U}(\mathcal{O})$ is finite if and only if $D = \mathbb{Q}$ or D is a quadratic imaginary extension of $\mathbb{Q}$. In [Ven94], Venkataramana extended the case $n = 2$ by including also 2-by-2 matrix algebras over a lot of division algebras.

Theorem 1.1.16 (Venkataramana)**.** *Let* $\mathcal{O}$ *be an order in a finite dimensional rational division algebra* D*. If* D *is different from* $\mathbb{Q}$*, a quadratic imaginary extension of* $\mathbb{Q}$ *and a totally definite quaternion algebra with centre* $\mathbb{Q}$*, then* $E_2(I)$ *has finite index in* $\mathrm{SL}_2(\mathcal{O})$ *for every non-zero ideal* I *in* $\mathcal{O}$*.*

If $n = 1$ and D is commutative, Dirichlet's Unit Theorem describes the group of units of an order in D. If D is not commutative, then very little is known. One of the few results is the following theorem which is based on [Kle00, Theorem 2]. A complete proof can be found in [JdR]. Before stating the theorem, we have to recall some basic definitions on quaternion algebras.

Definition 1.1.17. *If* F *is a field of characteristic different from* 2*, an* F*-algebra* A *is said to be a quaternion algebra over* F *if there exist* $a, b \in \mathcal{U}(F)$ *and an* F*-basis* $\{1, i, j, k\}$*, such that*

$$i^2 = a, \ j^2 = b, \ k = ij = -ji.$$

This algebra is denoted $\mathcal{H}\left(\frac{a,b}{F}\right)$*.*

A quaternion algebra $\mathcal{H}\left(\frac{a,b}{F}\right)$ over a number field F is said to be *totally definite* if F is totally real and a and b are totally negative. Equivalently $\mathcal{H}\left(\frac{a,b}{F}\right)$ is totally definite if every complex embedding of F is real and $\sigma(a), \sigma(b) < 0$ for every complex embedding of F.

Theorem 1.1.18 (Kleinert)**.** *Let D be a non-commutative finite dimensional simple rational algebra and $\mathcal{O}$ an order in D. The following properties are equivalent.*

1. *$\mathcal{O}^1$ is finite.*
2. *$\mathcal{U}(\mathcal{Z}(\mathcal{O}))$ has finite index in $\mathcal{U}(\mathcal{O})$.*
3. *$\mathcal{U}(\mathcal{O})$ has an abelian subgroup of finite index.*
4. *$\mathcal{U}(\mathcal{O})$ has a solvable subgroup of finite index.*
5. *D is a totally definite quaternion algebra.*

The last two theorems lead to the following definition.

Definition 1.1.19. *A finite dimensional simple algebra is said to be exceptional, if it is one of the following types:*

(1) a non-commutative division algebra different from a totally definite quaternion algebra,

(2) $M_2(\mathbb{Q})$,

(3) $M_2(K)$ with K a quadratic imaginary extension of $\mathbb{Q}$,

(4) $M_2(\mathcal{H})$ where $\mathcal{H}$ is a totally definite quaternion algebra with centre $\mathbb{Q}$, i.e. $\mathcal{H} = \mathcal{H}\left(\frac{a,b}{\mathbb{Q}}\right)$ with a and b negative integers.

We say that a finite dimensional semisimple algebra does not have exceptional components if none of its Wedderburn components is exceptional.

With this definition at hand, Theorem 1.1.15, Theorem 1.1.16 and Theorem 1.1.18 may be summarized as follows.

Theorem 1.1.20. *Let D be a finite dimensional rational division ring and let $\mathcal{O}$ be an order in D. Let n be a positive integer. If $n \geq 3$ or $n = 2$ and the simple algebra $M_2(D)$ is not exceptional of type (2)-(4), then $E_n(I)$ has finite index in $\mathrm{SL}_n(\mathcal{O})$ for every non-zero ideal I in $\mathcal{O}$. If $n = 1$ and D is not exceptional of type (1), then $\mathcal{O}^1$ is finite or $\mathcal{O}^1$ is commutative (and thus can be described by Dirichlet's Unit Theorem).*

Starting from this theorem, two different directions in the study of units may be taken: the study of the unit group of an order in a finite dimensional semisimple rational algebra having no exceptional components, or the study of exactly the algebras having exceptional components. We show now some of the results obtained in the framework of the first context. The second context is a much less studied area and constitutes the main subject of this thesis.

Based on Theorem 1.1.20, Jespers and Leal showed the following theorem.

Theorem 1.1.21. *[JL93, Proposition 3.2] Let G be a finite group and let $\mathcal{O}'$ be an order in a finite dimensional semisimple rational algebra A. Assume AG is semisimple, e is a primitive central idempotent of AG and $\mathcal{O}$ is an order in AGe. Assume the simple algebra AGe is not exceptional. If f is an idempotent of AG such that ef is non-central (in AGe) then $GBic^{\{f\}}(\mathcal{O}'G)$ contains a subgroup of finite index in $\mathcal{O}^1$, where $GBic^{\{f\}}(\mathcal{O}'G)$ is defined in (1.5).*

Observe that Theorem 1.1.21 gives a result up to commensurability. By adding the Bass units and restricting the types of algebras A (for example one may take $A = \mathbb{Q}G$ with G a finite group), one obtains a finite set of generators of $\mathcal{U}(\mathcal{O}'G)$ up to finite index. Indeed, it can be shown that the Bass units generate a subgroup of finite index in $\mathcal{U}(\mathbb{Z}G)$ for G a finite abelian group. This was already established by Bass and Milnor in 1965 in [Bas65].

Theorem 1.1.22. *Let G be a finite abelian group. The group generated by the Bass units is of finite index in $\mathcal{U}(\mathbb{Z}G)$.*

Including the Bass units in the Theorem 1.1.21 and considering $A = \mathbb{Q}(\xi)G$, where ξ is a root of unity, Jespers and Leal proved the following theorem.

Theorem 1.1.23. *[JL93, Theorem 3.3] Let G be a finite group and ξ a complex root of unity. Let ε be the set consisting of those primitive central idempotents e of $\mathbb{Q}(\xi)G$ for which $\mathbb{Q}(\xi)Ge$ is not a division ring. For each $e \in \varepsilon$ let f_e be a non-central idempotent in $\mathbb{Q}(\xi)G$ and let $\iota = \{f_e \mid e \in \varepsilon\}$. If $\mathbb{Q}(\xi)G$ does not have exceptional components then the group*

$$\langle GBic^{\iota}(\mathbb{Z}[\xi]G) \cup \mathcal{B}_1(\mathbb{Z}[\xi]G)\rangle$$

has finite index in $\mathcal{U}(\mathbb{Z}[\xi]G)$, with $GBic^{\iota}(\mathbb{Z}[\xi]G)$ and $\mathcal{B}_1(\mathbb{Z}[\xi]G)$ defined as in (1.5) and (1.3).

Remark 1.1.24. *The result in [JL93] is stated only for $\mathbb{Q}G$ and not $\mathbb{Q}(\xi)G$, but the proof may easily be generalized to $\mathbb{Q}(\xi)G$. For details, we refer to [JdR].*

In order to apply Theorem 1.1.21 and Theorem 1.1.23 one needs constructions of non-central idempotents for simple components of $\mathbb{Q}G$. Of course $\widehat{H} = \frac{1}{|H|}\widetilde{H}$ is an idempotent in $\mathbb{Q}G$, for H a subgroup of G. In [JL93], Jespers and Leal showed that these idempotents are sufficient for the Bass and bicyclic units to generate a subgroup of finite index, provided a supplementary condition on the group G holds. Recall herefore ([Pas68]) that a finite group G is said to be *fixed point free* if it has an (irreducible) complex representation ρ such that 1 is not an eigenvalue of $\rho(g)$ for all $1 \neq g \in G$.

Theorem 1.1.25. *[JL93] Let G be a finite group and suppose that $\mathbb{Q}G$ does not have exceptional components. If G has no non-abelian homomorphic image which is fixed point free, then the group*

$$\langle \mathcal{B}_1(\mathbb{Z}G) \cup \mathcal{B}_2(\mathbb{Z}G)\rangle$$

has finite index in $\mathcal{U}(\mathbb{Z}G)$, with $\mathcal{B}_1(\mathbb{Z}G)$ and $\mathcal{B}_2(\mathbb{Z}G)$ defined as in (1.3) and (1.4).

1.2 Discontinuous Group Actions on Hyperbolic Spaces

In this section we give some background on the theory of groups acting discontinuously on some topological space. We do this in Subsection 1.2.1 in the general context of a proper metric space. However, the results of this thesis are mainly established for actions on a hyperbolic space, respectively a direct product of hyperbolic spaces. Therefore we focus in Subsections 1.2.2 to 1.2.3 on hyperbolic spaces. The results presented in this section are mainly based on [Bea95], [EGM98] and [Rat06].

Throughout this section, we use the standard topological notation for a subset Y of a topological space $\mathbb{X}$:

$$\begin{aligned} \overline{Y} &= \text{Closure of } Y, \\ Y^\circ &= \text{Interior of } Y, \\ \partial Y &= \text{Boundary of } Y. \end{aligned}$$

1.2.1 Discontinuous Groups and Fundamental Domains

In this subsection we study the basic properties of discontinuous group actions and their link to group presentations in terms of generators and relations.

Before dealing with discontinuous groups, we recall the definition and some properties of discrete groups. Intuitively, a discrete group is a group whose inherited topology is the discrete one if seen as a topological space. A more formal definition is the following.

Definition 1.2.1. *A discrete group is a topological group Γ all of whose points are open.*

The notion of discrete group is not very interesting, as every topological group can be considered a discrete group by equipping it with the discrete topology. A more interesting concept is the one of *discrete subgroups*. A subgroup of a topological group is said to be discrete if the induced topology is reduced to the discrete one. Very basic but common examples are the group $\mathbb{Z}$ which is discrete in $\mathbb{R}$, or the Gaussian integers $\mathbb{Z}[i]$ which are discrete in $\mathbb{C}$. Examples that received a lot of attention are discrete subgroups of, for example, $\mathbb{R}^n$ or $\mathrm{GL}_n(\mathbb{C})$. The following results give alternative definitions of discreteness in case the group can also be equipped with a metric.

Lemma 1.2.2. *A metric space $\mathbb{X}$ is discrete if and only if every convergent sequence (x_n) in $\mathbb{X}$ is eventually constant.*

In this thesis we mostly consider discrete subgroups of $\mathrm{SL}_2(\mathbb{C})$. The following lemma gives an easy criterion for a subgroup to be discrete in $\mathrm{SL}_2(\mathbb{C})$. For $\gamma \in \mathrm{SL}_2(\mathbb{C})$, set

$$\|\gamma\|^2 = |a|^2 + |b|^2 + |c|^2 + |d|^2, \tag{1.6}$$

the norm of γ.

Lemma 1.2.3. *A subgroup* Γ *of* $\mathrm{SL}_2(\mathbb{C})$ *is discrete if and only if for each* $\lambda > 0$*, the set* $\{\gamma \in \Gamma \mid \|\gamma\| \leq \lambda\}$ *is finite.*

The following lemma shows when an order is discrete in $\mathbb{R}^n$. For a nice proof, we refer to [ST02, Theorem 6.1].

Lemma 1.2.4. *The additive subgroup* $\mathbb{Z}v_1 + \ldots + \mathbb{Z}v_m$ *is discrete in* $\mathbb{R}^n$ *if and only if the vectors* $v_1, \ldots, v_m$ *are linearly independent in* $\mathbb{R}^n$*.*

Definition 1.2.5. *A group* Γ *acts discontinuously on a topological space* $\mathbb{X}$ *if and only if* Γ *acts on* $\mathbb{X}$ *and for a compact subset* K *of* $\mathbb{X}$*, the set* $\{\gamma \in \Gamma \mid K \cap \gamma(K) \neq \emptyset\}$ *is finite.*

In the same sense, a group of homeomorphisms Γ on a topological space $\mathbb{X}$ is said to be discontinuous if and only if Γ acts discontinuously on $\mathbb{X}$. Poincaré was the first who introduced the term discontinuous. In [Poi81a], he defined a discontinuous group to be a group of linear fractional transformations of the complex plane that has no infinitesimal operations. He then defined a Fuchsian group to be a discontinuous group that leaves invariant a circle. Poincaré noticed that Fuchsian groups were equivalent to discrete groups of isometries of the hyperbolic plane. Later, in [Poi83], he defined a properly discontinuous group to be a group of linear fractional transformations of the complex plane that acts discontinuously on a non-empty open subset of the plane. He called such a group a Kleinian group. In [Poi81b], he knew that such a group acts as a discrete subgroup of isometries on the upper half-space model of hyperbolic 3-space. In modern terminology, a *Fuchsian group* is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$, respectively a discrete group of isometries of the upper half-plane. A *Kleinian group* is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$ or a discrete group of isometries of hyperbolic 3-space. In [Poi83], Poincaré generalized his observation on the link between discrete subgroups of isometries and discontinuous groups to subgroups of isometries of hyperbolic space. Recall that a metric space is *proper* if every closed ball is compact.

Theorem 1.2.6. *Let* $\mathbb{X}$ *be a proper metric space. Then, a group* Γ *of isometries of* $\mathbb{X}$ *is discrete if and only if* Γ *is discontinuous.*

The previous theorem was proved in greater generality by Siegel in [Sie43]. Koecher and Roelcke proved its consequences for explicit metric spaces, in [KR59]. This result is very useful as it often is easier to think in terms of discrete subgroups rather than in terms of groups acting discontinuously.

A subject that is largely studied, is the geometry of discrete groups of isometries of $\mathbb{R}^n$, $\mathbb{S}^n$ and $\mathbb{H}^n$, the n-dimensional Euclidean space, the n-dimensional unit sphere and the n-dimensional hyperbolic space respectively. An important tool in this study are fundamental domains for discontinuous groups of isometries of some metric space $\mathbb{X}$. Denote the group of isometries of $\mathbb{X}$ by $\mathrm{Iso}(\mathbb{X})$.

Definition 1.2.7. *A closed subset* $\mathcal{R}$ *of some metric space* $\mathbb{X}$ *is called a fundamental region for the discontinuous subgroup* Γ *of* $\mathrm{Iso}(\mathbb{X})$ *if the following conditions are satisfied*

1. $\mathbb{X} = \bigcup_{\gamma \in \Gamma} \gamma(\mathcal{R})$,

2. *the interior* $\mathcal{R}^\circ$ *meets each* Γ*-orbit at most once,*

3. *the boundary of* $\mathcal{R}$ *has Lebesgue measure zero.*

Definition 1.2.8. *A subset* $\mathcal{F}$ *of a metric space* $\mathbb{X}$ *is a fundamental domain for a discontinuous subgroup* Γ *of* $\mathrm{Iso}(\mathbb{X})$ *if and only if* $\mathcal{F}$ *is a connected fundamental region for* Γ*.*

An easy example to illustrate this concept, is the fundamental domain for the group consisting of discrete translational symmetries in two directions in Euclidean 2-space $\mathbb{R}^2$. Denote by $t_{\overrightarrow{v}}$ the translation in $\mathbb{R}^2$ by the vector $\overrightarrow{v}$ and let Γ be the group generated by $t_{\overrightarrow{v_1}}$ and $t_{\overrightarrow{v_2}}$, where $\overrightarrow{v_1} = (1,0)$ and $\overrightarrow{v_2} = (0,1)$. Then a fundamental domain $\mathcal{F}$ for Γ in $\mathbb{R}^2$ is shown in figure 1.1.

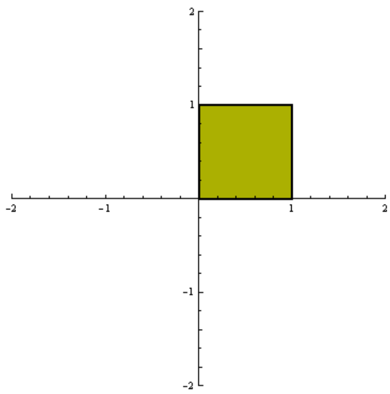


Figure 1.1: Fundamental domain for discrete translational symmetry in two dimensions.

The concept of fundamental domain is often linked with group presentations. In order to get finite presentations, it is necessary to construct locally finite fundamental domains. Recall that a collection $\mathcal{S}$ of subsets of a topological space $\mathbb{X}$ is said to be *locally finite* if and only if for each point $x \in \mathbb{X}$, there is an open neighbourhood U of $x \in \mathbb{X}$ such that U meets only finitely many members of $\mathcal{S}$. Equivalently, $\mathcal{S}$ is locally finite if and only if every compact subset of $\mathbb{X}$ meets only finitely many elements of $\mathcal{S}$.

Definition 1.2.9. *A fundamental domain* $\mathcal{F}$ *for a group* Γ *of isometries of a metric space* $\mathbb{X}$ *is locally finite if and only if* $\{\gamma(\mathcal{F}) \mid \gamma \in \Gamma\}$ *is a locally finite collection of subsets of* $\mathbb{X}$*.*

The following theorem gives a link between the fundamental domain for a discontinuous group Γ acting on a metric space $\mathbb{X}$ and a presentation of Γ.

Theorem 1.2.10. *Let* $\mathbb{X}$ *be a metric space and* $\Gamma \leq \mathrm{Iso}(\mathbb{X})$ *a discontinuous group. Let* $\mathcal{F} \subseteq \mathbb{X}$ *be a locally finite fundamental domain for* Γ*. Assume that there is an open connected set* $V \supseteq \mathcal{F}$ *such that for all* $\gamma, \tau \in \Gamma$*,*

- $\mathcal{F} \cap \gamma(\mathcal{F}) \neq \emptyset$ *if and only if* $V \cap \gamma(V) \neq \emptyset$*,*

- $\mathcal{F} \cap \gamma(\mathcal{F}) \cap \tau(\mathcal{F}) \neq \emptyset$ *if and only if* $V \cap \gamma(V) \cap \tau(V) \neq \emptyset$.

Let $\mathcal{S} = \{\gamma \in \Gamma \mid \mathcal{F} \cap \gamma(\mathcal{F}) \neq \emptyset\}$. *Then,* Γ *is finitely generated and* Γ *has the following presentation*

- Generators*:* $[\gamma]$ *with* $\gamma \in \mathcal{S}$*,*
- Relations*:* $[\gamma\tau] = [\gamma]\,[\tau]$ *if* $\gamma, \tau \in \mathcal{S}$ *and* $\mathcal{F} \cap \gamma(\mathcal{F}) \cap \gamma\tau(\mathcal{F}) \neq \emptyset$.

Theorem 1.2.10 was first proved by Macbeath in [Mac64]. He also proved the existence of the open connected set $V \supseteq \mathcal{F}$ in [Mac64, Theorem 2]. Later, Swan simplified the theorem a bit and used it in [Swa68, Swa71] for the study of Bianchi groups. A more modern version and proof may be found in [EGM98, Section 2.7]. The advantage of Theorem 1.2.10 is that the statement and the conditions are very simple. However, a disadvantage is that the presentation given by this theorem is very abundant. Poincaré's Polyhedron Theorem gives a more restrictive presentation result. We introduce here only the generator part of that theorem. The complete presentation part and more details on this are given in Chapter 2, in particular in Theorem 2.4.7. Unfortunately, Poincaré's theorem is not applicable to any locally finite fundamental domain and hence we need several more notions on fundamental domains. Recall that a *convex polyhedron* P in a metric space $\mathbb{X}$ is a non-empty, closed, convex subset of $\mathbb{X}$ such that the collection of its sides is locally finite in $\mathbb{X}$, where side is defined as a non-empty, maximal, convex subset of the border of P. In Chapter 2 (Definition 2.2.4), we will redefine these concepts clearly.

Definition 1.2.11. *Let* $\mathbb{X}$ *be a proper metric space. A convex fundamental polyhedron for a discrete group* $\Gamma \leq \mathrm{Iso}(\mathbb{X})$ *is a convex polyhedron* P *in* $\mathbb{X}$ *that is a locally finite fundamental domain for* Γ*.*

By the first two conditions in Definition 1.2.7, a fundamental polyhedron gives a sort of tiling of the space $\mathbb{X}$. This tiling is called *tessellation* of P and the formal definition is the following. Recall that a polyhedron P is called *thick* in a space $\mathbb{X}$ if the subspace generated by P is $\mathbb{X}$.

Definition 1.2.12. *A tessellation of a metric space* $\mathbb{X}$ *of dimension* n *is a set* $\mathcal{T}$ *consisting of convex thick polyhedra of* $\mathbb{X}$ *such that the following properties are satisfied:*

1. $\mathbb{X} = \cup_{P \in \mathcal{T}} P$,
2. $P^0 \cap Q^0 = \emptyset$ *for every two different members* P *and* Q *of* $\mathcal{T}$*,*
3. *the collection* $\mathcal{T}$ *is locally finite.*

The members of a tessellation are called tiles.

To state the generator part of Poincaré's Polyhedron Theorem in its easiest form, we need the concept of exactness for fundamental polyhedra and tessellations.

Definition 1.2.13. *A convex fundamental polyhedron P for a discontinuous group Γ is exact if and only if for each side S of P there is an element $\gamma \in \Gamma$ such that $S = P \cap \gamma(P)$. A tessellation $\mathcal{T}$ of $\mathbb{X}$ is exact if and only if each side S of a polyhedron P in $\mathcal{T}$ is a side of exactly two polyhedra P and Q in $\mathcal{T}$.*

Theorem 1.2.14. *Let P be an exact and convex fundamental polyhedron for a discontinuous group Γ acting on a proper metric space $\mathbb{X}$. Then, Γ is generated by the set*

$$\{\gamma \in \Gamma \mid P \cap \gamma(P) \text{ is a side of } P\}.$$

A detailed analysis and proof on this theorem will be given in Chapter 2 for $\mathbb{X}$ being the Euclidean space $\mathbb{R}^n$, the spherical space $\mathbb{S}^n$ or the hyperbolic space $\mathbb{H}^n$.

An easy corollary of Theorem 1.2.14 is the following.

Corollary 1.2.15. *If a discontinuous group Γ of isometries of a proper metric space $\mathbb{X}$ has a finite-sided, exact and convex fundamental polyhedron P (i.e. P has finitely many sides), then Γ is finitely generated.*

1.2.2 The Different Models of Hyperbolic Space and Their Group of Isometries

A hyperbolic space is a connected and simply connected Riemannian manifold with constant negative curvature. Historically, the theory of hyperbolic spaces appeared in 1829 with the work of Lobachevski on non-Euclidean geometry. His work was first published in some unknown Russian journal, but then the interest in the subject grew rapidly. Interesting references on the history of hyperbolic geometry are [Mil82] and [CFKP97]. Also the historical notes, appearing at the end of every chapter in [Rat06] give a nice overview of the historical development of this topic.

There are different models of hyperbolic space. The four most classical ones are the following:

1. the Upper Half-Space model $\mathbb{H}^n$,
2. the Unit Ball model $\mathbb{B}^n$,
3. the Klein model $\mathbb{K}^n$ and
4. the Hyperboloid model.

Every model has its own metric, geodesics and isometries. We will discuss in this subsection the three first ones, especially in dimension 2 and 3. We will also discuss their group of isometries and the link between the different models. Standard references on this subject are [Bea95, BH99, EGM98, Gro87, Rat06].

The first model is the upper half-space model, also called the Poincaré upper half-space,

$$\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}.$$

The metric ρ in this model is given by

$$\cosh \rho(a,b) = 1 + \frac{|a-b|^2}{2a_n b_n}, \tag{1.7}$$

where $|\;|$ denotes the Euclidean norm and $a = (a_1, \ldots, a_n),\ b = (b_1, \ldots, b_n) \in \mathbb{H}^n$. In particular, the hyperbolic ball

$$B_{\mathbb{H}^n}(a,r) = \{x \in \mathbb{H}^n \mid \rho(x,a) \leq r\}$$

with hyperbolic centre $a = (a_1, \ldots, a_n)$ and hyperbolic radius r is the Euclidean ball given by

$$(x_1 - a_1)^2 + \cdots + (x_{n-1} - a_{n-1})^2 + (x_n - a_n \cosh(r))^2 \quad \leq \quad (a_n \sinh(r))^2.$$

In other words

$$B_{\mathbb{H}^n}(a,r) = B_{\mathbb{R}^n}((a_1, \ldots, a_{n-1}, a_n \cosh(r)), a_n \sinh(r)).$$

Hence, the topology of $\mathbb{H}^n$ is that induced by the Euclidean topology of $\mathbb{R}^n$ and a subset of $\mathbb{H}^n$ is compact if and only if it is closed and bounded.

Before discussing the group of isometries of a hyperbolic space, we recall the notion of Möbius transformation. Denote by $\widehat{\mathbb{R}}^n$ the space $\mathbb{R}^n \cup \{\infty\}$. If $S(a,r)$ is a sphere in $\mathbb{R}^n$, given by

$$S(a,r) = \{x \in \mathbb{R}^n \mid |x-a| = r\},$$

then the *reflection* in $S(a,r)$ is the function σ defined by

$$\sigma(x) = a + \left(\frac{r}{|x-a|}\right)^2 (x-a), \tag{1.8}$$

for $x \in \widehat{R}^n$. If $S(a,r)$ is the unitary sphere $\mathbb{S}^n$, then we denote the image of $x \in \mathbb{R}^n$ by the reflection in $\mathbb{S}^n$ by x^*. If $P(a,t)$ is a plane, given by

$$P(a,t) = \{x \in \widehat{\mathbb{R}}^n \mid x \cdot a = t\} \cup \{\infty\},$$

where $a \in \mathbb{R}^n$, $x \cdot a$ is the usual scalar product and $t \in \mathbb{R}$, then the reflection σ in $P(a,t)$ is defined by

$$\sigma(x) = x - 2((x \cdot a) - t)a^*, \tag{1.9}$$

for $x \in \widehat{R}^n$. Let Σ denote a sphere or a plane. If y is the image of x by a reflection in Σ, then x and y are called *inverse points*.

Definition 1.2.16. *A Möbius transformation acting on $\widehat{\mathbb{R}}^n$ is a finite composition of reflections (in spheres and planes).*

Clearly, the Möbius transformations form a group, called the General Möbius group. One can show, see for instance [Bea95, Chapter 3], that a Möbius transformation is orientation-preserving if and only if it is the composition of an even number of reflections. The group of all orientation-preserving Möbius transformations is called the Möbius group. We now give some useful properties of Möbius transformations. The proofs may be found in [Bea95, Section 3.2].

Proposition 1.2.17. *Let ϕ be any Möbius transformation and Σ a sphere or a plane. Then $\phi(\Sigma)$ also is a sphere or a plane.*

Proposition 1.2.18. *Let x and y be inverse points with respect to Σ, with Σ a sphere or a plane, and let ϕ be a Möbius transformation. Then $\phi(x)$ and $\phi(y)$ are inverse points with respect to $\phi(\Sigma)$.*

Proposition 1.2.19. *The points x and y are inverse points with respect to the sphere or plane Σ if and only if every sphere or plane through x and y is orthogonal to Σ.*

Poincaré observed that each Möbius transformation ϕ acting on $\widehat{\mathbb{R}}^n$ could be extended to a Möbius transformation on $\widehat{\mathbb{R}}^{n+1}$. For $x = (x_1, \ldots, x_n) \in \widehat{\mathbb{R}}^n$, set

$$\tilde{x} = (x_1, \ldots, x_n, 0).$$

Based on this embedding of $\widehat{\mathbb{R}}^n$ onto $\widehat{\mathbb{R}}^{n+1}$, a reflection ϕ may be extended in the following way. If ϕ is a reflection in the sphere $S(a, r)$ with $a \in \mathbb{R}^n$ and $r \in \mathbb{R}^+$, then $\tilde{\phi}$ is the reflection in the sphere $S(\tilde{a}, r)$. Similarly, if ϕ is a reflection in $P(a, t)$, then $\tilde{\phi}$ is a reflection in $P(\tilde{a}, t)$. If ϕ is a Möbius transformation, then $\phi = \phi_1 \ldots \phi_n$ for ϕ_i reflections in spheres and planes, and hence we define $\tilde{\phi} = \tilde{\phi_1} \ldots \tilde{\phi_n}$. Observe that $\tilde{\phi}$ fixes the plane $x_{n+1} = 0$ and leaves the half-spaces $x_{n+1} > 0$ and $x_{n+1} < 0$ invariant. Using this, one can easily show that, if ϕ is a Möbius transformation, the extension $\tilde{\phi}$ is unique. Indeed suppose $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are two extensions of ϕ. Then $\tilde{\phi}_2^{-1}\tilde{\phi}_1$ fixes each point of the plane $x_{n+1} = 0$. Thus $\tilde{\phi}_2^{-1}\tilde{\phi}_1$ is the reflection in the plane $x_{n+1} = 0$ or the identity. As the first is impossible, $\tilde{\phi}_2^{-1}\tilde{\phi}_1$ is the identity and hence the claim is proven.

Definition 1.2.20. *The Poincaré extension of a Möbius transformation ϕ is the transformation $\tilde{\phi}$ as defined above.*

One may show that for two points $x \in \mathbb{R}^{n+1}$ and $y \in \mathbb{R}^{n+1}$, the transformation $\tilde{\phi}$ leaves $\frac{|x-y|^2}{x_{n+1}y_{n+1}}$ invariant. Hence by (1.7), the Poincaré extension of any Möbius transformation ϕ is an isometry of $\mathbb{H}^n$.

In particular, in dimension 2 and 3, Möbius transformations take a very special form. In fact the hyperbolic space becomes

$$\begin{aligned}\mathbb{H}^2 &= \{x + yi \mid x, y \in \mathbb{R}, y > 0\},\\ \mathbb{H}^3 &= \{z + rj \mid z \in \mathbb{C}, r \in \mathbb{R}^+\}.\end{aligned}$$

For convenience, we often think of $\mathbb{H}^2$ as a subset of the complex numbers $\mathbb{C}$, and of $\mathbb{H}^3$ as a subset of the Hamilton quaternions $\mathcal{H} = \mathcal{H}\left(\frac{-1,-1}{\mathbb{R}}\right)$. Recall that $\mathrm{Iso}(\mathbb{H}^2)$, respectively $\mathrm{Iso}(\mathbb{H}^3)$, denotes the group of isometries of $\mathbb{H}^2$ and $\mathbb{H}^3$ respectively. The group of orientation preserving isometries is denoted by $\mathrm{Iso}^+(\mathbb{H}^2)$ and $\mathrm{Iso}^+(\mathbb{H}^3)$ respectively. As we mainly will work in dimension 3, we analyse the isometries of $\mathbb{H}^3$ in details. The isometries of $\mathbb{H}^2$ are constructed in a similar way. We make a remark on this later. Let $P \in \mathbb{H}^3$. The group $\mathrm{PSL}_2(\mathbb{C})$ has a natural action on $\mathbb{H}^3$ which is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(P) = (aP+b)(cP+d)^{-1},$$

where $(aP+b)(cP+d)^{-1}$ is evaluated in the algebra $\mathcal{H}$.

Remark 1.2.21. *Throughout we will use the notation* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *both for an element of* $\mathrm{SL}_2(\mathbb{C})$ *as well as for its natural image in* $\mathrm{PSL}_2(\mathbb{C})$. *Moreover we will make abuse of notations, by using the same letter for both the matrix in* $\mathrm{SL}_2(\mathbb{C})$ *and the Möbius transformation acting on* $\mathbb{H}^3$.

Explicitly, if $P = z + rj$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $(aP+b)(cP+d)^{-1} = \frac{(aP+b)(\overline{P}\overline{c}+\overline{d})}{|cz+d|^2+|c|^2r^2}$, where $^-$ denotes the classical involution on the algebra $\mathcal{H}$, i.e. for $u = u_0 + u_1 i + u_2 j + u_3 k \in \mathcal{H}$,

$$\overline{u} = u_0 - u_1 i - u_2 j - u_3 k, \text{ the conjugate of } u. \tag{1.10}$$

Now, the numerator may be written as $(az+b)(\overline{cz}+\overline{d}) + a\overline{c}r^2 + arj(\overline{cz}+\overline{d}) - (az+b)rj\overline{c}$, where the last two terms may be written as $a(cz+d)rj - (az+b)crj$. Thus, if we write $\gamma(P) = \widehat{z} + \widehat{r}j$, then we obtain

$$\widehat{z} = \frac{(az+b)(\overline{cz}+\overline{d}) + a\overline{c}r^2}{|cz+d|^2+|c|^2r^2} \text{ and } \widehat{r} = \frac{r}{|cz+d|^2+|c|^2r^2}. \tag{1.11}$$

This action may be extended to $\widehat{\mathbb{H}}^3 = \mathbb{H}^3 \cup \partial\mathbb{H}^3$, where $\partial\mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$ in the following way:

- if $P \in \partial\mathbb{H}^3$ and P different from ∞ and $-\frac{d}{c}$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}(P) = (aP+b)(cP+d)^{-1}$, where the latter is simply evaluated in $\mathbb{C}$,
- if $P = -\frac{d}{c}$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}(P) = \infty$ and
- if $P = \infty$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\infty) = \frac{a}{c}$.

Remark 1.2.22. *One may show that transformations of the form*

$$z \mapsto \frac{az+b}{cz+d},$$

with a, b, c *and* d *complex numbers such that* $ad - bc \neq 1$ *are equivalent with orientation preserving Möbius transformations from* $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ *onto itself. For more details on this, see [Bea95, Chapter 4].*

It is well known (see for instance [EGM98]) that $\mathrm{Iso}^+(\mathbb{H}^3)$ is isomorphic with $\mathrm{PSL}_2(\mathbb{C})$. The following theorem sums up all the necessary information about $\mathrm{Iso}(\mathbb{H}^3)$.

Theorem 1.2.23. *[EGM98, Theorem 1.3] The hyperbolic metric is $\mathrm{PSL}_2(\mathbb{C})$-invariant. Thus, the hyperbolic distance and the hyperbolic volume are also $\mathrm{PSL}_2(\mathbb{C})$-invariant. Moreover, the determinant of the Jacobian matrix of any element of $\mathrm{PSL}_2(\mathbb{C})$ is positive. Hence $\mathrm{PSL}_2(\mathbb{C})$ is a group of orientation preserving motions for the hyperbolic geometry in $\mathbb{H}^3$. In fact, we identify $\mathrm{PSL}_2(\mathbb{C})$ with $\mathrm{Iso}^+(\mathbb{H}^3)$. Moreover, $\mathrm{PSL}_2(\mathbb{C})$ is of index 2 in the full group of isometries. A representative system of cosets of $\mathrm{PSL}_2(\mathbb{C})$ in the group of all isometries is given by the identiy and an arbitrary reflection Σ in a hemisphere perpendicular to $\mathbb{C}$. The group $\mathrm{Iso}(\mathbb{H}^3)$ is isomorphic to the semidirect product $\mathrm{PSL}_2(\mathbb{C}) \rtimes C_2$, where C_2 is the cyclic group of order 2 and its non-trivial element acts by complex conjugation on $\mathrm{PSL}_2(\mathbb{C})$.*

In the upper half-space model, the point j plays a particular role. The following lemma describes the stabilizer of this point. Recall that $\mathrm{SU}_2(\mathbb{C})$ denotes the *special unitary group*, which is the group of 2-by-2 unitary matrices of determinant 1. As, already suggested in remark 1.2.21, in the following lemma γ denotes as well the transformation in $\mathrm{PSL}_2(\mathbb{C})$ as its pre-image in $\mathrm{SL}_2(\mathbb{C})$.

Lemma 1.2.24. *The following statements are equivalent.*

(i) $\gamma \in \mathrm{SU}_2(\mathbb{C})$,

(ii) $\gamma(j) = j$,

(iii) $\|\gamma\|^2 = 2$, *with* $\|\gamma\|$ *defined as in (1.6).*

Proof. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$ and denote the Hermitian transpose of γ by γ^*. Then,

$$\gamma\gamma^* = \begin{pmatrix} |a|^2 + |b|^2 & a\overline{c} + b\overline{d} \\ \overline{a}c + \overline{b}d & |c|^2 + |d|^2 \end{pmatrix}.$$

Hence, if $\gamma \in \mathrm{SU}_2(\mathbb{C})$, then $\|\gamma\|^2 = 2$. So (i) implies (iii). By (1.7) and (1.11), one may easily show that

$$2\cosh \rho(j, \gamma(j)) = \|\gamma\|^2. \tag{1.12}$$

This shows that (iii) implies (ii). Finally, to show that (ii) implies (i), notice that, by (1.11), $\gamma(j) = j$ if an only if $|c|^2 + |d|^2 = 1$ and $a\overline{c} + b\overline{d} = 0$. This together with $ad - bc = 1$ implies that $\gamma = \begin{pmatrix} \overline{d} & -\overline{c} \\ c & d \end{pmatrix}$ and hence $\gamma \in \mathrm{SU}_2(\mathbb{C})$. ∎

The orientation-preserving isometries of $\mathbb{H}^2$ form a group isomorphic with $\mathrm{PSL}_2(\mathbb{R})$. The action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}^2$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(P) = \frac{aP + b}{cP + d},$$

where $P \in \mathbb{H}^2$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})$ and $\frac{aP+b}{cP+d}$ is evaluated in $\mathbb{C}$. Explicitly, if $P = x + yi$ and $\gamma(P) = \widehat{x} + \widehat{y}i$, then we have

$$\hat{x} = \frac{(ax+b)(cx+d) + acy^2}{(cx+d)^2 + c^2y^2} \text{ and } \hat{y} = \frac{y}{(cx+d)^2 + c^2y_1^2}. \tag{1.13}$$

The role of the point $j \in \mathbb{H}^3$ is played in $\mathbb{H}^2$ by i. Lemma 1.2.24 stays valid by replacing j by i and $\mathrm{SU}_2(\mathbb{C})$ by $\mathrm{SU}_2(\mathbb{R})$, the unitary matrices of determinant 1 over $\mathbb{R}$.

The second model for the hyperbolic n-space is the Unit Ball model

$$\mathbb{B}^n = \{a \in \mathbb{R}^n \mid |a| < 1\}.$$

The metric ρ is given by

$$\cosh \rho(a, b) = 1 + 2\frac{|a-b|^2}{(1-|a|^2)(1-|b|^2)}, \tag{1.14}$$

where $|\ |$ denotes, again, the Euclidean norm. An advantage of this model is that it is more symmetric than the upper half-space model. Intuitively this may be seen best by looking at its border $\partial\mathbb{B}^n = \{a \in \mathbb{R}^n \mid |a| = 1\}$. Indeed, in $\partial\mathbb{B}^n$ there is no point playing "a special role" as ∞ does in $\partial\mathbb{H}^n$. We will now give two propositions about Möbius transformations, leaving $\mathbb{B}^n$ invariant. Again, details and proofs may be found in [Bea95, Chapter 3]. We denote by $\mathbb{S}^{n-1}$ the unitary sphere $S(0,1)$ in $\mathbb{R}^n$. Let a be a point in $\mathbb{R}^n$. Recall that its inverse point by the reflection in $\mathbb{S}^{n-1}$ is denoted by a^*.

Proposition 1.2.25. *Let ϕ be a Möbius transformation with with $\phi(0) = 0$ and $\Phi(\mathbb{B}^n) = \mathbb{B}^n$. Then $\phi(x) = xA$ for some orthogonal matrix A.*

Proposition 1.2.26. *Let ϕ be a reflection in $S(a, r)$ in $\mathbb{R}^n$. Then the following properties are equivalent.*

(i) *$S(a, r)$ and $\mathbb{S}^{n-1}$ are orthogonal,*

(ii) *$\phi(a^*) = 0$,*

(iii) *$\phi(\mathbb{B}^n) = \mathbb{B}^n$.*

The following proposition describes the general form of a Möbius transformation.

Proposition 1.2.27. *Let ϕ be a Möbius transformation.*

(i) *If $\phi(\mathbb{B}^n) = \mathbb{B}^n$, then*

$$\phi(x) = (\sigma(x)) \cdot A,$$

where σ is a reflection in a sphere or a plane orthogonal to $\mathbb{S}^{n-1}$ and A is an orthogonal matrix.

(ii) *If* $\phi(\infty) = \infty$*, then*
$$\phi(x) = r(x \cdot A) + x_0,$$
where $r > 0$*,* $x_0 \in \mathbb{R}^n$ *and* A *is orthogonal.*

(iii) *If* $\phi(\infty) \neq \infty$*, then*
$$\phi(x) = r(\sigma(x) \cdot A) + x_0,$$
for some r*,* x_0*,* A *as in (ii) and some reflection* σ*.*

In dimension 2 and 3, the Unit Ball model is given by

$$\begin{aligned} \mathbb{B}^2 &= \{u = x + yi \in \mathbb{C} \mid |u| < 1\}, \\ \mathbb{B}^3 &= \{u = u_0 + u_1 i + u_2 j \in \mathcal{H} \mid |u| < 1\}, \end{aligned}$$

where $|\ |$ denotes the usual norm in $\mathbb{C}$ and $\mathcal{H} = \mathcal{H}\left(\frac{-1,-1}{\mathbb{R}}\right)$ respectively, that is, for $u = x+yi \in \mathbb{C}$, $|u|^2 = x^2+y^2$ and for $u = u_0+u_1i+u_2j+u_3k \in \mathcal{H}$, $|u|^2 = u_0^2+u_1^2+u_2^2+u_3^2$.

The unit ball model $\mathbb{B}^3$ may be identified in the same way, as $\mathbb{H}^3$, with a subset of $\mathcal{H}$. We analyse the orientation preserving isometries of the ball model $\mathbb{B}^3$. Therefore let $u = u_0 + u_1 i + u_2 j + u_3 k \in \mathcal{H}$ and define

$$\begin{aligned} u' &= u_0 - u_1 i - u_2 j + u_3 k, \\ u^* &= u_0 + u_1 i + u_2 j - u_3 k, \end{aligned} \tag{1.15}$$

and recall from (1.10), that $\overline{u} = u_0 - u_1 i - u_2 j - u_3 k$. Moreover if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{H})$, then we denote by $\overline{g}$ the matrix $\begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix} \in M_2(\mathcal{H})$. Define

$$\mathrm{SB}_2(\mathcal{H}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{H}) |\ d = a',\ b = c',\ a\overline{a} - c\overline{c} = 1 \right\}. \tag{1.16}$$

Note that if $f = \begin{pmatrix} a & c' \\ c & a' \end{pmatrix} \in \mathrm{SB}_2(\mathcal{H})$ then $f^{-1} = \begin{pmatrix} \overline{a} & -\overline{c} \\ -c^* & a^* \end{pmatrix}$. The following proposition gives the relation between the upper half-space model $\mathbb{H}^3$ and the ball model $\mathbb{B}^3$ and gives the group of orientation preserving isometries of the latter space, $\mathrm{Iso}^+(\mathbb{B}^3)$.

Proposition 1.2.28. *[EGM98, Proposition 1.2.3]*

(i) *For* $P \in \mathbb{H}^3$*, the quaternion* $-jP+1$ *is invertible in* $\mathcal{H}$ *and the map* $\eta_0 : \mathbb{H}^3 \longrightarrow \mathbb{B}^3$*, given by* $\eta_0(P) = (P - j)(-jP + 1)^{-1}$*, is an isometry. More precisely,* $\eta_0 = \mu\pi$*, where* π *is the reflection in the border of* $\mathbb{H}^3$ *and* μ *is the reflection in the Euclidean sphere with centre* j *and radius* $\sqrt{2}$*.*

(ii) *Let* $g = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & j \\ j & 1 \end{pmatrix} \in M_2(\mathcal{H})$*. The map* $\Psi : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SB}_2(\mathcal{H})$ *given by* $\Psi(\gamma) = \overline{g}\gamma g$ *is a group isomorphism.*

(iii) *For $u \in \mathbb{B}^3$ and $f = \begin{pmatrix} a & c' \\ c & a' \end{pmatrix} \in \mathrm{SB}_2(\mathcal{H})$, the quaternion $cu + a'$ is invertible in $\mathcal{H}$ and the transformation $f : \mathbb{B}^3 \longrightarrow \mathbb{B}^3$, defined by $f(u) = (au + c')(cu + a')^{-1}$ is an isometry of $\mathbb{B}^3$ and defines an action of $\mathrm{SB}_2(\mathcal{H})$ on $\mathbb{B}^3$. Again this action may be extended to the closure of $\mathbb{B}^3$, which we denote by $\overline{\mathbb{B}^3}$.*

(iv) *The group $Iso^+(\mathbb{B}^3)$ is isomorphic with $\mathrm{SB}_2(\mathcal{H})/\{1, -1\}$.*

(v) *The map η_0 is equivariant with respect to Ψ, that is $\eta_0(MP) = \psi(M)\eta_0(P)$, for $P \in \mathbb{H}^3$ and $M \in \mathrm{SL}_2(\mathbb{C})$.*

Note that item (i) clearly shows that the map η_0 is a Möbius transformation. Also note that an explicit formula for Proposition 1.2.28 (ii) is

$$\Psi(\gamma) = \frac{1}{2}\begin{pmatrix} a + \overline{d} + (b - \overline{c})j & b + \overline{c} + (a - \overline{d})j \\ c + \overline{b} + (d - \overline{a})j & \overline{a} + d + (c - \overline{b})j, \end{pmatrix} \tag{1.17}$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$. Hence $\|\Psi(\gamma)\|^2 = \|\gamma\|^2$, where the matrix norm was defined in (1.6).

Remark 1.2.29. *There is an analogue of Proposition 1.2.28 in dimension 2. Observe that the function η_0 in item (i) is in that case defined as follows*

$$\begin{aligned} \mathbb{H}^2 &\to \mathbb{B}^2 \\ P &\mapsto \frac{P - i}{-iP + 1}. \end{aligned}$$

The element g from item (ii) has to be replaced by

$$g = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

The third model for the hyperbolic n-space is the Klein model $\mathbb{K}^n$ whose underlying set also is the open unit ball but the distance between two points a and b is given by

$$\rho(a, b) = \frac{1}{2}\ln\frac{|a - b'||b - a'|}{|a - a'||b - b'|},$$

where $|\ |$ denotes again the Euclidean norm and the definition of a' and b' is illustrated in Figure 1.2.

The Klein model is of special use to prove incidence and convexity arguments, because in this model problems from hyperbolic geometry may be considered directly in Euclidean geometry. Indeed, one may show that the geodesics in $\mathbb{K}^n$ are the intersection of Euclidean lines with the unit ball. In $\mathbb{H}^n$ and $\mathbb{B}^n$, on the contrary, they are the intersection with Euclidean lines and circles orthogonal to the boundary. This is illustrated for dimension 2 in Figure 1.3.

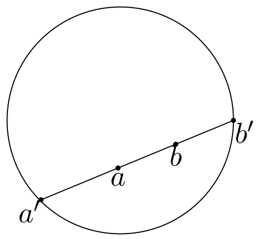


Figure 1.2: Distance in Klein model: $\rho(a,b) = \frac{1}{2}\ln\frac{|a-b'||b-a'|}{|a-a'||b-b'|}$.

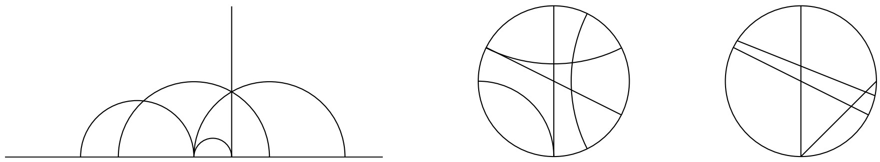


Figure 1.3: Geodesics in $\mathbb{H}^2$, $\mathbb{B}^2$ and $\mathbb{K}^2$.

1.2.3 Fundamental Polyhedra in Hyperbolic 2- and 3-Space

By Theorem 1.2.6 and Theorem 1.2.23, discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{C})$ have discontinuous actions on $\mathbb{H}^2$ and $\mathbb{H}^3$ respectively. Therefore, it makes sense to talk about fundamental domains for such discrete subgroups in hyperbolic 2- and 3-space. In this section we will give two known constructions of fundamental polyhedra in hyperbolic 2- and 3-space. We state all the results in dimension 3, but they may be reduced to dimension 2 without any problem.

Definition 1.2.30. *Let $\Gamma \leq \mathrm{Iso}(\mathbb{H}^3)$ be a discontinuous group and let $\mathcal{F}$ be a fundamental domain for Γ. One says that Γ is of finite covolume (or Γ is a cofinite group) if*

$$vol(\Gamma) = \int_{\mathcal{F}} dv < \infty,$$

where dv is the hyperbolic volume element. We call $vol(\Gamma)$ the covolume of Γ.

Observe that the element of hyperbolic volume of $\mathbb{H}^n$ is given by

$$\frac{dx_1 \dots dx_n}{(x_n)^n}. \tag{1.18}$$

For more details, we refer to [Rat06, Section 4.6].

Definition 1.2.31. *Let $\Gamma \leq \mathrm{Iso}(\mathbb{H}^3)$ be a discontinuous group and let $\mathcal{F}$ be a fundamental domain for Γ. We say that Γ is cocompact if Γ has a compact fundamental domain.*

Abusing notation, we say that a subgroup Γ of $\mathrm{SL}_2(\mathbb{C})$ is cofinite or cocompact if its image in $\mathrm{PSL}_2(\mathbb{C})$ has the respective property. Note that Definition 1.2.30 is justified by the following proposition. For a proof, we refer to [EGM98, Proposition 2.2.4].

Proposition 1.2.32. *Let $\Gamma \leq \mathrm{Iso}(\mathbb{H}^3)$ be a discontinuous group and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two fundamental domains for Γ. If $\int_{\mathcal{F}_1} dv < \infty$, then $\int_{\mathcal{F}_2} dv < \infty$ and $\int_{\mathcal{F}_1} dv = \int_{\mathcal{F}_2} dv$.*

The following proposition gives the covolume of the subgroup of a given discontinuous group. For a detailed proof, we refer to [Rat06, Theorem 6.7.3].

Proposition 1.2.33. *Let $\Gamma_2 \leq \Gamma_1$ be discrete subgroups of $\mathrm{PSL}_2(\mathbb{C})$. Then*

$$vol(\Gamma_2) = [\Gamma_1 : \Gamma_2] \cdot vol(\Gamma_1).$$

Remark 1.2.34. *Proposition 1.2.32 and Proposition 1.2.33 may be stated in a much more general context where Γ, Γ_1 and Γ_2 are discrete subgroups of the group of isometries of the Euclidean, spherical or hyperbolic space of dimension n.*

We now give the construction of the Dirchlet fundamental domain. So let $\Gamma \leq \mathrm{PSL}_2(\mathbb{C})$ be a discrete group and consider a point $P \in \mathbb{H}^3$. Denote the stabilizer of P in Γ by Γ_P. Let $\mathcal{F}_P$ be a fundamental polyhedron for Γ_P. It is convenient to choose P such that Γ_P is trivial or in a way that a fundamental domain may be easily determined. Often P is chosen to be the point $j \in \mathbb{H}^3$. Observe that as $P \in \mathbb{H}^3$ and Γ is discontinuous, Γ_P is finite and thus $\mathcal{F}_P$ is always more or less easy constructible. For every $\gamma \in \Gamma \setminus \Gamma_P$, set

$$D_\gamma(P) = \{u \in \mathbb{H}^3 \mid \rho(u, P) \leq \rho(u, \gamma(P))\}, \tag{1.19}$$

the half-space containing P, where ρ is defined in (1.7). The border of $D_\gamma(P)$ is the bisector of P and $\gamma(P)$, which we denote by $\mathrm{Bis}_\gamma(P)$, i.e.

$$\mathrm{Bis}_\gamma(P) = \{u \in \mathbb{H}^3 \mid \rho(u, P) = \rho(u, \gamma(P))\}. \tag{1.20}$$

Moreover we set

$$D_\gamma(P)' = \{u \in \mathbb{H}^3 \mid \rho(u, P) \geq \rho(u, \gamma(P))\}. \tag{1.21}$$

Then, put

$$\mathcal{F}_0 = \bigcap_{\gamma \in \Gamma \setminus \Gamma_P} D_\gamma(P). \tag{1.22}$$

Lemma 1.2.35. *Let $\mathcal{F}_0$ be defined as in (1.22). Then,*

$$\mathcal{F}_0 = \{u \in \mathbb{H}^3 \mid \rho(u, P) \leq \rho(\gamma(u), P) \text{ for every } \gamma \in \Gamma\}.$$

Proof. As $\gamma \in \mathrm{PSL}_2(\mathbb{C})$, γ is an isometry of $\mathbb{H}^3$ and so, for every $u \in \mathbb{H}^3$, $\rho(\gamma(u), P) = \rho(u, \gamma^{-1}(P))$. Hence, $u \in \{v \in \mathbb{H}^3 \mid \rho(v, P) \leq \rho(\gamma(v), P)$ for every $\gamma \in \Gamma\}$ if and only if $\rho(u, P) \leq \rho(u, \gamma^{-1}(P))$ for every γ, which is equivalent with $u \in D_\gamma(P)$ for every γ. Hence the result follows. ∎

Theorem 1.2.36. *The set*

$$\mathcal{F} = \mathcal{F}_0 \cap \mathcal{F}_P$$

is a fundamental polyhedron for Γ. The fundamental polyhedron so constructed is called the Dirichlet fundamental polyhedron with centre P (sometimes this is also called the Poincaré fundamental polyhedron).

Proof. By definition, $\mathcal{F}$ is closed. Moreover, as each $D_\gamma(P)$ is convex and contains P, $\mathcal{F}_0$ is convex and contains P. By the convexity of $\mathcal{F}_P$, $\mathcal{F}$ is convex.

We next prove that for every point $u \in \mathbb{H}^3$ there exists $\gamma \in \Gamma$ such that $\gamma(u) \in \mathcal{F}$. Consider the orbit of u under Γ and choose a point u_0 in this orbit, which is closest to P. We claim that this is possible because Γ is discontinuous. Indeed, take $r > 0$ such that the hyperbolic ball $B_{\mathbb{H}^3}(P,r)$ contains u. As $B_{\mathbb{H}^3}(P,r)$ is compact and Γ is discontinuous, $B_{\mathbb{H}^3}(P,r)\cap\gamma\left(B_{\mathbb{H}^3}(P,r)\right) \neq \emptyset$ for only finitely many $\gamma \in \Gamma$. Thus $B_{\mathbb{H}^3}(P,r)$ contains only finitely many elements of the orbit of u and hence this proves the claim. By Lemma 1.2.35, $u_0 \in \mathcal{F}_0$. If $u_0 \in \mathcal{F}_P$, we are done. If not, take $\tau \in \Gamma_P$ such that $\tau(u_0) \in \mathcal{F}_P$. As τ is an isometry of $\mathbb{H}^3$, $\rho(\tau(u_0), P) = \rho(u_0, \tau^{-1}(P)) = \rho(u_0, P)$, where the last equality is due to the fact that $\tau \in \Gamma_P$. Thus $\tau(u_0)$ is also a point in the orbit of u which is closest to P and hence $\tau(u_0) \in \mathcal{F}_0 \cap \mathcal{F}_P = \mathcal{F}$.

We now prove that $\mathcal{F}^\circ$ meets every Γ-orbit only once. So suppose that u and $\gamma_0(u)$ are in $\mathcal{F}^\circ$, for some $u \in \mathbb{H}^3$ and some $\gamma_0 \in \Gamma$. Then

$$\rho(u, \gamma_0^{-1}(P)) = \rho(\gamma_0(u), P) < \rho(\gamma_0(u), \gamma_0(P)) = \rho(u, P).$$

As $\rho(u,P) < \rho(u,\gamma(P))$ for every $\gamma \in \Gamma$, we have that $\gamma_0^{-1}(P) = P$ and thus $\gamma_0 \in \Gamma_P$. However this contradicts with the fact that $\mathcal{F}_P$ is a fundamental domain for Γ_P. Hence $\mathcal{F}^\circ$, meets every Γ-orbit only once.

Finally we prove that $\mathcal{F}$ is locally finite. Let $B_{\mathbb{H}^3}(P,r)$ be the hyperbolic ball with centre P and radius r. We show that $B_{\mathbb{H}^3}(P,r)$ meets $\gamma(\mathcal{F})$ for only finitely many $\gamma \in \Gamma$. If $\gamma \in \Gamma_P$, this is satisfied by definition. So suppose $\gamma \in \Gamma\setminus\Gamma_P$ and $B_{\mathbb{H}^3}(P,r)\cap\gamma(\mathcal{F}) \neq \emptyset$. Thus, there exists $u \in \mathcal{F}$ such that $\rho(\gamma(u), P) \leq r$. Then

$$\begin{aligned}
\rho(P,\gamma(P)) &\leq \rho(P,\gamma(u)) + \rho(\gamma(u),\gamma(P)) \\
&\leq r + \rho(u,P) \\
&\leq r + \rho(\gamma(u),P) \\
&\leq 2r.
\end{aligned}$$

As Γ is discontinuous, there are only finitely many γ which satisfy the last inequality. ∎

In order to apply Theorem 1.2.14, we need our fundamental polyhedron to have a finite number of sides. This gives rise to the following definition.

Definition 1.2.37. *A discrete group $\Gamma \leq \mathrm{PSL}_2(\mathbb{C})$ is called geometrically finite if there is a $P \in \mathbb{H}^3$ such that the Dirchlet fundamental polyhedron with centre P has finitely many sides.*

The above definition is taken from [EGM98, Definition 2.2.6]. Ratcliffe uses a different definition of geometrically finite in [Rat06]. Due to [Rat06, Theorem 12.4.6] both definitions coincide in $\mathbb{H}^n$ for $n = 1, 2$ or 3. The following theorem gives a criterion to guarantee geometrically finiteness.

Theorem 1.2.38. *[Wie77, Theorem 1] Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{C})$ be a discrete group of finite covolume. Then, there is $P \in \mathbb{H}^3$ such that the Dirichlet fundamental polyhedron with centre P has only finitely many sides. In other words, if $\Gamma \leq \mathrm{PSL}_2(\mathbb{C})$ is cofinite, then Γ is geometrically finite.*

This theorem also has been proven in a larger context in [GR70]. Together with the following theorem, they give a real sense to the concept of geometrically finite groups.

Theorem 1.2.39. *Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{C})$ be a discrete group of finite covolume. If Γ has a finite-sided Dirichlet polyhedron, then every Dirchlet polyhedron for Γ has finitely many sides.*

This theorem is due to Greenberg [Gre77]. Accessible proofs of Theorem 1.2.38 and Theorem 1.2.39, that are stated in a modern context, may be found in [Rat06, Chapter 12]. So Theorem 1.2.38 and Theorem 1.2.39 show that every cofinite discrete group of $\mathrm{PSL}_2(\mathbb{C})$ is geometrically finite and hence finitely generated, by Corollary 1.2.15. However, this result is not an equivalence. A finitely generated discontinuous group need not to be geometrically finite. An example of this is given in [Gre77].

By Theorem 1.2.38 and Theorem 1.2.39, the intersection appearing in the definition of $\mathcal{F}_0$ in (1.22) can be taken finite in some cases. There are also concrete methods to get this intersection finite. More on this will follow in Chapter 3. Another advantage of the construction of the Dirichlet fundamental polyhedron is that once one knows the finite intersection needed for its construction, one knows immediately a set of generates for the group Γ. This is explained in the following lemmas. These results are well-known, but as they are often somehow hidden in the texts and also for completeness' sake, we reprove them here.

Lemma 1.2.40. *Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{C})$ be a discrete group and let $\gamma \in \Gamma \setminus \Gamma_P$, for some $P \in \mathbb{H}^3$. Let $D_\gamma(P)$ and $Bis_\gamma(P)$ be defined as in (1.19) and (1.20). Then,*

$$\gamma(D_{\gamma^{-1}}(P)) = D_\gamma(P)' \text{ and } \gamma(Bis_{\gamma^{-1}}(P)) = Bis_\gamma(P).$$

Proof. Let $u \in D_{\gamma^{-1}}(P)$. Then $\rho(u,P) \leq \rho(u,\gamma^{-1}(P))$ and as γ is an isometry, $\rho(\gamma(u),\gamma(P)) \leq \rho(\gamma(u),P)$. Hence $\gamma(u) \in D_\gamma(P)'$. As $\partial D_\gamma(P) = \mathrm{Bis}_\gamma(P)$, the second part of the lemma follows trivially. ∎

Lemma 1.2.41. *Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{C})$ be a discrete group and let $\mathcal{F} = \mathcal{F}_0 \cap \mathcal{F}_P$ be given as in Theorem 1.2.36. Suppose, moreover, that $\mathcal{F}$ is exact and that there exists a finite set $X \subseteq \Gamma \setminus \Gamma_P$ such that*

$$\mathcal{F}_0 = \bigcap_{\gamma \in X} D_\gamma(P).$$

Then, Γ is generated by $X \cup X_P$, where X_P denotes a generating set for Γ_P.

Proof. Set

$$\tilde{X} = \{\gamma \in \Gamma \mid \mathcal{F} \cap \gamma(\mathcal{F}) \text{ is a side of } \mathcal{F}\}.$$

By Theorem 1.2.14, Γ is generated $\tilde{X}$. Hence we want to show that

$$\langle X \cup X_P \rangle = \langle \tilde{X} \rangle = \Gamma.$$

Clearly the left hand side is contained in the right hand side. So we have to show that $\langle X \cup X_P \rangle \supseteq \langle \tilde{X} \rangle$. If we denote by $\tilde{X}_P$ the intersection $\tilde{X} \cap \Gamma_P$, then by construction of $\mathcal{F}_P$ and Theorem 1.2.14, $\langle X_P \rangle = \langle \tilde{X}_P \rangle$. Let $\gamma \in \tilde{X} \setminus \tilde{X}_P$. Then $\gamma(\mathcal{F}) \subseteq \gamma(D_{\gamma^{-1}}(P)) = D_\gamma(P)'$. Thus $\mathcal{F} \cap \gamma(\mathcal{F}) \subseteq D_\gamma(P) \cap D_\gamma(P)' = \mathrm{Bis}_\gamma(P)$. As $\mathrm{Bis}_{\gamma_1}(P)$ and $\mathrm{Bis}_{\gamma_2}(P)$ are two totally geodesic hyperplanes and hence $\mathrm{Bis}_{\gamma_1}(P) \cap \mathrm{Bis}_{\gamma_2}(P)$ has co-dimension smaller or equal to 2 for $\gamma_1, \gamma_2 \in \Gamma \setminus \Gamma_P$ such that $\mathrm{Bis}_{\gamma_1}(P) \neq \mathrm{Bis}_{\gamma_2}(P)$, there exists $\gamma_0 \in X$ such that $\mathrm{Bis}_{\gamma_0}(P) = \mathrm{Bis}_\gamma(P)$. The latter is only possible if $\gamma_0(P) = \gamma(P)$, i.e. $\gamma_0^{-1}\gamma \in \Gamma_P$. Thus $\gamma = \gamma_0\tau$ for some $\tau \in \Gamma_P$ and hence $\gamma \in \langle X \cup X_P \rangle$. This proves the desired result. ∎

Another nice property of the Dirichlet fundamental domain is the following one.

Lemma 1.2.42. *Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{C})$ be a discrete subgroup and let $\gamma_0 \in \mathrm{PSL}_2(\mathbb{C})$. Moreover suppose that the stabilizer of P in Γ is trivial and let $\mathcal{F}$ be the Dirichlet fundamental domain for Γ with centre P. Then $\gamma_0(\mathcal{F})$ is the Dirichlet fundamental domain for the group $\gamma_0\Gamma\gamma_0^{-1}$ with centre $\gamma_0(P)$.*

Proof. As Γ_P is trivial, $\mathcal{F} = \bigcap_{1\neq\gamma\in\Gamma} D_\gamma(P)$. A point u is in $\mathcal{F}$ if and only if for every $\gamma \in \Gamma$,

$$\begin{aligned} & \rho(u, P) \leq \rho(u, \gamma(P)) \\ \Leftrightarrow \quad & \rho(\gamma_0(u), \gamma_0(P)) \leq \rho(\gamma_0(u), \gamma_0\gamma(P)) \\ \Leftrightarrow \quad & \rho(\gamma_0(u), \gamma_0(P)) \leq \rho(\gamma_0(u), (\gamma_0\gamma\gamma_0^{-1})\gamma_0(P)). \end{aligned}$$

The latter is true if and only if $\gamma_0(u)$ is in the Dirichlet fundamental domain for $\gamma_0\Gamma\gamma_0^{-1}$ with centre $\gamma_0(P)$. ∎

The above lemma may be stated in a more general context of a locally finite fundamental domain $\mathcal{F}$.

Lemma 1.2.43. *Let Γ be a group acting discontinuously on a metric space $\mathbb{X}$ and let $\gamma_0 \in \mathrm{Iso}(\mathbb{X})$. Let $\mathcal{F}$ be a locally finite fundamental domain for Γ. Then $\gamma_0(\mathcal{F})$ is a locally finite fundamental domain for $\gamma_0\Gamma\gamma_0^{-1}$.*

Proof. It is clear that $\gamma_0(\mathcal{F})$ is closed and its boundary has Lebesgue measure 0. It is also easy to see that if $\mathcal{F}$ is locally finite, $\gamma_0(\mathcal{F})$ is locally finite. We prove that $\gamma_0(\mathcal{F})$ contains a point of every orbit of $\gamma_0\Gamma\gamma_0^{-1}$. Let $x \in \mathbb{X}$ and consider $\gamma_0^{-1}(x) \in \mathbb{X}$. As $\mathcal{F}$ is a fundamental domain for Γ, there exists $\gamma \in \Gamma$ such that $\gamma\gamma_0^{-1}(x) \in \mathcal{F}$. Thus $\gamma_0\gamma\gamma_0^{-1}(x) \in \gamma_0(\mathcal{F})$. Suppose now that $x \in \gamma_0(\mathcal{F}^\circ)$ and $\gamma_0\gamma\gamma_0^{-1}(x) \in \gamma_0(\mathcal{F}^\circ)$ for some $1 \neq \gamma \in \Gamma$. This is equivalent with $\gamma_0^{-1}(x) \in \mathcal{F}^\circ$ and $\gamma\gamma_0^{-1}(x) \in \mathcal{F}^\circ$. As $\gamma_0^{-1}(x) \in \mathbb{X}$ and $\mathcal{F}$ is a fundamental domain for Γ, we have a contradiction. ∎

Observe that the construction of the Dirichlet fundamental polyhedron is valid for $\mathbb{H}^n$, for any $n \geq 2$. We have illustrated the construction in dimension 3, but this can be generalized to every dimension. Moreover it does not depend on hyperbolic space neither. If $\mathbb{X}$ is a metric space, then the construction of the Dirichlet fundamental polyhedron may be realized.

Another construction of a fundamental domain is that of the Ford fundamental domain (also called generalized Dirichlet domain in some references). Contrary to the construction of the Dirichlet fundamental polyhedron, this construction is based on Euclidean geometry. Let Γ be a group acting discontinuously on $\mathbb{H}^3$ and let Γ_∞ denote the stabilizer of $\infty \in \widehat{\mathbb{H}}^3$. Let $\gamma \in \Gamma \setminus \Gamma_\infty$. Observe that if $\Gamma \leq \mathrm{PSL}_2(\mathbb{C})$, then $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element of $\Gamma \setminus \Gamma_\infty$ if and only if $c \neq 0$. By Lemma 1.2.27

$$\gamma(x) = r(\sigma(x) \cdot A) + x_0,$$

where $r > 0$, $x_0 \in \mathbb{R}^3$, A orthogonal and σ a reflection in some sphere $S(a,t)$ with $a = \sigma^{-1}(\infty) = \gamma^{-1}(\infty)$. Hence, for $x, y \in \mathbb{H}^3$,

$$\begin{aligned} |\gamma(x) - \gamma(y)| &= r|\sigma x - \sigma y| \\ &= \frac{rt^2|x-y|}{|x-a||y-a|}. \end{aligned}$$

Now, suppose that $x, y \in S(a,k)$ for some $k > 0$. Then γ acts as a Euclidean isometry on $S(a,k)$ if and only if $k = t\sqrt{r}$. One may show that the sphere $S(a, t\sqrt{r})$ is the unique sphere in $\mathbb{R}^3$ on which γ acts as a Euclidean isometry. Therefore, $S(a, t\sqrt{r})$ is called the isometric sphere of γ and we denote it by I_γ. If γ is explicitly given by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, it is easy to see that

$$|\gamma(x) - \gamma(y)| = \frac{\frac{1}{|c|^2}|x-y|}{|x+\frac{d}{c}||y+\frac{d}{c}|},$$

and hence the isometric sphere is

$$I_\gamma = S\left(-\frac{d}{c}, \frac{1}{|c|}\right). \tag{1.23}$$

Observe that the centre $-\frac{d}{c}$ of the isometric sphere is contained in $\partial\mathbb{H}^3$ and so I_γ is a sphere orthogonal to $\partial\mathbb{H}^3$.

Remark 1.2.44. *The isometric sphere may also be defined in the ball model $\mathbb{B}^3$. Let $\gamma \in \mathrm{Iso}^+(\mathbb{B}^3)$ with $c \neq 0$. Again, by Lemma 1.2.27,*

$$\gamma(x) = \sigma(x) \cdot A,$$

where A is orthogonal and σ is a reflection in some sphere $S(a,t)$. By the same reasoning as above, we see that the isometric sphere is exactly the sphere defining the reflection σ.

We are now ready to define the Ford fundamental polyhedron. For this construction we suppose that Γ contains parabolic elements of the form $1 \neq \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, which fix the point ∞. Let $\mathcal{F}_\infty$ be a fundamental polyhedron for Γ_∞. Note that the elements of Γ_∞ are matrices of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ and thus Γ_∞ is an affine subgroup of Γ. Therefore finding a fundamental polyhedron is often easy.

We recall Shimizu's Lemma. For a proof see [EGM98, Section 2.3].

Lemma 1.2.45 (Shimizu's Lemma)**.** *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$*, and* $\tau = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ *where* $c \neq 0$ *and* $0 \neq t \in \mathbb{C}$*. If the group* $\langle \gamma, \tau \rangle$ *is discrete, then* $|ct| \geq 1$*.*

From this Lemma, we get that for every $\gamma \in \Gamma$ of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $|c|$ has a lower bound, i.e.

$$|c| \geq \frac{1}{|t|}, \tag{1.24}$$

for some $t \in \mathbb{R}$. Denote the exterior of I_γ by E_γ, i.e.

$$E_\gamma = \left\{ x \in \mathbb{H}^3 \mid |x + \frac{d}{c}| \geq \frac{1}{|c|} \right\} = \left\{ x \in \mathbb{H}^3 \mid |cx + d| \geq 1 \right\} \tag{1.25}$$

and denote the interior of I_γ by E'_γ, i.e.

$$E'_\gamma = \left\{ x \in \mathbb{H}^3 \mid |x + \frac{d}{c}| \leq \frac{1}{|c|} \right\}. \tag{1.26}$$

Define $\mathcal{F}_0$ as follows

$$\mathcal{F}_0 = \bigcap_{\gamma \in \Gamma \setminus \Gamma_\infty} E_\gamma = \{ x \in \mathbb{H}^3 \mid \forall\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \setminus \Gamma_\infty,\ |cx + d| \geq 1 \}. \tag{1.27}$$

For $u = z + rj \in \mathbb{H}^3$, we call r the *height* of u and we denote it by $h(u)$.

Lemma 1.2.46. *Let* $\mathcal{F}_0$ *be defined as in (1.27). Then,*

$$\mathcal{F}_0 = \{ u \in \mathbb{H}^3 \mid u \textit{ has maximal height in its orbit} \}.$$

Proof. We first claim that for a fixed $u = z + rj \in \mathbb{H}^3$, there exists an element of maximal height in its orbit. Indeed, for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, by (1.11), $h(\gamma(u)) > h(u)$ if and only if $|cu + d| < 1$. So u has maximal height in its orbit if and only if $|cu + d| \geq 1$ for every $c, d \in \mathbb{C}$, appearing on the second row of some $\gamma \in \Gamma$. The latter is the case if $r \geq \frac{1}{|c|}$. By (1.24), $\frac{1}{|c|} \leq |t|$ for some $t \in \mathbb{C}$. Thus if $r \geq |t|$, u has maximal height in its orbit. We now construct a sequence $(\gamma_n(u))$ with increasing height. This sequence will terminate in an element of maximal height in the orbit of u or this sequence is infinite. In the latter case every element has height strictly lower than $|t|$ and thus the sequence has an accumulation point, contradicting the discreteness of Γ. This proves the claim.

Consider an orbit and let u be an element in this orbit with maximal height. Hence for every $\gamma \in \Gamma$, $h(\gamma(u)) \leq h(u)$ and hence by (1.11), $|cu+d| \geq 1$, for every (c,d) such that (c,d) appears as the second row of a matrix in $\Gamma \setminus \Gamma_\infty$. Thus $u \in \mathcal{F}_0$.

To prove the other inclusion, let $u \in \mathcal{F}_0$. Then $|cu+d| \geq 1$, for every (c,d) appearing as the second row of a matrix in $\Gamma \setminus \Gamma_\infty$ and hence for every $\gamma \in \Gamma$, $h(\gamma(u)) \leq h(u)$. Thus every element of $\mathcal{F}_0$ reaches the maximum height in its orbit. ∎

This lemma at hand, we may prove that $\mathcal{F}_0 \cap \mathcal{F}_\infty$ is a fundamental domain for Γ.

Theorem 1.2.47. *Let Γ be a group acting discontinuously on $\mathbb{H}^3$ and suppose that Γ contains parabolic elements of the form $1 \neq \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. Then the set*

$$\mathcal{F} = \mathcal{F}_0 \cap \mathcal{F}_\infty$$

is a fundamental domain for Γ. The fundamental domain so constructed is called the Ford fundamental domain.

Proof. It is evident that $\mathcal{F}$ is closed and connected. We next show that $\mathcal{F}$ contains a point of every Γ-orbit. To prove this, consider an orbit and let u be an element in this orbit with maximal height. This point exists because of the proof of Lemma 1.2.46. There is $v \in \mathcal{F}_\infty$ and $\tau \in \Gamma_\infty$ such that $v = \tau(u)$. By (1.11), $h(v) = h(u)$ and hence we may assume that $u \in \mathcal{F}_\infty$. Thus, by Lemma 1.2.46, $u \in \mathcal{F}_0$ and so $u \in \mathcal{F}$.

Now we will show that two points of the same orbit cannot be in $\mathcal{F}^\circ$. Suppose that $u, \gamma(u) \in \mathcal{F}^\circ$, for some $1 \neq \gamma \in \Gamma$. As $\mathcal{F}_\infty$ is a fundamental domain, $\gamma \notin \Gamma_\infty$. Thus γ is realized with a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \neq 0$. Then,

$$h(u) = h(\gamma(u)) = \frac{h(u)}{|cu+d|}$$

which implies that $|cu+d| = 1$, which contradicts the fact that $u \in \mathcal{F}^\circ$. ∎

Remark 1.2.48. *In many cases it is possible to show that $\mathcal{F}$ is even a fundamental polyhedron, i.e. $\mathcal{F}$ is convex and locally finite. However we omit this proof here.*

Contrary to the Dirichlet fundamental polyhedron, this construction depends on the fact that the metric space $\mathbb{X}$ is the precise hyperbolic space $\mathbb{H}^n$. Moreover, whereas the Dirichlet fundamental domain is completely hyperbolic in nature, the Ford fundamental domain is rather based on Euclidean geometry. The main advantage of working with the Ford domain is that the formulas are much simpler because they are of Euclidean nature. However, mixing Euclidean and hyperbolic geometry also has disadvantages.

Transformations and mappings seem to work smoother in the Dirichlet domain because it is completely defined in hyperbolic terms.

Nevertheless we can prove a similar lemma as Lemma 1.2.42 in the context of a Ford fundamental domain.

Lemma 1.2.49. *Let* $\Gamma \leq \mathrm{PSL}_2(\mathbb{C})$ *be a discrete subgroup of* $\mathrm{PSL}_2(\mathbb{C})$ *and let* γ_0 *be an affine element of* $\mathrm{PSL}_2(\mathbb{C})$*. If* $\mathcal{F}$ *is a Ford fundamental domain for* Γ*, then* $\gamma_0(\mathcal{F})$ *is a Ford fundamental domain for the group* $\gamma_0\Gamma\gamma_0^{-1}$*.*

Proof. By definition $\mathcal{F} = \mathcal{F}_\infty \cap \mathcal{F}_0$, where $\mathcal{F}_\infty$ is a fundamental domain for Γ_∞ and $\mathcal{F}_0$ is defined in (1.27). So $\gamma_0(\mathcal{F}) = \gamma_0(\mathcal{F}_\infty) \cap \gamma_0(\mathcal{F}_0)$. By Lemma 1.2.43, $\gamma_0(\mathcal{F}_\infty)$ is a fundamental domain for $\gamma_0\Gamma_\infty\gamma_0^{-1}$. It is easy to see that the latter is exactly the stabilizer of ∞ in the group $\gamma_0\Gamma\gamma_0^{-1}$. We now claim that every element in $\gamma_0(\mathcal{F}_0)$ has maximal height in its $\gamma_0\Gamma\gamma_0^{-1}$-orbit. Indeed let $u \in \gamma_0(\mathcal{F}_0)$. Then $u = \gamma_0(v)$ for some $v \in \mathcal{F}_0$ and thus for every $\gamma_0\gamma\gamma_0^{-1} \in \gamma_0\Gamma\gamma_0^{-1}$, we have that

$$h(\gamma_0\gamma\gamma_0^{-1}(u)) = h(\gamma_0\gamma(v)) = h(\gamma(v)) \leq h(v) = h(\gamma_0^{-1}(u)) = h(u).$$

The second and the last equality are due to the fact that γ_0 is an affine transformation and the inequality comes from Lemma 1.2.46 and the fact that $v \in \mathcal{F}_0$. Hence the claim follows. By Lemma 1.2.46, $\gamma_0(\mathcal{F}_0)$ is the intersection of E_γ for every $\gamma \in \gamma_0\Gamma\gamma_0^{-1}$ such that γ does not stabilize ∞. Thus $\gamma_0(\mathcal{F})$ is the Ford fundamental domain for $\gamma_0\Gamma\gamma_0^{-1}$ and this proves the result. ∎

The best known fundamental polyhedron is the one for $\mathrm{PSL}_2(\mathbb{Z})$ acting on $\mathbb{H}^2$. In this case, the Dirichlet fundamental polyhedron of centre i and the Ford fundamental polyhedron coincide and are shown in Figure 1.4. This is not always the case. However, in Chapter 3, Theorem 3.1.2, we show that in the ball model both fundamental domains always coincide. In Chapter 3, we also give more information on groups having a Dirichlet and Ford fundamental polyhedron that coincide in the upper half-space model.

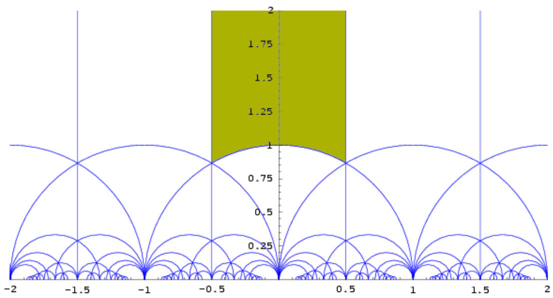


Figure 1.4: Fundamental domain for $\mathrm{PSL}_2(\mathbb{Z})$ acting on $\mathbb{H}^2$.

We can illustrate the use of Theorem 1.2.14 on this fundamental polyhedron. So, denote the coloured zone in Figure 1.4 by $\mathcal{F}$. Then $\mathcal{F}$ is a fundamental domain for $\mathrm{PSL}_2(\mathbb{Z})$ and it has three sides given by $x = \frac{1}{2}$, $x = -\frac{1}{2}$, and $x^2 + y^2 = 1$. Call them S_1,

S_2 and S_3. Set $\gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ and $\gamma_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then it is easy to see that $\mathcal{F} \cap \gamma_1(\mathcal{F}) = S_1$, $\mathcal{F} \cap \gamma_2(\mathcal{F}) = S_2$ and $\mathcal{F} \cap \gamma_3(\mathcal{F}) = S_3$. As $\gamma_2 = \gamma_1^{-1}$, $\mathrm{PSL}_2(\mathbb{Z})$ is generated by γ_1 and γ_3 by Theorem 1.2.14.

An equivalent lemma to Lemma 1.2.41, showing that finitely many generators can be derived from a finite-sided Ford domain, may be shown. However, we will not do this here. A more general result about this will be proven in Chapter 4.

1.2.4 From Quaternion Algebras to Groups Acting Discontinuously on Hyperbolic Space

In Subsection 1.2.2, we have seen that every discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$, or $\mathrm{PSL}_2(\mathbb{R})$ respectively, has a discontinuous action on $\mathbb{H}^3$, or $\mathbb{H}^2$ respectively. In this subsection, we give a more concrete example of such a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$. We use quaternion algebras to construct discontinuous actions on $\mathbb{H}^3$ and $\mathbb{H}^2$. Let K be an algebraic number field and let $\mathcal{O}_K$ be an order in K. For a and b non-zero elements of K, recall the definition of the quaternion algebra $\mathcal{H}\left(\frac{a,b}{K}\right)$ from Definition 1.1.17. In the particular case in which $a = b = -1$, we simply denote this algebra as $\mathcal{H}(K)$ and as $\mathcal{H}$ if furthermore $K = \mathbb{R}$. The set $\{1, i, j, k = ij\}$ is an additive K-basis of $\mathcal{H}\left(\frac{a,b}{K}\right)$. If $a, b \in \mathcal{O}_K$ then we denote by

$$\mathcal{H}\left(\frac{a,b}{\mathcal{O}_K}\right) = \mathcal{O}_K + \mathcal{O}_K i + \mathcal{O}_K j + \mathcal{O}_K k,$$

a subring of $\mathcal{H}\left(\frac{a,b}{K}\right)$. By N we denote the usual norm on $\mathcal{H}\left(\frac{a,b}{K}\right)$, that is

$$N(u_0 + u_1 i + u_2 j + u_3 k) = u_0^2 - au_1^2 - bu_2^2 + abu_3^2.$$

In the special case of $\mathcal{H}$ we simply denote $N(x)$ as $|x|^2$. Let

$$\mathrm{SL}_1\left(\mathcal{H}\left(\frac{a,b}{\mathcal{O}_K}\right)\right) = \left\{x \in \mathcal{H}\left(\frac{a,b}{\mathcal{O}_K}\right) \mid N(x) = 1\right\}.$$

By Lemma 1.1.13, the group $\langle \mathcal{U}(\mathcal{O}_K), \mathrm{SL}_1\left(\mathcal{H}\left(\frac{a,b}{\mathcal{O}_K}\right)\right)\rangle$ has finite index in the unit group $\mathcal{U}\left(\mathcal{H}\left(\frac{a,b}{\mathcal{O}_K}\right)\right)$ of $\mathcal{H}\left(\frac{a,b}{\mathcal{O}_K}\right)$. Since Dirichlet's Unit Theorem deals with the structure of $\mathcal{U}(\mathcal{O}_K)$, if one wants to investigate on the unit group $\mathcal{U}\left(\mathcal{H}\left(\frac{a,b}{\mathcal{O}_K}\right)\right)$ of $\mathcal{H}\left(\frac{a,b}{\mathcal{O}_K}\right)$, one needs to deal with $\mathrm{SL}_1\left(\mathcal{H}\left(\frac{a,b}{\mathcal{O}_K}\right)\right)$ (this is similar to Remark 1.1.14).

As in the case of a classical quaternion algebra, for $u \in \mathcal{H}\left(\frac{a,b}{K}\right)$, we define $\overline{u}$, u' and u^* as in (1.10) and (1.15). Observe that the mapping $u \mapsto u'$ defines an algebra isomorphism of $\mathcal{H}\left(\frac{a,b}{K}\right)$ and both $u \mapsto u^*$ and $u \mapsto \overline{u}$ define involutions of $\mathcal{H}\left(\frac{a,b}{K}\right)$. If L

is a field extension of K with $\sqrt{a}, \sqrt{b} \in L$, then define for $u = u_0 + u_1 i + u_2 j + u_3 k$,

$$\begin{aligned} \theta : \mathcal{H}\left(\frac{a,b}{K}\right) &\to M_2(L) \\ u \mapsto \gamma_u &= \begin{pmatrix} u_0 + u_1\sqrt{a} & u_2\sqrt{b} + u_3\sqrt{ab} \\ u_2\sqrt{b} - u_3\sqrt{ab} & u_0 - u_1\sqrt{a} \end{pmatrix}. \end{aligned} \tag{1.28}$$

The map θ is an injective homomorphism of algebras and

$$N(u) = det(\gamma_u),$$

for all $u \in \mathcal{H}\left(\frac{a,b}{K}\right)$. For more details on this map, we refer to [EGM98, Chapter X].

The following theorem shows that the group $\mathrm{SL}_1\left(\mathcal{H}\left(\frac{a,b}{\mathcal{O}_K}\right)\right)$ acts discontinuously on $\mathbb{H}^3$ via this embedding θ. The theorem is stated and proved in its most generality in [EGM98, Theorem 10.1.2]. We use here a weaker version from [CJLdR04, Theorem 21].

Theorem 1.2.50. *Let $\mathcal{H}\left(\frac{a,b}{K}\right)$ be a quaternion algebra over a number field K and assume that the following conditions hold*

- *K has exactly one pair of complex embeddings,*
- *$\mathcal{H}\left(\frac{a,b}{\sigma(K)}\right) \otimes_K \mathbb{R}$ is a division ring (necessarily the ring of Hamiltonian quaternions $\mathcal{H}$) for every real embedding σ of K.*

Then for every order Γ in $\mathcal{H}\left(\frac{a,b}{K}\right)$, we have the following properties

1. *$\mathrm{SL}_1(\Gamma)$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$,*
2. *$\mathrm{SL}_1(\Gamma)$ has finite covolume,*
3. *$\mathrm{SL}_1(\Gamma)$ is cocompact.*

Note that, by Lemma 1.2.38 and Lemma 1.2.39, the second item implies automatically that every Dirichlet fundamental polyhedron for $\mathrm{SL}_1(\Gamma)$ has finitely many sides. The kernel of θ is $\mathrm{I}(a,b,K) = \mathrm{SL}_1\left(\mathcal{H}\left(\frac{a,b}{\mathcal{O}_K}\right)\right) \cap \mathcal{U}(\mathcal{O}_K)$, a finite group. We denote the quotient $\mathrm{SL}_1\left(\mathcal{H}\left(\frac{a,b}{\mathcal{O}_K}\right)\right)/\mathrm{I}(a,b,K)$ by $\mathrm{PSL}_1\left(\mathcal{H}\left(\frac{a,b}{\mathcal{O}_K}\right)\right)$. Hence finding a set of generators for (respectively, for a subgroup of finite index in) $\mathrm{SL}_1\left(\mathcal{H}\left(\frac{a,b}{\mathcal{O}_K}\right)\right)$ amounts to finding a set of generators for (respectively, for a subgroup of finite index in) $\mathrm{PSL}_1\left(\mathcal{H}\left(\frac{a,b}{\mathcal{O}_K}\right)\right)$.

The following theorem gives a criterion, when a quaternion algebra is a division algebra. For a proof, we refer to [Pie82, Section 1.6]

Theorem 1.2.51. *The following are equivalent for a quaternion algebra $A = \mathcal{H}\left(\frac{a,b}{K}\right)$.*

1. *A is a division algebra.*

2. *for every* $u \in A \setminus \{0\}$, $N(u) \neq 0$.
3. *if* $(x, y, z) \in K^3$ *satisfies* $ax^2 + by^2 = z^2$, *then* $x = y = z = 0$.

From this we get the following useful corollary.

Corollary 1.2.52. *If* $K = \mathbb{Q}(\sqrt{-d})$ *with* d *a positive square-free integer, then* $\mathcal{H}\left(\frac{-1,-1}{K}\right)$ *is a division algebra if and only if* $d \equiv 7 \mod 8$.

Proof. Consider the equation $x^2 + y^2 = -z^2$ in $\mathbb{Q}(\sqrt{-d})$. As $\mathbb{Q}(\sqrt{-d})$ is a field, there is a non-trivial solution for $x^2 + y^2 = -z^2$ if and only if there is a non-trivial solution for $x^2 + y^2 = -1$. By [FGS71], the latter equation has no solution in $\mathbb{Q}(\sqrt{-d})$, with d positive and square-free, if and only if $d \equiv 7 \mod 8$ and hence the result follows by Theorem 1.2.51. ∎

2

Poincaré's Polyhedron Theorem

À ce moment, je quittai Caen, où j'habitais alors, pour prendre part à une course géologique entreprise par l'École des Mines. Les péripéties du voyage me firent oublier mes travaux mathématiques ; arrivés à Coutances, nous montâmes dans un omnibus pour je ne sais quelle promenade ; au moment où je mettais le pied sur le marchepied, l'idée me vint sans que rien dans mes pensées antérieures parût m'y avoir préparé, que les transformations dont j'avais fait usage pour définir les fonctions fuchsiennes étaient identiques à celles de la géométrie non euclidienne. Je ne fis pas la vérification : je n'en aurais pas eu le temps, puisque, à peine assis dans l'omnibus, je repris la conversation commencée, mais j'eus tout de suite une entière certitude. De retour à Caen, je vérifiais le résultat à tête reposée pour l'acquit de ma conscience.

Henri Poincaré

In this chapter, we give a new self-contained proof of Poincaré's Polyhedron Theorem on presentations of discontinuous groups of isometries of a Riemann variety of constant curvature. The proof is not based on the theory of covering spaces, but only makes use of basic geometric concepts. Most of the used concepts are either well-known or were introduced in Chapter 1. Through this method, one hence obtains a proof that is of a more constructive nature than most known proofs.

In Section 2.2 we give the necessary definitions and background on well known facts on geometry, hyperplanes and polyhedra. Section 2.3 discusses tessellations as a preparation to Section 2.4 where the main theorem on presentations (Theorem 2.4.7) is proven.

2.1 Poincaré's Polyhedron Theorem in Literature

Poincaré's Polyhedron Theorem is a widely known and an often used result. We quickly state a sketch of the theorem. Let, throughout this Chapter,

$\mathbb{X}$ be a space of constant curvature, i.e. the hyperbolic space $\mathbb{H}^n$ (or $\mathbb{B}^n$), the spherical space $\mathbb{S}^n$ or the Euclidean space $\mathbb{R}^n$.

Let P be a convex polyhedron and assume that for each side S of P, there exists a side S^* and an element $\gamma_S \in \text{Iso}(\mathbb{X})$ such that

(i) $\gamma_S^{-1}(S) = S^*$,

(ii) $\gamma_{S^*} = \gamma_S^{-1}$.

By sides, we understand the faces of codimension 1 of P. The isometries γ_S are called *(side-)pairing transformations.* One considers a third condition:

(iii) $\gamma_S(P^\circ) \cap P^\circ = \emptyset$.

Observe that if S is a side such that $S = S^*$, then by condition (ii), $\gamma^2 = 1$. This relation is called a *reflection relation.*

The paring transformations induce a relation on P, where, because of condition (iii), each point in P° is only equivalent to itself. Denote the set of equivalence classes of this relation by $\widetilde{P}$ and endow it with the usual topology, such that the projection $\pi : P \to \widetilde{P}$ is continuous and open. The next condition on P one considers is

(iv) for every point $u \in P$, $\pi^{-1}(u)$ is a finite set.

The next two conditions concern the edges of P, i.e. the faces of codimension 2. In fact one can show that, by conditions (i)-(iv), the edges of P come in finite cycles $(E_1, E_2, \ldots, E_n)$. Based on these finite cycles one constructs the so-called *cycle relations.* This is explained in greater detail in Section 2.4. The first condition based on the edges is the following.

(v) For every edge E, the cycle transformation associated to E has finite order t.

Let $\alpha(E)$ be the angle at E, measured form inside P. Then, we require that

(vi) $\sum_{i=1}^{n} \alpha(E_i) = \frac{2\pi}{t}$,

where $E_1, \ldots, E_n$ are the edges contained in the cycles associated to E and t is the finite order of the cycle transformation.

Finally the last condition states that

(vii) $\widetilde{P}$ is complete.

These conditions at hand, we are ready to state Poincaré's Polyhedron Theorem.

Theorem 2.1.1 (Poincaré's Polyhedron Theorem)**.** *Let $P \subseteq \mathbb{X}$ be a convex polyhedron with pairing transformations satisfying conditions (i) through (vii). Then Γ, the group generated by the pairing transformations is discrete, P is a fundamental polyhedron for Γ, and the reflection and cycle relations form a complete set of relations for Γ.*

Poincaré first published the theorem for dimension two in [Poi81a], then one year later also for dimension three in [Poi83]. Since then, a lot has appeared on this theorem in the literature. Among the most known references on this theorem, are the following books [Bea95, BH99, EGM98, Mas88, Rat06]. Several articles, such as for example [EP94, Mas71, dR71], give new proofs of this theorem, others, such as [AG11], try to generalize it.

Because of our aim in this thesis, we are mainly interested in the presentation part of this, i.e. a method to obtain a presentation of a discontinuous subgroup of the group of isometries of a Riemann variety of constant curvature from a fundamental polyhedron for the group (Theorem 2.4.7). The presentation part is in fact given as a corollary of Poincaré's Polyhedron Theorem.

Corollary 2.1.2. *Let $P \subseteq \mathbb{X}$ be a convex fundamental polyhedron for the discrete group Γ. Then the pairing transformations of P generate Γ, and the reflection and cycle relations form a complete set of relations for Γ.*

As already stated in [EP94, Section 9], most of the published proofs on Poincaré's theorem are rather unsatisfactory. In [EP94], Epstein and Petronio give a long list of references on this topic and describe clearly why each of them is unsatisfactory to their eyes. Here, we only sketch the problems in the main references. The two original versions written by Poincaré ([Poi82] and [Poi83]) are written in an old-fashioned way and do not give rigorous proofs. Moreover, for the proof of the three-dimensional case, one simply refers for a large part to the two-dimensional case, and this without really proving that the results used are still valid. In more modern versions, as for example [Bea95], only the 2-dimensional case is dealt with. The main problem is that in two dimensions, edges are reduced to points, and this simplifies the proofs a lot. Ratcliffe [Rat06] gives a nice and clear overview of the topic. However, he adds the condition of exactness (see Definition 1.2.13 and Theorem 1.2.14), which simplifies the proofs, but also restricts the cases to which the theorem may be applied. Finally, the proofs in [Mas88] and [EP94] focus mostly on that part of Poincaré's Theorem which states that if a polyhedron satisfies conditions (i) through (vii) then it is a fundamental domain for a discontinuous group. As a consequence, the presentation part of the theorem is obtained somehow indirectly and the intuition on the presentation part is hidden in these proofs. In [Mac64], an analogous, but different, presentation result is proven for topological transformation groups. The generators obtained by Poincaré's Polyhedron Theorem can be derived from this result. It should also be possible to derive the relations given by Poincaré, but here a lot more work has to be done. Moreover, geometrically seen, Poincaré's Theorem is more intuitive. That is why, in this chapter, we give a new and self-contained proof of the presentation part of Poincaré's Theorem, which we hope contributes into a deeper understanding of this important result.

2.2 Hyperplanes, Half-Spaces and Polyhedra

In this section, we give some facts about geometry. These facts are well-known but for the clarity, we recall them and settle all our definitions. There is quite a bit of overlap with the beginning of [EP94]. For the sake of clearness and completeness, we reprove all the lemmas needed for this chapter. As our definition of cell is different from the one given in [EP94, Definition 2.8], from Section 2.4 onwards both results are independent. Moreover, the proof of lemma 2.2.8 is more complete than the one given in [EP94]. We first give some notational conventions for this chapter. Recall the different models of hyperbolic space from Subsection 1.2.2.

If x is a point of $\mathbb{S}^n$ then by x' we denote its *antipode*. If x and y are two different points of $\mathbb{X}$ then there is a unique geodesic containing x and y unless $\mathbb{X}$ is spherical and x and y are antipodal, in which case all the geodesics containing x contain y too.

A *subspace* of $\mathbb{X}$ is a subset that contains the entire geodesic passing through any two of its non-antipodal points and that is closed under taking antipodes. If $\mathbb{X} = \mathbb{R}^n$ then the subspaces are the affine varieties. If $\mathbb{X} = \mathbb{S}^n$ then the subspaces are the intersection with $\mathbb{X}$ of the affine subspaces of $\mathbb{R}^{n+1}$ containing 0. In $\mathbb{H}^n$ and $\mathbb{B}^n$, the subspaces are intersections with $\mathbb{X}$ of affine subspaces and half-spheres orthogonal to the boundary of $\mathbb{X}$. Finally, the subspaces of $\mathbb{K}^n$ are the intersections of $\mathbb{K}^n$ with subspaces of $\mathbb{R}^n$. Every subspace S of $\mathbb{X}$ is a Riemann variety with the same constant curvature than $\mathbb{X}$. The codimension of S in $\mathbb{X}$ is the difference between the dimensions of $\mathbb{X}$ and S. For every non-empty subset U of $\mathbb{X}$ there is a unique minimal subspace of $\mathbb{X}$ containing U, it is called the *subspace generated* by U. If S is a subspace of $\mathbb{X}$ of dimension n and $x \in \mathbb{X} \setminus S$ then the subspace generated by $S \cup \{x\}$ is of dimension $n + 1$.

Let x and y be two different points of $\mathbb{X}$ and assume that x and y are not antipodal in case $\mathbb{X}$ is spherical. Then, $[x, y]$ denotes the geodesic of $\mathbb{X}$ joining x and y and $(x, y) = [x, y] \setminus \{x, y\}$. A subset S of $\mathbb{X}$ is said to be *convex* if $[x, y] \subseteq S$ for all distinct non-antipodal $x, y \in S$.

Definition 2.2.1. *Let $x \in \mathbb{X}$ and let U be a subset of $\mathbb{X}$, such that if $\mathbb{X}$ is spherical then U does not contain the antipode of x. Then the* cone *based on U with vertex x is $\bigcup_{u\in U}[x, u]$.*

A hyperplane is a codimension 1 subspace. If H is a hyperplane of $\mathbb{X}$ then $\mathbb{X} \setminus H$ has two connected components called the *open half-spaces* defined by H. If U is one of these then the other open half-space defined by H is denoted U' and we have

$$\begin{aligned} H &= \partial U = \partial\overline{U} = \partial U' = \partial\overline{U'}, \\ \overline{U} &= U \cup H, \\ \overline{U}^{\circ} &= U, \\ U' &= \mathbb{X} \setminus \overline{U} \text{ and} \\ \overline{U} &= \mathbb{X} \setminus U'. \end{aligned}$$

The sets $\overline{U}$ and $\overline{U'}$ are called the closed half-spaces defined by H. Moreover, if Z is one of the two closed half-spaces defined by H then the other is denoted Z'.

It is easy to see that the set formed by the non-empty intersections of finitely many open half-spaces is a basis for the topology of $\mathbb{X}$.

Lemma 2.2.2. *Let $x \in \mathbb{X}$ and $U \subseteq \mathbb{X}$ and assume that one of the following conditions holds.*

1. *U is an open subset of a hyperplane H of $\mathbb{X}$ and $x \notin H$.*
2. *U is an open subset of $\mathbb{X}$ and if $\mathbb{X}$ is spherical then U does not contain the antipode of x.*

Then $\bigcup_{u \in U}(x, u)$ is an open subset of $\mathbb{X}$ and it is dense in the cone based on U with vertex x.

Proof. (1) Assume that U is an open subset of a hyperplane H and $x \notin H$. First observe that, in the spherical case, H is closed under taking antipodes and therefore it does not contain the antipode of x. Let $C = \cup_{u \in U}(x, u)$. We have to prove that C is open. We first reduce the statement to the Euclidean case. This is clear for the hyperbolic geometry by using the Klein model. To reduce the spherical case to the Euclidean case, consider $\mathbb{S}^n$ as a subset of $\mathbb{R}^{n+1}$ and let V be the half-space of $\mathbb{S}^n$ with border H and containing x and let E be the hyperplane of $\mathbb{R}^{n+1}$ tangent to the sphere in the point of V whose tangent in $\mathbb{S}^n$ is parallel to H. The stereographic projection from the centre of the sphere is a bijection $V \to E$ mapping the intersections of V with the geodesics of the sphere to the Euclidean geodesics of E. Hence the statement for the Euclidean geometry implies the statement for the hyperbolic and spherical geometries.

So we only have to prove the statement for $\mathbb{X} = \mathbb{R}^n$. Making use of some affine transformations we may, without loss of generality, assume that x is the origin and H is given by the equation $x_n = 1$. As U is a union of squares of the form $(a_1, b_1) \times \cdots \times (a_{n-1}, b_{n-1}) \times \{1\}$ it is enough to prove the statement under the assumption that U is one of these squares. Hence, again making use of some linear transformations if needed, we may assume that $U = (-1, 1)^{n-1} \times \{1\}$. Then $C = \{(x_1, \ldots, x_n) \mid 0 < x_n < 1 \text{ and } |x_i| < x_n \text{ for each } 1 < i < n\}$, which is clearly an open subset.

(2) Assume now that U is open and, in case $\mathbb{X}$ is spherical it does not contain the antipode of x. Every $u \in \mathbb{X} \setminus \{x\}$ belongs to some hyperplane H_u such that $x \notin H_u$. Thus, by part (1), $\cup_{u \in U}(x, u) = \cup_{u \in U} \cup_{v \in U \cap H_u} (x, v)$ is open and its closure contains $U \cup \{x\}$. So, it is dense in the cone based on U with vertex x. ∎

Lemma 2.2.3. *[EP94, Lemma 2.3] If D is an intersection of closed half-spaces in $\mathbb{X}$ then either $D^\circ \neq \emptyset$ or D is contained in a hyperplane of $\mathbb{X}$. Moreover, if $D^\circ \neq \emptyset$ then D° is dense in D.*

Proof. We may assume that D is not empty. Let $\mathcal{S}$ be the set whose elements are the subspaces S of $\mathbb{X}$ with the property that $D \cap S$ has a non-empty interior, say V, as a subset of S and such that V is dense in $D \cap S$. Clearly, if $x \in D$ then $\{x\} \in \mathcal{S}$. So $\mathcal{S} \neq \emptyset$. Let S be a maximal element of $\mathcal{S}$. It is enough to show that $D \subseteq S$. Indeed, if this holds then either S is contained in a hyperplane, and hence so is D, or $\mathbb{X} = S \in \mathcal{S}$ and thus D° is dense in D.

Assume that $D \not\subseteq S$ and let $x \in D \setminus S$. Let V be the interior of $D \cap S$ considered as a subset of S. By definition, V is a non-empty open subset of S. Let T be the subspace generated by $S \cup \{x\}$. Then S is a hyperplane of T. By Lemma 2.2.2, $C_x = \cup_{v \in V}(x, v)$ is an open subset of T contained in D and $x \in \partial C_x$. As this property holds for every

$x \in (D \cap T) \setminus S$, we obtain that $\cup_{x \in (D \cap T) \setminus S} C_x$ is open in T and dense in $D \cap T$. Therefore $T \in \mathcal{S}$, contradicting the maximality of S. Thus $D \subseteq S$, as desired. ∎

Definition 2.2.4. *A non-empty subset P of $\mathbb{X}$ is said to be a polyhedron of $\mathbb{X}$ if $P = \cap_{Z \in \mathcal{S}} Z$ for a family $\mathcal{S}$ of closed half-spaces such that $\{\partial Z \mid Z \in \mathcal{S}\}$ is locally finite.*

In this case, one says that $\mathcal{S}$ defines the polyhedron P. For example, a subspace S is a polyhedron because $S = \cap_{i=1}^k H_i$ for some hyperplanes $H_1, \ldots, H_k$ and then $S = \cap_{i=1}^k Z_i \cap Z_i'$ where Z_i and Z_i' are the two closed subspaces containing H_i. Let P be a polyhedron and let S be the subspace generated by P. By Lemma 2.2.3, P contains a non-empty open subset of S (and it is dense in S). We then say that P is *thick* in S (note that S is the unique subspace of $\mathbb{X}$ in which S is thick). If P is thick in $\mathbb{X}$ we simply say that P is *thick*. The *dimension* (respectively, *codimension*) of P is by definition the dimension of S (respectively, the codimension of S in $\mathbb{X}$).

Definition 2.2.5. *The relative interior of P, denoted P^r, is the interior of P in the subspace generated by P.*

Lemma 2.2.6. *[EP94, Proposition 2.3] Let P be a thick polyhedron and let $\mathcal{S}$ be a family of closed half-spaces defining P. Then $\partial P = \bigcup_{Z \in \mathcal{S}} P \cap \partial Z$ and the following conditions are equivalent for a closed half-space Z_0 of $\mathbb{X}$:*

1. *$P \neq \bigcap_{Z \in \mathcal{S} \setminus \{Z_0\}} Z$.*
2. *$P \subseteq Z_0$ and $P \cap \partial Z_0$ is thick in ∂Z_0.*

Proof. The inclusion $\partial P \supseteq \bigcup_{Z \in \mathcal{S}} P \cap \partial Z$ is clear. For the converse inclusion assume that $x \in P \setminus \partial Z$ for every $Z \in \mathcal{S}$. As $\{\partial Z \mid Z \in \mathcal{S}\}$ is locally finite, any ball of $\mathbb{X}$ with centre x intersects only finitely many ∂Z with $Z \in \mathcal{H}$ and hence x has a neighbourhood not intersecting any ∂Z. This implies that $x \in P^\circ$ and thus $x \notin \partial P$. Since P is closed it follows that $\partial P = \bigcup_{Z \in \mathcal{S}} P \cap \partial Z$.

Let $P_0 = \cap_{Z \in \mathcal{S} \setminus \{Z_0\}} Z$.

(1) implies (2). Assume that $P \neq P_0$. Clearly $Z_0 \in \mathcal{S}$ and therefore $P \subseteq Z_0$. Let $x \in P_0 \setminus P$. As P is thick there is a non-empty open subset U of $\mathbb{X}$ contained in P such that if $\mathbb{X}$ is spherical then the antipode of x is not in U. Then $x \in \mathbb{X} \setminus Z_0 = (Z_0')^\circ$ and therefore every open segment (x, u) with $u \in U$ intersects ∂Z_0. By Lemma 2.2.2, $C = \cup_{u \in U} (x, u)$ is an open subset of $\mathbb{X}$ contained in P_0 and hence $C \cap \partial Z_0$ is a non-empty open subset of ∂Z_0 contained in P. Therefore $P \cap \partial Z_0$ is thick in ∂Z_0.

(2) implies (1). Assume that Z_0 satisfies (2). Let x be an element of the interior of $P \cap \partial Z_0$ in ∂Z_0. It easily is verified that then $x \notin \partial Z$ for every $Z \in \mathcal{S} \setminus \{Z_0\}$. Hence, $x \in (P_0)^\circ$. Clearly $x \notin (Z_0)^\circ$ as $x \in \partial Z_0$. Because $P^\circ \subseteq (Z_0)^\circ$ it follows that $x \notin P^\circ$. Thus $P \neq P_0$. ∎

Observe that condition (2) of Lemma 2.2.6 does not depend on $\mathcal{S}$, but only depends on P. A closed half-space Z_0 of $\mathbb{X}$ satisfying the equivalent conditions of Lemma 2.2.7 is called an *essential half-space* of P and ∂Z_0 is called an *essential hyperplane* of P.

Lemma 2.2.7. *[EP94, Lemma 2.4 and Proposition 2.5] Every thick polyhedron of $\mathbb{X}$ is the intersection of its essential half-spaces and, in particular, ∂P is the union of the intersection of P with the essential hyperplanes of P.*

Proof. Let P be a thick polyhedron of $\mathbb{X}$ and let $\mathcal{S}$ be a set of closed half-spaces defining P. Let $\mathcal{S}_1$ be the set of essential closed half-spaces of P and let $P_1 = \cap_{Z\in\mathcal{S}_1} Z$. As $\mathcal{S}_1 \subseteq \mathcal{S}$ we have that $P \subseteq P_1$. Assume that this inclusion is strict and take $x \in P_1 \setminus P$. Let $Z_1, \dots, Z_k$ be the elements of $\mathcal{S}$ whose boundaries contain x. Then, there is a ball U of $\mathbb{X}$ centred in x such that $U \cap \partial Z = \emptyset$ for every $Z \in \mathcal{S} \setminus \{Z_1, \dots, Z_k\}$. Let l be a non-negative integer such that $l \leq k$ and Z_i is an essential half-space of P if and only if $i \leq l$. Then, as $P \subseteq Z_i$ and $\partial Z_i \cap P$ is not thick in ∂Z_i, for $l < i \leq k$, we have by Lemma 2.2.6 that $P = \cap_{Z\in\mathcal{S}\setminus\{Z_{l+1},\dots,Z_k\}} Z$. Hence $x \in \cap_{i=1}^{l} Z_i \cap U = P_1 \cap U = \cap_{Z\in\mathcal{S}\setminus\{Z_{l+1},\dots,Z_k\}} Z \cap U = P \cap U$, a contradiction.

The last part of the statement of the lemma follows from Lemma 2.2.6. ∎

Lemma 2.2.8. *[EP94, Lemma 2.7] Let Z_1, Z_2 and Z_3 be closed half-spaces of $\mathbb{X}$ such that $\partial Z_1 \cap \partial Z_2 \cap \partial Z_3$ has codimension 2 and $Z_1 \cap Z_2 \cap Z_3$ is thick. Then $Z_1 \cap Z_2 \cap Z_3 = Z_i \cap Z_j$ for some $i, j \in \{1, 2, 3\}$.*

Proof. We may assume that Z_1, Z_2 and Z_3 are pairwise different. Then ∂Z_1, ∂Z_2 and ∂Z_3 are pairwise different for otherwise $Z_1 \cap Z_2 \cap Z_3$ is not thick.

We first prove the result for $\mathbb{X} = \mathbb{R}^n$, the Euclidean space. Then each ∂Z_i is a Euclidean hyperplane in $\mathbb{R}^n$ and $\partial Z_1 \cap \partial Z_2 \cap \partial Z_3$ is a codimension 2 affine subspace of $\mathbb{R}^n$. Applying some Euclidean transformation if needed, we may assume that $Z_1 = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0\}$ and $Z_2 = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_2 \geq 0\}$. Then, $\partial Z_1 \cap \partial Z_2 \cap \partial Z_3 = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_1 = x_2 = 0\}$ and $\partial Z_3 = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid a_1x_1 + a_2x_2 = 0\}$ with $a_1a_2 \neq 0$ and $a_1 > 0$.

Assume $a_2 > 0$. If $Z_3 = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid a_1x_1 + a_2x_2 \leq 0\}$ then $Z_1 \cap Z_2 \cap Z_3 \subseteq \partial Z_1 \cap \partial Z_2$ contradicting the thickness of $Z_1 \cap Z_2 \cap Z_3$. So, $Z_3 = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid a_1x_1 + a_2x_2 \geq 0\}$ and $Z_1 \cap Z_2 = Z_1 \cap Z_2 \cap Z_3$.

To finish the proof for $\mathbb{X} = \mathbb{R}^n$, it remains to deal with $a_2 < 0$. If

$$Z_3 = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid a_1x_1 + a_2x_2 \leq 0\}$$

then $Z_1 \cap Z_3 = Z_1 \cap Z_2 \cap Z_3$. Otherwise, $Z_3 = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid a_1x_1 + a_2x_2 \geq 0\}$ and thus $Z_2 \cap Z_3 = Z_1 \cap Z_2 \cap Z_3$. This finishes the proof in the Euclidean case.

In case $\mathbb{X} = \mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ each $Z_i = \mathbb{S}^n \cap Y_i$ with Y_i a closed half-space of $\mathbb{R}^{n+1}$ such that $0 \in \partial Y_i$. As $Z_1 \cap Z_2 \cap Z_3$ is thick in $\mathbb{S}^n$ and $Y_1 \cap Y_2 \cap Y_3$ contains the Euclidean cone with centre 0 and base $Z_1 \cap Z_2 \cap Z_3$, we deduce that $Y_1 \cap Y_2 \cap Y_3$ is thick in $\mathbb{R}^{n+1}$. Then, from the Euclidean case we deduce that $Y_1 \cap Y_2 \cap Y_3 = Y_i \cap Y_j$ for some $i, j \in \{1, 2, 3\}$ and hence $Z_1 \cap Z_2 \cap Z_3 = Z_i \cap Z_j$.

To prove the result in the hyperbolic case we use the Klein model $\mathbb{K}^n$ seen as subset of $\mathbb{R}^n$. Then the hyperplanes are the intersection of Euclidean hyperplanes with $\mathbb{K}^n$ and the result follows again from the Euclidean case. ∎

Lemma 2.2.9. *Let $\mathcal{S}$ be a countable set of proper subspaces of $\mathbb{X}$. Then*

1. $\mathbb{X} \neq \bigcup_{S\in\mathcal{S}} S$.
2. *If each $S \in \mathcal{S}$ has codimension at least 2 then for $x, y \in \mathbb{X} \setminus \cup_{S\in\Gamma} S$ there is $z \in \mathbb{X} \setminus \{x, y, x', y'\}$ such that $([x, z] \cup [z, y]) \cap \cup_{S\in\Gamma} S = \emptyset$. In particular, $\mathbb{X} \setminus \cup_{S\in\mathcal{S}} S$ is path-connected.*

Proof. (1) Without loss of generality, we may assume that each element of $\mathcal{S}$ is a hyperplane. We argue by induction on the dimension n of $\mathbb{X}$. The statement is clear for dimension 1 because $\mathbb{X}$ is uncountable while for $n = 1$ every element of $\mathcal{S}$ has cardinality 1. Since the number of hyperplanes is uncountable, $\mathbb{X}$ has a hyperplane H with $H \notin \mathcal{S}$. The induction hypothesis implies that $H \neq \bigcup_{S\in\mathcal{S}} H \cap S$ and therefore $\mathbb{X} \neq \bigcup_{S\in\mathcal{S}} S$.

(2) For $S \in \mathcal{S}$ and $x \in \mathbb{X}$ let S_x denote the subspace of $\mathbb{X}$ generated by $S \cup \{x\}$. Because of the assumption, each S_x is a proper subspace of $\mathbb{X}$. Assume $x, y \in \mathbb{X} \setminus \bigcup_{S\in\mathcal{S}} S$. By (1), there exists $z \in \mathbb{X} \setminus \bigcup_{S\in\mathcal{S}} (S_x \cup S_y)$. If $u \in (x, z) \cap S$ for some $S \in \mathcal{S}$ then x and u are different and non-antipodal points in S_x and hence the geodesic containing both x and u is contained in S_x, contradicting the fact that $z \notin S_x$. Therefore, the concatenation of the segments $[x, z]$ and $[z, y]$ is a path joining x and y contained in $\mathbb{X} \setminus \bigcup_{S\in\mathcal{S}} S$. Hence, (2) follows. ∎

2.3 Tessellations

Recall the definition of tessellation from Definition 1.2.12. If only the second and the third condition of Definition 1.2.12 are satisfied then we call $\mathcal{T}$ a partial tessellation of $\mathbb{X}$. Members of partial tessellations are also called tiles. It is easy to see that $P \cap Q^0 = \emptyset$ for any two distinct tiles P and Q. In particular, $P \cap Q = \partial P \cap \partial Q$. One readily verifies that, because of locally finiteness, every partial tessellation has to be countable. If T is a tile of a tessellation $\mathcal{T}$ then $\partial T = \cup_{R\in(\mathcal{T}\setminus\{T\})} T \cap R$. This is not necessarily true if $\mathcal{T}$ is a partial tessellation.

Definition 2.3.1. *A* cell *C of a partial tessellation $\mathcal{T}$ of $\mathbb{X}$ is a non-empty intersection of tiles satisfying the following property: if $T \in \mathcal{T}$ then either $C \subseteq T$ or $C^r \cap T = \emptyset$.*

Clearly every cell of a partial tessellation $\mathcal{T}$ is a polyhedron and it is contained in only finitely many tiles of $\mathcal{T}$. As the intersection of two different tiles is contained in the boundary of both, the cells of codimension 0 are precisely the tiles and hence the codimension of the intersection of two different tiles is at least 1. We define a side and an edge of a partial tessellation or a tile in function of a cell.

Definition 2.3.2. *A* side *of $\mathcal{T}$ is a cell of codimension 1 and an* edge *of $\mathcal{T}$ is a cell of codimension 2.*

Definition 2.3.3. *If T is a tile of $\mathcal{T}$ and C is a cell (respectively, side, edge) of $\mathcal{T}$ contained in T then we say that C is a cell (respectively, side, edge) of T in $\mathcal{T}$. In case the tessellation is clear from the context we simply say that C is a cell, side or edge of T.*

Note that this new definition of side is different from the usual definition of a side of a convex polyhedron (which was given in Section 1.2). In fact a side, as defined in the usual sense, is the union of sides as defined in Definition 2.3.2.

Lemma 2.3.4. *Let $\mathcal{T}$ be a tessellation of $\mathbb{X}$. If $x \in \mathbb{X}$ and $C = \bigcap_{T\in\mathcal{T}, x\in T} T$ then C is a cell of $\mathcal{T}$ and $x \in C^r$.*

Proof. Let $x \in \mathbb{X}$ and let $T_1, \ldots, T_k$ be the tiles of $\mathcal{T}$ containing x. Hence, $C = T_1 \cap \cdots \cap T_k$ and $T_1, \ldots, T_k$ are the only tiles containing C.

To prove that C is a cell we need to show that if Q is a tile different from any T_i then $Q \cap C^r = \emptyset$. We first consider the case where $\mathbb{X}$ is spherical and $x' \in C$. In this case we prove that $\mathbb{X} = T_1 \cup \cdots \cup T_k$, which of course implies the desired statement. Let U be an open convex neighbourhood of x such that $U \cap T = \emptyset$ for every $T \in \mathcal{T} \setminus \{T_1, \ldots, T_k\}$. Let $z \in \mathbb{X}$. If $z = x$ or x' then, by assumption, $z \in T_i$ for some (all) i. Otherwise the geodesic containing x' and z also contains x and therefore it intersects $U \setminus \{x\}$. In fact there exist non-antipodal elements x_1 and x_2 in U that both belong to the complete geodesic containing x and z and are such that $x \in (x_1, x_2)$ and $z \notin (x_1, x_2)$. Then z belongs to either $[x', x_1]$ or $[x', x_2]$. By symmetry we may assume that $z \in [x', x_1]$. Moreover, as $x_1 \in U \subseteq T_1 \cup \cdots \cup T_k$, there is $i = 1, \ldots, k$ with $x_1 \in T_i$. Then $z \in T_i$, as desired.

So we may assume that either $\mathbb{X}$ is not spherical or $x' \notin C$ and we argue by contradiction. Thus, suppose that there exist $Q \in \mathcal{T} \setminus \{T_1, \ldots, T_k\}$ and $y \in Q \cap C^r$. In particular, $y \neq x$ and if $\mathbb{X}$ is spherical then $y \neq x'$. Let U be an open convex neighbourhood of x such that $U \cap T = \emptyset$ for every $T \in \mathcal{T} \setminus \{T_1, \ldots, T_k\}$ and $y \notin U$. Hence, we can take the geodesic g containing x and y and take a point $y_1 \in C^r$ such that $y \in (x, y_1)$. This point exists because g is contained in the subspace generated by C and hence y is an interior point of $g \cap C$. By Lemma 2.2.2, $W = \cup_{u\in U}(y_1, u)$ is an open subset of $\mathbb{X}$. Since $y \in W \cap Q$ and Q is thick, we get that W contains a point $z \in Q^\circ$. Let $u \in U$ be such that $z \in (y_1, u)$. As $U \subseteq T_1 \cup \cdots \cup T_k$, $u \in T_i$ for some i, $(y_1, u) \subseteq T_i$ and therefore $z \in T_i \cap Q^\circ$. However, this contradicts with the fact that T_i and Q are different tiles of the the tessellation $\mathcal{T}$. So, in this case, C indeed is a cell.

To prove the second part, assume that $x \notin C^r$ and let L be the subspace generated by C. Clearly, the dimension of L is positive and $k > 1$. Therefore $C \subseteq \partial T_i$ for every i. Consider C as a thick polyhedron of L. As $x \notin C^r$, it follows from Lemma 2.2.6, that x belongs to one of the essential hyperplanes of C, as thick polyhedra of L. Fix $y \in C^r$ and an open interval (y, z) containing x. Then $(y, x) \subseteq C^r$ and $(x, z) \cap C = \emptyset$. Therefore $z \notin T_i$ for some i. Renumbering the T_i's and replacing (x, z) by a smaller interval if necessary, one may assume that $(x, z) \cap T_1 = \emptyset$. We claim that $H \cap [y, z] = \{x\}$ (equivalently $y \notin H$) for some essential hyperplane H of T_1. Otherwise y belongs to all the essential hyperplanes of T_1 containing x. Then (x, z) is contained in all these essential hyperplanes. If V is an open neighbourhood of x only intersecting the essential hyperplanes of T_1 containing x then $(x, z) \cap V$ is a non-empty subset contained in T_1, contradicting the construction. This proves the claim. Note that $x \in U \cap H \cap T_1$. Hence, $U \cap H \cap T_1$ is a non-empty open subset of $H \cap T_1$. So, by the second part of Lemma 2.2.3, there exists $w \in (U \cap H \cap T_1) \cap (H \cap T_1)^r$. Hence, H is the only essential hyperplane of T_1 containing w.

We claim that $(w, y) \subseteq T_1^\circ$. Indeed, for suppose the contrary, then there exists $u \in (w, y) \cap \partial T_1$. So $u \in H_1$ for some essential hyperplane H_1 of T_1. If $H_1 \neq H$ and Z_1 is the closed half-space of $\mathbb{X}$ with $\partial Z_1 = H_1$ and $T_1 \subseteq Z_1$ then $w, y \in Z_1^\circ$. Then $u \in (w, y) \subseteq Z_1^\circ$, a contradiction. So, $H_1 = H$ and $u \in (w, y) \cap H$. Now $y \notin H$ and $w \in T_1$. Hence a reasoning as above (interchanging the role of w and y and replacing H_1 by H) yields that $(w, y) \cap H = \emptyset$, a contradiction.

Because of the claim and since $w \in U$ and U is open in $\mathbb{X}$, there exists $z_1 \in U$ such that $w \in (y, z_1)$. As $w \in H$ and $y \notin H$, we have $z_1 \notin T_1$. However $U \subseteq T_1 \cup \cdots \cup T_k$ and hence $z_1 \in T_i$ for some $i \geq 2$. Then $[y, z_1] \subseteq T_i$ and we conclude that $\emptyset \neq (y, w) \subseteq T_1^\circ \cap T_i$ with $i > 1$, a contradiction. ∎

The cell of $\mathcal{T}$ formed by the intersection of the tiles containing x is the smallest cell containing x and we call it the *cell* of $\mathcal{T}$ *generated* by x. By Lemma 2.3.4, the relative interiors of the cells of $\mathcal{T}$ form a partition of $\mathbb{X}$. The cell generated by x is the unique cell of $\mathcal{T}$ whose relative interior contains x.

Lemma 2.3.5. *Let $\mathcal{T}$ be a partial tessellation of $\mathbb{X}$. Let T_1 and T_2 be two tiles of $\mathcal{T}$ and let $S = T_1 \cap T_2$. Assume S has codimension 1 and let H be the hyperplane generated by S. Then the following properties hold:*

1. *H is an essential hyperplane of both T_1 and T_2.*
2. *T_1 and T_2 are contained in different closed half-spaces defined by H.*
3. *$S^r \cap T = \emptyset$ for every tile T different from both T_1 and T_2.*
4. *S is a side of $\mathcal{T}$.*

Proof. Let $x \in S^r$ and let U be an open neighbourhood of x such that $U \cap H \subseteq S$. As $x \in \partial T_1$, from Lemma 2.2.7 we obtain that $x \in H_1$, for some essential hyperplane H_1 of T_1. If $H_1 \neq H$ then $H \cap U$ intersects non-trivially the two open half-spaces defined by H_1, contradicting the fact that T_1 does not intersect one of these open half-spaces. Therefore $H = H_1$. Hence (1) follows. It also proves that H is the only essential hyperplane of T_1 containing x and, by symmetry, it also is the only essential hyperplane of T_2 containing x. So there is an open ball B of $\mathbb{X}$ centred in x and not intersecting any essential hyperplane of T_1 or T_2 different from H. Let Z be the closed half-space with boundary H and containing T_1. Then $B \cap Z^\circ$ is one of the two non-empty connected components of $B \setminus H$ and it is contained in T_1°. Since $T_1^\circ \cap T_2 = \emptyset$ it follows that $T_2 \subseteq Z'$. This proves (2). The same argument then shows that $B \cap Z' \subseteq T_2$. Therefore $B \subseteq T_1 \cup T_2$. If

T is any tile such that $x \in T$ then B contains a point in T°. So $T^\circ \cap (T_1 \cup T_2) \neq \emptyset$ and therefore $T = T_1$ or $T = T_2$. Hence (3) follows. Clearly (4) is a consequence of (3). ∎

The following proposition follows at once from Lemma 2.3.5.

Proposition 2.3.6. *Every side of a partial tessellation $\mathcal{T}$ is contained in exactly two tiles and it is the intersection of these tiles.*

Lemma 2.3.7. *Let $\mathcal{T}$ be a partial tessellation of $\mathbb{X}$, let H be a hyperplane of $\mathbb{X}$ and let Z be a closed half-space defined by H. Then*

$$\mathcal{T}_Z = \{T \cap H \mid T \in \mathcal{T} \text{ and } Z \text{ is an essential closed half-space of } T\}$$

is a partial tessellation of H.

Proof. Let T_1 and T_2 be different elements of $\mathcal{T}_Z$. If $(T_1 \cap H)^r \cap (T_2 \cap H)^r \neq \emptyset$ then $T_1 \cap T_2$ is a side of $\mathcal{T}$ and H is the subspace generated by this side. Then, by Lemma 2.3.5, T_1 and T_2 are in different closed half-spaces defined by H, a contradiction. This proves that $\mathcal{T}_Z$ is a partial tessellation. As $\mathcal{T}$ is locally finite, so is $\mathcal{T}_Z$. ∎

Proposition 2.3.8. *If T is a tile of a tessellation $\mathcal{T}$ then ∂T is the union of the sides of T in $\mathcal{T}$.*

Proof. Let $x \in \partial T$ and let $T_1, \dots, T_k$ be the tiles containing x and that are different from T. Let U be an open neighbourhood of x such that $T, T_1, \dots, T_k$ are the only tiles intersecting U. By Lemma 2.2.7, there exists an essential hyperplane H of T in $\mathbb{X}$ such that $x \in H$. Then $H \cap T$ is a thick polyhedron of H. Therefore, by Lemma 2.2.3, $U \cap (H \cap T)^r$ is a non-empty open subset of H. Because $\mathcal{T}$ is a tessellation and $T \cap H \subseteq \partial T$, we get that $U \cap T \cap H \subseteq \cup_{i=1}^k T \cap T_i$. So $U \cap (H \cap T)^r$ is non-empty open subset of H contained in $\cup_{i=1}^k T \cap T_i \cap H$. Hence, $T \cap T_i \cap H$ is thick in H for some i. So, by Lemma 2.3.5, $T \cap T_i$ is a side of T. Therefore x belongs to a side of T. This proves one of the inclusions of the statement. The other one is obvious. ∎

Proposition 2.3.9. *Let T be a tile of a tessellation $\mathcal{T}$ of $\mathbb{X}$. If E is an edge of T in $\mathcal{T}$ then E is contained in exactly two sides of T in $\mathcal{T}$.*

Proof. Let E be an edge of T in $\mathcal{T}$. First we prove by contradiction that E cannot be contained in three different sides of T in $\mathcal{T}$. So, assume that E is contained in three distinct sides, say S_1, S_2 and S_3 of T in $\mathcal{T}$. Because of Proposition 2.3.6, $S_i = T \cap T_i$ with T_i a tile different from T. Let H_i denote the hyperplane generated by S_i. Because of Lemma 2.3.5, each H_i is an essential hyperplane of T. Let Z_i denote the closed half-space defined by H_i such that $T \subseteq Z_i$. By Lemma 2.3.5, $T_i \subseteq Z_i'$. Furthermore, by Lemma 2.2.8, we may assume that $Z_1 \cap Z_2 \cap Z_3 = Z_2 \cap Z_3$. As each H_i is an essential hyperplane of T we deduce that $H_1 \in \{H_2, H_3\}$ and hence, we may assume that $H_1 = H_2$. Thus $Z_1 = Z_2$. If $H_3 = H_1$ then $Z_1 = Z_3$ and S_1, S_2 and S_3 are tiles of the partial tessellation $\mathcal{T}_{Z_1}$ of H_1 defined as in Lemma 2.3.7. Since E has codimension 1 in H_1 and because it is contained in each S_i we get that $S_i \cap S_j$ is a side of $\mathcal{T}_{Z_1}$ for every $i \neq j$. Consequently, S_1, S_2 and S_3 are distinct tiles of $\mathcal{T}_{Z_1}$ containing points in the relative interior of E and hence in the relative interior of a side of $\mathcal{T}_{Z_1}$. This contradicts with Lemma 2.3.5. So $H_3 \neq H_1$ and thus $H_1 \cap H_3$ is the subspace generated by E. Moreover, S_1 and S_2 are tiles of $\mathcal{T}_{Z_1}$ and $S_1 \cap S_2$ is a side of $\mathcal{T}_{Z_1}$. By Lemma 2.3.5, S_1 and S_2 are in different closed half-spaces of H_1 defined by $H_1 \cap H_3$. These closed half-spaces are $H_1 \cap Z_3$ and $H_1 \cap Z_3'$. By symmetry we also may assume that $S_2 \subseteq Z_3'$. Hence, $T \cap T_2 = S_2 \subseteq Z_1 \cap Z_3 \cap Z_1' \cap Z_3' \subseteq H_1 \cap H_3$, in contradiction with the fact that S_2 has codimension 1.

It remains to prove that E is contained in two different sides of T. Let $x \in E^r$. Then $x \in \partial T$ and therefore $x \in S$ for some side S of T by Proposition 2.3.8. Hence, by the definition of a cell, $E \subseteq S$. By Lemma 2.3.5, $S = T \cap T_1$ with T_1 a tile of $\mathcal{T}$ different from T. Let H be the hyperplane of $\mathbb{X}$ generated by S. By Lemma 2.3.5, H is an essential hyperplane of both T and T_1. Furthermore, T and T_1 are included in different closed half-spaces defined by H. Note that, because of Lemma 2.2.7, a point $y \in S$ which is not in any essential hyperplane of T or T_1 different from H has a neighbourhood contained in $T \cup T_1$.

We claim that there is a hyperplane H_1 of $\mathbb{X}$ different from H such that H_1 is an essential hyperplane of either T or T_1 and it intersects E^r non-trivially. Indeed, for otherwise, for every $y \in E^r$ there is a neighbourhood U_y of y in $\mathbb{X}$ such that $U_y \subseteq T \cup T_1$. Then $\cup_{y \in E^r} U_y$ does not intersect any tile of $\mathcal{T}$ different from both T and T_1. In particular, the only tiles intersecting E^r are T and T_1. So T and T_1 are the only tiles containing E and hence $E = S$, a contradiction. This proves the claim.

So let H_1 be a hyperplane different from H such that $E^r \cap H_1 \neq \emptyset$ and H_1 is an essential hyperplane of either T or T_1. We claim that $E^r \subseteq H_1$. Otherwise E has positive dimension and the subspace L generated by E is not contained in H_1. Hence $H_1 \cap L$ is a hyperplane of L, since $\emptyset \neq L \cap H_1 \neq L$. As E^r is an open subset of L of dimension at least 1, it has points in the two open half-spaces of L defined by $H_1 \cap L$. This implies that E^r has points in the two open half-spaces of $\mathbb{X}$ defined by H_1. This contradicts with the facts that $E^r \subseteq T \cap T_1$ and either T or T_1 is contained in one closed half-space defined by H_1, because it is an essential hyperplane of either T or T_1. So we have proved that $E^r \subseteq H_1$. Lemma 2.2.8 therefore yields that if H_1 is essential in T (respectively, T_1) then H and H_1 are the only essential hyperplanes of T (respectively, T_1) intersecting E^r non-trivially.

Let $x \in E^r$ and let $T, T_1, R_1, \ldots, R_k$ be the different tiles of $\mathcal{T}$ containing x. Then $E \subseteq T \cap T_1 \cap (\cap_{i=1}^k R_i)$, by the definition of a cell.

Assume that H_1 is an essential hyperplane of T. Consider an open ball centred at x such that $U \cap R = \emptyset$ for every tile $R \notin \{T, T_1, R_1, \ldots, R_k\}$ and $U \cap H_2 = \emptyset$ for every essential hyperplane H_2 of T with $H_2 \notin \{H, H_1\}$. Let V be the open half-space of $\mathbb{X}$ defined by H and containing T°. Then $V \cap U \cap H_1$ is a non-empty open subset of H_1 contained in the boundary of T and hence it is also contained in $\cup_{i=1}^k T \cap R_i$, because $V \cap T_1 = \emptyset$. Thus $T \cap R_i$ has codimension 1 for some i and hence it is a side of T

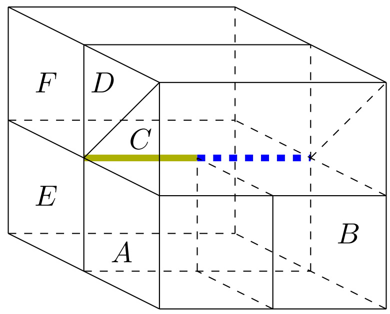


Figure 2.1: The intersection of the sides $C \cap D$ and $D \cap F$ of D is the union of the edges $A \cap D$ (green, fat and continuous) and $B \cap D$ (blue, fat and dashed). None of these two edges is the intersection of two sides of D while both are the intersection of two sides of C.

containing E.

Finally assume that H_1 is an essential hyperplane of T_1 and not of T. In this case we consider an open ball U in $\mathbb{X}$ with centre x such that $U \cap R = \emptyset$ for every tile $R \notin \{R_1, \dots, R_k\}$ and $U \cap H_2 = \emptyset$ for every essential hyperplane H_2 of T with $H_2 \neq H$. Moreover, let Z_1 be the open half-space defined by H_1 not containing T_1. Then $Z_1 \cap U \cap H$ is a non-empty open subset of H contained in the boundary of T. Hence $Z_1 \cap U \cap H \subseteq \cup_{i=1}^{k} T \cap R_i$ and therefore $T \cap R_i$ is a side of T containing E, for some i.

In both cases E is contained in two different sides of T containing E, namely S and $T \cap R_i$, as desired. ∎

Note that if the dimension of $\mathbb{X}$ is at least 3 then an edge of a tile T in $\mathcal{T}$ is not necessarily the intersection of two sides of T, although it is contained in exactly two distinct sides. Moreover, even if an edge E is the intersection of two sides of a tile, it could be properly contained in the intersection of two sides of another tile (see Figure 2.1).

Let E, S and T be respectively an edge, a side and a tile of $\mathcal{T}$ with $E \subseteq S \subseteq T$. We define recursively two sequences: one of tiles $(T_0, T_1, \dots)$ and another of sides $(S_1, S_2, \dots)$ by setting

$$\begin{aligned} T_0 &= T, \\ S_1 &= S, \\ T_i &= \text{tile containing } S_i \text{ and different from } T_{i-1}, \text{ and} \\ S_{i+1} &= \text{side of } T_i \text{ containing } E \text{ and different from } S_i. \end{aligned} \tag{2.1}$$

This is well defined by Proposition 2.3.6 and Proposition 2.3.9 and we have for $i \geq 1$:

$$\begin{gathered} T_{i-1} \text{ and } T_i \text{ are the only tiles containing } S_i \text{ and} \\ S_i \text{ and } S_{i+1} \text{ are the only sides of } T_i \text{ containing } E. \end{gathered}$$

For example, assume that Figure 2.1 represents part of a tessellation of $\mathbb{R}^3$ and take $T = A$, $S = A \cap C$ and $E = A \cap C$, the red, fat, continuous segment. Then, the sequence of tiles is periodic of period 5 starting with (A, C, D, F, E). If $T = B$, $S = B \cap C$ and E is the blue, fat, dashed segment then again the sequence of tiles is periodic of period 5 starting with (B, C, D, F, E). If one considers the edge $E = A \cap B \cap C$ then, with an appropriate side and tile selection, we obtain a sequence of tiles of period 3 starting with (A, B, C). We will show that this behaviour is general.

As every edge is contained in finitely many tiles, the sequences only have finitely many different elements. Moreover, if k is minimum such that $T_k = T_m$ for some $m > k$ then $k = 0$. Indeed, S_k and S_{k+1} are the only sides of T_k containing E and the same happens for S_m and S_{m+1}. Therefore either $S_k = S_m$ and $S_{k+1} = S_{m+1}$ or $S_k = S_{m+1}$ and $S_{k+1} = S_m$. In the former case T_{m-1} contains $S_m = S_k = T_{k-1} \cap T_k$ and it is different from $T_m = T_k$. Therefore, if $k \neq 0$ then $T_{m-1} = T_{k-1}$. In the latter case, if $k \neq 0$ then T_{m+1} contains $S_{m+1} = S_k = T_{k-1} \cap T_k$ and it is different form $T_m = T_k$. Hence, $T_{m+1} = T_{k-1}$. In both cases we obtain a contradiction with the minimality of k.

Let m be the minimal positive integer with $T_0 = T_m$, then

$$(T_0, T_1, \ldots, T_m) \tag{2.2}$$

is called an *edge loop* of E. This definition depends on the choice of the tile T and the side S. However, if S is replaced by another side S^* containing E and included in T (there is only one option by Proposition 2.3.9) then the edge loop obtained is $(T_m, T_{m-1}, \ldots, T_2, T_1, T_0)$. If we replace T by one of the tiles T_i and S by one of the sides of T_i containing E, i.e. either S_{i+1} or S_i then the edge loop is either

$$(T_i, T_{i+1}, \ldots, T_m, T_1, \ldots, T_{i-1}, T_i)$$

or

$$(T_i, T_{i-1}, \ldots, T_1, T_m, \ldots, T_{i+1}, T_i).$$

The next lemma shows that there are no other alternatives.

Lemma 2.3.10. *If E is an edge and $(T_0, T_1, \ldots, T_m)$ is an edge loop of E then $T_1 \ldots, T_m$ are precisely the tiles containing E.*

Proof. Note that, as stated before, there are only finitely many tiles containing E. We can order them in such a way that $T_0, \ldots, T_n$ are all these tiles and $T_0, \ldots T_m$ are the tiles forming the edge loop. Let $z \in E^r$. Then $z \in T_i$ for every $0 \leq i \leq n$ and there exists $\lambda > 0$ such that the ball $B = B(z, \lambda)$ intersects a tile T if and only $T = T_i$ for for some $0 \leq i \leq n$. Moreover

$$B = B_1 \cup B_2$$

with

$$B_1 = \bigcup_{i=0}^{m} B \cap T_i \quad \text{and} \quad B_2 = \bigcup_{i=m+1}^{n} B \cap T_i.$$

We prove the result by contradiction. So suppose that $n > m$. As T_i is thick and T_i^0 is dense in T_i, $B \cap T_i$ contains an open subset of $\mathbb{X}$ for every $0 \leq i \leq n$. Thus both B_1 and B_2 contain an open subset of $\mathbb{X}$. As the sets T_i° are disjoint for $0 \leq i \leq n$, $B_1 \not\subseteq B_2$, nor the inverse. Clearly B_1 and B_2 are closed and hence, by Lemma A.1.2, $dim(B_1 \cap B_2) \geq n-1$. As $B_1 \cap B_2 \subseteq \bigcup_{0\leq i\leq m, m+1\leq j\leq n} (T_i \cap T_j)$, $dim(B_1 \cap B_2) = n-1$ and hence there exists $0 \leq i \leq m$ and $m+1 \leq j \leq n$ such that $T_i \cap T_j$ is of codimension 1. Thus it is a side containing E by Proposition 2.3.6. Denote this side by S^*. By the definition of an edge loop, $S_i = T_{i-1} \cap T_i$ and $S_{i+1} = T_i \cap T_{i+1}$ are two different sides contained in T_i and containing E (indices are interpreted modulo m.) Moreover $T_{i-1} \neq T_j$ and $T_{i+1} \neq T_j$ and hence E is contained in three different sides, which contradicts Lemma 2.3.9. ∎

2.4 Group Presentations

Throughout this section Γ is a discontinuous subgroup of the group of isometries of $\mathbb{X}$. Using that $\mathbb{X}$ is a proper metric space it is easy to see that every discontinuous subgroup of $\text{Iso}(\mathbb{X})$ is countable. The action of Γ on $\mathbb{X}$ induces a dimension preserving action on the set of subspaces of $\mathbb{X}$. In particular, Γ acts on the set of polyhedra of $\mathbb{X}$.

We show in Theorem 2.4.7 that the presentation of a group may be established based on the tessellation given by a fundamental polyhedron. The main idea of the generating part is the following. Given a non-trivial element $\gamma \in \Gamma$, one considers a path from a point inside the fundamental polyhedron P for Γ to a point inside $\gamma(P)$. The path can be chosen such that it intersects only intersections of images of P of codimension 1. Every such intersection corresponds to a (side-)pairing transformation (see Section 2.1) of Γ and it may then be shown that γ can be written as a product of those pairing transformations. Theorem 2.4.7 also gives the relations between the different generators. We will give more details about this later.

Throughout this section

P is a fundamental polyhedron for Γ (as defined in Definition 1.2.11) and $\mathcal{T} = \mathcal{T}_P$.

When we refer to cells, tiles, sides or edges it is always with respect to $\mathcal{T}$. Since every cell is contained in only finitely many tiles, the stabilizer of one cell is finite.

Definition 2.4.1. *If S is a side of P then, by Proposition 2.3.6, there is a unique $\gamma \in \Gamma \setminus \{1\}$ such that $S = P \cap \gamma(P)$ and $S^r \cap \tau(P) = \emptyset$ for every $\tau \in \Gamma \setminus \{1, \gamma\}$. We denote this γ as γ_S. It is called a* (side-)pairing transformation.

As already mentioned in Section 2.1,

$$S = P \cap \gamma_S(P).$$

If γ is a pairing transformation then $P \cap \gamma^{-1}(P) = \gamma^{-1}(P \cap \gamma(P))$ also is a side of P and hence γ^{-1} is a pairing transformation as well. In this case one denotes

$$S_\gamma = P \cap \gamma(P)$$

and one says then that S_γ and $S_{\gamma^{-1}}$ are paired sides. If S is a side then the side paired with S is denoted S^*. So,

$$S^* = \gamma_S^{-1}(S).$$

If E is an edge of P then, by Proposition 2.3.9, it is contained in exactly two sides, say S_γ and S_{γ_1}. Because Γ permutes edges, $\gamma^{-1}(E)$ is an edge of the tessellation and it is contained in $S_{\gamma^{-1}}$. Therefore $\gamma^{-1}(E)$ and $\gamma_1^{-1}(E)$ are edges of P.

Some relations amongst the side-pairing transformations can be deduced. A first type of relations is easily obtained.

Definition 2.4.2. *If S_1 and S_2 are two paired sides then $\gamma_{S_1} = \gamma_{S_2}^{-1}$. Such a relation is called a pairing relation. In case S is a side paired with itself then the pairing relation takes the form $\gamma_S^2 = 1$ and such a relation usually is called a reflection relation.*

To define the second type of relations, we introduce the following definition.

Definition 2.4.3. *A* loop *of Γ with respect to P (or simply a loop of Γ, if the polyhedron is clear from the context) is a finite ordered list $(\gamma_0, \gamma_1, \dots, \gamma_n)$ of elements of Γ such that $\gamma_0 = \gamma_n$ and $\gamma_{i-1}(P) \cap \gamma_i(P)$ is a side for each $1 \le i \le n$ (equivalently, each $\gamma_{i-1}^{-1}\gamma_i$ is a pairing transformation).*

Example 2.4.4.

1. *If $\gamma, \tau \in \Gamma$ and $\gamma^{-1}\tau$ is a pairing transformation then (γ, τ, γ) is a loop of Γ.*
2. *If $(T_0 = \gamma_0(P), T_1 = \gamma_1(P), \dots, T_m = \gamma_m(P))$ is an edge loop then $(\gamma_0, \gamma_1, \dots, \gamma_m)$ is a loop of Γ.*
3. *If $\gamma_{S_1} \cdots \gamma_{S_m} = 1$ for sides $S_1, \dots, S_n$ of P then*

$$(1, \gamma_{S_1}, \gamma_{S_1}\gamma_{S_2}, \dots, \gamma_{S_1} \cdots \gamma_{S_{n-1}}, \gamma_{S_1} \cdots \gamma_{S_m} = 1)$$

is a loop. Conversely, if $(\gamma_0, \gamma_1, \dots, \gamma_m)$ is a loop of Γ then $S_i = P \cap \gamma_{i-1}^{-1}\gamma_i(P)$ is a side of P and $\gamma_{i-1}^{-1}\gamma_i = \gamma_{S_i}$ for every $i \in \{1, \dots, m\}$. Furthermore, $\gamma_{S_1} \cdots \gamma_{S_m} = 1$.

By Example 2.4.4.(3), relations amongst pairing transformations are completely determined by loops. In case a loop $(\gamma_0, \dots, \gamma_m)$ is determined by an edge loop, as in Example 2.4.4.(2), then the resulting relation $\gamma_{S_1} \cdots \gamma_{S_m} = 1$ is called an *edge loop relation.* Note that the pairing relations are the relations corresponding to the loops (γ, τ, γ) for $\gamma^{-1}\tau$ a pairing transformation.

We give an alternative interpretation of the edge loop relations. Let E be an edge of P and choose one of the two sides S of P containing E. Recursively one obtains a sequence $E_1, E_2, \dots$ of edges of P and a sequence of sides $S_1, S_2, \dots$, with $E_i \subseteq S_i$ for each i, and which is uniquely determined by the following rules:

$$\begin{array}{l} E_1 = E, \quad S_1 = S, \quad E_{n+1} = \gamma_{S_n}^{-1}(E_n) \quad \text{and} \\ S_{n+1} \text{ and } S_n^* = \gamma_{S_n}^{-1}(S_n) \text{ are the two sides of } P \text{ containing } E_{n+1}. \end{array} \tag{2.3}$$

Let $\gamma_n = \gamma_{S_1} \cdots \gamma_{S_n}$ for every $n \geq 0$ (in particular, we agree that $\gamma_0 = 1$). Observe that $\gamma_n^{-1}(E) = E_{n+1} \subseteq P$. In particular $E \subseteq \gamma_n(P)$ and hence E is contained in the tiles

$$T_0 = \gamma_0(P) = P, T_1 = \gamma_1(P), T_2 = \gamma_2(P), \dots$$

and $T_{n-1} \cap T_n$ is a side of $\mathcal{T}$ for every $n \geq 1$. Moreover $\gamma_n^{-1}(T_{n-1} \cap T_n) = S_n^* \neq S_{n+1} = \gamma_n^{-1}(T_n \cap T_{n+1})$. Therefore $T_{n-1} \cap T_n$ and $T_n \cap T_{n+1}$ are the two sides of T_n containing E. This proves that $T_0 = P, T_1, T_2, \dots$ is a sequence of tiles as defined in (2.1). We know this is a periodic sequence and if it has period m then $(T_0, T_1, \dots, T_m)$ is the edge loop defined by E,S and P and $(\gamma_0 = 1, \gamma_1, \dots, \gamma_m)$ is the loop of Γ associated to this edge loop. As T_i determines γ_i, the sequence $\gamma_0, \gamma_1, \dots$ also is periodic of period m. As $E_i = \gamma_i^{-1}(E)$ and $S_i = P \cap \gamma_{i-1}^{-1}\gamma_i(P)$, the sequence of pairs (E_i, S_i) also is periodic, say of period k and let $t = \frac{m}{k}$. Then t is a positive integer and the edge loop relation associated to the loop $(\gamma_0, \gamma_1, \dots, \gamma_m)$ takes the form $1 = \gamma_m = \gamma_{kt} = (\gamma_{S_1} \cdots \gamma_{S_k})^t$.

Definition 2.4.5. *If E is an edge contained in a side S, then the associated relation $(\gamma_{S_1} \cdots \gamma_{S_k})^t = 1$, constructed as above, is called a cycle relation.*

This is the second type of relations we need for the Poincaré result. Observe that cycle relation and edge loop relation are synonymous concepts.

Since m is the minimum integer so that $\gamma_m = \gamma_0 = 1$ and $\gamma_{kt} = (\gamma_{S_1} \cdots \gamma_{S_k})^t$, we deduce that t is the order of $\gamma_{S_1} \cdots \gamma_{S_k}$. An alternative way to see that $\gamma_{S_1} \dots \gamma_{S_k}$ has finite order is by observing that it stabilizes the edge E and the stabilizer of every cell is finite.

Some of the cycle relations are redundant. For example, if S and R are the two sides of P containing the edge E and if $(T_0 = P, T_1, T_2, \dots, T_{m-1}, T_m = P)$ is the edge loop obtained by applying the above procedure to E and S then the edge loop obtained by applying the procedure to E and R is $(T_0 = P, T_{m-1}, \dots, T_2, T_1, T_0 = P)$. These two loops give rise to equivalent cycle relations: $(\gamma_{S_1} \dots \gamma_{S_k})^t = 1$ and $(\gamma_{S_k}^{-1} \dots \gamma_{S_1}^{-1})^t = 1$. This is because if the period of the list (E_i, S_i) obtained from E and S is k then $\gamma_{S_{k-1}}^{-1}(S_{k-1})' = R$ and hence the list of pairs of edges and sides starting with (E, R) is (E, R), (E_{k-1}, S'_{k-1}), $\dots$, (E_1, S'_1), $\dots$. On the other hand, if we replace E by one of the edges E_i then the sequence of pairs of edges and sides obtained is a shift of the list obtained with E and S or R. Then, the cycle relation obtained with E_i is a conjugate of the cycle relation associated with E.

The edges in the list $E_1, \dots, E_k$ form a *cycle of edges* of P. Clearly, the non-equivalent cycles of edges of P define a partition of the edges of P.

Example 2.4.6. *Let $n \geq 3$ and let D_{2n} be the group of isometries of a regular polygon of the Euclidean plane with n sides. Then, the acute wedge P between the two half-lines $S_1 = \{(x, 0) \mid x > 0\}$ and $S_2 = \{(x, x\tan(\frac{\pi}{n}) \mid x > 0\}$ is a fundamental polyhedron for D_{2n}. Let γ_i be the reflection in the line containing S_i. Then $S_i = P \cap \gamma_i(P)$ and $\gamma_i^2 = 1$. So S_1 and S_2 are the two sides of P, as fundamental polyhedron for Γ and the pairing relations are the reflection relations $\gamma_{S_1}^2 = \gamma_{S_2}^2 = 1$. The only edge is the vertex consisting of the single point $(0, 0)$. The sequence of edges and sides starting with E*

and S_1 is periodic of period 2. Clearly the sequence $\gamma_0 = 1, \gamma_1 = \gamma_{S_1}, \gamma_2 = \gamma_{S_1}\gamma_{S_2}, \ldots$ is periodic of period $2n$. Hence $t = \frac{2n}{2} = n$ and the cycle relation is $(\gamma_{S_1}\gamma_{S_2})^n = 1$. This is also easy to see because $\gamma_{S_1}\gamma_{S_2}$ is the rotation around $(0,0)$ of angle $\frac{2\pi}{n}$. Therefore it has finite order and its order is indeed n.

We are ready to state Poincaré's Theorem on presentations of discontinuous groups.

Theorem 2.4.7 (Poincaré)**.** *Let $\mathbb{X}$ be either a Euclidean, hyperbolic or spherical space. Let P be a fundamental polyhedron for a discontinuous group of isometries on the space $\mathbb{X}$. The pairing transformations generate Γ, that is*

$$\Gamma = \langle \gamma \in \Gamma \mid P \cap \gamma(P) \text{ is a side of } P \rangle,$$

and the pairing and cycle relations form a complete set of relations for Γ.

We first prove that the pairing transformations generate Γ. The proof that the pairing and cycle relations form a complete set of relations requires much more work and will be postponed until the end of the section.

Proof of Theorem 2.4.7 (Generators). Let $\mathcal{T}$ be the tessellation of $\mathbb{X}$ formed by the polyhedra $\gamma(P)$ with $\gamma \in \Gamma$. As Γ is countable, $\mathcal{T}$ is countable and, as every non-empty cell is the intersection of finitely many tiles (those containing a relative interior point of the cell), the number of cells of $\mathcal{T}$ is countable. Let Ω be the complement in $\mathbb{X}$ of the union of the cells of codimension at least 2. Then, by Lemma 2.2.9, Ω is path-connected.

Let $\sigma \in \Gamma$ and let $x \in P^\circ$. So $\sigma(x) \in \sigma(P)^\circ$ and $x, \sigma(x) \in \Omega$. Because Ω is path-connected, there exists a continuous function $\alpha : [0,1] \to \Omega$ with $\alpha(0) = x$ and $\alpha(1) = \sigma(x)$. Let $A = \{\gamma \in \Gamma \mid \alpha([0,1]) \cap \gamma(P) \neq \emptyset\}$. As $\alpha([0,1])$ is compact and as P is locally finite, A is finite. We define recursively a sequence of subsets of A by setting $A_0 = \{1\}$ and if $i \geq 1$ then

$$A_i = \{\tau \in A \mid \tau(P) \cap \gamma(P) \text{ is a side for some } \gamma \in A_{i-1} \setminus \bigcup_{j<i-1} A_j\}.$$

Let $B = \bigcup_{i \geq 0} A_i$. We claim that $B = A$. Otherwise let $a = \min\{t \in [0,1] \mid \alpha(t) \in \gamma(P)$ for some $\gamma \in A \setminus B\}$. This minimum exists because $A \setminus B$ is non-empty and $\bigcup_{\gamma \in A \setminus B} \gamma(P)$ is closed. Moreover $a > 0$ because $\alpha(0) = x \in P^\circ$ and $1 \in B$, so that $x \notin \gamma(P)$ for each $\gamma \in A \setminus B$. Then $\alpha([0,a)) \subseteq \bigcup_{\gamma \in B} \gamma(P)$ and as this union is closed, $\alpha(a) \in \gamma(P) \cap \tau(P)$ for some $\gamma \in B$ and $\tau \in A \setminus B$. As $\alpha(a) \in \Omega$, $\gamma(P) \cap \tau(P)$ has dimension 3 and hence it is a side. This contradicts the definition of B. Hence $A = B$ and in particular $\sigma \in B$. By using the sets A_i, we create a sequence $\gamma_0 = 1, \gamma_1, \ldots, \gamma_k = \sigma$ such that for every $1 \leq j$, $\gamma_{j-1}(\mathcal{F}) \cap \gamma_j(\mathcal{F})$ is a side.

So, there exists a continuous function $\alpha : [0,1] \to \Omega$ and a sequence $0 = a_0 < a_1 < \ldots < a_k = 1$ such that $\alpha(a_0) = x$, $\alpha(a_k) = \sigma(x)$ and $\alpha([a_{i-1}, a_i]) \subseteq T_i$ with T_i tile of $\mathcal{T}$ for each $1 \leq i \leq k$. Clearly $T_0 = P$. Write $T_i = \gamma_i(P)$ with $\gamma_i \in \Gamma$. Then, $\gamma_1 = 1$, $\gamma_k = \sigma$ and $\alpha(a_i) \in \gamma_i(P) \cap \gamma_{i+1}(P)$ for every $i \in \{1, 2, \ldots, k-1\}$. Since each $\gamma_i(P) \cap \gamma_{i+1}(P)$ is a side of $\gamma_i(P)$, $P \cap \gamma_i^{-1}\gamma_{i+1}(P)$ is a side of P. Hence $\tau_i = \gamma_i^{-1}\gamma_{i+1}$ is a pairing transformation for each $1 \leq i < k$. Finally $\sigma = \gamma_k = (\gamma_1^{-1}\gamma_2)(\gamma_2^{-1}\gamma_3)\cdots(\gamma_{k-1}^{-1}\gamma_k) = \tau_1 \cdots \tau_{k-1}$. Hence, σ belongs to the subgroup of Γ generated by the pairing transformations. Because σ is an arbitrary element of Γ, the result follows. ∎

Remark 2.4.8. *Note that the previous proof can be simplified by using [Ale96]. Indeed at the end of the proof of [Ale96, Theorem 1], the author proves that "every segment in $\mathbb{R}^n$ can be covered by a sequence of polyhedra in which every two consecutive polyhedra are adjacent at an $(n-1)$-dimensional face". Using this, one can then easily construct a sequence $0 = a_0 < a_1 < \ldots < a_k = 1$ such that $\alpha(a_0) = x$, $\alpha(a_k) = \sigma(x)$, $\alpha([a_{i-1}, a_i]) \subseteq T_i$ with T_i tile of $\mathcal{T}$ and such that $T_i \cap T_{i+1}$ is a side of T_i for each $1 \leq i \leq k$. The proof then finishes in the same way as above. However, since our goal is to make the proof self-contained, and avoid ambiguities, the proof given above seemed best suited.*

Let Δ denote the group given by the presentation of Theorem 2.4.7. More precisely,

$$\Delta = F/N,$$

where F is the free group with basis the symbols $[\gamma]$, one for each pairing transformation γ, and N is the normal closure of the subgroup of F generated by the set X consisting of the pairing and cycle relations, i.e. X is formed by the products $[\gamma_S][\gamma_{S^*}]$ with S and S^* paired sides of P (pairing relations) and the elements of the form $([\gamma_{S_1}]\ldots[\gamma_{S_k}])^t$, where $(E_1 = E, S_1 = S, E_2, S_2, \ldots)$ is the list defined by (2.3) for an edge E and a side S containing E, k is the period of the list and t is the order of $\gamma_{S_1}\ldots\gamma_{S_k}$ (cycle relations).

It is clear that the function

$$\begin{array}{rcl} \varphi : \Delta & \rightarrow & \Gamma \\ {[\gamma]} & \mapsto & \gamma \end{array}$$

is surjective. So, in order to prove the relation part of Theorem 2.4.7, we just have to prove the injectivity of φ, i.e. we have to prove that if $\gamma_1\gamma_2\ldots\gamma_n = 1$ for some $\gamma_i \in \Gamma$, then $[\gamma_1][\gamma_2]\ldots[\gamma_n] = 1$. The main idea therefore is to link products in Δ with loops in the space on which Γ is acting. We then show that if $\gamma_1\gamma_2\ldots\gamma_n = 1$, then the loop associated to $[\gamma_1][\gamma_2]\ldots[\gamma_n]$ is homotopic to the trivial loop, i.e. a point, and hence its value is 1.

Recall that the edge loop relations and the cycle relations are synonymous concepts, so we may replace the cycle relations by the edge loop relations

$$[\gamma_0^{-1}\gamma_1][\gamma_1^{-1}\gamma_2]\cdots[\gamma_{n-1}^{-1}\gamma_n],$$

where $(\gamma_0, \gamma_1, \ldots, \gamma_{n-1}, \gamma_n)$ is an edge loop. Abusing notation, we will consider the symbols $[\gamma]$, with γ a pairing transformation, as elements of Δ. Hence,

$$[\gamma_S][\gamma_{S^*}] = 1, \text{ for every side } S \text{ of } P$$

(so $[\gamma_S]^{-1} = [\gamma_S^{-1}]$) and

$$[\gamma_0^{-1}\gamma_1][\gamma_1^{-1}\gamma_2]\cdots[\gamma_{n-1}^{-1}\gamma_n] = 1, \text{ for every edge loop } (\gamma_0, \gamma_1, \ldots, \gamma_n) \text{ of } P.$$

Let $\gamma, \tau \in \Gamma$ and let C be a cell of $\mathcal{T}$ of codimension m at most 2 that is contained in $\gamma(P) \cap \tau(P)$. We define $\kappa_C(\gamma, \tau) \in \Delta$ as follows.

- If $m = 0$ then $\kappa_C(\gamma, \tau) = 1$.
- If $m = 1$ then $\kappa_C(\gamma, \tau) = [\gamma^{-1}\tau]$.
- If $m = 2$ then C is an edge contained in $\gamma(P) \cap \tau(P)$ and thus, by Lemma 2.3.10, γ and τ belong to the edge loop of C. Up to a cyclic permutation, we can write the edge loop of C as $(\gamma = \sigma_0, \ldots, \sigma_t = \tau, \sigma_{t+1}, \ldots, \sigma_m = \gamma)$ (or the equivalent edge loop $(\gamma = \sigma_m, \sigma_{m-1}, \ldots, \sigma_t = \tau, \sigma_{t-1}, \ldots, \sigma_1, \sigma_0 = \gamma)$) and we set
$$\kappa_C(\gamma, \tau) = [\sigma_0^{-1}\sigma_1][\sigma_1^{-1}\sigma_2] \cdots [\sigma_{t-1}^{-1}\sigma_t] = [\sigma_m^{-1}\sigma_{m-1}] \cdots [\sigma_{t+1}^{-1}\sigma_t].$$

Observe that $\kappa_C(\gamma, \gamma) = 1$ in the three cases.

Lemma 2.4.9. *Let $\gamma, \tau \in \Gamma$ and let C be a cell of $\mathcal{T}$ of codimension $m \leq 2$ that is contained in $\gamma(P) \cap \tau(P)$. The following properties hold.*

1. *$\kappa_C(\gamma, \tau) = \kappa_C(\tau, \gamma)^{-1}$.*
2. *If D is cell of $\mathcal{T}$ contained in C and of codimension at most 2 then $\kappa_D(\gamma, \tau) = \kappa_C(\gamma, \tau)$.*
3. *If $\gamma_1, \ldots, \gamma_n \in \Gamma$ and $C \subseteq \bigcap_{i=1}^n \gamma_i(P)$ then $\kappa_C(\gamma_1, \gamma_n) = \kappa_C(\gamma_1, \gamma_2) \ldots \kappa_C(\gamma_{n-1}, \gamma_n)$.*

Proof. (1) If $m = 0$, then $\gamma = \tau$ and there is nothing to prove. If $m = 1$ then $\gamma^{-1}\tau = \gamma_{S_{\gamma^{-1}\tau}}$ and $\tau^{-1}\gamma = \gamma_{S_{\tau^{-1}\gamma}} = \gamma_{S'_{\gamma^{-1}\tau}}$ and hence $\kappa_C(\gamma, \tau)\kappa_C(\tau, \gamma) = [\gamma_{S_{\gamma^{-1}\tau}}][\gamma_{S^*_{\gamma^{-1}\tau}}] = 1$, a pairing relation. Finally if $m = 2$, then we can write the edge loop of C as $(\gamma = \sigma_0, \ldots, \sigma_t = \tau, \sigma_{t+1}, \ldots, \sigma_m = \gamma)$ and thus
$$\begin{aligned} \kappa_C(\gamma, \tau) &= [\sigma_0^{-1}\sigma_1][\sigma_1^{-1}\sigma_2] \cdots [\sigma_{t-1}^{-1}\sigma_t], \\ \kappa_C(\tau, \gamma) &= [\sigma_t^{-1}\sigma_{t+1}][\sigma_{t+1}^{-1}\sigma_{t+2}] \cdots [\sigma_{m-1}^{-1}\sigma_m]. \end{aligned}$$
It is now easy to see that $\kappa_C(\gamma, \tau)\kappa_C(\tau, \gamma) = 1$ and hence the result follows.

(2) If $C = D$ then there is nothing to prove. So assume that $C \neq D$. If C is a side then D is an edge and γ and τ are two consecutive elements of the edge loop of D. Then $\kappa_D(\gamma, \tau) = [\gamma^{-1}\tau] = \kappa_C(\gamma, \tau)$. Otherwise, C is a tile and hence $\gamma = \tau$. Thus $\tau_D(\gamma, \tau) = 1 = \tau_C(\gamma, \tau)$.

(3) By induction it is enough to prove the statement for $n = 3$. So assume $n = 3$. If either $\gamma_1 = \gamma_2$ or $\gamma_2 = \gamma_3$ then the desired equality is obvious. So assume that $\gamma_1 \neq \gamma_2$ and $\gamma_2 \neq \gamma_3$. If C is an edge then, up to a cyclic permutation, possibly reversing the order and making use of Lemma 2.3.10, the edge loop of C is of the form $(\gamma_1 = \sigma_0, \ldots, \gamma_2 = \sigma_t, \ldots, \gamma_3 = \sigma_l, \ldots, \sigma_m)$. Then,
$$\begin{aligned} \kappa_C(\gamma_1, \gamma_3) &= [\sigma_0^{-1}\sigma_1][\sigma_1^{-1}\sigma_2] \cdots [\sigma_{l-1}^{-1}\sigma_l] \\ &= ([\sigma_0^{-1}\sigma_1][\sigma_1^{-1}\sigma_2] \cdots [\sigma_{t-1}^{-1}\sigma_t])\,([\sigma_t^{-1}\sigma_{t+1}] \cdots [\sigma_{l-1}^{-1}\sigma_l]) \\ &= \kappa_C(\gamma_1, \gamma_2)\kappa_C(\gamma_2, \gamma_3) \end{aligned}$$

Otherwise, $S=\gamma_1^{-1}(C)$ is a side of P, $\gamma_S=\gamma_1^{-1}\gamma_2$, $\gamma_{S^*}=\gamma_2^{-1}\gamma_1$ and $\gamma_1=\gamma_3$. Then,

$$\kappa_C(\gamma_1,\gamma_3)=1=[\gamma_S][\gamma_{S^*}]=\kappa_C(\gamma_1,\gamma_2)\kappa_C(\gamma_2,\gamma_3).$$

∎

We denote by $\mathbb{X}_{\mathcal{T}}$ the complement in $\mathbb{X}$ of the union of the cells of $\mathcal{T}$ of codimension at least three. By Lemma 2.3.4, every element of $\mathbb{X}_{\mathcal{T}}$ is either in the interior of a tile or in the relative interior of a side or an edge. The first ones are those that belong to exactly one tile, the elements of the relative interior of one side belong to exactly two tiles (Lemma 2.3.5) and the remaining elements belong to at least three tiles.

If $x\in\mathbb{X}_{\mathcal{T}}$, $\gamma,\tau\in\Gamma$ and $x\in C\subseteq\gamma(P)\cap\tau(P)$ for some cell C then the codimension of C is at most 2 and we define

$$\kappa_x(\gamma,\tau)=\kappa_C(\gamma,\tau). \tag{2.4}$$

This is well defined because if D is another cell containing x and contained in $\gamma(P)\cap\tau(P)$ with $\kappa_C(\gamma,\tau)\neq\kappa_D(\gamma,\tau)$ then $\gamma\neq\tau$ and $C\neq D$. Hence neither C nor D is a tile and either C or D is a side. Therefore, $\gamma(P)\cap\tau(P)$ is a side and hence, by Lemma 2.4.9.(3), $\kappa_C(\gamma,\tau)=[\gamma^{-1}\tau]=\kappa_D(\gamma,\tau)$, a contradiction. This proves that indeed $\kappa_x(\gamma,\tau)$ is well defined. By Lemma 2.4.9 we have $\kappa_x(\gamma,\tau)=\kappa_x(\tau,\gamma)^{-1}$ and if $x\in\cap_{i=1}^n\gamma_i(P)$ with $\gamma_1,\dots,\gamma_n\in\Gamma$ then

$$\kappa_x(\gamma_1,\gamma_n)=\kappa_x(\gamma_1,\gamma_2)\cdots\kappa_x(\gamma_{n-1},\gamma_n) \tag{2.5}$$

Lemma 2.4.10. *Let $\alpha:[0,1]\to\mathbb{X}_{\mathcal{T}}$ be a continuous function and let $0\le a<b<c\le 1$ and $\gamma,\tau,\sigma\in\Gamma$ be such that $\alpha(a)\in\gamma(P)$, $\alpha((a,c))\subseteq C^r$ for a cell C of $\sigma(P)$ and $\alpha((a,b))\subseteq D^r$ for a cell D of $\tau(P)$. Then*

$$\kappa_{\alpha(a)}(\gamma,\sigma)=\kappa_{\alpha(a)}(\gamma,\tau)\;\kappa_{\alpha(b)}(\tau,\sigma).$$

Proof. First of all, observe that $\alpha(a),\alpha(b)\in C\cap D$, $\alpha(a)\in\gamma(P)\cap\sigma(P)\cap\tau(P)$, $\alpha(b)\in\tau(P)\cap\sigma(P)$, $C\subseteq\tau(P)$ and $D\subseteq\sigma(P)$, because α is continuous and every cell is the closure of its relative interior. The desired equality is clear if $\tau=\sigma$. So, assume that $\tau\neq\sigma$. Then C and D are not tiles because they are included in $\tau(P)\cap\sigma(P)$. If $C\subseteq\gamma(P)$ then $\kappa_{\alpha(a)}(\gamma,\sigma)=\kappa_C(\gamma,\sigma)=\kappa_C(\gamma,\tau)\;\kappa_C(\tau,\sigma)=\kappa_{\alpha(a)}(\gamma,\tau)\;\kappa_{\alpha(b)}(\tau,\sigma)$, by Lemma 2.4.9.(3). Assume that $C\not\subseteq\gamma(P)$. In particular γ,τ and σ are pairwise different and $\alpha(a)\notin C^r$. Hence C is a side, because Lemma 2.3.4 implies that $\mathbb{X}_{\mathcal{T}}\cap E=E^r$ for every edge E. Again by Lemma 2.3.4, we obtain that $\alpha(a)$ belongs to the relative interior of a cell E of $\gamma(P)$ and E is properly contained in C. Therefore E is an edge and τ and σ appears consecutively in an edge loop of E, i.e. after a cyclic permutation or a reversing in the ordering, an edge loop of E takes the form $(\sigma_1=\gamma,\dots,\sigma_i=\tau,\sigma_{i+1}=\sigma,\dots,\sigma_m)$. Then, $\kappa_{\alpha(a)}(\gamma,\sigma)=\kappa_E(\gamma,\sigma)=([\sigma_1^{-1}\sigma_2]\cdots[\sigma_{i-1}^{-1}\sigma_i])\;[\sigma_i^{-1}\sigma_{i+1}]=\kappa_E(\gamma,\tau)\kappa_C(\tau,\sigma)=\kappa_{\alpha(a)}(\gamma,\tau)\kappa_{\alpha(b)}(\tau,\sigma)$, as desired. ∎

Definition 2.4.11. *Let $\alpha : [a,b] \to \mathbb{X}_{\mathcal{T}}$ be a continuous function on a compact interval $[a,b]$. An ordered list $\mathcal{L} = (a_0, \gamma_1, a_1, \gamma_2, \dots, \gamma_n, a_n)$ such that $a = a_0 < a_1 < \dots < a_n = b$ and $\alpha(a_{i-1}, a_i)$ is contained in the relative interior of a cell of $\gamma_i(P)$ for all $1 \leq i \leq n$, is called an α-*adapted list*.*

Given an α-adapted list $\mathcal{L}$, we define

$$\Phi(\mathcal{L}) = \kappa_{\alpha(a_1)}(\gamma_1, \gamma_2)\, \kappa_{\alpha(a_2)}(\gamma_2, \gamma_3) \cdots \kappa_{\alpha(a_{n-1})}(\gamma_{n-1}, \gamma_n),$$

unless $n = 1$, where we set $\Phi(\mathcal{L}) = 1$.

Observe that, if $i \neq 0, n$ then $\alpha(a_i)$ belongs to the boundaries of both $\alpha((a_{i-1}, a_i))$ and $\alpha((a_i, a_{i+1}))$. Hence, $\alpha(a_i) \in \gamma_i(P) \cap \gamma_{i+1}(P)$ and thus $\kappa_{\alpha(a_i)}(\gamma_i, \gamma_{i+1})$ is well defined.

Lemma 2.4.12. *Let $\alpha : [a,b] \to \mathbb{X}_{\mathcal{T}}$ be a continuous function such that both $\alpha(a)$ and $\alpha(b)$ belong to the interior of some tile. If $\mathcal{L}$ and $\mathcal{L}'$ are α-adapted lists then $\Phi(\mathcal{L}) = \Phi(\mathcal{L}')$.*

Proof. Without loss of generality, we may assume that $[a,b] = [0,1]$. Let $\mathcal{L} = (a_0, \gamma_1, a_1, \dots, \gamma_n, a_n)$ and $\mathcal{L}' = (a'_0, \gamma'_1, a'_1, \dots, \gamma'_m, a'_m)$. First observe that $\gamma_1(P) = \gamma'_1(P)$ is the only tile containing $\alpha(0)$. Thus $\gamma_1 = \gamma'_1$. If $n = 1$ then $\alpha([0,1)) \subseteq \gamma_1(P)^\circ$ and therefore $\gamma'_i = \gamma_1$ for every $i \in \{1, \dots, m\}$. Thus $\Phi(\mathcal{L}) = 1 = \Phi(\mathcal{L}')$. Similarly, if $m = 1$ then $\Phi(\mathcal{L}) = 1 = \Phi(\mathcal{L}')$. In the remainder of the proof we assume that $n, m > 1$.

We construct an α-adapted list $\mathcal{D}'_\alpha$ containing $\mathcal{L}'$ and all the a_i's as follows. For every $1 \leq i \leq i' < n$ and $j \in \{1, \dots, m-1\}$ such that $a_{i-1} \leq a'_{j-1} < a_i \leq a_{i'} < a'_j \leq a_{i'+1}$, we insert in $\mathcal{L}'$ the sublist $(\gamma_i, a_i, \dots, \gamma_{i'}, a_{i'})$ between a'_{j-1} and γ'_j. Similarly, we construct another list $\mathcal{D}_\alpha$ containing $\mathcal{L}$ and all the a'_i's. We can consider the transition from $\mathcal{L}'$ to $\mathcal{D}'_\alpha$ (or from $\mathcal{L}$ to $\mathcal{D}_\alpha$) as the result of inserting finitely many pairs (γ_i, a_i). On the other hand, we can consider the transition from $\mathcal{D}_\alpha$ to $\mathcal{D}'_\alpha$ as the result of replacing finitely many group elements. Therefore, to prove the lemma, it is enough to deal with the following two cases: (1) $\mathcal{L}'$ is obtained by inserting in $\mathcal{L}$ one pair (τ, b) between a_{i-1} and γ_i for $a_{i-1} < b < a_i$ and $\tau \in \Gamma$; (2) $\mathcal{L}'$ is obtained by replacing in $\mathcal{L}$ one γ_i by τ, and in both cases $\alpha((a_{i-1}, a_i))$ is contained in the relative interior of a cell contained in $\gamma_i(P) \cap \tau(P)$ and $\tau \neq \gamma_i$.

(1) Assume $\mathcal{L}'$ is obtained by inserting one pair (τ, b) between a_{i-1} and γ_i. We consider separately the cases when $i > 1$ or $i = 1$. If $i > 1$ then $a = a_{i-1}$, b, $c = a_i$, $\gamma = \gamma_{i-1}$, τ and $\sigma = \gamma_i$ satisfy the hypothesis of Lemma 2.4.10 and therefore

$$\begin{aligned}
\Phi(\mathcal{L}) &= \kappa_{\alpha(a_1)}(\gamma_1, \gamma_2) \cdots \kappa_{\alpha(a_{n-1})}(\gamma_{n-1}, \gamma_n) \\
&= \kappa_{\alpha(1)}(\gamma_1, \gamma_2) \cdots \kappa_{\alpha(a_{i-2})}(\gamma_{i-2}, \gamma_{i-1}) \kappa_{\alpha(a_{i-1})}(\gamma_{i-1}, \tau) \kappa_{\alpha(b)}(\tau, \gamma_i) \\
&\quad\ \kappa_{\alpha(a_i)}(\gamma_i, \gamma_{i+1}) \dots \kappa_{\alpha(a_{n-1})}(\gamma_{n-1}, \gamma_n) \\
&= \Phi(\mathcal{L}').
\end{aligned}$$

If $i = 1$ then $\tau = \gamma_1$ because $\alpha(a_0)$ is in the relative interior of a tile. Thus $\kappa_{\alpha(b)}(\tau, \gamma_1) = 1$

and hence

$$\begin{aligned}\Phi(\mathcal{L}) &= \kappa_{\alpha(a_1)}(\gamma_1,\gamma_2)\cdots\kappa_{\alpha(a_{n-1})}(\gamma_{n-1},\gamma_n)\\ &= \kappa_{\alpha(b)}(\tau,\gamma_1)\kappa_{\alpha(a_1)}(\gamma_1,\gamma_2)\cdots\kappa_{\alpha(a_{n-1})}(\gamma_{n-1},\gamma_n)\\ &= \Phi(\mathcal{L}').\end{aligned}$$

(2) Assume $\mathcal{L}'$ is obtained by replacing γ_i with $\tau \neq \gamma_i$ in $\mathcal{L}$. By definition, $\alpha((a_{i-1}, a_i)) \subseteq E^r$ for some cell E contained in $\gamma_i(P) \cap \tau(P)$. Since $\gamma_i \neq \tau$, clearly E is either an edge or a side. As both $\alpha(0)$ and $\alpha(1)$ belong to the interior of tiles we have $i \neq 1, n$. To prove $\Phi(\mathcal{L}) = \Phi(\mathcal{L}')$, it is enough to show

$$\kappa_{\alpha(a_{i-1})}(\gamma_{i-1},\tau)\kappa_{\alpha(a_i)}(\tau,\gamma_{i+1}) = \kappa_{\alpha(a_{i-1})}(\gamma_{i-1},\gamma_i)\kappa_{\alpha(a_i)}(\gamma_i,\gamma_{i+1}).$$

Let D_1 and D_2 be the cells generated by $\alpha(a_{i-1})$ and $\alpha(a_i)$ respectively. Then $D_1, D_2 \subseteq E$. If E is an edge then $D_1 = D_2 = E$ and hence

$$\begin{aligned}\kappa_{\alpha(a_{i-1})}(\gamma_{i-1},\tau)\kappa_{\alpha(a_i)}(\tau,\gamma_{i+1}) &= \kappa_E(\gamma_{i-1},\tau)\kappa_E(\tau,\gamma_{i+1})\\ &= \kappa_E(\gamma_{i-1},\gamma_{i+1})\\ &= \kappa_E(\gamma_{i-1},\gamma_i)\kappa_E(\gamma_i,\gamma_{i+1})\\ &= \kappa_{\alpha(a_{i-1})}(\gamma_{i-1},\gamma_i)\kappa_{\alpha(a_i)}(\gamma_i,\gamma_{i+1}),\end{aligned}$$

by Lemma 2.4.9.(3). Otherwise, E is a side. If D_1 is a side then $D_1 = E$ and $\gamma_{i-1} \in \{\gamma_i, \tau\}$. Thus

$$\begin{aligned}\kappa_{\alpha(a_{i-1})}(\gamma_{i-1},\tau)\kappa_{\alpha(a_i)}(\tau,\gamma_{i+1}) &= \kappa_E(\gamma_{i-1},\tau)\kappa_{D_2}(\tau,\gamma_{i+1})\\ &= \kappa_{D_2}(\gamma_{i-1},\tau)\kappa_{D_2}(\tau,\gamma_{i+1})\\ &= \kappa_{D_2}(\gamma_{i-1},\gamma_{i+1})\\ &= \kappa_{D_2}(\gamma_{i-1},\gamma_i)\kappa_{D_2}(\gamma_i,\gamma_{i+1})\\ &= \kappa_E(\gamma_{i-1},\gamma_i)\kappa_{D_2}(\gamma_i,\gamma_{i+1})\\ &= \kappa_{\alpha(a_{i-1})}(\gamma_{i-1},\gamma_i)\kappa_{\alpha(a_i)}(\gamma_i,\gamma_{i+1}),\end{aligned}$$

by statements (2) and (3) of Lemma 2.4.9. The case where $E = D_2$ is proved similarly. Finally, assume that E is a side and $D_1 \neq E$ and $E \neq D_2$. Then D_1 and D_2 are edges and γ_i and τ are consecutive members of the edge loops of D_1 and D_2. After some reordering the edge loop of D_1 takes the form $(\sigma_1 = \gamma_{i-1}, \sigma_2, \ldots, \sigma_t = \gamma_i, \sigma_{t+1} = \tau, \ldots, \sigma_m)$ and the edge loop of D_2 takes the form $(\tau_1 = \gamma_i, \tau_2 = \tau, \ldots, \tau_l = \gamma_{i+1}, \ldots, \tau_n)$. Then $[\sigma_t^{-1}\sigma_{t+1}] = [\gamma_i^{-1}\tau] = [\tau_1^{-1}\tau_2]$ and

$$\begin{aligned}&\kappa_{\alpha(a_{i-1})}(\gamma_{i-1},\tau)\ \kappa_{\alpha(a_i)}(\tau,\gamma_{i+1})\\ &\qquad = ([\sigma_1^{-1}\sigma_2]\ldots[\sigma_{t-1}^{-1}\sigma_t][\sigma_t^{-1}\sigma_{t+1}])\ ([\tau_2^{-1}\tau_3]\ldots[\tau_{l-1}^{-1}\tau_l])\\ &\qquad = ([\sigma_1^{-1}\sigma_2]\ldots[\sigma_{t-1}^{-1}\sigma_t])\ ([\tau_1^{-1}\tau_2][\tau_2^{-1}\tau_3]\ldots[\tau_{l-1}^{-1}\tau_l])\\ &\qquad = \kappa_{\alpha(a_{i-1})}(\gamma_{i-1},\gamma_i)\ \kappa_{\alpha(a_i)}(\gamma_i,\gamma_{i+1}).\end{aligned}$$

This finishes the proof. ∎

To finish the proof of Theorem 2.4.7, Lemma 2.4.16 will be crucial. To prove this lemma in a smooth way, we need to construct piecewise geodesic simple paths. Therefore we introduce the definition in the following paragraph along with Lemma 2.4.14 and Lemma 2.4.15. This argumentation could probably be simplified by applying typical generic position/transversality arguments. However for the sake of completeness we prefer to give a detailed self-contained proof of such a construction.

Definition 2.4.13. *A parametrization of a geodesic segment $[x, y]$ in $\mathbb{X}$ is a surjective continuous function $\alpha : [a, b] \to [x, y]$ with $\alpha(a) = x$, $\alpha(b) = y$ and such that the map $t \to d(x, \alpha(t))$ is not decreasing (here d denotes the distance function on $\mathbb{X}$).*

For such a map, if P is a polyhedron then $\alpha^{-1}(P)$ is a closed interval (maybe empty or of 0 length). Indeed, it is closed because so is P and α is continuous. To prove that $\alpha^{-1}(P)$ is an interval, let $a \le t_0 < t_1 \le b$, with $\alpha(t_0), \alpha(t_1) \in P$ then $\alpha([t_0, t_1])$ is the geodesic segment $[\alpha(t_0), \alpha(t_1)]$ and hence it is contained in P because P is convex. Therefore $[t_0, t_1] \subseteq \alpha^{-1}(P)$. This proves that $\alpha^{-1}(P)$ indeed is an interval.

By

$$\mathcal{C}([a, b], \mathbb{X}_{\mathcal{T}})$$

we denote the set consisting of the continuous functions $\alpha : [a, b] \to \mathbb{X}_{\mathcal{T}}$ for which there is a finite ascending sequence $a = b_0 < b_1 \cdots < b_m = b$ such that, for every $i \in \{1, \ldots, m\}$, the restriction of α to $[b_{i-1}, b_i]$ is a parametrization of a geodesic segment.

Lemma 2.4.14. *If $\alpha \in \mathcal{C}([a, b], \mathbb{X}_{\mathcal{T}})$ then there is an α-adapted list.*

Proof. Without loss of generality, we may assume that $[a, b] = [0, 1]$. By restricting to the intervals $[b_{i-1}, b_i]$, one may assume without loss of generality that α is a parametrization of a geodesic segment. Then, for every cell C, $\alpha^{-1}(C)$ is a closed interval of $[0, 1]$ (maybe empty or of length 0) and the image of α intersects finitely many tiles. We claim that for every $t \in [0, 1)$ there is a cell C and an $\epsilon > 0$ such that $\alpha((t, t + \epsilon)) \subseteq C^r$ and for every $t \in (0, 1]$ there is a cell D and an $\epsilon > 0$ such that $\alpha((t - \epsilon, t)) \subseteq D^r$. By symmetry, we only prove the first statement. So, fix $t \in [0, 1)$ and assume that $T_1, \ldots, T_k$ are the only tiles containing $\alpha(t)$. For every $i \in \{1, \ldots, k\}$, let $\epsilon_i = \max\{\epsilon \ge 0 \mid \epsilon \le 1 - t,\ \alpha(t + \epsilon) \in T_i\}$. Let U be a neighbourhood of $\alpha(t)$ not intersecting any tile different from every $T_1, \ldots, T_k$. Then there is an $\epsilon > 0$ such that $\alpha([t, t + \epsilon]) \subseteq U$ and therefore $\alpha(t + \epsilon) \in T_i$ for some i. Hence, $\epsilon_i > 0$ for some $i \in \{1, \ldots, k\}$. Renumbering if necessary, we may assume that m is a positive integer such that $m \le k$ and $\epsilon_i > 0$ if and only if $i \le m$. Let $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_m\}$. Then $\alpha((t, t + \epsilon)) \subseteq T_1 \cap \cdots \cap T_m$ and $T_1, \ldots, T_m$ are the unique tiles containing an element of $\alpha((t, t + \epsilon))$. This implies that $T_1 \cap \cdots \cap T_m$ is a cell, say C, and $\alpha((t, t + \epsilon))$ is contained in the relative interior of C by Lemma 2.3.4. This proves the claim.

Let c be the supremum of all $t \in [0, 1]$ with the property that there exists an α_t-adapted list where α_t denotes the restriction of α to $[0, t]$. By the claim $c > 0$. It remains to be shown that $c = 1$. Otherwise, by the claim, there exists $\epsilon > 0$ such that $\alpha((c-\epsilon,c))$ is contained in the relative interior of a cell C and $\alpha((c,c+\epsilon))$ is contained in the relative interior of a cell D. Let $\tau,\sigma\in\Gamma$ be such that $C\subseteq\tau(P)$ and $D\subseteq\sigma(P)$. If $(a_0=0,\gamma_1,a_1,\dots,\gamma_n,a_n=c-\epsilon)$ is an $\alpha_{c-\epsilon}$-adapted list then $(a_0=0,\gamma_1,a_1,\dots,\gamma_n,c-\epsilon,\tau,c,\sigma,c+\epsilon)$ is an $\alpha_{c+\epsilon}$-adapted list, contradicting the maximality of c. ∎

Let $x,y\in\mathbb{X}$ and let

$$\mathcal{C}_{x,y}([a,b],\mathbb{X}_{\mathcal{T}})=\{\alpha\in\mathcal{C}([a,b],\mathbb{X}_{\mathcal{T}})\mid\alpha(a)=x,\alpha(b)=y\}$$

On $\mathcal{C}_{x,y}([a,b],\mathbb{X}_{\mathcal{T}})$ we consider the metric d defined as follows. If

$$\alpha,\beta\in\mathcal{C}([a,b],\mathbb{X}_{\mathcal{T}})\text{ then }d(\alpha,\beta)=\max\{d(\alpha(c),\beta(c))\mid c\in[a,b]\}.\tag{2.6}$$

Assume also that both x and y belong to the interior of some tile of $\mathcal{T}$. By Lemma 2.4.12 and Lemma 2.4.14 there is a well defined map

$$\Phi:\Gamma\to\Delta$$

given by

$$\Phi(\alpha)=\Phi(\mathcal{L}),$$

for $\mathcal{L}$ an α-adapted list. The next aim is proving that the map

$$\Phi:\mathcal{C}_{x,y}([0,1],\mathbb{X}_{\mathcal{T}})\to\Delta$$

is constant. To do so, we first prove a strong simply connected property on $\mathbb{X}_{\mathcal{T}}$ with respect to the elements of $\mathcal{C}_{x,y}([a,b],\mathbb{X}_{\mathcal{T}})$. Recall that if $\alpha,\beta\in\mathcal{C}_{x,y}([a,b],\mathbb{X}_{\mathcal{T}})$ then a homotopy from α to β is a continuous function $H:[a,b]\times[a,b]\to\mathbb{X}_{\mathcal{T}}$ such that $H(a,t)=\alpha(t)$, $H(b,t)=\beta(t)$, $H(t,0)=x$ and $H(t,1)=y$ for every $t\in[0,1]$. We say that α and β are *strongly homotopic* if there is an homotopy from α to β such that $H(t,-)\in\mathcal{C}_{x,y}([a,b],\mathbb{X}_{\mathcal{T}})$ for every $t\in[a,b]$. Clearly, this defines an equivalence relation on $\mathcal{C}_{x,y}([a,b],\mathbb{X}_{\mathcal{T}})$. Moreover, this equivalence relation is preserved by concatenation. More precisely, if $\alpha\in\mathcal{C}_{x,y}([a,b],\mathbb{X}_{\mathcal{T}})$ and $\beta\in\mathcal{C}_{y,z}([a,b],\mathbb{X}_{\mathcal{T}})$ then the concatenation of α and β is the function $\alpha\oplus\beta:[a,b]\to\mathbb{X}_{\mathcal{T}}$ defined by

$$(\alpha\oplus\beta)(t)=\begin{cases}\alpha(x+2(t-a)(\frac{y-x}{b-a})), & \text{if } a\le t\le\frac{a+b}{2};\\ \beta(y+2(t-\frac{a+b}{2})(\frac{y-z}{b-a})), & \text{if } \frac{a+b}{2}\le t\le b.\end{cases}$$

If $\alpha_i,\beta_i\in\mathcal{C}_{x,y}([a,b],\mathbb{X}_{\mathcal{T}})$ are so that α_i and β_i are strongly homotopic for $i\in\{1,2\}$ then $\alpha_1\oplus\beta_1$ and $\alpha_2\oplus\beta_2$ are strongly homotopic.

Lemma 2.4.15. *All the elements of* $\mathcal{C}_{x,y}([a,b],\mathbb{X}_{\mathcal{T}})$ *are strongly homotopic.*

Proof. Let V and W be two subspaces of $\mathbb{X}$ of dimension m and n respectively and let $\langle V,W\rangle$ denote the smallest subspace of $\mathbb{X}$ containing V and W. If $V\cap W\neq\emptyset$ then the dimension of $\langle V,W\rangle$ is at most $m+n$. Otherwise the dimension of $\langle V,W\rangle$ is at most

$m+n+1$. Indeed, let $w \in W$ and let V_1 be the smallest subspace of $\mathbb{X}$ containing V and w. Then V_1 has dimension $m+1$, $\langle V, W\rangle = \langle V_1, W\rangle$ and the dimension of this space is at most $m+n+1$, because $V_1 \cap W \neq \emptyset$.

Let L be a geodesic line of $\mathbb{X}$ and let V be a geodesic subspace of $\mathbb{X}$ of codimension at least 3. Then, by the previous, $\langle L, V\rangle$ has positive codimension. Therefore, if $\{L_i \mid i \in I\}$ is a countable family of geodesic lines and $\{V_j \mid j \in J\}$ is a countable family of subspaces of codimension at least 3 then, by Lemma 2.2.9, $\cup_{i\in I, j\in J}\langle L_i, V_j\rangle$ is a proper subset of $\mathbb{X}$. Using this for the case when $\{V_j \mid j \in J\}$ is the family of subspaces generated by the cells of tiles of $\mathcal{T}$ of codimension at least 3, we deduce that for every countable family $S = \{[x_i, y_i] \mid i \in I\}$ of geodesic segments contained in $\mathbb{X}_{\mathcal{T}}$, there exists $w \in \mathbb{X}_{\mathcal{T}} \setminus \cup_{i\in I, j\in J}\langle L_i, V_j\rangle$, where L_i denotes the geodesic line containing $[x_i, y_i]$. This implies that the intersection of $\langle L_i, V_j\rangle$ with the geodesic plane containing both L_i and w is contained in L_i. Thus, the cone of $[x_i, y_i]$ with vertex w does not intersect any V_j and hence it is contained in $\mathbb{X}_{\mathcal{T}}$. (Observe that if $\mathbb{X}$ is spherical then the antipode of w is not in any $[x_i, y_i]$ because $w \notin L_i$.)

Let $x', y' \in \mathbb{X}$ with $x' \neq y'$ and let $\rho = d(x', y')$. Let $\alpha_0 : [0,1] \to [x', y']$ be the parametrization of a segment $[x', y']$ of constant speed, that is α_0 is the inverse of the map $z \in [x', y'] \mapsto \frac{d(x',z)}{\rho} \in [0,1]$. Assume α is an arbitrary parametrization of the segment $[x', y']$. Consider the function $H : [0,1] \times [0,1] \to [x', y']$ defined by

$$H(s,t) = \alpha_0\left(\frac{d(x', \alpha(t))}{\rho} + s\left(t - \frac{d(x', \alpha(t))}{\rho}\right)\right).$$

As both the distance function d and α are continuous, H is a continuous function and $H(0,t) = \alpha(t)$ and $H(1,t) = \alpha_0(t)$. Furthermore $H(s,0) = \alpha_0(0) = x'$ and $H(s,1) = \alpha_0(1) = y'$. Moreover, as α is a parametrization, the function $t \mapsto d(x', \alpha(t))$ is non-decreasing. Therefore, for every $s \in [0,1]$, the function $t \mapsto \frac{d(x',\alpha(t))}{\rho} + s\left(t - \frac{d(x',\alpha(t))}{\rho}\right) = st + \frac{(1-s)d(x',\alpha(t))}{\rho}$ is non-decreasing. Hence

$$t \mapsto d(x', H(s,t)) = d\left(x', \alpha_0\left(\frac{d(x', \alpha(t))}{\rho} + s\left(t - \frac{d(x', \alpha(t))}{\rho}\right)\right)\right)$$

is non-decreasing too. Therefore $H(s,-)$ is a parametrization of $[x', y']$, for every $s \in [0,1]$. As the image of H is $[x', y']$, we have showed that H is a strong homotopy between α and α_0.

Without loss of generality, we may assume that $[a,b] = [0,1]$. Let $\alpha, \beta \in \mathcal{C}_{x,y}([0,1], \mathbb{X}_{\mathcal{T}})$. We need to show that α and β are strongly homotopic. Clearly, there is an ascending finite list $0 = a_0 < a_1 < \dots a_n = 1$ such that the restrictions to $[a_{i-1}, a_i]$ of α and β are both parametrizations of segments. By the previous paragraph, we may assume without loss of generality that the restriction to each segment $[a_{i-1}, a_i]$ of α and β has constant speed. We now argue by induction on n. If $n = 1$ then $\alpha = \beta$ (because of the constant speed) and hence there is nothing to prove. Let $x_i = \alpha(a_i)$ and $y_i = \beta(a_i)$ for $i \in \{0, 1, \dots, n\}$. By the discussion in the second paragraph of the proof, there exists $w \in \mathbb{X}_{\mathcal{T}}$ such that all the cones determined by the segments $[x_{i-1}, x_i]$ and $[y_{i-1}, y_i]$ and centred in w are contained in $\mathbb{X}_{\mathcal{T}}$. Let α_i denote the restriction of α to $[a_{i-1}, a_i]$ and let β_i denote the restriction of β to $[a_{i-1}, a_i]$. Further, let $\alpha' : [0,1] \to \mathbb{X}_{\mathcal{T}}$ be such that it agrees with α on $[0, a_{n-2}]$, the restriction of α' to $[a_{n-2}, a_{n-1}]$ is a parametrization of the interval $[x_{n-2}, w]$ of constant speed and the restriction of α' to $[a_{n-1}, a_n]$ is a parametrization of $[w, y]$ of constant speed. Similarly, let $\beta' : [0,1] \to \mathbb{X}_{\mathcal{T}}$ agree with β on $[0, a_{n-2}]$, its restriction to $[a_{n-2}, a_{n-1}]$ is a parametrization of the geodesic interval $[y_{n-2}, w]$ of constant speed and the restriction of β' to $[a_{n-1}, 1]$ is a parametrization of $[w, y]$ of constant speed. By the induction hypothesis, the restrictions of α' and β' to $[0, a_{n-1}]$ are strongly homotopic. Furthermore α' and β' coincide on $[a_{n-1}, 1]$ and hence α' and β' are strongly homotopic. It remains to prove that α and α' are strongly homotopic, and that so are β and β'. For this it is enough to prove that the restrictions of α and α' (respectively, β and β') to $[a_{n-2}, a_n]$ are strongly homotopic. This reduces the problem to the case where $n = 2$ and the two geodesic triangles $x\alpha(a_1)\beta(a_1)$ and $\alpha(a_1)\beta(a_1)y$ are contained in $\mathbb{X}_{\mathcal{T}}$. Let $\gamma : [0,1] \to [\alpha(a_1), \beta(a_1)]$ be a parametrization of $[\alpha(a_1), \beta(a_1)]$ of constant speed. For every $s \in [0, a_1]$ let $t \to H(s,t)$ be the parametrization of $[x, \gamma(s)]$ of constant speed and, for $s \in [a_1, 1]$, let $t \to H(s,t)$ be the parametrization of $[\gamma(s), y]$ of constant speed. In other words, if $0 \le t \le a_1$ then $H(s,t)$ belongs to the geodesic segment $[x, \gamma(s)]$ and

$$d(x, H(s,t)) \quad = \quad d(x, \gamma(s))\frac{t}{a_1}. \tag{2.7}$$

On the other hand, if $a_1 \le t \le 1$ then $H(s,t)$ belongs to the geodesic segment $[\gamma(s), b]$ and

$$d(\gamma(s), H(s,t)) \quad = \quad d(\gamma(s), y)\frac{t - a_1}{1 - a_1}. \tag{2.8}$$

Clearly $H(s, -) \in \mathcal{C}_{x,y}([0,1], \mathbb{X}_{\mathcal{T}})$ for every $s \in [0,1]$ and $H(0,-) = \alpha$ and $H(1,-) = \beta$. Finally, it is easy to see that the function H is continuous. ∎

Lemma 2.4.16. *If both x and y belong to the interior of some tile then*

$$\Phi : \mathcal{C}_{x,y}([a,b], \mathbb{X}_{\mathcal{T}}) \to \Delta$$

is a constant mapping.

Proof. Again, without loss of generality, we may assume that $[a,b] = [0,1]$. We claim that it is sufficient to show that Φ is locally constant. Indeed, assume this is the case and let $\alpha, \beta \in \mathcal{C}_{x,y}([0,1], \mathbb{X}_{\mathcal{T}})$. By Lemma 2.4.15 there is a strong homotopy H from α to β. Let c denote the supremum of the $s \in [0,1]$ for which $\Phi(H(s,-)) = \Phi(\alpha)$. Since, by assumption, Φ is constant in a neighbourhood of $H(x,-)$, it easily follows that $c = 1$ and thus $\Phi(\alpha) = \Phi(\beta)$.

To prove that Φ is locally constant, we show that for every $\alpha \in \mathcal{C}_{x,y}([0,1], \mathbb{X}_{\mathcal{T}})$ and for every α-adapted list

$$\mathcal{L} = (a_0, \gamma_1, a_1, \ldots, \gamma_n, a_n)$$

(which exists because of Lemma 2.4.14), there is a positive real number δ such that for every $\beta \in \mathcal{C}_{x,y}([0,1],\mathbb{X}_\mathcal{T})$ with $d(\alpha,\beta) < \delta$, where d was defined in (2.6), there is a β-adapted list $\mathcal{D} = (b_0,\tau_1,b_1,\dots,\tau_m,b_m)$ and an increasing sequence of integers $j_0 = 0 < j_1 < j_2 < \cdots < j_{n-1} < j_n = m$ such that

$$\mathcal{L}' = (a_0,\tau_{j_1},a_1,\tau_{j_2},a_2,\dots,\tau_{j_{i-1}},a_{i-1},\tau_{j_n},a_n)$$

is an α-adapted list and, for every $i \in \{0,1,\dots,n\}$, $\Phi((\mathcal{L}')_i) = \Phi((\mathcal{D})_i)$, where $(\mathcal{L}')_i = (a_0,\tau_1,a_1,\dots,\tau_i,a_i)$ and $(\mathcal{D})_i = (b_0,\tau_1,b_1,\dots,\tau_{j_i},b_{j_i})$. In particular, by Lemma 2.4.12, we have $\Phi(\alpha) = \Phi(\mathcal{L}) = \Phi(\mathcal{L}') = \Phi((\mathcal{L}')_n) = \Phi((\mathcal{D})_n) = \Phi(\mathcal{D}) = \Phi(\beta)$, as desired.

Since $\mathcal{T}$ is locally finite, there is $\delta_1 > 0$ such that for every $i \in \{1,\dots,n-1\}$ and every $\gamma \in \Gamma$, if $B(\alpha(a_i),2\delta_1)) \cap \gamma(P) \neq \emptyset$ then $\alpha(a_i) \in \gamma(P)$. Since α is continuous there is $\epsilon < \min\left\{\frac{a_i-a_{i-1}}{2} \mid i \in \{1,\dots,n\}\right\}$ such that, for every $i \in \{0,1,\dots,n\}$, $d(\alpha(t),\alpha(a_i)) < \delta_1$ for every t with $|t-a_i| < \epsilon$. For every $i \in \{1,\dots,n-1\}$, let $a_i' = a_i - \epsilon$ and $a_i'' = a_i + \epsilon$. We also set $a_n' = 1$ and $a_0'' = 0$. Observe that $a_{i-1}'' \le a_i'$ for every $i \in \{1,\dots,n\}$. Each $\alpha([a_i'',a_{i+1}'])$ is compact and it is contained in the relative interior of a cell C_i contained in $\gamma_i(P)$. Using again that $\mathcal{T}$ is locally finite we obtain a positive number δ_2 such that $d(\alpha(t),\gamma(P)) > \delta_2$ for every $t \in [a_i'',a_{i+1}']$ and every $\gamma \in \Gamma$ with $C_i \not\subseteq \gamma(P)$. Let $\delta = \min\{\delta_1,\delta_2\}$. We will prove that δ satisfies the desired property. Let $\beta \in \mathcal{C}_{x,y}([0,1],\mathbb{X}_\mathcal{T})$ with $d(\alpha,\beta) < \delta$. Then $d(\alpha(t),\beta(t)) < \delta$ for every $t \in [0,1]$. In particular,

$$\text{if } t \in (a_i',a_i'') \text{ and } \beta(t) \in \gamma(P) \text{ then } \alpha(a_i) \in \gamma(P) \tag{2.9}$$

because $d(\beta(t),\alpha(a_i)) < 2\delta_1$. Moreover,

$$\text{if } t \in [a_{i-1}'',a_i'] \text{ and } \beta(t) \in \gamma(P) \text{ then } C_i \subseteq \gamma(P), \tag{2.10}$$

since $d(\alpha(t),\beta(t)) < \delta_2$. Furthermore

$$\text{if } C_i \text{ is a tile then } \beta([a_{i-1}'',a_i']) \subseteq C_i^\circ = \gamma_i(P)^\circ. \tag{2.11}$$

Indeed, as $\alpha([a_{i-1}'',a_i']) \subseteq \alpha((a_{i-1},a_i)) \subseteq C_i^\circ$, it follows that $C_i = \gamma_i(P)$ is the only tile intersecting $\alpha([a_{i-1}'',a_i'])$ and therefore it also is the only tile intersecting $\beta([a_{i-1}'',a_i'])$. Then (2.11) follows.

Let $\mathcal{D} = (b_0,\tau_1,b_1,\dots,\tau_m,\gamma_m)$ be a β-adapted list. We enlarge $\mathcal{D}$ by inserting each a_i' and a_i''. More precisely, if we rename the list

$$(a_1',a_1'',a_2',a_2'',\dots,a_{n-1}',a_{n-1}'') = (c_1,c_2,\dots,c_{2(n-1)}),$$

then we insert in $\mathcal{D}$ the sequence $(\tau_j,c_i,\tau_j,c_{i+1},\dots,\tau_j,c_k)$ between b_{j-1} and τ_j for every i whenever $c_{i-1} \le b_{j-1} < c_i < \cdots < c_k < b_j \le c_{k+1}$. So we may assume without loss of generality that there is an ascending sequence $0 = j_0' < j_1 < j_1' < j_2 < j_2' < \cdots < j_{n-1} < j_{n-1}' < j_n = m$ such that for every $i \in \{1,\dots,n\}$ $a_i' = b_{j_i}$ and for every $i \in \{0,1,\dots,n-1\}$ we have $a_i'' = b_{j_i'}$. We claim that $\mathcal{L}' = (a_0,\tau_{j_1},a_1,\dots,\tau_{j_n},a_n)$ is an α-adapted list. For that observe that $[b_{j_i-1},b_{j_i}] \subseteq [a_{i-1}'',a_i']$ and $\beta((b_{j_i-1},b_j))$ is contained in $\tau_{j_i}(P)$. Therefore $C_i \subseteq \tau_{j_i}(P)$, by (2.10).

It remains to prove that $\Phi((\mathcal{L}')_i) = \Phi((\mathcal{D})_i)$ for every $i \in \{1, \ldots, n\}$. We argue by induction. As $\alpha(0) \in \gamma_1(P)^\circ$, necessarily $C_1 = \gamma_1(P)$. Using (2.11) it is easy to prove that $\beta([a''_0 = 0, b_{j_1} = a'_1]) \subseteq \gamma_1(P)$. Hence $\tau_j = \gamma_1$ for every $j \in \{1, \ldots, j_1\}$. Therefore $\Phi((\mathcal{L}')_1) = 1 = \Phi((\mathcal{D})_1)$. Assume that $i > 1$ and $\Phi((\mathcal{L}')_{i-1}) = \Phi((\mathcal{D})_{i-1})$. Let E be the cell generated by $\alpha(a_{i-1})$, i.e. the unique one whose relative interior contains $\alpha(a_{i-1})$. For every $j \in \{1, \ldots, m\}$ let E_j be the cell generated by $\beta(b_j)$. Then $E \subseteq C_i$, $\alpha(a_{i-1}) \in E_j$ for every $j_{i-1} \leq j \leq j'_{i-1}$, by (2.9) and C_i is contained in every cell intersecting some $\beta([a''_{i-1}, a'_i])$, by (2.10). Therefore, for every $j'_{i-1} \leq j \leq j_i$, C_i is contained in every tile containing $\beta(b_j)$ and hence $E \subseteq C_i \subseteq E_j$. We conclude that $E \subseteq E_j$ for every $j_{i-1} \leq j \leq j_i$. Using that $E_j \subseteq \tau_j(P) \cap \tau_{j+1}(P)$ we have

$$\begin{aligned}
\kappa_{\alpha(a_{i-1})}(\tau_{j_{i-1}}, \tau_{j_i}) &= \kappa_E(\tau_{j_{i-1}}, \tau_{j_i}) = \kappa_E(\tau_{j_{i-1}}, \tau_{j_{i-1}+1}) \cdots \kappa_E(\tau_{j_i-1}, \tau_{j_i}) \\
&= \kappa_{E_{j_{i-1}}}(\tau_{j_{i-1}}, \tau_{j_{i-1}+1}) \cdots \kappa_{E_{j_i-1}}(\tau_{j_i-1}, \tau_{j_i}) \\
&= \kappa_{\beta(b_{j_{i-1}})}(\tau_{j_{i-1}}, \tau_{j_{i-1}+1}) \cdots \kappa_{\beta(\tau_{j_i-1})}(\tau_{j_i-1}, \tau_{j_i})
\end{aligned}$$

by Lemma 2.4.9.(3). Then

$$\begin{aligned}
\Phi((\mathcal{L}')_i) &= \Phi((\mathcal{L}')_{i-1})\kappa_{\alpha(a_{i-1})}(\tau_{j_{i-1}}, \tau_{j_i}) \\
&= \Phi((\mathcal{D})_{i-1})\kappa_{\beta(b_{j_{i-1}})}(\tau_{j_{i-1}}, \tau_{j_{i-1}+1}) \cdots \kappa_{\beta(b_{j_i-1})}(\tau_{j_i-1}, \tau_{j_i}) \\
&= \Phi((\mathcal{D})_i),
\end{aligned}$$

as desired. ∎

Proof of Theorem 2.4.7 (Relations). Let $\gamma_1, \ldots, \gamma_n$ be a list of pairing transformations such that $\gamma_1 \cdots \gamma_n = 1$. We have to show that $[\gamma_1] \cdots [\gamma_n] = 1$. We may assume that $\gamma_1 \cdots \gamma_n = 1$ is a minimal relation, i.e. $\gamma_i \cdots \gamma_j \neq 1$ for every $1 \leq i < j \leq n$ with $(i, j) \neq (1, n)$. For every $i \in \{1, \ldots, n\}$ let $S_i = \gamma_1 \cdots \gamma_{i-1}(P) \cap \gamma_1 \cdots \gamma_i(P)$, a side of both $\gamma_1 \cdots \gamma_{i-1}(P)$ and $\gamma_1 \cdots \gamma_i(P)$.

Fix $c_0 \in P^\circ$ and for every $i \in \{1, \ldots, n\}$ let $b_i \in (P \cap \gamma_i(P))^r$ and $c_i = \gamma_1\gamma_2 \cdots \gamma_{i-1}(b_i)$. Observe that each $c_i \in S_i$ for $i \geq 1$. If the sides S_i and S_{i+1} are contained in different essential hyperplanes of the tile $\gamma_1\gamma_2 \cdots \gamma_i(P)$ then $(c_i, c_{i+1}) \subseteq (\gamma_1\gamma_2 \cdots \gamma_i)(P)^\circ$. However, they might be in the same essential hyperplane, and therefore we introduce some additional elements of $\mathbb{X}$. For each $i \in \{1, 2, \ldots, n-1\}$, choose $c'_i \in \gamma_1\gamma_2 \cdots \gamma_i(P)^\circ$. Consider the geodesic segments,

$$[c_0, c_1 = b_1],\ [c_1 = b_1, c'_1],\ [c'_1, c_2 = \gamma_1(b_2)],\ [c_2, c'_2],\ [c'_2, c_3 = \gamma_1 g_2(b_3)], \ldots,$$

$$[c_{n-1} = \gamma_1 \cdots \gamma_{n-2}(b_{n-1}), c'_{n-1}]\ [c'_{n-1}, c_n = \gamma_1\gamma_2 \cdots \gamma_{n-1}(b_n)],\ [c_n, c_{n+1} = c_0].$$

By construction, for each $1 \leq i < n$,

$$(c_0, c_1) \subseteq P^\circ, (c_i, c'_i) \subseteq \gamma_1 \cdots \gamma_i(P)^\circ, (c'_i, c_{i+1}) \subseteq \gamma_1 \cdots \gamma_i(P)^\circ, (c_n, c_0) \subseteq P^\circ.$$

Furthemore, the closure of each of the listed geodesic segments is contained in $\mathbb{X}_{\mathcal{T}}$. Let $\alpha : [0, 1] \to \mathbb{X}_{\mathcal{T}}$ be the continuous function whose graph is obtained by concatenating all these segments. Then there exists an ascending sequence

$$a_0 < a_1 < a'_1 < a_2 < a'_2 < a_3 < \cdots < a_{n-1} < a'_{n-1} < a_n < a_{n+1} = 1$$

with $\alpha(a_i) = c_i$ and $\alpha(a'_i) = c'_i$ for each i. Furthermore,

$$\alpha((a_{i-1}, a'_i)) = (c_i, c'_i) \subseteq \gamma_1 \cdots \gamma_i(P)^\circ$$

and

$$\alpha((a'_i, a_{i+1})) = (c'_i, c_{i+1}) \subseteq \gamma_1 \cdots \gamma_i(P)^\circ.$$

Therefore,

$$\begin{aligned}\mathcal{L} &= (a_0, 1, a_1, \gamma_1, a'_1, \gamma_1, a_2, \gamma_1\gamma_2, a'_2, \gamma_1\gamma_2, a_2, \gamma_1\gamma_2\gamma_3 a'_3, \\ &\quad \cdots, \gamma_1\gamma_2\cdots\gamma_{n-1}, a'_{n-1}, \gamma_1\cdots\gamma_{n-1}, a_n, \gamma_1\cdots\gamma_n, a_{n+1})\end{aligned}$$

is an α-adapted list. Thus,

$$\begin{aligned}\Phi(\alpha) &= [\gamma_1]\,[\gamma_1^{-1}\gamma_1]\,[\gamma_1^{-1}(\gamma_1\gamma_2)]\,[(\gamma_1\gamma_2)^{-1}(\gamma_1\gamma_2)]\,[(\gamma_1\gamma_2)^{-1}(\gamma_1\gamma_2\gamma_3)] \\ &\quad \cdots [(\gamma_1\cdots\gamma_{n-1})^{-1}(\gamma_1\cdots\gamma_n - 1)]\,[(\gamma_1\cdots\gamma_{n-1})^{-1}(\gamma_1\cdots\gamma_n)] \\ &= [\gamma_1][\gamma_2]\cdots[\gamma_n].\end{aligned}$$

On the other hand, let $\beta \in \mathcal{C}_{c_0,c_0}([0,1], \mathbb{X}_\mathcal{T})$ denote the constant path, i.e. $\beta(t) = c_0$ for every $t \in [0,1]$. Then $(0,1,1)$ is a β-adapted list and $\Phi(\beta) = 1$. Lemma 2.4.16 yields that $[\gamma_1]\cdots[\gamma_n] = \Phi(\alpha) = \Phi(\beta) = 1$, as desired. ∎

Remark 2.4.17. *Observe that, by defining a side of a fundamental polyhedron as a cell of codimension* 1 *(see Definition 2.3.2), where cell is defined in Definition 2.3.1, we do not need the condition of exactness from Definition 1.2.13 in the statement of Poincaré's Polyhedron Theorem in Theorem 2.4.7. One may then easily show that this condition can also be dropped in the statement of Lemma 1.2.41.*

3

A First Application of Poincaré's Polyhedron Theorem to Units in Integral Group Rings

L'algèbre n'est qu'une géométrie écrite;
la géométrie n'est qu'une algèbre figurée.
Sophie Germain

In this chapter, we describe a finite and easy implementable algorithm (called DAFC) to compute a finite set of generators, up to finite index, for a discrete subgroup Γ of $\mathrm{PSL}_2(\mathbb{C})$, respectively $\mathrm{PSL}_2(\mathbb{R})$, of finite covolume. This algorithm uses the construction of a Dirichlet fundamental domain. This may then be applied to the search of generators, up to commensurability, of the unit group $\mathcal{U}(\mathbb{Z}G)$ of the integral group ring of a finite group. Compared to earlier known results, we do now allow exceptional simple components of type (2) and (3), but also some exceptional components of the form (1) (see Definition 1.1.19). We also point out that exceptional components of type (2) have already been dealt with in [DJK10], using the theory of Farey Symbols, which are also closely related to fundamental domains. In Section 3.1, we describe the algorithm. We also give an algorithm which establishes the Dirichlet fundamental polyhedron for Γ in the case Γ is cocompact. In Section 3.2, we show how this algorithm may be applied to unit groups of orders in division algebras and matrix algebras. We also give some concrete examples. As mentioned above, in Section 3.3, we show how these results can be used in the investigation on the unit group of an order in the group ring $\mathbb{Q}G$. In particular, we deal with the case when G is finite nilpotent. Finally, in Section 3.4, we show how the results used to construct the DAFC may be used to improve results and get new information on DF domains and Double Dirichlet domains.

3.1 Towards an Algorithm for Computing a Finite Set of Generators

The main purpose of this section is to give a finite algorithm to compute generators for a given discrete subgroup Γ of $\mathrm{PSL}_2(\mathbb{C})$ (respectively $\mathrm{PSL}_2(\mathbb{R})$), up to finite index, that is of finite covolume. The algorithm is based on the construction of a Dirichlet fundamental polyhedron of centre j in $\mathbb{H}^3$ (respectively of centre i in $\mathbb{H}^2$) and thus we need explicit formulas for the bisectors defining the Dirichlet fundamental polyhedron associated to a discrete group in two and three dimensional hyperbolic space. We then give some lemmas that simplify the algorithm. Finally, we give an explicit criterion that determines the finite number of steps the algorithm has to go through. Calculations are done in dimension 3. In [CJLdR04] an algorithm was obtained in the case Γ is cocompact. Our algorithm is a refinement and also applies to the non-cocompact case. Recall, from Remark 1.2.21, that we make an abuse of notation and denote elements of $\mathrm{SL}_2(\mathbb{C})$ and their natural image in $\mathrm{PSL}_2(\mathbb{C})$ by the same element γ and the same notation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{C}$.

Let $\mathbb{B}^3$ be the ball model of hyperbolic space, as defined in Subsection 1.2.2, let $0 \in \mathbb{B}^3$ be the origin and let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) \text{ and } \Psi(\gamma) = \begin{pmatrix} A & C' \\ C & A' \end{pmatrix} \in \mathrm{SB}_2(\mathcal{H}),$$

where $\mathrm{SB}_2(\mathcal{H})$ is defined in (1.16) and Ψ is given in Proposition 1.2.28 and in (1.17). Recall that the isometric sphere associated to the transformation γ, respectively $\Psi(\gamma)$, is the unique sphere on which γ, respectively $\Psi(\gamma)$ acts as a Euclidean isometry. By (1.23), if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \neq 0$, then the isometric sphere of γ has centre $-\frac{d}{c}$ and radius $\frac{1}{|c|}$. A similar characterization may be defined in the ball model $\mathbb{B}^3$.

Lemma 3.1.1. *Let $\Psi(\gamma) = \begin{pmatrix} A & C' \\ C & A' \end{pmatrix} \in \mathrm{SB}_2(\mathcal{H})$ with $C \neq 0$ and let $\Sigma_{\Psi(\gamma)}$ be the sphere with centre $P_{\Psi(\gamma)} = -C^{-1}A'$ and radius $R_{\Psi(\gamma)} = \frac{1}{|C|}$. Then $\Sigma_{\Psi(\gamma)}$ is the isometric sphere of $\Psi(\gamma)$ in $\mathbb{B}^3$. Moreover the centre $P_{\Psi(\gamma)}$ of $\Sigma_{\Psi(\gamma)}$ and $\Psi(\gamma^{-1})(0)$ are inverse points with respect to $\mathbb{S}^2 = \partial\mathbb{B}^3$.*

Proof. We claim that 0, $P_{\Psi(\gamma)}$ and $\Psi(\gamma^{-1})(0)$ are collinear. Indeed, one may easily show that $P_{\Psi(\gamma)} \cdot (\Psi(\gamma^{-1})(0))^{-1} = |A|^2|C|^{-2} \in \mathbb{R}$ (as $C \neq 0$ by the assumption that $\gamma \notin \mathrm{SU}_2(\mathbb{C})$), which proves the claim. Also,

$$|P_{\Psi(\gamma)}| \cdot |\Psi(\gamma^{-1})(0)| = |-C^{-1}A'| \cdot |-\overline{C}A^{*-1}| = 1.$$

Thus $P_{\Psi(\gamma)}$ and $\Psi(\gamma^{-1})(0)$ are inverse points with respect to $\mathbb{S}^2 = \partial\mathbb{B}^3$. Recall, from Subsection 1.2.2, that the inverse point of a point P with respect to $\mathbb{S}^2$ is denoted by P^*. Thus $P^*_{\Psi(\gamma)} = \Psi(\gamma^{-1})(0)$. This proves the second part. Moreover

$$1 + R^2_{\Psi(\gamma)} = \frac{1 + |C|^2}{|C|^2} = \frac{|A|^2}{|C|^2} = |P_{\Psi(\gamma)}|^2$$

and hence $\Sigma_{\Psi(\gamma)}$ and $\mathbb{S}^2$ are orthogonal spheres. Denote the reflection in $\Sigma_{\Psi(\gamma)}$ by σ. By Proposition 1.2.26, $\sigma(\mathbb{B}^3) = \mathbb{B}^3$ and $\sigma \circ \Psi(\gamma^{-1})(0) = 0$. Consequently, by Proposition 1.2.25, $A = \sigma \circ \Psi(\gamma^{-1})$ is an orthogonal matrix and thus $\Psi(\gamma) = A \circ \sigma$ (i.e. σ followed by A) as in Proposition 1.2.27. Hence by Remark 1.2.44, $\Sigma_{\psi(\gamma)}$ is the isometric sphere of $\Psi(\gamma)$. ∎

So the isometric sphere in the ball model is given by

$$\Sigma_{\Psi(\gamma)} = S(P_{\Psi(\gamma)}, R_{\Psi(\gamma)}).$$

Note that in fact the isometric sphere of the ball model is strictly speaking only the part of the Euclidean sphere $\Sigma_{\Psi(\gamma)}$ which intersects the ball model, i.e. $\mathbb{B}^3 \cap \Sigma_{\Psi(\gamma)}$. However throughout this chapter we make some abuse of notation and denote the Euclidean sphere with centre $-C^{-1}A'$ and radius $\frac{1}{|C|}$ as well as the isometric sphere by $\Sigma_{\Psi(\gamma)}$. The following theorem shows that in the ball model the concepts of isometric sphere of $\Psi(\gamma)$ and bisector of 0 and $\Psi(\gamma^{-1})(0)$ are the same. This theorem is well-known, but not always clearly stated in all references. In [Bea95, Theorem 9.5.2], Beardon shows it for dimension 2. We give here an independent proof in dimension 3 (which is of course adaptable to dimension 2). Therefore recall, by Lemma 1.2.24, that $\gamma \in \mathrm{SU}_2(\mathbb{C})$ if and only if $\gamma(j) = j$ or, equivalently, $\Psi(\gamma)(0) = 0$. In the latter case the bisector of 0 and $\Psi(\gamma^{-1})(0)$ does not exist as both are the same points. Also $\Psi(\gamma)(0) = 0$ if and only if $C = 0$ and hence the isometric sphere does not exist neither. Therefore, in the following theorem we exclude the case $\gamma \in \mathrm{SU}_2(\mathbb{C})$. Also, recall that another equivalent condition for $\gamma \in \mathrm{SU}_2(\mathbb{C})$ is that $\|\gamma\|^2 = 2$ and vice-versa.

Theorem 3.1.2. *Let $\gamma \in \mathrm{SL}_2(\mathbb{C})$ with $\gamma \notin \mathrm{SU}_2(\mathbb{C})$. Then, in the ball model, the isometric sphere associated to $\Psi(\gamma)$ equals the bisector of the geodesic segment linking 0 and $\Psi(\gamma^{-1})(0)$.*

Proof. Let $\Psi(\gamma) = \begin{pmatrix} A & C' \\ C & A' \end{pmatrix}$. So $P_{\Psi(\gamma)} = -C^{-1}A'$. By Lemma 3.1.1, 0, $P_{\Psi(\gamma)}$ and $\Psi(\gamma^{-1})(0)$ are collinear and $P_{\Psi(\gamma)}$ and $\Psi(\gamma^{-1})(0)$ are inverse points with respect to $\mathbb{S}^2 = \partial\mathbb{B}^3$.

Now let r be the ray through $P_{\Psi(\gamma)}$ and put $M = r \cap \Sigma_{\Psi(\gamma)}$. Clearly $|M| = \frac{|A|-1}{|C|}$. By (1.14), it is easy to verify that the hyperbolic metric, ρ, in $\mathbb{B}^3$ satisfies $\rho(0,u) = \ln(\frac{1+|u|}{1-|u|})$. Thus we have

$$\rho(0, M) = \ln\left(\frac{1 + \frac{|A|-1}{|C|}}{1 - \frac{|A|-1}{|C|}}\right) = \ln\left(\frac{|C| + |A| - 1}{|C| - |A| + 1}\right),$$

and

$$\rho(0, \Psi(\gamma^{-1})(0)) = \ln\left(\frac{|A| + |C|}{|A| - |C|}\right).$$

Moreover, using the fact that $|A|^2-|C|^2=1$, one easily calculates that $2\cdot\ln(\frac{|C|+|A|-1}{|C|-|A|+1}) = \ln(\frac{|A|+|C|}{|A|-|C|})$, and hence $2\rho(0,M)=\rho(0,\Psi(\gamma^{-1})(0))$. As the ray r is orthogonal to $\Sigma_{\Psi(\gamma)}$, $\Sigma_{\Psi(\gamma)}$ is the bisector of the geodesic segment linking 0 and $\Psi(\gamma^{-1})(0)$. ∎

In the upper half-space model, the result from the above theorem is not necessarily true. The bisector of the geodesic segment linking 0 and $\Psi(\gamma^{-1})(0)$ being a pure hyperbolic notion, its projection $\eta_0^{-1}(\Sigma_{\Psi(\gamma)})$, where η_0 is defined in Proposition 1.2.28, to the upper half-space model $\mathbb{H}^3$ is still a bisector. In fact it is the bisector of the geodesic segment linking $\eta_0^{-1}(0)=j$ and $\eta_0^{-1}(\Psi(\gamma^{-1})(0))=\gamma^{-1}(j)$. Note that the latter equality comes from Proposition 1.2.28(v). However the isometric sphere associated to a transformation γ is a purely Euclidean concept and hence the projection of the isometric sphere of $\Psi(\gamma)$ by η_0^{-1} is no longer an isometric sphere. So, if we denote, as in Subsection 1.2.3, the isometric sphere in $\mathbb{H}^3$ associated to γ by I_γ, then in general we do not have that $I_\gamma=\eta_0^{-1}(\Sigma_{\Psi(\gamma)})$. We put

$$\Sigma_\gamma = \eta_0^{-1}(\Sigma_{\Psi(\gamma)}). \tag{3.1}$$

In fact, $\Sigma_\gamma\cap\mathbb{H}^3$ is nothing else than $\mathrm{Bis}_{\gamma^{-1}}(j)$ from (1.20). Note that, as η_0 is an isometry, Σ_γ is either a Euclidean sphere with centre in $\partial\mathbb{H}^3$ or a vertical plane orthogonal to $\partial\mathbb{H}^3$. In case it is a Euclidean sphere, we denote its centre by P_γ and its radius by R_γ.

Lemma 3.1.3. *Let* $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{C})$ *and* $\gamma\notin \mathrm{SU}_2(\mathbb{C})$.

1. *Σ_γ is a Euclidean sphere if and only if $|a|^2+|c|^2\neq 1$. In this case, its centre and its radius are respectively given by $P_\gamma=\frac{-(\overline{a}b+\overline{c}d)}{|a|^2+|c|^2-1}$ and $R_\gamma^2=\frac{1+|P_\gamma|^2}{|a|^2+|c|^2}$.*

2. *Σ_γ is a plane if and only if $|a|^2+|c|^2=1$. In this case $Re(\overline{v}z)+\frac{|v|^2}{2}=0$, $z\in\mathbb{C}$ is a defining equation of Σ_γ in $\mathbb{H}^3$, where $v=\overline{a}b+\overline{c}d$.*

Note that the equation in the second item, defining the plane Σ_γ does not involve the third component of a point $u\in\mathbb{H}^3$. This is due to the fact that this plane is orthogonal to $\partial\mathbb{H}^3$.

Proof. Suppose first that Σ_γ is a Euclidean sphere. By Lemma 3.1.1, $P^*_{\Psi(\gamma)}=\Psi(\gamma)^{-1}(0)$ and the two spheres $\Sigma_{\Psi(\gamma)}$ and $\mathbb{S}^2$ are orthogonal. Hence 0 and $\Psi(\gamma^{-1})(0)$ are inverse points with respect to $\Sigma_{\Psi(\gamma)}$. By Proposition 1.2.28, η_0 is a Möbius transformation and by Proposition 1.2.18, it hence follows that $j=\eta_0^{-1}(0)$ and $\gamma^{-1}(j)=\eta_0^{-1}(\Psi(\gamma)^{-1}(0))$ are inverse points with respect to $\Sigma_\gamma=\eta_0^{-1}(\Sigma_{\Psi(\gamma)})$. So, if Σ_γ is not a vertical plane, then j, $\gamma^{-1}(j)$ and P_γ are collinear points. By (1.11),

$$\gamma^{-1}(j) = -\frac{(\overline{a}b+\overline{c}d)}{|a|^2+|c|^2}+\frac{1}{|a|^2+|c|^2}j$$

and hence for the three points to be collinear we must have that $|a|^2+|c|^2\neq 1$. In that case it follows that $P_\gamma=l\cap\partial\mathbb{H}^3$, where l is the Euclidean line determined by j and

$\gamma^{-1}(j)$. A simple calculation gives the formula of P_γ. Since j and $\gamma^{-1}(j)$ are inverse points with respect to Σ_γ, $R_\gamma^2 = |j - P_\gamma| \cdot |\gamma^{-1}(j) - P_\gamma|$. This gives the formula of R_γ^2 and proves the first item.

If $|a|^2 + |c|^2 = 1$, the line l determined by j and $\gamma^{-1}(j)$ is parallel to the border of the upper half-space $\partial\mathbb{H}^3$. So Σ_γ cannot be a Euclidean sphere and hence is a vertical plane. Conversely, if Σ_γ is a vertical plane, j and $\gamma^{-1}(j)$ have to be at the same height and hence $|a|^2 + |c|^2 = 1$. In this case,

$$\gamma^{-1}(j) = -(\overline{a}b + \overline{c}d) + j$$

and hence $v = j - \gamma^{-1}(j) = \overline{a}b + \overline{c}d$ is orthogonal to Σ_γ. From this one obtains the mentioned defining equation of Σ_γ, hence the second item. ∎

The next lemma gives some more information on the bisectors in the ball model.

Lemma 3.1.4. *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$ *and* $\Psi(\gamma) = \begin{pmatrix} A & C' \\ C & A' \end{pmatrix} \in \mathrm{SB}_2(\mathcal{H})$. *Suppose that* $\gamma \notin \mathrm{SU}_2(\mathbb{C})$. *Then the following properties hold.*

1. $|A|^2 = \frac{2+\|\gamma\|^2}{4}$ *and* $|C|^2 = \frac{\|\gamma\|^2-2}{4}$,
2. $P_{\Psi(\gamma)} = \frac{1}{-2+\|\gamma\|^2} \cdot [\ -2(\overline{a}b + \overline{c}d) + [(|b|^2 + |d|^2) - (|a|^2 + |c|^2)]j\ ]$,
3. $\Psi(\gamma^{-1})(0) = P^*_{\Psi(\gamma)} = \frac{1}{2+\|\gamma\|^2} \cdot [\ -2(\overline{a}b + \overline{c}d) + [(|b|^2 + |d|^2) - (|a|^2 + |c|^2)]j\ ]$,
4. $|P_{\Psi(\gamma)}|^2 = \frac{2+\|\gamma\|^2}{-2+\|\gamma\|^2}$,
5. $R^2_{\Psi(\gamma)} = \frac{4}{-2+\|\gamma\|^2}$,
6. $\Sigma_{\Psi(\gamma)} = \Sigma_{\Psi(\gamma_1)}$ *if and only if* $\gamma_1 = \gamma_0\gamma$ *for some* $\gamma_0 \in \mathrm{SU}(2,\mathbb{C})$.

Proof. The proof of the five items is straightforward using the explicit formulas for $\Psi(\gamma)$, A and C given by equation (1.17) and knowing that $P_{\Psi(\gamma)} = -C^{-1}A'$.

We now prove the last item. Suppose $\Sigma_{\Psi(\gamma)} = \Sigma_{\Psi(\gamma_1)}$. As $\Psi(\gamma)$ and $\Psi(\gamma_1)$ are Möbius transformations, by Proposition 1.2.27 and Remark 1.2.44, $\Psi(\gamma) = A \circ \sigma$ and $\Psi(\gamma_1) = A_1 \circ \sigma_1$ for A and A_1 two orthogonal maps and σ and σ_1 reflections in the spheres $\Sigma_{\Psi(\gamma)}$ and $\Sigma_{\Psi(\gamma_1)}$ respectively. Thus $\sigma = \sigma_1$ and thus $\Psi(\gamma_1) = A_1 \circ \sigma_1 = A_1 \circ \sigma = A_1 A^{-1} \circ \Psi(\gamma)$. Put $\Psi(\gamma_0) = A_1 A^{-1}$ and one implication is proved. To prove the inverse implication suppose $\gamma_1 = \gamma_0\gamma$ for some $\gamma_0 \in \mathrm{SU}(2,\mathbb{C})$. Then $\Psi(\gamma) = A \circ \sigma$ and $\Psi(\gamma_1) = A_1 \circ \sigma_1$ for A and A_1 two orthogonal maps and σ and σ_1 reflections in some spheres. We have that $\Psi(\gamma_1) = \Psi(\gamma_0)\Psi(\gamma) = \Psi(\gamma_0)A\sigma$ and as $\Psi(\gamma_0)A$ is an orthogonal map, $\Psi(\gamma_0)A = A_1$ and $\sigma = \sigma_1$. Consequently $\Sigma_{\Psi(\gamma)} = \Sigma_{\Psi(\gamma_1)}$. ∎

We will now give a lemma which will be crucial in the implementation of the algorithm. Let $\gamma \in \mathrm{SL}_2(\mathbb{C})$ and $\gamma \notin \mathrm{SU}_2(\mathbb{C})$ and let r be the ray through the centre of $\Sigma_{\Psi(\gamma)}$. Denote by M and N, respectively, the intersection of r with $\Sigma_{\Psi(\gamma)}$ and $\mathbb{S}^2$. Denote the

Euclidean distance from M to N by ρ_γ. Explicitly we have that $\rho_\gamma = 1+R_{\Psi(\gamma)} - |P_{\Psi(\gamma)}|$. Our next result shows that ρ_γ is a strictly decreasing function of $\|\gamma\|^2$. Note that the Euclidean volume of the intersection of the interior of $\Sigma_{\Psi(\gamma)}$ with $\mathbb{B}^3$ is a function of ρ_γ.

Lemma 3.1.5. *Let Γ be a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$ acting on $\mathbb{B}^3$. Then ρ_γ is a strictly decreasing function of $\|\gamma\|^2$ on $\Gamma \setminus \mathrm{PSU}_2(\mathbb{C})$.*

Proof. By Lemma 3.1.4, one obtains that $\rho_\gamma = 1 - \left(\frac{\|\gamma\|^2+2}{\|\gamma\|^2-2}\right)^{\frac{1}{2}} + 2\left(\|\gamma\|^2 - 2\right)^{-\frac{1}{2}}$. Observe that

$$\begin{aligned} \|\gamma\|^2 - 2 &= |a|^2 + |b|^2 + |c|^2 + |d|^2 - 2\det(\gamma) \\ &\geq |a|^2 + |b|^2 + |c|^2 + |d|^2 - 2(|ad| + |bc|) \\ &\geq (|a| - |d|)^2 + (|b| - |c|)^2 \\ &\geq 0. \end{aligned}$$

By Lemma 1.2.24, the latter is an equality if and only if $\gamma \in \mathrm{SU}_2(\mathbb{C})$. Consider now the continuous function $f : \left]\sqrt{2}, +\infty\right[\longrightarrow \mathbb{R}$ given by $f(x) = 1 - (\frac{x^2+2}{x^2-2})^{\frac{1}{2}} + 2(x^2-2)^{\frac{-1}{2}}$. Then $f'(x) = -2x(x^2-2)^{-3/2}(x^2+2)^{-1/2}[-2+\sqrt{x^2+2}]$, which shows that f is a strictly decreasing function. From this the result follows. ∎

We now come to our algorithm to compute a finite set of generators for a subgroup of finite index in a discrete subgroup Γ of $\mathrm{Iso}^+(\mathbb{H}^3)$ which is of finite covolume. We describe this algorithm first under the assumption that the stabilizer $\Psi(\Gamma)_0$ of the point 0 in $\mathbb{B}^3$ is trivial. In the next section we avoid this assumption and explain how to change the algorithm slightly depending on Γ being the unit group of an order in a division algebra or a matrix algebra. Because of the concrete formulas obtained in Lemma 3.1.4, one may compute a fundamental domain in $\mathbb{B}^3$. Using the map η_0 one can then convert this to a fundamental domain in $\mathbb{H}^3$ which is more suitable for visualization. By Lemma 3.1.3, one may also directly compute a fundamental domain in $\mathbb{H}^3$. For $\gamma \in \mathrm{SL}_2(\mathbb{C})$, let

$$B(\gamma) = \overline{B(P_{\Psi(\gamma)}, R_{\Psi(\gamma)})} \cap \overline{\mathbb{B}^3},$$

the intersection of the Euclidean ball $\overline{B(P_{\Psi(\gamma)}, R_{\Psi(\gamma)})}$ with the closed unit ball $\overline{\mathbb{B}^3}$ (so this full closed ball determined by the sphere $\Sigma_{\Psi(\gamma)}$). Let $f : \Gamma \to \mathbb{C}$ be the map defined by $f(\gamma) = \|\gamma\|^2$. Then we order the elements of $\mathrm{Im}(f)$ in a strictly increasing sequence r_i for $i \geq 1$. Note that this is possible because of the discreteness of Γ. For $n \geq 1$, we define the sets $\mathcal{F}_n$ recursively in the following way:

$$\mathcal{F}_1 = \bigcup_{\gamma \in \Gamma} \{B(\gamma) \mid \text{ and } \|\gamma\|^2 = r_1 \text{ and } \gamma \neq 1\}$$

and for every $n \geq 1$ define

$$\mathcal{F}_n = \bigcup_{i=r_1}^{r_{n-1}} \mathcal{F}_i \cup \bigcup_{\gamma \in \Gamma} \{B(\gamma) \mid \|\gamma\|^2 = r_n \text{ and } B(\gamma) \not\subseteq \cup_{i=r_1}^{r_{n-1}} \mathcal{F}_i\}.$$

Later in all the examples we consider, the sequence r_i with $i \geq 1$ may be taken inside the set of natural numbers. Using these definitions, the following proposition describes the *Dirichlet Algorithm of Finite Covolume* (DAFC). Recall the definition of $D_\gamma(j)$ from (1.19). Moreover, we recall some concepts of hyperbolic geometry. Details may be found in [Rat06].

Definition 3.1.6. *A horosphere Σ of $\mathbb{B}^3$, based at a point P of $\mathbb{S}^2$, is the intersection with $\mathbb{B}^3$ of a Euclidean sphere in $\overline{\mathbb{B}^3}$ tangent to $\mathbb{S}^2$ in P. The interior of a horosphere is called a horoball.*

Definition 3.1.7. *Let P be a convex polyhedron in $\mathbb{H}^n$ or $\mathbb{B}^n$ respectively. An ideal point of P is a point of $\overline{P} \cap \mathbb{H}^n$, respectively of $\overline{P} \cap \mathbb{B}^n$, where $\overline{P}$ is the closure of P in $\mathbb{R}^n$.*

Proposition 3.1.8 (DAFC)**.** *Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{C})$ be a discrete group of finite covolume and with $\Psi(\Gamma)_0$ trivial. Then the following algorithm computes in a finite number of steps in $\mathbb{H}^3$ a finite set of generators of a subgroup of finite index of Γ.*

Step 1: Compute $\mathcal{F}_1$, $\mathcal{F}_2$, ... in this given order.

Step 2: Set N the minimum such that $\partial\mathbb{B}^3 \subseteq \mathcal{F}_N$.

Step 3: Set $\mathcal{F} = \overline{\mathbb{B}^3 \setminus \mathcal{F}_N} \subseteq \mathbb{B}^3$.

Step 4: Set $P = \eta_0^{-1}(\mathcal{F}) \subseteq \mathbb{H}^3$.

Then P is a finite-sided convex polyhedron given by $P = \bigcap_{\gamma \in L} D_{\gamma^{-1}}(j)$, where $L = \{\gamma \in \Gamma \mid B(\gamma) \in \mathcal{F}_N\}$ (a finite set). Moreover, P contains the Dirichlet fundamental polyhedron of centre j for Γ and $\langle \gamma \mid \gamma \in L \rangle$ is a subgroup of finite index in Γ.

Proof. By Lemma 1.2.3, the set $\{B(\gamma) \mid \|\gamma\|^2 = n\}$ is finite. So the sets $\mathcal{F}_i$ for $i \geq 1$ are also finite and thus they are computable in a finite number of steps. Since Γ is of finite covolume, Lemma 1.2.38 and Lemma 1.2.39 imply that the Dirichlet fundamental domain for Γ has finitely many sides. By Lemma 3.1.5, there exists $M > 0$ such that

$$\bigcap_{\gamma; \|\gamma\|^2 \leq M} D_{\gamma^{-1}}(j)$$

is the Dirichlet fundamental polyhedron of centre j for Γ. In that case $\partial\mathbb{B}^3 \subseteq \mathcal{F}_M$ and thus the DAFC stops in a finite number of steps (at the latest at $\mathcal{F}_M$, but maybe earlier). Set $L = \{\gamma \in \Gamma \mid B(\gamma) \in \mathcal{F}_N\}$. By the construction of $\mathcal{F}_N$, $P = \bigcap_{\gamma \in L} D_{\gamma^{-1}}(j)$ and this is clearly a finite-sided convex polyhedron, which contains the Dirichlet fundamental polyhedron with centre j for Γ.

We now show that $\langle \gamma \mid \gamma \in L \rangle$ is a subgroup of finite index in Γ. We claim that P has finite covolume in $\mathbb{H}^3$. Indeed, this is trivial if P is compact. So, suppose P is not compact and let $I = \{p_1, \ldots, p_m\}$ be the set containing the ideal points of P that are different from ∞. The set I is indeed finite, because P has only finitely many sides. At every $p_i \neq \infty$, draw a horoball Σ_i based at p_i, that is only intersecting the sides of

P that intersect p_i and such that the vertical projection of Σ_i to $\partial\mathbb{H}^3$ does not contain any other ideal point p_i. At the ideal point ∞ (if it exists), draw a horoball of the form $\{z+rj \in \mathbb{H}^3 \mid z \in C_\infty, r > \epsilon_\infty\}$, for some compact subset C_∞ of $\partial\mathbb{H}^3$ containing $P \cap \partial\mathbb{H}^3$ and some $\epsilon_\infty > 0$. Call this horoball Σ_∞. It is easy to see that

$$Vol(P) \leq Vol\left(P \setminus \left(\bigcup_{i=1}^{m} \Sigma_i \cup \Sigma_\infty\right)\right) + \sum_{i=1}^{m} Vol(P \cap \Sigma_i) + Vol(\Sigma_\infty).$$

As $P \setminus (\bigcup_{i=1}^{m} \Sigma_i \cup \Sigma_\infty)$ is compact, $Vol\left(P \setminus (\bigcup_{i=1}^{m} \Sigma_i \cup \Sigma_\infty)\right) < \infty$. By (1.18),

$$Vol(\Sigma_\infty) = \mu(C_\infty) \int_{\epsilon_\infty}^{\infty} \frac{1}{r^3} dr,$$

where μ is the classical Lebesgue measure. This is clearly finite.

For $1 \leq i \leq m$, consider $Vol(P \cap \Sigma_i)$. Observe that $P \cap \Sigma_i$ is the hyperbolic cone of base $P \cap \partial\Sigma_i$ and vertex p_i. Let $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ be a transformation sending p_i to the point ∞. Then $\gamma(P \cap \Sigma_i)$ is the cone based at $\gamma(P \cap \partial\Sigma_i)$ and with vertex ∞. By the construction of Σ_i, $P \cap \partial\Sigma_i$ is contained in a compact set C and thus $\gamma(P \cap \Sigma_i)$ is contained in the cone based at $\gamma(C)$ and with vertex ∞. Hence there exists a compact set C' in $\partial\mathbb{H}^3$ and an $\epsilon > 0$ such that

$$\gamma(P \cap \Sigma_i) \subseteq \{z + rj \in \mathbb{H}^3 \mid z \in C', r > \epsilon\}.$$

Denote the latter set by A. As before, by (1.18),

$$vol(A) = \mu(C') \int_{\epsilon}^{\infty} \frac{1}{r^3} dr,$$

which is finite. Thus this proves the claim.

Clearly the Dirichlet fundamental polyhedron for $\langle \gamma \mid \gamma \in L \rangle$ is contained in P and hence it has finite volume. Thus, by Proposition 1.2.33, $\langle \gamma \mid \gamma \in L \rangle$ has finite index in Γ. This finishes the proof of the algorithm. ∎

Remark 3.1.9. *The fact that P has finite volume could also have been deduced from [Vin85, Proposition 4.2].*

Note that the generating set computed by the DAFC is not necessarily the generating set of the complete group Γ. In fact the algorithm stops as soon as it has found a polyhedron of finite volume and this just guarantees a generating set of a subgroup of finite index in Γ. However, this is sufficient for our main purpose: obtaining finitely many generators up to commensurability in $\mathcal{U}(\mathbb{Z}G)$. Nevertheless, in case the group Γ is cocompact, the next proposition, which is based on [CJLdR04, Proposition 3.1 and Proposition 3.2], gives a finite algorithm to compute the generating set of the complete group Γ. Moreover, in that case, the resulting polyhedron P is the Dirichlet fundamental polyhedron for Γ.

Proposition 3.1.10 (Refined DAFC)**.** *Let* $\Gamma \leq \mathrm{PSL}_2(\mathbb{C})$ *be a cocompact discrete group with* $\Psi(\Gamma)_0$ *trivial. Suppose* $P = \eta_0^{-1}(\overline{\mathbb{B}^3 \setminus \mathcal{F}_N})$ *is the polyhedron in* $\mathbb{H}^3$ *given by the DAFC. Then the following is an algorithm which is computable in a finite number of steps.*

Step 1: Compute the finite number of vertices V_i *of* P.

Step 2: Compute $k = cosh^{-1}(\frac{r_N}{2})$, $r = max\{\frac{k}{2}, \rho(j, V_i) \mid V_i \text{ vertex of } P\}$ *and* $\tilde{N} = 2cosh(2r)$.

Step 3: Set $\tilde{\mathcal{F}} = \eta_0^{-1}(\overline{\mathbb{B}^3 \setminus \mathcal{F}_{\tilde{N}}})$.

Set $L = \{\gamma \in \Gamma \mid B(\gamma) \subseteq \mathcal{F}_{\tilde{N}}\}$. *Then* L *is a generating set for* Γ *and* $\tilde{F} = \bigcap_{\gamma \in L} D_{\gamma^{-1}}(j)$ *is the Dirichlet fundamental polyhedron of centre* j *for* Γ.

Proof. By construction of $\tilde{\mathcal{F}}$, it is clear that $\tilde{F} = \bigcap_{\gamma \in L} D_{\gamma^{-1}}(j)$. We now prove that this is the Dirichlet fundamental polyhedron of centre j for Γ. First, observe that, by (1.12),

$$k = \max\{\rho(\gamma(j), j) \mid \|\gamma\|^2 \leq r_N\}.$$

By contradiction, suppose that $\tilde{\mathcal{F}}$ is not the Dirichlet fundamental polyhedron for Γ. Then there exists $u \in \tilde{\mathcal{F}}$ such that $u \notin D_{\gamma_0}(j)$ for some γ_0, with $\|\gamma_0\|^2 > \tilde{N}$ (the latter is due to the construction of $\tilde{\mathcal{F}}$). As $u \notin D_{\gamma_0}(j)$, $\rho(u, \gamma_0(j)) < \rho(u, j)$. As $u \in \tilde{\mathcal{F}}$, $\rho(u, j) \leq r$. By (1.12), we have

$$2cosh\rho(j, \gamma_0(j)) = \|\gamma_0\|^2 > \tilde{N} = 2cosh(2r),$$

and thus we get the following inequality

$$2r < \rho(j, \gamma_0(j)) \leq \rho(j, u) + \rho(u, \gamma_0(j)) < 2\rho(u, j) \leq 2r.$$

The latter is of course a contradiction and this proves that $\tilde{\mathcal{F}}$ is indeed the Dirichlet fundamental polyhedron for Γ.

By Lemma 1.2.41 and Remark 2.4.17, $\Gamma = \langle \gamma \mid \gamma \in L \rangle$, which proves the result. ∎

Note that for our application to the unit group of integral group rings, the groups we are working with will be discrete subgroups of unit groups of orders in quaternion division algebras or some discrete subgroups of $\mathrm{SL}_2(\mathbb{C})$. The DAFC cannot be directly applied to such groups, but only to their projections in $\mathrm{PSL}_2(\mathbb{C})$. More concretely, if one is interested in finding generators, up to finite index, for a discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{C})$, one may use the DAFC to get generators for the projection of Γ in $\mathrm{PSL}_2(\mathbb{C})$. The generators of Γ will then be the pre-images in $\mathrm{SL}_2(\mathbb{C})$ of the discovered units together with the (finite) kernel of the action of Γ on $\mathbb{H}^3$. If, on the other hand, Γ is the discrete group $\mathrm{SL}_1\left(\mathcal{H}\left(\frac{a,b}{\mathcal{O}_K}\right)\right)$ determined by a (division) quaternion algebra $\mathcal{H}\left(\frac{a,b}{\mathcal{O}_K}\right)$, then one has to proceed as explained in Subsection 1.2.4: first one has to embed $\mathrm{SL}_1\left(\mathcal{H}\left(\frac{a,b}{\mathcal{O}_K}\right)\right)$ in $\mathrm{SL}_2(\mathbb{C})$ and then consider its image in $\mathrm{PSL}_2(\mathbb{C})$. To get a set of generators, one has to add the kernel of the action (i.e. $\mathrm{I}(a,b,K)$) to the pre-images of the set of generators given by the DAFC.

As will be shown in several applications, often Dirichlet fundamental polyhedra for discrete groups contain symmetries. These can be used to shorten the DAFC and to list the generators of the discrete group in a more compact manner (see the next section for details). The following proposition that describes some isomorphisms and some involutions of $\mathrm{PSL}_2(\mathbb{C})$ will be useful to describe some symmetries for the applications under consideration.

Proposition 3.1.11. *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) \setminus \mathrm{SU}_2(\mathbb{C})$ *with* $|a|^2 + |c|^2 \neq 1$ *(so* Σ_γ *is a Euclidean sphere by Lemma 3.1.3). Denote by* σ *the conjugation by the matrix* $\begin{pmatrix} \sqrt{i} & 0 \\ 0 & \sqrt{-i} \end{pmatrix}$*, by* δ *the conjugation by the matrix* $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$*. Let* $\tau(\gamma) = \overline{\gamma}$ *denote complex conjugation of the entries of* γ *and define* $\phi = \sigma^2 \circ \delta \circ \tau$*. Then in* $\mathbb{H}^3$

1. $P_{\phi(\gamma)}$ *is the reflection of* P_γ *in* $\mathbb{S}^2$*,*
2. τ *induces a reflection in the plane spanned by* 1 *and* j*, i.e.* $P_{\tau(\gamma)} = \overline{P_\gamma}$ *and* $R_{\tau(\gamma)} = R_\gamma$*,*
3. σ^2 *induces a reflection in the axis* $x = y = 0$*, i.e.* $P_{\sigma^2(\gamma)} = -P_\gamma$ *and* $R_{\sigma^2(\gamma)} = R_\gamma$*,*
4. σ *restricted to* $\partial\mathbb{H}^3 = \{z \in \mathbb{C}\}$ *induces a rotation of ninety degrees around the point of origin, i.e.* $P_{\sigma(\gamma)} = iP_\gamma$ *and* $R_{\sigma(\gamma)} = R_\gamma$*.*

Proof. To prove the first item, first note that if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\phi(\gamma) = \begin{pmatrix} \overline{d} & \overline{c} \\ \overline{b} & \overline{a} \end{pmatrix}.$$

By Lemma 3.1.4, $P_{\Psi(\phi(\gamma))} = \pi(P_{\Psi(\gamma)})$, where π is the reflection in the plane $\{(x,y,z) \in \mathbb{R}^3 \mid z = 0\}$. Since $\eta_0 : \mathbb{H}^3 \to \mathbb{B}^3$ is equivariant, it follows that $P_{\phi(\gamma)}$ is the reflection of P_γ in $\mathbb{S}^2$, but the radius is not necessarily maintained.

The three following items follow straightforward from Lemma 3.1.3. ∎

Remark 3.1.12. *In* $\mathbb{H}^2$*,* ϕ *is a reflection in* $\mathbb{S}^1$*,* τ *is a reflection in the imaginary axis and* σ^2 *has the same action on* P_γ *as* τ *does, for every* $\gamma \in \mathrm{SL}_2(\mathbb{C})$*.*

Finally the following result will be useful when implementing the algorithm for a given group. Note that it gives an easily verifiable criteria for cocompactness. Its proof follows easily from Lemma 3.1.4.

Lemma 3.1.13. *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$ *and* $\gamma \notin \mathrm{SU}_2(\mathbb{C})$*. Then*

1. $0 \notin \Sigma_{\Psi(\gamma)}$.
2. $j \in \Sigma_{\Psi(\gamma)}$ *if and only if* $|a|^2 + |c|^2 = 1$.
3. $j \in B(\gamma)$ *if and only if* $|a|^2 + |c|^2 < 1$.
4. $-j \in \Sigma_{\Psi(\gamma)}$ *if and only if* $|b|^2 + |d|^2 = 1$.
5. $-j \in B(\gamma)$ *if and only if* $|b|^2 + |d|^2 < 1$.

Note that all the above lemmas and propositions may be established in the two dimensional model. Let $\gamma \in \mathrm{SL}_2(\mathbb{R})$. As a Möbius transformation, γ acts on $\mathbb{H}^2$. Similar, as in Proposition 1.2.28, the map $\eta_0 : \mathbb{H}^2 \longrightarrow \mathbb{B}^2$, given by $\eta_0(P) = (P - i)(-iP + 1)^{-1}$ gives an isometry between the two models. Proceeding as in the 3-dimensional model, we obtain explicit formulas for the bisectors. The role of j is played by i and, since we are in the commutative setting, calculations are easier.

3.2 Applications

In this section we apply the DAFC to several examples. We divide this section in two subsections. First we treat examples coming from division algebras and hence ameliorate [CJLdR04]. In fact in [CJLdR04], the authors were only able to treat small examples, because of the absence of concrete formulas. Then, in the second subsection we apply our method to matrix algebras and hence show that our method, to determine generators up to finite index, does not require the condition of cocompactness, which was required in [CJLdR04].

3.2.1 Division Algebras

All examples given are discrete and cocompact subgroups of $\mathrm{Iso}^+(\mathbb{B}^n)$, $n = 2, 3$. In fact we could just apply the DAFC to the examples. However as these groups are cocompact, we get a sort of "starting point", which will make the implementation much easier. To simplify notations, we will from now on denote by Γ as well the group Γ as its embedding in $\mathrm{SL}_2(\mathbb{C})$. One encounters two situations.

Case I: $\Psi(\Gamma)_0$ is trivial. In this case we may just apply the DAFC such as it is stated in Proposition 3.1.8 (respectively 3.1.10). However Lemma 3.1.13 and the fact that the group is cocompact gives us a way of finding a "special" bisector with which we may start. In fact as the group is cocompact, there has to exist $\gamma_0 \in \Gamma$ whose bisector $\Sigma_{\Psi(\gamma_0)}$ separates j and the origin 0 in $\mathbb{B}^3$. Because of Lemma 3.1.13, $\gamma_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $|a|^2 + |c|^2 < 1$. Thus we may look for such a γ_0 with smallest norm and set

$$V = \mathrm{Exterior}(\Sigma_{\Psi(\gamma_0)}) \cap \partial\mathbb{B}^3.$$

Then the DAFC stops when $V \subseteq \mathcal{F}_N$, for some $N \in \mathbb{N}$. As stated before this is absolutely not necessary for the DAFC to run. However if one considers the situation in the upper half-space model, instead of the ball model, one notices that, because of cocompactness, there has to exist one bisector separating the points j and ∞. In fact the upper half-space model being not as symmetric as the ball model, this special bisector represents some kind of "upper dome" (see Figures 3.1, 3.2 or 3.5 for examples) which guarantees the fundamental domain to be compact.

Case II:$\Psi(\Gamma)_0$ **is non-trivial**. In this case we first determine a fundamental domain $\mathcal{F}_0$ for $\Psi(\Gamma)_0$, a polyhedron with the origin on its boundary. As Γ is discontinuous and $0 \in \mathbb{B}^3$, the group $\Psi(\Gamma)_0$ is finite. In all the applications we consider, the stabilizer $\Psi(\Gamma)_0$ is a small group and hence $\mathcal{F}_0$ is relatively easy to determine. As the fundamental domain is given by the intersection of some construction, based on a Dirichlet fundamental domain, with $\mathcal{F}_0$ (see Theorem 1.2.36), we can modify the DAFC in such a way that it does not stop when the whole border of $\mathbb{B}^n$ is covered by the different F_i for $i \geq 2$, but earlier. In fact let

$$V = \partial(\mathcal{F}_0) \cap \partial\mathbb{B}^n.$$

Moreover the definition of the $\mathcal{F}_i$ has to be adapted in such a way that one only considers $B(\gamma)$ for $\gamma \in \Gamma \backslash \Psi^{-1}(\Psi(\Gamma)_0)$, and not in the whole group Γ. Then we will find a generating set for a subgroup of finite index of Γ by letting the DAFC stop when $V \subseteq \mathcal{F}_N$, for $N \in \mathbb{N}$. By Lemma 1.2.41 and Remark 2.4.17, the output is $\langle G_0, \gamma_1, \cdots, \gamma_n \rangle$ which has finite index in Γ and where G_0 denotes a generating set for $\Psi(\Gamma)_0$ and γ_i for $1 \leq i \leq n$ are such that $B(\gamma_i) \in \mathcal{F}_N$.

Generators for Γ: Once obtained the polyhedron P, given by the DAFC, one finds a fundamental domain for Γ by applying the refined DAFC of Proposition 3.1.10 and hence also a set of generators.

We now have all the tools to implement the DAFC in a way that the reader can follow and easily reproduce. Our first example is a generalized quaternion division algebra $\mathcal{H}\left(\frac{a,b}{K}\right)$ with $K = \mathbb{Q}(\sqrt{-d})$, with d a positive square-free integer. By Theorem 1.2.50, the unit group of any order of $\mathcal{H}\left(\frac{a,b}{K}\right)$ is a cocompact discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$. In particular, we revisit the work of [CJLdR04], we consider Example 8 of chapter X of [EGM98] (page 474), and one example in dimension 2.

To implement the DAFC, we find an additive basis for the ring of integers of K. This, together with the fact that we only consider reduced norm one elements in the unit group, leads to a system of Diophantine equations whose solution set is Γ. We get a sieve parameterizing the system by the matrix norm of the elements. For small values of d the algorithm easily can be done by hand, and for larger d, we made use of the software package Mathematica, to get a polyhedron that contains the Dirichlet fundamental domain. Hence we also get a set of generators (up to finite index). Examples of this algorithm may be found on the PhD candidate's homepage.

We first revisit [CJLdR04]. In this case we get a particular nice system of Diophantine equations one of which is to write a number as the sum of four squares. Consider

$\mathcal{H}(K) = \mathcal{H}\left(\frac{-1,-1}{K}\right)$, with $K = \mathbb{Q}(\sqrt{-d})$, with d a positive square-free integer. Let $\mathcal{O}_K$ be the ring of integers of K. We will consider the cases $d = 15$ and $d = 23$. In fact, only for $d \equiv 7 \bmod 8$, by Corollary 1.2.52, the quaternion algebras $\mathcal{H}(K)$ is a division algebra. Thus, by Theorem 1.2.50, $\Gamma = \mathrm{SL}_1\left(\mathcal{H}\left(\frac{-1,-1}{\mathcal{O}_K}\right)\right) = \{x \in \mathcal{H}\left(\frac{-1,-1}{\mathcal{O}_K}\right) \mid N(x) = 1\}$, acts discretely and cocompactly on $\mathbb{H}^3$.

We have that $\Psi(\Gamma)_0 = \Psi(\Gamma_j)$ and it is easily seen that $\Gamma_j = \langle i, j\rangle \cong Q_8$, the quaternion group of order 8 and that $\eta_0^{-1}(\mathcal{F}_0)$ can be taken to be that part of the unit ball centred at the origin whose projection on $\partial\mathbb{H}^3$ is the upper half of the unit circle, i.e. $\{z \in \mathbb{C} \mid |z| \leq 1, Im(z) \geq 0\}$.

We have that $\mathcal{O}_K = \mathrm{span}_{\mathbb{Z}}[1, w]$, where $w = \frac{1+\sqrt{-d}}{2}$. Write $u = u_0 + u_1 i + u_2 j + u_3 k \in \mathcal{H}(K)$ and $u_t = x_t + y_t w$. Define $x = (x_0, x_1, x_2, x_3)$ and $y = (y_0, y_1, y_2, y_3)$. So u is determined by the vector (x, y). Furthermore, on the contrary to (1.28), in this example we use the embedding

$$u \mapsto \gamma_u = \begin{pmatrix} u_0 + u_1 i & u_2 + u_3 i \\ -u_2 + u_3 i & u_0 - u_1 i \end{pmatrix}$$

as used in [CJLdR04].

The next lemma gives formulas to compute $\|\gamma_u\|$ as well as the bisector associated to an element γ_u. These concepts are necessary to implement the DAFC.

Lemma 3.2.1. *Let $u = u_0 + u_1 i + u_2 j + u_3 k \in \mathcal{H}(K)$ and $N(u) = 1$. Then*

$$\begin{cases} |x|^2 - (\frac{d+1}{4})|y|^2 = 1, \\ 2\langle x|y\rangle + |y|^2 = 0. \end{cases} \tag{3.2}$$

Moreover, the following hold.

(i) $\|\gamma_u\|^2 = 2 + d|y|^2 \in 2\mathbb{Z}$,

(ii) $|P_{\Psi(\gamma)}|^2 = \frac{4+d|y|^2}{d|y|^2}$,

(iii) $R^2_{\Psi(\gamma)} = \frac{4}{d|y|^2}$.

Proof. The first part of the statement follows by using the integral basis and the condition on the reduced norm. The second part follows from Lemma 3.1.4. ∎

By Lemma 3.2.1 item (i), the norm of γ_u only depends on $\|y\|^2$ and hence we may order the different sets $\mathcal{F}_i$ appearing in the DAFC by $\|y\|^2$ instead of by the norm of γ_u. Also because of the first two defining equations of Lemma 3.2.1, we may define the sequence r_n used in the definition of the sets $\mathcal{F}_n$ appearing in Proposition 3.1.8 as follows: for $n \geq 1$, $r_n = 2n$. So the definition of the new sets $\mathcal{F}'_n$ will be as follows.

$$\mathcal{F}'_1 = \bigcup_{\gamma\in\Gamma} \{B(\gamma) \mid \|y\|^2 = 2 \text{ and } \gamma \neq 1\}, \tag{3.3}$$

$$\mathcal{F}'_n = \bigcup_{i=r_1}^{r_{n-1}} \mathcal{F}'_i \cup \bigcup_{\gamma\in\Gamma} \{B(\gamma) \mid \|y\|^2 = 2n \text{ and } B(\gamma) \not\subseteq \cup_{i=r_1}^{r_{n-1}} \mathcal{F}'_i\}. \tag{3.4}$$

Let $J(x) = (x_1, -x_0, -x_3, x_2)$ and $S(x) = (-x_3, x_2, -x_1, x_0)$ and $J(y)$ is analogously defined. Then J and S are skew orthogonal linear maps and $\langle S, J\rangle$ is isomorphic to Q_8.

Lemma 3.2.2. *Let* $u \in \mathcal{H}(K), N(u) = 1$ *and* $\gamma = \begin{pmatrix} u_0 + u_1 i & u_2 + u_3 i \\ -u_2 + u_3 i & u_0 - u_1 i \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *Then*

1. $|a|^2 + |c|^2 = 1 + \frac{d}{2}|y|^2 + \langle J(x)|y\rangle \cdot \sqrt{d}$,
2. $|b|^2 + |d|^2 = 1 + \frac{d}{2}|y|^2 - \langle J(x)|y\rangle \cdot \sqrt{d}$,
3. $\overline{a}b + \overline{c}d = [\langle -S(x)|y\rangle + i\langle SJ(x)|y\rangle] \cdot \sqrt{d}$,
4. $|\overline{a}b + \overline{c}d|^2 = (1 + \frac{d}{2}|y|^2)^2 - d\langle J(x)|y\rangle^2 - 1$,
5. $R_{\Psi(\gamma)} = \frac{2\sqrt{d}}{d|y|}$,
6. $P_{\Psi(\gamma)} = \frac{2\sqrt{d}}{d|y|^2} \cdot [\langle S(x)|y\rangle - \langle S(x)|J(y)\rangle i - \langle J(x)|y\rangle j\ ]$.

We omit the proof as it follows by straightforward computations. The formulas show that the centres of the bisectors in the ball model, up to a scalar in $\mathbb{Z}[\sqrt{d}]$, belong to $\mathbb{Z}^3$ and in the upper half-space model they belong to $\mathbb{Z}[i]$, also up to a scalar in $\mathbb{Z}[\sqrt{d}]$. The group of symmetries of a fundamental domain for Γ contains $\langle \sigma, \tau\rangle$, where σ and τ are as defined in Proposition 3.1.11, and $\langle \Gamma, \sigma, \tau\rangle$ is a discrete group. Note that Proposition 3.1.11 also indicates another symmetry which is inversion in $\mathbb{S}^2$ and which is denoted by ϕ. This symmetry also acts on the tessellation of $\mathbb{H}^3$ induced by Γ. However $\mathbb{S}^2 \cap \mathbb{H}^3$ is part of the boundary of the fundamental domain for Γ_j and hence also part of the boundary of $\mathcal{F}$. Hence in this case this symmetry is lost. In the next example we will work with a group Γ having a trivial stabilizer Γ_j and hence the symmetry ϕ will show up.

With all this information we are ready to implement the DAFC. We do this for the cases $d = 15$ and $d = 23$. We analyse the case $d = 15$ in details. As stated above, $\Psi(\langle i, j\rangle) = \Psi(\Gamma)_0$ and $\eta_0^{-1}(\mathcal{F}_0)$ is as described above. We define $V = \partial\mathcal{F}_0 \cap \partial\mathbb{B}^3$ and hence the DAFC searches for N minimal such that $V \subseteq \mathcal{F}_N$. Moreover by using the symmetries we can make the DAFC even faster. In the upper half-space model this may be seen in the following way: $V \subseteq \mathcal{F}_N$ means in the upper half-space model that the "base" of $\eta_0^{-1}(\mathcal{F}_0)$ is covered by Euclidean spheres. Mathematically this means that for every point $P \in \{z + rj \in \mathbb{H}^3 \mid r = 0,\ |z|^2 \leq 1 \text{ and } Im(z) \geq 0\}$, there exists $\gamma \in \Gamma$ such that $P \in \eta_0^{-1}(B(\gamma))$. We compute the sets F'_n for $n \geq 1$. However because of the symmetries we do not need to "cover" the whole "base" of $\eta_0^{-1}(\mathcal{F}_0)$, but only one quarter (as shown in figure 3.3(a)). In fact, by Proposition 3.1.11, if we denote the set of spheres shown in Figure 3.3(a) by X_{15}, then $X_{15} \cup \sigma(X_{15}) \cup \tau\sigma^2(X_{15} \cup \sigma(X_{15}))$ "covers" the whole $\eta_0^{-1}(\mathcal{F}_0) \cap \partial\mathbb{H}^3$. Hence, if we denote the quarter of $\eta_0^{-1}(\mathcal{F}_0)$ depicted in Figure 3.3(a) by Δ, as a supplementary condition in the definition of $\mathcal{F}'_n$ we set that $B(\gamma) \cap (\eta_0(\Delta) \cap \partial\mathbb{B}^3) \neq \emptyset$. At $N = 4$, the algorithm stops. Moreover including all the conditions, we obtain 5 different $B(\gamma)$ in $\mathcal{F}_1'$, no $B(\gamma)$ in $\mathcal{F}_2'$ nor in $\mathcal{F}_3'$ and finally 4 more $B(\gamma)$ in $\mathcal{F}_4'$. These give us the 9 different bisectors shown in Figure 3.3(a). Thus the set $S_{15} = \Psi^{-1}(\Psi(\Gamma)_0) \cup \{g(\gamma) \mid g \in \langle \sigma, \tau \rangle,\ \eta_0^{-1}(B(\gamma)) \in X_{15}\}$ gives a generating set for a subgroup of finite index in Γ. We may also apply the refined DAFC. Therefore we first compute $\max\{\rho(j, V_i) \mid V_i \text{ vertex of } \mathcal{F}\}$ which gives us ~ 3.33. We also compute k. By Proposition 3.1.10, the definition of k is based on the maximal value $\|\gamma\|^2$ takes. In this case the maximal value is $N = 4$, which gives a maximal value $r_N = 8$, which gives the maximal value for $\|y\|^2$. By Lemma 3.2.1 we get a maximal value $\|\gamma\|^2 = 2+15\cdot 8 = 122$. This gives $k = cosh^{-1}(\frac{122}{2}) \approx 4.8$. Thus $r = \max\{\{\frac{k}{2}\} \cup \{\rho(j, V_i) \mid V_i \text{ vertex of } \mathcal{F}\} \approx 3.33$ and $\tilde{N} = 2cosh(2 \cdot r) \approx 780.6$. As $2 + 15 \cdot 52 = 782$, we have to compute $\mathcal{F}_n'$ for $5 \leq n \leq \frac{52}{2} = 26$. However we find that all $\mathcal{F}_n'$, for $5 \leq n \leq 26$, are empty and hence S_{15} is a generating set for the whole group and $\mathcal{F}$ is a fundamental domain for it. All this and the case $d = 23$ is summed up in the following theorem.

Note that in the next theorem $\Psi^{-1}(\Psi(\Gamma)_0) = \langle i, j \rangle$.

Theorem 3.2.3. *Let $\Gamma = \mathrm{SL}_1(\mathcal{H}(-1, -1, \mathbb{Z}[\frac{1+\sqrt{-d}}{2}]))$ and let Y_d be a finite set of units γ such that $B(\gamma) \in \mathcal{F}_N$ and let $S_d = \Psi^{-1}(\Psi(\Gamma)_0) \cup \{g(\gamma) \mid g \in \langle \sigma, \tau \rangle,\ \gamma \in Y_d\}$.*

1. *If $d = 15$ then $\Gamma = \langle S_d \rangle$, where*

$$\begin{aligned} Y_{15} = \{&2 + (-1+\omega)i + 2j + \omega k, 2 + 2i + (1-\omega)j + \omega k, \\ &-2 + (-2+\omega)i + (1+\omega)j, 2 + (-1+\omega)i + \omega j - 2k, \\ &(-2+\omega) + (1+\omega)i + 2j, (-4+2\omega) + 2i + 3j + \\ &(-2-2\omega)k, (-4+2\omega) + 3i + 2j + (-2-2\omega)k, \\ &(-4+2\omega) + 3i - 2j + (-2-2\omega)k, (-4+2\omega) + \\ &(2+2\omega)i + 3j - 2k\}\,. \end{aligned}$$

2. *If $d = 23$ then $\langle S_d \rangle$ has finite index in Γ, where*

$$\begin{aligned} Y_{23} = \{&(-3+\omega) + (2+\omega)i, (-2+\omega) - 2i + (1+\omega)j - 2k, \\ &(-3+\omega) - (2+\omega)k, -2 - 2i + (-2+\omega)j - (1+\omega)k, \\ &-2 + (-2+\omega)i + (1+\omega)j - 2k, -4 + 2i - 2j + \\ &(1-2\omega)k, -4 + (-1+2\omega)i + 2j + 2k, (-4+2) - 2i + \\ &(1+\omega)j - (4+\omega)k, (-5+\omega) + (2-\omega)i - 2j - (2+2\omega)k, \\ &(-4+2\omega) + (1+\omega)i + 2j - (4+\omega)k, (-4-\omega) + (1+\omega)i \\ &-2j + (4-2\omega)k, (-5+\omega) + 2i + (-2+\omega)j - (2+2\omega)k, \\ &(-4-\omega) + (-4+2\omega)i + (1+\omega)j + 2k, (-5+\omega) + \\ &(2+2\omega)k + (2-\omega)j + 2k, -6 - 3i + (-2+2\omega)j - 2\omega k, \\ &-5 + (2+2\omega)i + 2j + (4-2\omega)k, -6 + 2\omega i + 3j + \\ &(2-2\omega)k, -5 - 2i + (-4+2\omega)j - (2+2\omega)k, -5 + \\ &(-4+2\omega)i + (2+2\omega)j - 2k, -6 + 2\omega i - 3j + (2-2\omega)k\}\,. \end{aligned}$$

Figures 3.3(a) and 3.3(b) show the projection on $\partial\mathbb{H}^3$ of the bisectors of the elements in Y_d. Figure 3.1 and 3.2 show the polyhedron given by the DAFC that contains a fundamental domain for $\langle S_d \rangle$ for $d = 15$ and $d = 23$ respectively. Observe, that by the computations above, the polyhedron depicted in Figure 3.1 is the Dirichlet fundamental polyhedron of centre j for $\mathrm{SL}_1\left(\mathcal{H}\left(\frac{-1,-1}{\mathbb{Z}[\frac{1+\sqrt{-15}}{2}]}\right)\right)$.

Remark 3.2.4. *In the context of a division algebra, Pell and Gauss units were constructed in [JPSF09, JSF13]. For some of these units γ, say, we have that $P_\gamma = 0$ and these are typically associated to hyperbolic elements. Together, these units seem to generate a subgroup of finite index. If so, these can be used as generic units in the group ring problem. Still in the same context, in [JPSF09] it was proved that* $\mathrm{SL}_1\left(\mathcal{H}\left(\frac{-1,-1}{\mathcal{O}_K}\right)\right)$ *is a hyperbolic group for all positive* $d \equiv 7 \pmod 8$*. This line of classification was introduced and initiated in [JPP05].*

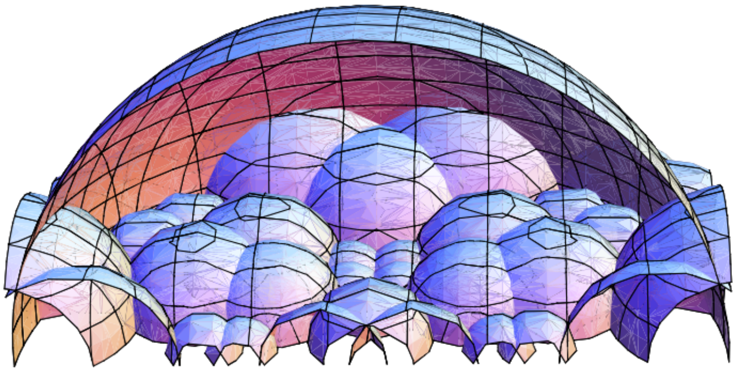


Figure 3.1: Polyhedron associated to S_{15}

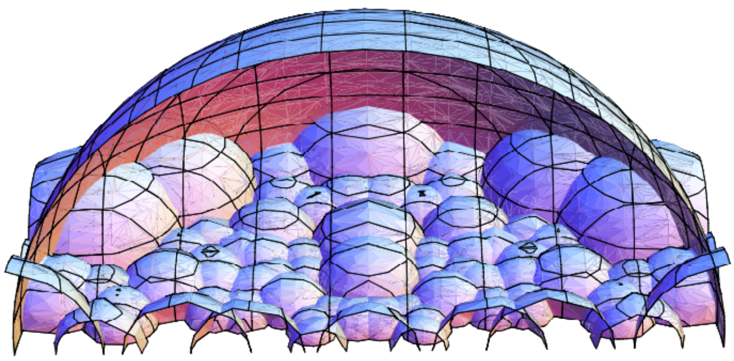
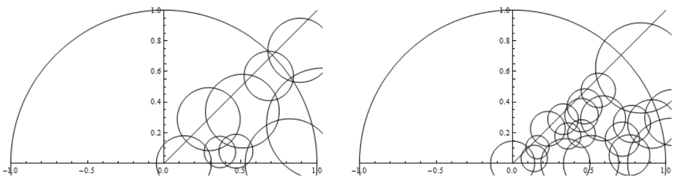


Figure 3.2: Polyhedron associated to S_{23}



(a) Projection of bisectors in Y_{15}. (b) Projection of bisectors in Y_{23}.

Figure 3.3:

The next cases we treat are division algebras of the form $\mathcal{H}\left(\frac{a,b}{\mathbb{Q}(i')}\right)$, $(i' = \sqrt{-1})$, with $0 < a < b$ integers and $\Gamma = \mathrm{SL}_1\left(\mathcal{H}\left(\frac{a,b}{\mathbb{Z}[i']}\right)\right)$. For $a = 2$ and $b = 5$, this can be found in [EGM98, Chapter 10], and it is commensurable with $\mathrm{PO}_4(\mathbb{Z}, q)$, where $q(x, y, z, w) = -2x^2 - 5y^2 - 10z^2 + w^2$.

Let $u \in \Gamma$ and write $u = u_0 + u_1 i + u_2 j + u_3 k$, with $u_t \in \mathbb{Z}[i']$. In this case

$$\gamma_u = \begin{pmatrix} u_0 + u_1\sqrt{a} & u_2\sqrt{b} + u_3\sqrt{ab} \\ u_2\sqrt{b} - u_3\sqrt{ab} & u_0 - u_1\sqrt{a} \end{pmatrix},$$

as stated in (1.28). Let $u_t = x_t + y_t i'$, $x = (x_0, x_1, x_2, x_3)$, $y = (y_0, y_1, y_2, y_3)$, $q(x) = x_0^2 - ax_1^2 - bx_2^2 + abx_3^2$ and let $B(x, y)$ be the bilinear form associated to $q(x)$. The next lemma describes Γ as a subgroup of $\mathrm{PSL}_2(\mathbb{C})$ in terms of a system of Diophantine equations.

Lemma 3.2.5. *Let* $u = u_0 + u_1 i + u_2 j + u_3 k \in \mathrm{SL}_1\left(\mathcal{H}\left(\frac{a,b}{\mathbb{Z}[i']}\right)\right)$*, and let* γ_u *and the vectors* (x_0, x_1, x_2, x_3) *and* (y_0, y_1, y_2, y_3) *be as described above. Then*

$$\begin{cases} q(x) - q(y) = 1, \\ B(x, y) = 0. \end{cases}$$

Moreover if we set $x_0^2 + ay_1^2 + by_2^2 + abx_3^2 = n \in \mathbb{N}$*, we obtain the following system.*

$$\begin{cases} x_0^2 + ay_1^2 + by_2^2 + abx_3^2 = n, \\ y_0^2 + ax_1^2 + bx_2^2 + aby_3^2 = n - 1, \\ B(x, y) = 0, \\ \|\gamma_u\|^2 = 4n - 2. \end{cases} \tag{3.5}$$

Proof. The first set of equations follows from the fact that the determinant of γ_u is 1, Γ being a subgroup of $\mathrm{PSL}_2(\mathbb{C})$. For the three first equations of the second set we replace $x_0^2 + ay_1^2 + by_2^2 + abx_3^2$ by n in the first set. The last equation is just mere calculation. ∎

Here $\Gamma_j = 1$ and so we are in Case I. Taking $a = 2, b = 5$ and $n = 2$ in system (3.5), we find u such that $\gamma_u = \begin{pmatrix} i' - i'\sqrt{2} & 0 \\ 0 & i' + i'\sqrt{2} \end{pmatrix}$ and thus $|a|^2 + |c|^2 = 3 - 2\sqrt{2} < 1$ and hence the bisector associated to this unit gives a starting point to run the DAFC (as explained in Case I). Moreover Proposition 3.2.5 shows that the sequence r_n of Proposition 3.1.8 may be taken as $r_n = 4n - 2$. Using Mathematica, we compute $N = 102$, which means that the units needed in a generating set all have norm smaller than $4 \cdot 102 - 2$. In this case the fundamental domain contains again the symmetries given by the group $\langle \sigma, \tau \rangle$ described in Proposition 3.1.11. Moreover the symmetry ϕ from Proposition 3.1.11 may also be taken into account. Indeed, as stated in the proof of Proposition 3.1.11, in $\mathbb{B}^3$ the map $\Psi(\phi)$ is the reflection π in the plane $\{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$. As $\Psi(\Gamma)_0$ is trivial here, we have to cover the whole $\partial\mathbb{B}^3$ (recall that if $\Psi(\Gamma)_0$ is not trivial, we only have to cover the part $\partial\mathcal{F}_0 \cap \partial\mathbb{B}^3$) and hence the symmetry π and thus also ϕ may be taken into account. Thus the group $\langle \sigma, \tau, \phi \rangle$ is contained in the group of symmetries of Γ.

Theorem 3.2.6. *In* $\mathrm{SL}_1\left(\mathcal{H}\left(\frac{2,5}{\mathbb{Z}[i']}\right)\right)$*, the subgroup*

$$S_{2,5}(\mathbb{Z}[i]) = \langle -1, g(\gamma) \mid g \in \langle \sigma^2, \tau, \phi \rangle, \gamma \in Y \rangle,$$

where Y *is given below, has finite index.*

$$\begin{aligned} Y = \{ & i' - i'i, 2i' - i'j, 2 - i'i - j, 2 - 2i + i'j, 3i' + k, 3 + i'i - i'k, \\ & 2 + i'i - i'j - i'k, 4i' - i'i - i'j + k, 3 - 3i + k, 2 - 3i + i'j + k, \\ & 3 - i'i - 2j - k, 6 - i'j - 2i'k, 2 + 4i'i - i'j - 2i'k, 6 - 3j - k, \\ & 2 - 4i'i - 3j - k, (1 - 4i') + (4 + 2i')i + (2 + 2i')j + 2k, \\ & (1 + 4i') + (4 - 2i')i + (-2 - 2i')j - 2k, 3 - 7i + 4i'j + k, \\ & (8 + 3i') + (-4 + 2i')i + (-2 - 2i')j + (1 - 2i')k \}. \end{aligned}$$

Figure 3.4 shows the projection on $\partial\mathbb{H}^3$ of the polyhedron given by the DAFC for $\mathrm{SL}_1\left(\mathcal{H}\left(\frac{2,5}{\mathbb{Z}[i']}\right)\right)$. The bisectors coming from the elements in Y are drawn in green and bold face. Figure 3.5 shows a part of that polyhedron in $\mathbb{H}^3$.

We finish this section with an example of orders in division algebras over $\mathbb{Q}$, i.e. an example in dimension 2. Consider a division algebra of the type $\mathcal{H}\left(\frac{a,b}{\mathbb{Q}}\right)$ with $a > 0, b > 0$ integers (for example with $a = 2$ and $b = 5$). We consider the following order $\mathcal{H}\left(\frac{a,b}{\mathbb{Z}}\right)$. For $u = x_0 + x_1 i + x_2 j + x_3 k \in \mathcal{H}\left(\frac{a,b}{\mathbb{Z}}\right)$, we have

$$\gamma_u = \begin{pmatrix} x_0 + x_1\sqrt{a} & x_2\sqrt{b} + x_3\sqrt{ab} \\ x_2\sqrt{b} - x_3\sqrt{ab} & x_0 - x_1\sqrt{a} \end{pmatrix}.$$

Letting $x = (x_0, x_1, x_2, x_3)$, $q_1(x) = x_0^2 - ax_1^2 - bx_2^2 + abx_3^2$ and $q_2(x) = x_0^2 + ax_1^2 + bx_2^2 + abx_3^2$, we obtain the following lemma.

Lemma 3.2.7. *Let* $u = x_0 + x_1 i + x_2 j + x_3 k \in \mathcal{H}\left(\frac{a,b}{\mathbb{Z}}\right)$ *and let* $x = (x_0, x_1, x_2, x_3)$, $q_1(x)$, $q_2(x)$ *and* γ_u *be as described above. Then*

$$\begin{cases} q_1(x) = 1, \\ 2q_2(x) = \|\gamma\|^2. \end{cases} \tag{3.6}$$

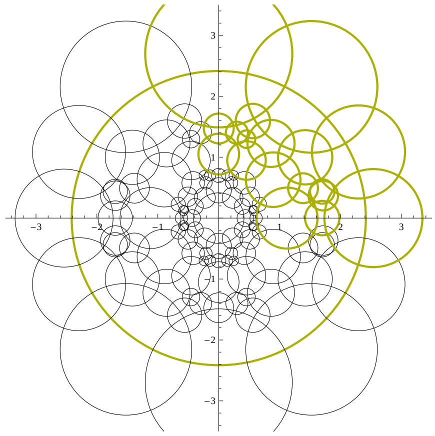


Figure 3.4: Projection of the polyhedron associated to $S_{2,5}(\mathbb{Z}[i])$.

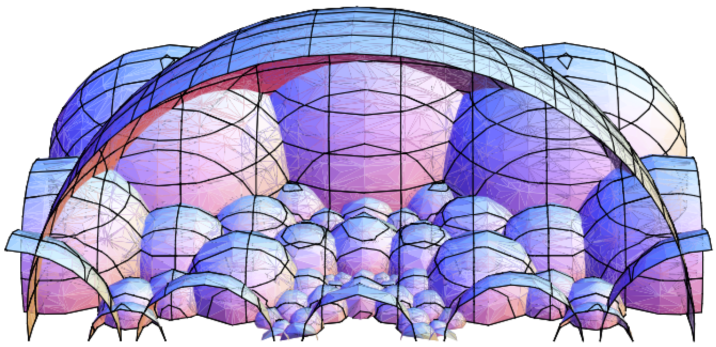


Figure 3.5: Part of the polyhedron associated to $S_{2,5}(\mathbb{Z}[i])$

Moreover if $q_2(x_0,0,0,x_3)=n$ *we obtain*

$$\begin{cases} x_0^2+abx_3^2=n,\\ ax_1^2+bx_2^2=n-1,\\ \|\gamma\|^2=4n-2, n\in\mathbb{N}. \end{cases} \tag{3.7}$$

This lemma being similar to Lemma 3.2.5, we omit the proof. Taking $a=2, b=5$ we have that $n=1$ or $n\geq 9$. Clearly the stabilizer of i, Γ_i, is trivial (note that as we are in dimension 2 the role of j is played by i.) For $n=1$ we find the identity matrix and for $n=9$ we find an element whose bisector Σ separates i and 0 in the ball model, which gives a starting point for the DAFC as described in Case I. The sequence r_n needed for the DAFC is defined here by $r_n=4n-2$ and $N=46$. Hence we obtain a set of generators for a subgroup of finite index containing units of norm smaller than $4\cdot 46-2$. Figure 3.6 shows the resulting polyhedron.

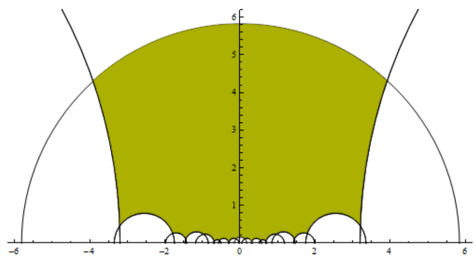


Figure 3.6: Polyhedron associated to $S_{2,5}(\mathbb{Z})$.

Theorem 3.2.8. *The group* $S_{2,5}(\mathbb{Z})=\langle -1, g(\gamma)\mid g\in\langle\sigma^2,\phi\rangle, \gamma\in Y\}$*, where* $Y=\{3+2i, 2+2i+j+k, 2+2i+j-k, 3+3i+k, 6+3j+k\}$*, is a subgroup of finite index of* $\mathrm{SL}_1\left(\mathcal{H}\left(\frac{2,5}{\mathbb{Z}}\right)\right)$.

3.2.2 Matrix Algebras

In this section we consider groups Γ of finite covolume and with at least one ideal vertex (so Γ is not cocompact). First we give some definition.

Definition 3.2.9. *An ideal vertex of a convex polyhedron* $\mathcal{F}$ *in* $\mathbb{B}^3$ *is a point* P *of* $\mathcal{F}\cap\partial\mathbb{B}^3$ *for which there is a closed horoball* C *of* $\overline{\mathbb{B}^3}$ *based at* P *such that* C *meets just the sides of* $\mathcal{F}$ *incident to* P *and such that* $\partial C\cap\mathcal{F}$ *is compact.*

More details on these concepts may be found in [Rat06, Section 6.4]. To describe how to implement the DAFC and obtain a fundamental domain we work in $\mathbb{B}^3$ and suppose that j is an ideal vertex (which is also the case in all the examples we consider).

Case I: $\Psi(\Gamma)_0$ is trivial. As in subsection 3.2.1, in this case again, we may just run the DAFC as stated in Proposition 3.1.8. However again because of visualisation in the upper half-space model, we are first looking for some "starting point". So we find a

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, with $|a|^2 + |c|^2 = 1$. By Lemma 3.1.13, these are exactly the units associated to bisectors that contain the point $j \in \mathbb{B}^3$ (note that since $\Psi(\Gamma)_0$ is trivial we have that $\gamma \notin \mathrm{SU}_2(\mathbb{C})$). Since Γ has finite covolume, a finite number of them, those with smallest matrix norm, $\gamma_1, \cdots, \gamma_m$ say, can be chosen such that there exists a neighbour V_j of j in $\mathbb{S}^2$ such that $(V_j \setminus j) \subseteq \bigcup_{1 \leq k \leq m} \mathrm{Interior}(\Sigma_{\Psi(\gamma_k)})$. To know the exact value of m one has to know the link of j. Recall that the link of the ideal vertex j is defined to be the set $\Sigma_j \cap \mathcal{F}$, where Σ_j is a horosphere based at j that meets just the sides of $\mathcal{F}$ incident with j. Set

$$V = (\mathbb{S}^2 \setminus \{j\}) \cap \bigcap_{1 \leq k \leq m} \mathrm{Exterior}(\Sigma_{\Psi(\gamma_k)}).$$

Since Γ is of finite covolume, we may use the DAFC to find $N \in \mathbb{N}$ such that V is contained in $\mathcal{F}_N$. The output is $\langle \gamma_1, \cdots \gamma_n \rangle$, a subgroup of finite index in Γ, where $\{\gamma_{m+1} \cdot \cdots \gamma_n\}$ is the set of all γ_i such that $B(\gamma_i) \in \mathcal{F}_N$.

Case II: $\Psi(\Gamma)_0$ **non-trivial.** In this case, proceed as in Case I with

$$V = (\partial \mathcal{F}_0) \cap (\mathbb{S}^2 \setminus \{j\}) \bigcap_{1 \leq k \leq m} \mathrm{Exterior}(\Sigma_{\Psi(\gamma_k)}),$$

where $\mathcal{F}_0$ is a fundamental domain for $\Psi(\Gamma)_0$. Then $\langle G_0, \gamma_1, \cdots \gamma_n \rangle$ is of finite index in Γ, where G_0 is a generating set of $\Psi(\Gamma)_0$.

The examples we give here are the Bianchi Groups.

Definition 3.2.10. *Let $\mathbb{Q}(\sqrt{-k})$, with $k > 0$, be an imaginary quadratic number field and let $\mathcal{O}_k$ be its ring of integers. A Bianchi group is a group of the form $\mathrm{PSL}_2(\mathcal{O}_k)$ for k a square-free positive integer.*

In [EGM98, Theorem 7.3.4], a Ford Fundamental domain, as defined in (1.27) and Theorem 1.2.47, is given for these groups. Here we describe a Dirichlet fundamental polyhedron for all k. Note that the Bianchi groups can also be handled as groups commensurable with the unit group of an order in the split quaternion algebra $\mathcal{H}(K)$, $K = \mathbb{Q}(\sqrt{-k})$ and $k \equiv 1, 2 \bmod 4$ or $d \equiv 3 \bmod 4$ and $\mathcal{H}(K)$ not a division ring. All this can be handled as in the previous section (the division assumption in the previous section was only used to guarantee that the groups were cocompact and hence of finite covolume).

To ensure, that the groups we will use have finite covolume, we state a theorem from [EGM98].

Theorem 3.2.11. *[EGM98, Theorem 7.1.1] Let $\mathbb{Q}(\sqrt{-k})$ be an imaginary quadratic field and let $\mathcal{O}_k$ be its ring of integers. Then the group $\mathrm{PSL}_2(\mathcal{O}_k)$ has the following properties:*

1. *$\mathrm{PSL}_2(\mathcal{O}_k)$ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$.*

2. $\mathrm{PSL}_2(\mathcal{O}_k)$ *has finite covolume, but is not cocompact.*

3. $\mathrm{PSL}_2(\mathcal{O}_k)$ *has a fundamental domain* $\mathcal{F}$ *bounded by finitely many geodesic surfaces.*

4. $\mathrm{PSL}_2(\mathcal{O}_k)$ *is finitely presented.*

Note that item 3 follows straightforward from item 2 and Theorem 1.2.38. Item 4 then follows from Corollary 1.2.15.

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma = \mathrm{PSL}_2(\mathcal{O}_k)$. Let $\omega = \sqrt{-k}$ if $k \equiv 1, 2 \ mod\ 4$ and $\omega = \frac{1+\sqrt{-k}}{2}$ if $k \equiv 3 \ mod\ 4$. $\mathcal{O}_k$ is defined as $\mathbb{Z}[\omega]$. By Theorem 3.2.11, Γ is not cocompact. We have that $\Psi^{-1}(\Psi(\Gamma)_0) = \Gamma \cap \mathrm{SU}_2(\mathbb{C})$. By Lemma 1.2.24, this is equivalent with $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 2$ and $ad - bc = 1$. If $k = 2$ or $k > 3$, then for every element $a \in \mathcal{O}_k$, $|a|^2 > 1$ except if $a = \pm 1$ or $a = 0$ and thus the only element in $\Gamma \cap \mathrm{SU}_2(\mathbb{C})$ in those cases is $\gamma_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. So $\Psi(\Gamma)_0 = \langle \Psi(\gamma_0) \rangle$. However if $k = 1$, one easily computes that $\Psi(\Gamma)_0 = \langle \Psi(\gamma_1), \Psi(\gamma_2) \rangle \cong C_2 \times C_2$, where $\gamma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $\gamma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. If $k = 3$, $\Psi(\Gamma)_0 = \langle \Psi(\gamma_1), \Psi(\gamma_2) \rangle \cong S_3$, where $\gamma_1 = \begin{pmatrix} \omega & 0 \\ 0 & \overline{\omega} \end{pmatrix}$ and $\gamma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Since $\Psi(\gamma_0) \in \Psi(\Gamma)_0$, for all k, we have that a fundamental domain $\mathcal{F}$ for Γ is a subset of $\{(x, y, z) \in \mathbb{B}^3 \mid z \geq 0\}$ in the ball model and of $\{u \in \mathbb{H}^3 \mid |u| \geq 1\}$ in the upper half-space model. Since j is an ideal vertex of Γ, we have to find the elements γ such that $j \in \Sigma_{\Psi(\gamma)}$, as it is explained in Case I. By Lemma 3.1.13, this is the case if and only if $|a|^2 + |c|^2 = 1$. As $|a| \geq 1$ for every $0 \neq a \in \mathcal{O}_k$, $a = 0$ or $c = 0$. We may suppose that $c = 0$. Indeed, set $\gamma_0\gamma = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. Then if $c \neq 0$, $a' \neq 0$, and hence $c' = 0$. As, by the last item of Lemma 3.1.4, $\Sigma_{\Psi(\gamma_0\gamma)} = \Sigma_{\Psi(\gamma)}$, we can hence suppose c to be 0. If $c = 0$, then γ fixes the point $\infty \in \partial\mathbb{H}^3$ and hence $\Psi(\gamma) \in \Psi(\Gamma)_j$ in $\mathbb{B}^3$. Denoting by $\mathcal{F}_j$ the Dirichlet fundamental domain for $\Psi(\Gamma)_j$ acting on $\mathbb{B}^3$, we have that $\mathcal{F} \subseteq \mathcal{F}_0 \cap \mathcal{F}_j \cap \{(x, y, z) \in \mathbb{B} \mid z \geq 0\}$. So, referring to Case II, we may take $\{\Psi(\gamma_1), \cdots, \Psi(\gamma_m)\} \subseteq \Psi(\Gamma)_j$ and $V = (\partial\mathcal{F}_j) \cap (\partial\mathcal{F}_0) \cap (\mathbb{S}^2 \setminus \{j\})$.

Using $\eta_0 : \mathbb{H}^3 \to \mathbb{B}^3$ we transfer this information to $\mathbb{H}^3$. In this model $\eta_0^{-1}(\mathcal{F}_j) = \mathcal{F}_\infty$ and

$$\Gamma_\infty = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid ad = 1, a, b, d \in \mathbb{Z}[\omega] \right\}.$$

So let $\mathcal{F}_\infty$ be the Dirichlet fundamental domain for Γ_∞ and let $\hat{\mathcal{F}}_\infty$ be the projection of $\mathcal{F}_\infty$ on $\mathbb{C} = \partial\mathbb{H}^3$. Then, for all k, $\mathcal{F}_\infty = \{z + rj \in \mathbb{H}^3 \mid z \in \hat{\mathcal{F}}_\infty\}$. The following lemma gives more details on $\mathcal{F}_\infty$. As it is proved by easy computations based on Lemma 3.1.3 we omit the proof.

Lemma 3.2.12. *1. If $1 < k \equiv 1,2$ mod 4 then $\mathcal{F}_\infty = \{z + rj \in \mathbb{H}^3 \mid -\frac{1}{2} \le Re(z) \le \frac{1}{2},\ -\frac{\sqrt{k}}{2} \le Im(z) \le \frac{\sqrt{k}}{2}\}$*

2. *If $3 < k \equiv 3$ mod 4 then $\mathcal{F}_\infty = \{z + rj \in \mathbb{H}^3 \mid -\frac{1}{2} \le Re(z) \le \frac{1}{2},\ -\frac{1+k}{4} \le Re(z) + \sqrt{k}Im(z) \le \frac{1+k}{4},\ -\frac{1+k}{4} \le Re(z) - \sqrt{k}Im(z) \le \frac{1+k}{4}\}$.*

3. *If $k = 3$, then $\mathcal{F}_\infty = \{z + rj \in \mathbb{H}^3 \mid 0 \le Re(z) \le \frac{1}{2},\ 0 \le Re(z) + \sqrt{3}Im(z) \le 1\}$.*

4. *If $k = 1$ then $\mathcal{F}_\infty = \{z + rj \in \mathbb{H}^3 \mid -\frac{1}{2} \le Re(z) \le \frac{1}{2},\ 0 \le Im(z) \le \frac{1}{2}\}$.*

For $k \equiv 1,2$ mod 4, $\hat{\mathcal{F}}_\infty$ is a rectangle with vertices $\pm\frac{1}{2} \pm \frac{\sqrt{k}}{2}i$ and for $k \equiv 3 \mod 4$ it is a hexagon with vertices $\pm\frac{(k+1)\sqrt{k}}{4k}i$ and $\pm\frac{1}{2} \pm \frac{(k-1)\sqrt{k}}{4k}i$. For $k \neq 3$, all vertices of this hexagon lie on the circle centred at 0 with radius $\frac{(k+1)\sqrt{k}}{4k}$. Hence, for $k \in \{1,2,3,7,11\}$, $\mathcal{F}_\infty \cap \partial\mathbb{H}^3$ is included in the interior of $\mathbb{S}^2 \cap \partial\mathbb{H}^3$ and hence ∞ is the only ideal vertex of Γ in $\widehat{\mathbb{H}}^3$, respectively j is the only ideal vertex in $\widehat{\mathbb{B}}^3$.

We implemented the DAFC for some Bianchi groups for $k \equiv 3 \mod 4$. Note that the implementation for Bianchi groups for $k \equiv 1,2 \mod 4$ is done in the same way, the only difference lies in the definition of ω and $\mathcal{F}_\infty$. So let $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\omega])$ where $\omega = \frac{1+\sqrt{-k}}{2}$ for $k \equiv 3 \mod 4$. The following lemma describes the group Γ in terms of Diophantine equations.

Lemma 3.2.13. *Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $a = x_0 + y_0\omega$, $b = x_1 + y_1\omega$, $c = x_2 + y_2\omega$, $d = x_3 + y_3\omega$, $x = (x_0, x_1, x_2, x_3) \in \mathbb{Z}^4$, $y = (y_0, y_1, y_2, y_3) \in \mathbb{Z}^4$, $\det(x) = x_0x_3 - x_1x_2$, $J(x) = (x_3, -x_2, -x_1, x_0)$ and $\det(y)$ and $J(y)$ are analogously defined as $det(x)$ and $J(x)$. Then*

$$\begin{cases} \det(x) - (\frac{k+1}{4})\det(y) = 1, \\ \det(y) + \langle x|J(y)\rangle = 0, \\ \|\gamma\|^2 = |x|^2 + (\frac{k+1}{4})|y|^2 + \langle x|y\rangle \in \mathbb{N}. \end{cases} \tag{3.8}$$

Proof. If we compute the determinant of γ, we get that

$$det(x) - \frac{k-1}{4}det(y) + \frac{1}{2}\langle x|J(y)\rangle + (det(y) + \langle x|J(y)\rangle)\frac{\sqrt{-k}}{2} = 1.$$

Hence we have that

$$det(x) - \frac{k-1}{4}det(y) + \frac{1}{2}\langle x|J(y)\rangle = 1$$

and

$$det(y) + \langle x|J(y)\rangle = 0.$$

Replacing $\langle x|J(y)\rangle$ by $-det(y)$ in the first equation, we get the two first equations of the lemma. The third equation comes from mere computations of $\|\gamma\|^2$. ∎

By Lemma 1.2.2, the equation $\|\gamma\|^2 = n$ has a finite number of solutions (x, y). Consequently, for each $n \in \mathbb{N}$ the system above has a finite number of solutions (x, y) such that $\|\gamma\|^2 = n$. To implement the DAFC we choose the sequence $r_n = n$, for $n \geq 1$, because $\|\gamma\|^2 \in \mathbb{N}$. The next theorem gives the outcome of the DAFC for three examples.

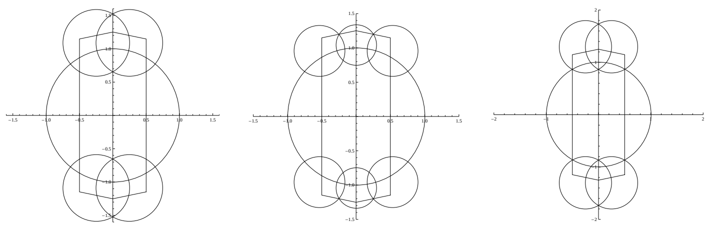
Theorem 3.2.14. *For $k \in \{19, 23, 27\}$, the subgroup $\Gamma_{(k)} = \langle \Psi^{-1}(\Psi(\Gamma)_0),\ g(Y_k) \mid g \in \langle \sigma^2, \tau \rangle \rangle$ is of finite index in $\mathrm{PSL}_2(\mathbb{Z}[\omega])$, where $\omega = \frac{1+\sqrt{-k}}{2}$ and Y_k is given below.*

$$Y_{19} = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -\omega \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1-\omega & 2 \\ 2 & \omega \end{pmatrix} \right\},$$

$$Y_{23} = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -\omega \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -2+\omega & 3 \\ -1-\omega & -3 \end{pmatrix}, \begin{pmatrix} -3+\omega & 2+\omega \\ -2-\omega & -3+\omega \end{pmatrix} \right\},$$

$$Y_{27} = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -\omega \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -\omega \\ 1-\omega & -3 \end{pmatrix} \right\}.$$

Figures 3.7(a), 3.7(b) and 3.7(c) show the projection of the polyhedron associated to Γ_k on $\partial\mathbb{H}^3$ for $k = 19$, $k = 23$ and $k = 27$ respectively. Figures 3.8, 3.9 and 3.10 show the polyhedron given by the DAFC associated to Γ_k, rotated over ninety degrees and seen from "above".



(a) $\Gamma_{(19)}$ (b) $\Gamma_{(23)}$ (c) $\Gamma_{(27)}$

Figure 3.7:

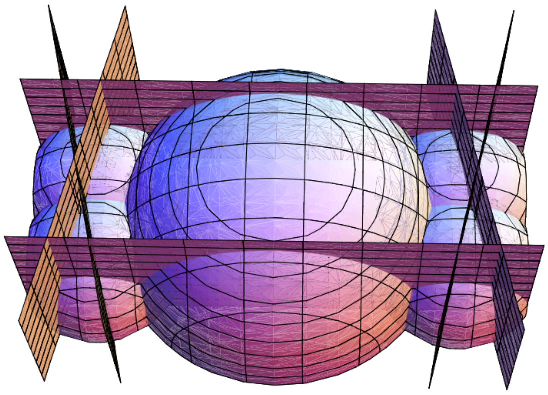


Figure 3.8: $\Gamma_{(19)}$

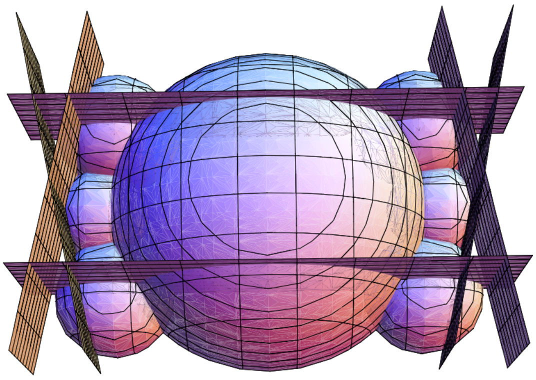


Figure 3.9: $\Gamma_{(23)}$

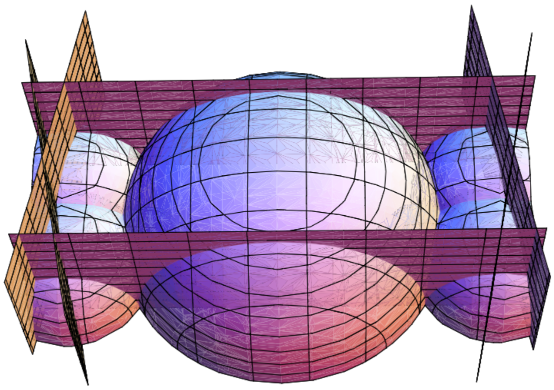


Figure 3.10: $\Gamma_{(27)}$

3.3 Application to Units in Group Rings

Let G be a finite group and consider the order $\mathbb{Z}G$ in the group ring $\mathbb{Q}G$. Because of the algorithm given in Section 3.2, we are now able to handle exceptional components of type (2) and (3) and also some of the exceptional components of type (1) (recall that exceptional components have been defined in Definition 1.1.19).

This together with the results mentioned in Subsection 1.1.3 allow us to describe, up to commensurability, finitely many generators in $\mathcal{U}(\mathbb{Z}G)$ for many more groups than previously known.

In case G is a nilpotent finite group, we are now able to give a finite set of elements of $\mathcal{U}(\mathbb{Z}G)$ that generates a subgroup of finite index, this only with some restrictions on exceptional components of type (1) and (4). We will just sketch the reasoning leading to Theorem 3.3.1. Giving all the details would be beyond the scope of this thesis. So suppose the finite group G is nilpotent. In order to state the result we need to recall some background. Set

$$\widehat{X} = \frac{1}{|X|}\sum_{x\in X} x \in \mathbb{Q}G = \frac{1}{|X|}\widetilde{X},$$

where $\widetilde{X}$ was defined in (1.1). If X is a subgroup then $\widehat{X}$ is an idempotent of $\mathbb{Q}G$. Furthermore, if X is a normal subgroup, then $\widehat{X}$ is a central idempotent of $\mathbb{Q}G$. In [JOdR12] the rational representations of a finite nilpotent group G have been explicitly described by exhibiting a set of matrix units of each simple component of $\mathbb{Q}G$. In particular, a precise description is given when a simple component $\mathbb{Q}Ge$ (e a primitive central idempotent) is of exceptional type (this also follows from [JL95] or [Ban88]). These simple components are one of the following algebras:

- $\mathcal{H}(\mathbb{Q}(\xi_{2^{m-1}} + \xi_{2^{m-1}}^{-1}, \xi_n))$ with $1 \neq n$ a positive odd integer such that 2 has odd order modulo n and an integer $m \geq 3$,
- $M_2(\mathbb{Q})$,
- $M_2(\mathbb{Q}(\sqrt{-2}))$,
- $M_2(\mathbb{Q}(i))$,
- $M_2(\mathbb{Q}(\sqrt{-3}))$ or
- $M_2(\mathcal{H}(\mathbb{Q}))$.

Let $G_e = \{g \in G \mid ge = e\}$ denote the stabilizer of e in G. Note that $\mathbb{Q}Ge$ is a simple component of $\mathbb{Q}G\widehat{G} \cong \mathbb{Q}(G/G_e)$. For each simple algebra $\mathbb{Q}Ge$, a description of G/G_e is given and, in case $\mathbb{Q}Ge$ is a matrix algebra, a complete set of matrix units is also given (see also [JL95]). The exceptional simple components $\mathbb{Q}Ge$ of the type $\mathcal{H}(\mathbb{Q}(\xi_{2^{m-1}} + \xi_{2^{m-1}}^{-1}, \xi_n))$ are determined by groups G/G_e of the type $\mathbb{Q}_{2^m} \times C_n$ with $1 \neq n$ a positive integer such that 2 has odd order modulo n and an integer $m \geq 3$. The exceptional simple components of the type $M_2(\mathcal{H}(\mathbb{Q}))$ are determined by some specific 2-groups of order at most 64. The exceptional simple components $\mathbb{Q}Ge$ of the type $M_2(F)$ are determined by groups G/G_e of the following type:

- D_8 the dihedral group of order 8,
- $D_{16}^- = \langle a, b \mid a^8 = b^2 = 1, ba = a^3 b\rangle$,
- $D_{16}^+ = \langle a, b \mid a^8 = b^2 = 1, ba = a^5 b\rangle$,
- $\mathcal{D} = \langle a, b, c \mid a^2 = b^2 = c^4 = 1, ac = ca, bc = cb, ba = c^2 ab\rangle$,
- $D_8 \times C_3$,
- $Q_8 \times C_3$,
- $D^+ = \langle a, b, c \mid a^4 = 1, b^2 = 1, c^4 = 1, ca = ac, cb = bc, ba = ca^3 b\rangle$.

In all of these cases, the idempotent e has been explicitly described and an explicit set of matrix units $E_{11}, E_{22}, E_{12}, E_{21}$ is given in [JOdR12, Theorem 4.5]. It is this set of units that we will use to describe the following congruence subgroup of level m in $\mathbb{Q}Ge$ (with m a positive integer):

$$\Gamma_{2,N}(m\mathcal{O}) = \left(1 + \sum_{1 \leq i,j \leq 2} (m\mathcal{O})E_{ij}\right) \cap \mathrm{SL}_2(\mathcal{O}) = (1 + M_2(m\mathcal{O})) \cap \mathrm{SL}_2(\mathcal{O}).$$

With the DAFC algorithm one can calculate a finite set of generators for a subgroup of finite index in such a group. If one chooses the integer m appropriately then

$$1 - e + \Gamma_{2,N}(m\mathcal{O}) \subseteq \mathbb{Z}G,$$

where e is explicitly given in [JOdR12]. Generators of these groups are the units that are used in part 3 of the following result. The units listed in part 4 are to deal with orders determined by fixed point free groups. Because we exclude simple components that are division algebras that are not totally definite quaternion algebras, the only non-commutative fixed point free epimorphic images of G that can occur are $Q_8 \times C_n$ with the order of 2 modulo n even. We then use the matrix idempotents (as part of a set of matrix units) determined in [JOdR12]. So, all the above together with the results stated in the preliminaries give us the following result.

Theorem 3.3.1. *Let G be a nilpotent finite group of nilpotency class n. Assume that the rational group algebra $\mathbb{Q}G$ does not have simple components of the type $\mathcal{H}(\mathbb{Q}(\xi_n, \xi_{2^{m-1}} + \xi_{2^{m-1}}^{-1}))$ with n an odd integer so that the order of $2 \bmod n$ is odd and an integer $m \geq 3$, or $M_2(\mathcal{H}(\mathbb{Q}))$ (equivalently G does not have epimorphic images of the type $Q_8 \times C_n$ for such n nor some special 2-groups of order at most 64.) Then the group generated by the following units is of finite index in $\mathcal{U}(\mathbb{Z}G)$:*

(i) *$b_{(n)}$, with b a Bass cyclic unit in $\mathbb{Z}G$ and $b_{(n)}$ are units constructed inductively based on b,*

(ii) *the bicylic units in $\mathbb{Z}G$,*

(iii) *generators for the groups $\Gamma_{2,N}(m\mathcal{O})$ (where for the respective congruence groups we use the matrix units described in [JOdR12]) with N a normal subgroup of G so that*

   (a) $\mathcal{O} = \mathbb{Z}$, $m = 8|N|$ *and* $G/N = D_8$,
   (b) $\mathcal{O} = \mathbb{Z}[sqrt{-}2]$, $m = 8|N|$ *and* $G/N = D_{16}^{-}$,
   (c) $\mathcal{O} = \mathbb{Z}[i]$, $m = 8|N|$ *and* $G/N = D_{16}^{+}$,
   (d) $\mathcal{O} = \mathbb{Z}[i]$, $m = 2|N|$ *and* $G/N = \mathcal{D}$,
   (e) $\mathcal{O} = \mathbb{Z}[\sqrt{-3}]$, $m = 24|N|$ *and* $G/N = D_8 \times C_3$,
   (f) $\mathcal{O} = \mathbb{Z}[\sqrt{-3}]$, $m = 24|N|$ *and* $G/N = Q_8 \times C_3$,
   (g) $\mathcal{O} = \mathbb{Z}[i]$, $m = 32|N|$ *and* $G/N = D^{+}$,

(iv) *$u_{g,N} = 1 + mE_{11}gE_{22}$ and $u'_{g,N} = 1 + mE_{22}gE_{11}$, with $m = |N|2n$, $g \in G$ and N a normal subgroup of G so that $G/N = Q_8 \times C_n = \langle a, b, c \mid a^4 = 1, a^2 = b^2, ba = a^{-1}b, ca = ac, cb = bc\rangle$ such that 2 has even order in $\mathcal{U}(\mathbb{Z}_n)$, where $E_{11} = e\frac{1}{2}(1 + xa + yab)$ and $E_{22} = e\frac{1}{2}(1 - xa - yab)$. Moreover $e = \widetilde{N}\frac{1}{2}(1 - a^2)\left(1 - \frac{1}{n}(\hat{c})\right)$, where $x = \frac{1}{2}(\alpha + b\alpha b^{-1})$, $y = b^3\frac{1}{2}(\alpha - b\alpha b^{-1}) \in \mathbb{Z}\langle c^{n/p}\rangle$, and $\alpha = \prod_{k=0}^{m-1}\left(1 + bc^{(n/p)2^k}\right)$ and p is a prime divisor of n so that $2^p \equiv -1 \mod p$.*

In [GS95] different units were used for (iv). In general, the result fails if one does not include the units listed in (iii) (we refer the reader to [Seh93, Section 25]).

By Theorem 3.2.11 and Proposition 1.2.33, the groups listed in part (iii) of Theorem 3.3.1 are of finite covolume (coarea) but not cocompact and, up to finite index, a set of generators can be calculated using the DAFC. However, for most of them it gives a too large set of generators to be listed here. As a matter of example, we therefore restrict ourselves to determine a set of generators for a subgroup of finite index for each of the following groups:

- $\Gamma_2(8\mathbb{Z})$,
- $\Gamma_2(2\mathbb{Z}[\sqrt{-2}])$ and
- $\Gamma_2(2\mathbb{Z}[i])$.

Let $n_0 \in \mathbb{N}$. In $\mathrm{PSL}_2(\mathbb{Z})$ consider the discrete Fuchsian subgroup

$$\Gamma_2(n_0\mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} 1+n_0a & n_0b \\ n_0c & 1+n_0d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z} \right\}.$$

The following lemma defines the group $\Gamma_2(n_0\mathbb{Z})$ in terms of a system of algebraic equations.

Lemma 3.3.2. *Let $\gamma \in \Gamma_2(n_0\mathbb{Z})$. Write $\gamma = 1 + n_0\hat{\gamma}$, where $\hat{\gamma} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If we set $\|\hat{\gamma}\|^2 - 2\det(\hat{\gamma}) = n$, then we get the following system of equations.*

$$\begin{cases} \|\hat{\gamma}\|^2 - 2\det(\hat{\gamma}) = n, \\ \mathrm{tr}(\hat{\gamma}) + n_0\det(\hat{\gamma}) = 0, \\ \|\gamma\|^2 = 2 + n_0^2 n, n \in \mathbb{N}. \end{cases} \tag{3.9}$$

Proof. The norm of γ is given by $\|\gamma\|^2 = 2 + n_0^2(\|\hat{\gamma}\|^2 + \frac{2}{n_0}\mathrm{tr}(\hat{\gamma}))$. One easily computes that $\det(\gamma) = 1$ if and only if $\mathrm{tr}(\hat{\gamma}) + n_0\det(\hat{\gamma}) = 0$. Hence, it follows that $\|\gamma\|^2 = 2 + n_0^2(\|\hat{\gamma}\|^2 - 2\det(\hat{\gamma}))$. Set $\|\hat{\gamma}\|^2 - 2\det(\hat{\gamma}) = n$ and the result follows. ∎

Next, working in $\mathbb{B}^2$, we look for elements γ whose isometric circle passes through i. In this case, these are exactly the elements stabilizing i. Thus, for such elements γ, we have that $(1+n_0a)^2 + n_0^2c^2 = 1$ and therefore $\gamma = \begin{pmatrix} 1 & n_0b \\ 0 & 1 \end{pmatrix}$. Hence, their defining bisectors, in $\mathbb{H}^2$, which are vertical lines, are given by the equation $Re(n_0bz) + \frac{n_0^2b^2}{2} = 0$ or equivalently $x + \frac{n_0b}{2} = 0$. The bisector corresponding to such an element of smallest norm is the line $x + \frac{n_0}{2} = 0$. We use the DAFC to cover the compact region $[\frac{-n_0}{2} \leq x \leq \frac{n_0}{2}]$, up to a finite number of points, which are precisely the remaining ideal vertices of Γ. For $n_0 = 8$, we get the compact region $[-4, 4]$ which is covered using solutions of the above system with $n \leq 2304$. This is depicted in Figure 3.11. As this example gives a lot of generators, we do not write them as matrices as usual, but under a more compact form (the three columns give the entries a, b, c and d of the 19 generating matrices).

Theorem 3.3.3. *$\tilde{\Gamma}_2(8\mathbb{Z}) = \langle g(\gamma) \mid g \in \langle \sigma^2, \phi \rangle, \gamma \in X \rangle < \Gamma_2(8\mathbb{Z})$, where the transformations σ^2 and ϕ are given in Lemma 3.1.11 and where X is given by the table below, is of finite index in $\Gamma_2(8\mathbb{Z})$.*

a	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*
0	0	1	0	-3	1	-9	3	8	-6	-11	8
-1	1	-1	1	5	-2	-8	3	13	-8	8	-5
2	-1	4	-2	-10	7	-3	2	8	-3	-22	8
-7	2	3	-1	6	-4	9	-6	16	-26	-10	16
-4	3	5	-4	11	-4	8	-3	21	-8	34	-13
7	-2	-4	1	-4	1	15	-4	-16	6	42	-16
-5	2	7	-3								

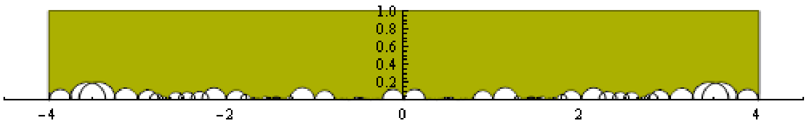


Figure 3.11: Polyhedron associated to $\tilde{\Gamma}_2(8\mathbb{Z})$.

We now move on to the groups $\Gamma_2(2\mathbb{Z}[\sqrt{-2}])$ and $\Gamma_2(2\mathbb{Z}[i])$. The defining system of Diophantine equations to describe the groups (and to obtain generators for these groups) in the next theorem are exactly the same as in Lemma 3.3.2, i.e.,

$$\begin{cases} \|\hat{\gamma}\|^2 - 2\det(\hat{\gamma}) = n, \\ \operatorname{tr}(\hat{\gamma}) + n_0 \det(\hat{\gamma}) = 0, \\ \|\gamma\|^2 = 2 + n_0^2 n, n \in \mathbb{N}. \end{cases} \tag{3.10}$$

only the entries of $\hat{\gamma}$ are now in $\mathbb{Z}[\sqrt{-k}]$. This is the case for any congruence subgroup which we still have to deal with.

Theorem 3.3.4. *1. The subgroup $\tilde{\Gamma}_2(2\mathbb{Z}[\sqrt{-2}]) = \langle -1, g(X_1) \mid g \in \langle \sigma^2 \circ \tau, \phi \rangle \rangle < \Gamma_2(2\mathbb{Z}[\sqrt{-2}])$, where X_1 is given below, is of finite index .*

2. The subgroup $\tilde{\Gamma}_2(2\mathbb{Z}[i]) = \langle -1, g(X_2) \mid g \in \langle \sigma^2 \circ \tau, \phi \rangle \rangle < \Gamma_2(2\mathbb{Z}[i])$, where X_2 is given below, is of finite index .

$$X_1 = \left\{ \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -2\sqrt{-2} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ -2\sqrt{-2} & -1 \end{pmatrix}, \right.$$
$$\begin{pmatrix} -3 & 2\sqrt{-2} \\ -2\sqrt{-2} & -3 \end{pmatrix}, \begin{pmatrix} 1-2\sqrt{-2} & 2\sqrt{-2} \\ -2\sqrt{-2} & -1-\sqrt{-2} \end{pmatrix},$$
$$\begin{pmatrix} -1+2\sqrt{-2} & 2 \\ 4 & -1-\sqrt{-2} \end{pmatrix}, \begin{pmatrix} -1+2\sqrt{-2} & 4 \\ 2 & -1-\sqrt{-2} \end{pmatrix},$$
$$\begin{pmatrix} -3+2\sqrt{-2} & 4 \\ 4 & -3-2\sqrt{-2} \end{pmatrix}, \begin{pmatrix} -3-2\sqrt{-2} & 2\sqrt{-2} \\ -4\sqrt{-2} & -3+2\sqrt{-2} \end{pmatrix},$$
$$\left. \begin{pmatrix} -3-2\sqrt{-2} & 4\sqrt{-2} \\ -2\sqrt{-2} & -3+2\sqrt{-2} \end{pmatrix}, \begin{pmatrix} 5-2\sqrt{-2} & 4\sqrt{-2} \\ -4\sqrt{-2} & 5+2\sqrt{-2} \end{pmatrix} \right\},$$
$$X_2 = \left\{ \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -2i \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ -2i & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} -1+2i & 2 \\ 2 & -1-2i \end{pmatrix}, \begin{pmatrix} -1-2i & 2i \\ -2i & -1+2i \end{pmatrix} \right\}.$$

The polyhedron given by the DAFC and its projection on $\partial\mathbb{H}^3$ for the two groups of the theorem above is given in Figure 3.12, Figure 3.13 and Figure 3.14 respectively.

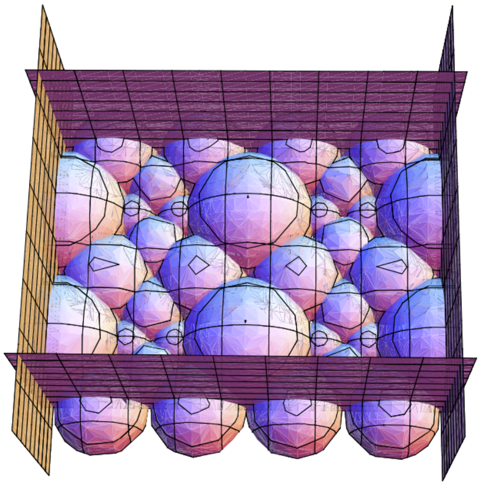


Figure 3.12: Polyhedron associated to $\tilde{\Gamma}_2(2\mathbb{Z}[\sqrt{-2}])$

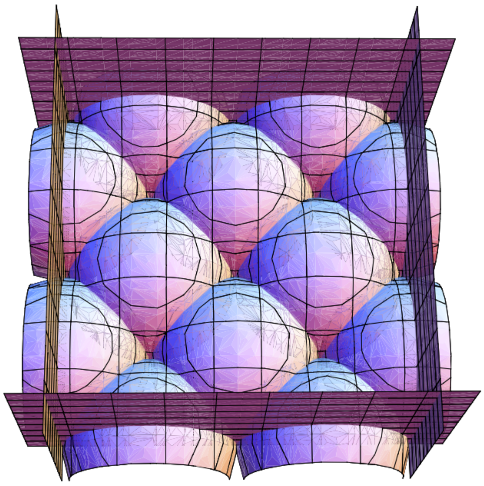
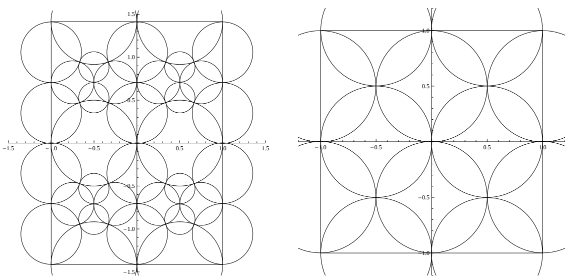


Figure 3.13: Polyhedron associated to $\tilde{\Gamma}_2(2\mathbb{Z}[i])$



(a) $\tilde{\Gamma}_2(2\mathbb{Z}[\sqrt{-2}])$ (b) $\tilde{\Gamma}_2(2\mathbb{Z}[i])$

Figure 3.14: Projections of polyhedron given by DAFC

We finish this section with two remarks. First observe that a similar result as Theorem 3.3.1 could also be established for finite groups which are not necessarily nilpotent. By a recent result in [EKVG14] the exceptional components different from division algebras, hence of type (2)-(4), as well as their associated groups, are known. So the units in (iii) in Theorem 3.3.1 could be replaced by the results appearing in [EKVG14, Table 2]. However, on the contrary to the case of a nilpotent group, the idempotents e needed for the construction of these units are not known explicitly. So their is no unified generic theorem for non-nilpotent groups, but the method may just be applied by a case-by-case analysis. Another problem appearing in the case of non-nilpotent groups is that the exceptional components of type (1) are in a much greater number as in the nilpotent case. This is also why we illustrate the method in the nilpotent case.

In the previous section, we have constructed an algorithm which gives a finite set of generators of a subgroup Γ of finite index of a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$ or $\mathrm{PSL}_2(\mathbb{R})$. Note that, once one knows a fundamental domain for Γ, Theorem 2.4.7 also gives a method to find the relations between those generators. So, in theory by the DAFC, we do not only get generators but also relations for the group Γ, in case it is cocompact. However, to obtain these relations one has to go through many long and tedious calculations. The outcome often is a very long list of generators and relations. For this reason we have not ventured into this in the examples.

3.4 Digression: DF Domains and Double Dirichlet Domains

In this section we show how the concrete formulas obtained in Lemma 3.1.3 and Lemma 3.1.4 may be used to improve and get new results on discrete groups having a DF domain or a double Dirichlet Domain. The following definitions are taken from [Lak12]. As we are working with Ford domains in this chapter, we suppose throughout that the group Γ in question contains parabolic elements of the form $1 \neq \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, which fix the point ∞ (see Subsection 1.2.3).

Definition 3.4.1. *A Dirichlet fundamental domain which is also a Ford domain in* $\mathbb{H}^n$ *is called a DF domain.*

Definition 3.4.2. *A Dirichlet fundamental domain which has multiple centres is called a double Dirichlet Domain.*

In particular, we revisit [Lak12] and complement its results. We give an algebraic criterion that is easy to be checked. Throughout the whole chapter, in the lemmas and theorems, we consider the Dirichlet domain of centre i in $\mathbb{H}^2$ and of centre j in $\mathbb{H}^3$. Therefore we assume, without loss of generality, that the stabilizer of i, or j respectively, in Γ, is trivial. We claim that this is possible. Denote by $\mathcal{A}$ the affine subgroup of $\mathrm{PSL}_2(\mathbb{R})$, or $\mathrm{PSL}_2(\mathbb{C})$ respectively, i.e. $\mathcal{A}$ is the subgroup of upper triangular matrices. Consider the conjugated group $\tau\Gamma\tau^{-1}$ of Γ for some $\tau \in \mathcal{A}$ and let $P_0 \in \{i, j\}$, according to the space $\mathbb{H}^2$ or $\mathbb{H}^3$ respectively. Then $(\tau\Gamma\tau^{-1})_{P_0} = \tau\Gamma_{\tau^{-1}(P_0)}\tau^{-1}$ and thus if the stabilizer of $\tau^{-1}(P_0)$ is trivial in Γ, the stabilizer of P_0 is trivial in $\tau\Gamma\tau^{-1}$. Let $\mathcal{F}$ be some fundamental domain for Γ. By definition every point in the interior of $\mathcal{F}$ has trivial stabilizer. As $\mathcal{A}$ acts transitively on the upper half-plane, there exists $\tau \in \mathcal{A}$ such that $\Gamma_{\tau^{-1}(P_0)}$ is trivial. Moreover, if Γ contains a parabolic element of the form $1 \neq \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, then the conjugate $\tau\Gamma\tau^{-1}$ also contains such a parabolic element. So, instead of proving the results for Γ, we will prove them for a group conjugated to Γ with trivial stabilizer of P_0. By Lemma 1.2.42, if $\tau\Gamma\tau^{-1}$ has a double Dirichlet domain, Γ has a double Dirichlet domain. Similarly by Lemma 1.2.49, if $\tau\Gamma\tau^{-1}$ has a DF domain, Γ has a DF domain. This proves the claim.

In the following lemma we use the restriction of Proposition 3.1.11 to the space $\mathbb{H}^2$, as explained in Remark 3.1.12. Abusing notations, in the following lemma σ denotes the linear operator represented by the matrix $\begin{pmatrix} \sqrt{i} & 0 \\ 0 & \sqrt{-i} \end{pmatrix}$ and τ denotes the map sending $z \in \mathbb{H}^2$ to $\overline{z}$.

Lemma 3.4.3. *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \setminus \mathrm{SU}_2(\mathbb{R})$. *Then the following are equivalent.*

1. $a = d$.
2. $\gamma = \sigma^2 \circ \tau \circ \sigma_\gamma$, *where* σ_γ *is the reflection in* Σ_γ.
3. Σ_γ *is the bisector of the geodesic linking* ti *and* $\gamma^{-1}(ti)$, *for every* $t > 0$.
4. *There exists* $t_0 \neq 1$ *such that* Σ_γ *is the bisector of the geodesic segment linking* $t_0 i$ *and* $\gamma^{-1}(t_0 i)$.

Proof. From the definitions of the linear operators σ and τ, one may easily deduce that $\sigma^2 \circ \tau$ denotes the reflection in the imaginary axes, i.e., $\sigma^2 \circ \tau(z) = -\overline{z}$. We first prove that the two first items are equivalent.

Suppose that $a = d$ and $c \neq 0$. Then, by Lemma 3.1.3,

$$P_\gamma = -\frac{d(b+c)}{d^2 + c^2 - 1} = -\frac{d(b+c)}{ad - 1 + c^2} = -\frac{d}{c},$$

and $R_\gamma^2 = \frac{1}{c^2}$. Thus $\Sigma_\gamma = I_\gamma$ and hence, by (1.8), the reflection σ_γ in Σ_γ is given by

$$\sigma_\gamma(z) = P_\gamma - (|c|^2 \sigma^2 \circ \tau(z - P_\gamma))^{-1} = -\frac{d}{c} + \frac{1}{c(c\overline{z} + d)} = \sigma^2 \circ \tau(\gamma(z)).$$

If $c = 0$, then, again by Lemma 3.1.3, σ_γ is the reflection in the line $x = \mp \frac{b}{2}$ and, by (1.9),

$$\sigma_\gamma(z) = -\overline{z} \mp b = \sigma^2 \circ \tau(z \pm b) = \sigma^2 \circ \tau(\gamma(z)).$$

Hence in either case we have that $\gamma = \sigma^2 \circ \tau \circ \sigma_\gamma$. Now suppose that $\gamma = \sigma^2 \circ \tau \circ \sigma_\gamma$. We first suppose that P_γ exists, i.e., Σ_γ is a Euclidean sphere. In this case we have that $\gamma(z) = \sigma^2 \circ \tau \circ \sigma_\gamma(z) = \frac{-P_\gamma z + P_\gamma^2 - R_\gamma^2}{z - P_\gamma}$, from which it follows straightforwardly that $a = d$. If Σ_γ is a vertical line, $x = x_0$ say, then $\sigma_\gamma(z) = -\overline{z} + 2x_0$. Hence $\gamma(z) = \sigma^2 \circ \tau \circ \sigma_\gamma(z) = z - 2x_0$ and hence $a = 1 = d$.

To prove that the second item implies the third, suppose that $\gamma = \sigma^2 \circ \tau \circ \sigma_\gamma$ and let $u \in \Sigma_\gamma$. Then, for every $t > 0$,

$$\rho(\gamma^{-1}(ti), u) = \rho(\sigma_\gamma \circ \sigma^2 \circ \tau(ti), u) = \rho(\sigma_\gamma(ti), u) = \rho(ti, \sigma_\gamma(u)) = \rho(ti, u)$$

and hence Σ_γ is the bisector of the geodesic linking ti and $\gamma^{-1}(ti)$, for every $t > 0$. Obviously the third item implies the fourth.

We now prove that the fourth item implies the first. Let $u \in \Sigma_\gamma$. Then we have that $\rho(t_0 i, u) = \rho(u, \gamma^{-1}(t_0 i))$ and hence $\rho(t_0 i, u) = \rho(t_0 i, \gamma(u))$. By (1.11), we have that

$\mathrm{Im}(\gamma(z)) = |\gamma'(z)|\mathrm{Im}(z)$, where $\mathrm{Im}(z)$ denotes the imaginary part of z. Then by (1.7), we obtain that $|\gamma'(u)||t_0 i - u|^2 = |t_0 i - \gamma(u)|^2$. It follows that

$$\mathrm{Re}(u)^2|\gamma'(u)| - \mathrm{Re}(\gamma(u))^2 = (|\gamma'(u)| - 1)t_0^2 + (1 - |\gamma'(u)|)|\gamma'(u)|\mathrm{Im}(u)^2.$$

We may write this as an equation of the type $\alpha t^2 = \beta$ having $t = t_0$ as a solution. However as $u \in \Sigma_\gamma$, by definition $\rho(u,i) = \rho(u, \gamma^{-1}(i))$ and hence $t = 1$ is also solution of the given equation. Thus we have that $\alpha = \beta$ and $\alpha(t_0^2 - 1) = 0$. As the equation also admits a solution $t_0 \neq 1$, it follows that $\alpha = 0$ and thus $|\gamma'(u)| = 1$, for all $u \in \Sigma_\gamma$, i.e. $\Sigma_\gamma = I_\gamma$. Applying Lemma 3.1.3, we obtain that $a = d$. ∎

Before we state the following corollary, we recall some notions from geometry and group theory.

Definition 3.4.4. *An angle α is a submultiple of an angle β if and only if either there is a positive integer n such that $\alpha = \frac{\beta}{n}$ or $\alpha = 0$.*

Definition 3.4.5. *Let P be an exact, convex fundamental polyhedron for a discrete group Γ acting on $\mathbb{H}^n$. If every side-pairing transformation γ_S associated to a side S is the reflection in $\mathbb{H}^n$ in the hyperplane generated by S, then Γ is a called a discrete reflection group.*

We will make use of the following theorem. For a proof, we refer to [Rat06, Theorem 7.1.3]

Theorem 3.4.6. *Let P be a finite-sided, n-dimensional, convex polyhedron in $\mathbb{H}^n$ of finite volume all of whose dihedral angles are submultiples of π. Then the group Γ generated by the reflections of $\mathbb{H}^n$ in the sides of P is a discrete reflection group with respect to the polyhedron P.*

Lemma 3.4.7. *Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ be a discrete subgroup with trivial stabilizer of $i \in \mathbb{H}^2$ and let $\mathcal{F}$ be its Dirichlet fundamental polyhedron with centre i. Let $\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ be the side-pairing transformations of $\mathcal{F}$ for $1 \leq i \leq n$. If, for every $1 \leq i \leq n$, $a_i = d_i$, then Γ is the subgroup of the orientation preserving isometries of a discrete reflection group.*

Proof. First note that, as $a_i = d_i$, we have by Lemma 3.1.3 that $\Sigma_{\gamma_i} = I_{\gamma_i}$ (as we suppose Γ_i to be trivial, Σ_{γ_i} always exists) and more precisely $\Sigma_{\gamma_i^{-1}}$ has the same radius as Σ_{γ_i} and their centres are the same in absolute value, but have opposite sign. Hence if Σ_{γ_i} intersects the imaginary axis Σ, say, then so does $\Sigma_{\gamma_i^{-1}}$ and the angle between Σ and Σ_{γ_i} is half the the angle between Σ_{γ_i} and $\Sigma_{\gamma_i^{-1}}$. Consider the polyhedron P, say, whose sides are Σ and the Σ_{γ_i}'s with $P_{\gamma_i} \geq 0$. As $\mathcal{F}$ is a fundamental Dirichlet polyhedron, its dihedral angles are submultiples of π. Hence the dihedral angles of P are submultiples of π and thus, by Theorem 3.4.6, the group $\tilde{\Gamma} = \langle \sigma^2 \circ \tau, \sigma_{\gamma_k} \mid P_{\gamma_i} \geq 0 \rangle$, where σ_i denotes the reflection in Σ_{γ_i}, is a discrete reflection group with respect to P. The result then follows by Lemma 3.4.3. ∎

In [Lak12, Theorem 5.3], conditions are given for a Fuchsian group Γ to have a fundamental domain $\mathcal{F}$ which is both a Ford domain and a Dirichlet domain (a DF domain): *A finitely generated, finite coarea Fuchsian group* Γ *admits a DF domain* $\mathcal{F}$*, if and only if* Γ *is an index 2 subgroup of a reflection group.* It is also proved that in the Kleinian case Γ has a generating set consisting of elements whose traces are real ([Lak12, Theorem 6.3].) Our next theorem complements this nice result.

Theorem 3.4.8. *Let* $\Gamma \leq \mathrm{PSL}_2(\mathbb{C})$ *be a finitely generated discrete group, acting on* $\mathbb{H}^n, n \in \{2,3\}$*, and* $P_0 \in \{i,j\}$*, according to* Γ *being Fuchsian or Kleinian and suppose that the stabilizer of* P_0 *in* Γ *is trivial. Then* Γ *admits a DF domain* $\mathcal{F}$ *with centre* P_0 *if and only if for every side-pairing transformation* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *of* $\mathcal{F}$ *we have that* $d = \overline{a}$*. In particular,* $\mathrm{tr}(\gamma) \in \mathbb{R}$ *for all these elements. Moreover, if* Γ *is Fuchsian, then* $\tilde{\Gamma} = \langle \sigma^2 \circ \tau, \Gamma \rangle$ *is a reflection group,* $\hat{\Gamma} = \langle \sigma^2, \Gamma \rangle$ *is a Coxeter group and both contain* Γ *as a subgroup of index two.*

Proof. Let $\mathcal{F}$ be a DF domain, in $\mathbb{H}^n$ with $n = 2,3$, for Γ with centre $P_0 \in \{i,j\}$. Let Φ_0 be a set of side-pairing transformations, i.e., Φ_0 consists of those elements of Γ whose isometric circles (respectively isometric spheres) and vertical lines (respectively vertical planes) form the boundary of $\mathcal{F}$.

As we suppose the stabilizer of P_0 in Γ to be trivial, there exists a bijection $f : \Phi_0 \to \Phi_0$ such that if $\gamma \in \Phi_0, \gamma \notin \Gamma_\infty$ then $I_\gamma = \Sigma_{f(\gamma)}$. Since $\mathcal{F}$ is a Ford domain we have that $\mathcal{F} \cap \gamma^{-1}(\mathcal{F}) = I_\gamma$. Since $\mathcal{F}$ is also a Dirichlet domain we have that $\mathcal{F} \cap \gamma^{-1}(\mathcal{F}) = \Sigma_\gamma$. Consequently $I_\gamma = \Sigma_\gamma$, and thus, by Lemma 3.1.3, $d = \overline{a}$.

If Σ_γ is a Euclidean line or plane then $\Sigma_{\gamma^{-1}}$ is also a line or a plane. Indeed suppose that $\Sigma_{\gamma^{-1}}$ is not a Euclidean line or plane. Then $\Sigma_{\gamma^{-1}} = I_{\gamma^{-1}}$ and $\gamma^{-1} \notin \Gamma_\infty$, which implies that $\gamma \notin \Gamma_\infty$ and hence $\Sigma_\gamma = I_\gamma$ is a Euclidean sphere, a contradiction. From this it follows that $|a| = |d| = 1 = ad$ and thus $d = \overline{a}$ and $c = 0$. Hence, in the Kleinian case, $\mathrm{tr}(\gamma) \in \mathbb{R}$, for all $\gamma \in \Phi_0$.

To prove the converse, suppose that $d = \overline{a}$. Then, by Lemma 3.1.3, $I_\gamma = \Sigma_\gamma$, for all side-pairing transformation γ whose bisector is not a Euclidean line or plane. If Σ_γ is a Euclidean line or plane, then $|a|^2 + |c|^2 = 1$, by Lemma 3.1.3. As $d = \overline{a}$, we also have that $|a|^2 - bc = 1$ and hence $c(\overline{c} + b) = 0$. This implies that $c = 0$ and hence the result also follows in this case.

By Lemma 3.4.7 we have that $\tilde{\Gamma} = \langle \sigma^2 \circ \tau, \Phi \rangle$ is a reflection group containing Γ as a subgroup of index 2. Since $o(\sigma^2\gamma) = 2$, $\mathrm{tr}(\sigma^2\gamma) = 0$, for all $\gamma \in \Phi_0$ it follows that $\hat{\Gamma} = \langle \sigma^2, \Phi_0 \rangle$ is a Coxeter group with $[\hat{\Gamma} : \Gamma] = 2$. ∎

Note that a presentation of $\tilde{\Gamma}$ and $\hat{\Gamma}$ can be obtained using [EGM98, Theorem 2.7.5]. Combining all the results obtained, we obtain the following Theorem.

Theorem 3.4.9. *Let* Γ *be a Fuchsian group with trivial stabilizer of* $i \in \mathbb{H}^2$*. Then the following are equivalent.*

1. *Γ is the subgroup of orientation preserving isometries of a Fuchsian reflection group containing the reflection in the imaginary axis.*

2. *Γ has a DF domain of centre i.*

3. *Γ has a Dirichlet fundamental domain $\mathcal{F}$ with centre i such that, for every side-pairing transformation $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a = d$.*

4. *Γ has a Dirichlet fundamental domain $\mathcal{F}$ with i and $t_0 i$ as centres for some $1 \neq t_0 > 0$.*

5. *Γ has a Dirichlet fundamental domain $\mathcal{F}$ such that all the points of geodesic through i and ti for $t > 0$ are centres of $\mathcal{F}$.*

Proof. Theorem 3.4.8 shows that the second and the third item are equivalent. The equivalence of items 3, 4 and 5 is given by Lemma 3.4.3. Moreover by Lemma 3.4.7 the third item implies the first. We show that the first item implies the fourth. Fix a polyhedron P for the reflection group such that one of the sides is the imaginary axis Σ. Then $\sigma^2 \circ \tau$ is the reflection in Σ. Let Σ_i be a side of P, denote by σ_{γ_i} the reflection in Σ_i and let $\gamma_i = \sigma^2 \circ \tau \circ \sigma_{\gamma_i}$. Then the result follows from Lemma 3.4.3. ∎

The previous result can be generalized, by conjugating the group Γ as explained in the beginning of the section. We then get the following result, where the Dirichlet fundamental domain has arbitrary centre $P \in \mathbb{H}^2$. However, in that case, the third item has to be dropped.

Corollary 3.4.10. *Let Γ be a Fuchsian group with trivial stabilizer of $P \in \mathbb{H}^2$. Then the following are equivalent.*

1. *Γ is the subgroup of orientation preserving isometries of a Fuchsian reflection group containing the reflection in the vertical line through P.*

2. *Γ has a DF domain with centre P.*

3. *Γ has a Dirichlet fundamental domain $\mathcal{F}$ such that all the points of geodesic through P and $P + i$ are centres of $\mathcal{F}$.*

It follows, from Theorem 3.4.9, that all examples of [EGM98, Section 7.3] are groups whose Ford domain is also a Dirichlet domain. Note that this does not follow from the results of [Lak12].

Another interesting example is the group inducing the unique hyperbolic structure of the trice-punctured sphere. This is the free group on two generators $\Gamma(2) = \langle z+2, \frac{z}{2z+1} \rangle$, the congruence group at level 2. Its fundamental polyhedron is the ideal square in $\mathbb{H}^2$ depicted in Figure 3.15 and its orbifold is the trice-punctured sphere (see for example [Rat06, Theorem 9.8.8]). By the previous theorem it has a DF domain. The same can be deduced for $\mathrm{PSL}_2(\mathbb{Z})$. Hence we see that $\langle \Gamma(2), \sigma^2 \circ \tau \rangle \leq \langle \mathrm{PSL}_2(\mathbb{Z}), \sigma^2 \circ \tau \rangle$

are both reflection groups and hence $\langle \Gamma(2), \sigma^2 \circ \tau \rangle$ is not a maximal reflection group. In this case this is due to one more symmetry in the fundamental domain for $\Gamma(2)$. In [Lak12, Section 6] a maximal reflection group, constructed in this way, is exhibited thus answering a question of [ABSW08]. The fundamental domain does not have the symmetric property exhibited by the fundamental domain for $\Gamma(2)$.

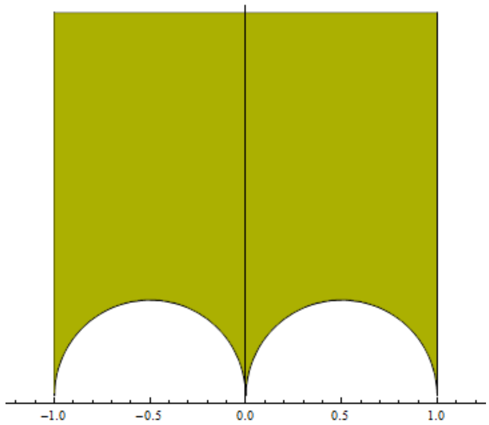


Figure 3.15: Fundamental domain for $\Gamma(2)$

4

Towards the Unit Group of the Integral Group Ring of the Group $Q_8 \times C_7$

On a dit souvent que la géométrie est l'art
de bien raisonner sur des figures mal faites.
Ce n'est pas une boutade,
c'est une vérité qui mérite qu'on y réfléchisse.
Henri POINCARÉ

The problem of determining finitely many generators for the unit group $\mathcal{U}(\mathbb{Z}G)$ of the integral group ring of a finite group G has, up to commensurability, been reduced to orders in the simple components of the rational group algebra $\mathbb{Q}G$. If G does not have non-commutative fixed point free homomorphic images, then the Bass units together with the bicyclic units cover subgroups of finite index in the unit group of orders of the simple components that are not exceptional (see Definition 1.1.19). For the exceptional simple components new methods had to be developed.

In Chapter 3, we have shown how to handle the exceptional components of type (2) and (3), i.e. we can deal with two-by-two matrices over the rationals or a quadratic imaginary extension of the rationals. We described the DAFC algorithm do deal with these components. We showed that this algorithm also handles exceptional components of type (1) that are quaternion division algebras over a quadratic imaginary extension of $\mathbb{Q}$. However the class of exceptional components of type (1) is much larger. For example, quaternion algebras over a cyclotomic field $\mathbb{Q}(\xi_n)$ are also conceivable. Here ξ_n is a n-th primitive root of unity. The group G of smallest possible order for which such an algebra is a simple component of $\mathbb{Q}G$ is the group $Q_8 \times C_7$, where Q_8 is the quaternion group and C_7 is the cyclic group of order 7. The Wedderburn-Artin decomposition of $\mathbb{Q}(Q_8 \times C_7)$ is

$$\mathbb{Q}(Q_8 \times C_7) \cong 4\mathbb{Q} \oplus \mathbb{Q}(\xi_7) \oplus \mathcal{H}(\mathbb{Q}) \oplus \mathcal{H}(\mathbb{Q}(\xi_7)).$$

Indeed, it is well known and easily verified that $\mathbb{Q}Q_8 \cong 4\mathbb{Q} \oplus \mathcal{H}(\mathbb{Q})$ and $\mathbb{Q}C_7 \cong \mathbb{Q} \oplus \mathbb{Q}(\xi_7)$. As $\mathbb{Q}(Q_8 \times C_7) \cong \mathbb{Q}Q_8 \otimes \mathbb{Q}C_7$, the claim follows.

Hence, by Remark 1.1.14 and Theorem 1.1.20, determining finitely many generators for a subgroup of finite index in the unit group of $\mathbb{Z}(Q_8 \times C_7)$ in $\mathbb{Q}(Q_8 \times C_7)$ comes down to study the group $\mathrm{SL}_1(\mathcal{H}(\mathbb{Z}[\xi_7]))$. The embedding (1.28) shows that this group can be considered as a subgroup of the group $\mathrm{SL}_2(\mathbb{C})$. However $\mathrm{SL}_1(\mathcal{H}(\mathbb{Z}[\xi_7]))$ is not discrete in $\mathrm{SL}_2(\mathbb{C})$ and hence by Theorem 1.2.6, $\mathrm{SL}_1(\mathcal{H}(\mathbb{Z}[\xi_7]))$ does not act discontinuously on $\mathbb{H}^3$. Hence we cannot apply Theorem 2.4.7 directly to get a presentation of $\mathrm{SL}_1(\mathcal{H}(\mathbb{Z}[\xi_7]))$. Nevertheless, one can in fact show that $\mathrm{SL}_1(\mathcal{H}(\mathbb{Z}[\xi_7]))$ has a discrete embedding in $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ and thus, by Theorem 1.2.6, it acts discontinuously on $\mathbb{H}^3 \times \mathbb{H}^3 \times \mathbb{H}^3$. Inspired by the classical case of the norm 1 subgroups of quaternion algebras which are discrete in $\mathrm{SL}_2(\mathbb{C})$, the idea is to generalize Theorem 2.4.7 to groups acting discontinuously on direct products of hyperbolic spaces. Obverse that we have proved Theorem 2.4.7 in Chapter 2 for groups acting discontinuously on spaces of constant curvature. This is not the case for direct products of hyperbolic spaces and hence some generalization is necessary. In the next section we give a general description of discontinuous actions on hyperbolic space. As the problem of generalizing Theorem 2.4.7 to groups acting discontinuously on direct products of hyperbolic spaces seems huge, we first want to handle a test case. The smallest possible test case is the one of groups acting discontinuously on a direct product of 2 hyperbolic spaces of dimension 2. The Hilbert Modular group (see Definition 4.2.1) is an example of such a group. In this chapter we give a generalization of Theorem 2.4.7 to the Hilbert Modular group. In Section 4.2, we give all the necessary definitions. In Section 4.3, we describe a fundamental domain for the Hilbert Modular group and in Section 4.4, we give some topological and geometrical information on this fundamental domain. In Section 4.5, we prove the generating part of the generalization of Theorem 2.4.7 and in Section 4.6, we prove the relation part. Finally in the last section, we make some conclusions and give ideas for further working on this topic.

4.1 Actions on Direct Products of Hyperbolic Spaces

In this subsection we show how one may construct a discontinuous action on a direct product of hyperbolic spaces. The ideas are based on [Maa40] and [Pet08]. Consider for every non-negative integers r and s the metric space

$$\mathbb{H}_{r,s} = \mathbb{H}^2 \times \overset{(r)}{\cdots} \times \mathbb{H}^2 \times \mathbb{H}^3 \times \overset{(s)}{\cdots} \times \mathbb{H}^3. \tag{4.1}$$

This space is endowed with the metric ρ given by

$$\rho(X,Y)^2 = \sum_{i=1}^{n} \rho^{(i)}(X_i,Y_i)^2, \tag{4.2}$$

where ρ^i is the hyperbolic metric given in (1.7). We construct the group

$$\mathcal{G}_{r,s} = \mathrm{PSL}_2(\mathbb{R}) \times \overset{(r)}{\cdots} \times \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{C}) \times \overset{(s)}{\cdots} \times \mathrm{PSL}_2(\mathbb{C}), \tag{4.3}$$

a direct product of r copies of $\mathrm{PSL}_2(\mathbb{R})$ and s copies of $\mathrm{PSL}_2(\mathbb{C})$. An element $g \in \mathcal{G}_{r,s}$ will be written as an $(r+s)$-tuple $(g^{(1)}, \ldots, g^{(r+s)})$ and an element of $\mathbb{H}_{r,s}$ will be written as an $(r+s)$-tuple, say $Z = (Z_1, \ldots, Z_{r+s})$. The action of $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{C})$ on $\mathbb{H}^2$ and $\mathbb{H}^3$, described in Subsection 1.2.2, induces componentwise an action of $\mathcal{G}_{r,s}$ by isometries on $\mathbb{H}^{r,s}$:

$$g \cdot Z = (g^{(1)} \cdot Z_1, \ldots, g^{(r+s)} \cdot Z_{r+s}).$$

By Theorem 1.2.6, a discrete subgroup of $\mathcal{G}_{r,s}$ induces a discontinuous action on $\mathbb{H}_{r,s}$.

For a concrete example of this, consider a simple algebra A that is a quaternion algebra over a number field K. Every field homomorphism $\sigma : K \to \mathbb{C}$ naturally extends to a homomorphism $\sigma : A \to M_2(\mathbb{C})$, via $a \mapsto 1 \otimes a$ and to an isomorphism $\mathbb{C} \otimes_{\sigma(K)} A \cong M_2(\mathbb{C})$. This homomorphism maps $\mathrm{SL}_1(A)$, the group consisting of reduced norm one elements, into $\mathrm{SL}_2(\mathbb{C})$. This naturally leads to a homomorphisms from $\mathrm{SL}_1(A)$ to $\mathrm{PSL}_2(\mathbb{C})$. We can compose this with the action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathbb{H}^3$, by the Poincaré extension of Möbius transformations, explained in Subsection 1.2.2. This yields an action of $\mathrm{SL}_1(A)$ on $\mathbb{H}^3$. Note that if $\sigma(K) \subseteq \mathbb{R}$ and A is unramified in σ (i.e. $\mathbb{R} \otimes_{\sigma(K)} A \cong M_2(\mathbb{R})$), then there is a homomorphism from $\mathrm{SL}_1(A)$ to $\mathrm{PSL}_2(\mathbb{R})$ and hence σ restricts to an action on $\mathbb{H}^2$. More precisely, let $\sigma_1, \ldots, \sigma_t$ be representatives (modulo complex conjugation) of the homomorphisms from K to $\mathbb{C}$. If r is the number of real embeddings of K on which A is unramified and $s = t - r$, then the action of σ_i on the i-th component gives an action of $\mathrm{SL}_1(A)$ on $\mathbb{H}_{r,s}$. Explicitly the action is given by

$$a \cdot (x_1, \ldots, x_t) = (\sigma_1(a) \cdot x_1, \ldots, \sigma_t(a) \cdot x_t),$$

where $a \in \mathrm{SL}_1(A)$ and $x_1, \ldots, x_r \in \mathbb{H}_{r,s}$. Let $\mathcal{O}$ be an order in K. Then $\mathcal{O}$ is generated by at most $r + 2s$ vectors, that are linearly independent over $\mathbb{R}$. The quaternion algebra over $\mathcal{O}$ is an order in A and is generated by at most $4(r + 2s)$ vectors, that are again linearly independent over $\mathbb{R}$. By Lemma 1.2.4, the image of the quaternion algebra over $\mathcal{O}$ in $M_2(\mathbb{R})^r \times M_2(\mathbb{C})^s$ is discrete, and thus, by Theorem 1.2.6, the action of $\mathrm{SL}_1(\mathcal{O})$ on $\mathbb{H}_{r,s}$ is discontinuous.

4.2 The Hilbert Modular Group and its Action on $\mathbb{H}^2 \times \mathbb{H}^2$

In this Chapter we study a presentation of the Hilbert Modular group via its action on $\mathbb{H}^2 \times \mathbb{H}^2$.

Definition 4.2.1. *The Hilbert Modular group is the special linear group of degree* 2 *of the ring of integers of real quadratic extensions.*

So throughout k is a square-free positive integer greater than 1 and

$$\omega = \frac{1 + \sqrt{k}}{k_0}, \text{ with } k_0 = \begin{cases} 1, & \text{if } k \not\equiv 1 \mod 4, \\ 2, & \text{if } k \equiv 1 \mod 4. \end{cases} \tag{4.4}$$

Moreover, $K = \mathbb{Q}(\sqrt{k})$ and $R = \mathbb{Z}[\omega]$, the ring of integers of K. For $\alpha \in K$, let α' denote the algebraic conjugate of α. Then $\omega' = \frac{1-\sqrt{k}}{k_0}$ and if $\alpha = \alpha_0 + \alpha_1\omega$ with $\alpha_0, \alpha_1 \in \mathbb{Q}$, then $\alpha' = \alpha_0 + \alpha_1\omega'$. Let N denote the norm map of K over $\mathbb{Q}$ and ϵ_0 denote the fundamental unit of K, i.e. ϵ_0 is the smallest unit ϵ_0 of R greater than 1. Then, by Dirichlet's Unit Theorem (Theorem 1.1.6), $\mathcal{U}(R) = \{\alpha \in R \mid N(\alpha) = \pm 1\} = \pm\langle\epsilon_0\rangle$, where ϵ_0 is a fundamental unit in $\mathbb{Q}(\sqrt{d})$.

We consider $\mathrm{SL}_2(R)$ as a discrete subgroup of $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ by identifying a matrix $A \in \mathrm{SL}_2(R)$ with the pair (A, A'), where A' is the result of applying the algebraic conjugate $'$ in each entry of A. Thus we consider $\mathrm{SL}_2(R)$ acting on $\mathbb{H}^2 \times \mathbb{H}^2$. More precisely, if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(K)$, $Z = (x_1 + y_1 i, x_2 + y_2 i) \in \mathbb{H}^2 \times \mathbb{H}^2$ and $\gamma(Z) = (\hat{x}_1 + \hat{y}_1 i, \hat{x}_2 + \hat{y}_2 i)$ then a straightforward calculation yields

$$\hat{x}_1 = \frac{(ax_1 + b)(cx_1 + d) + acy_1^2}{(cx_1 + d)^2 + c^2y_1^2}, \quad \hat{y}_1 = \frac{y_1}{(cx_1 + d)^2 + c^2y_1^2}, \tag{4.5}$$

and

$$\hat{x}_2 = \frac{(a'x_2 + b')(c'x_2 + d') + a'c'y_2^2}{(c'x_2 + d')^2 + c'^2y_2^2}, \quad \hat{y}_2 = \frac{y_2}{(c'x_2 + d')^2 + c'^2y_2^2}. \tag{4.6}$$

If $Z = (Z_1, Z_2) \in \mathbb{H}^2 \times \mathbb{H}^2$ then we write

$$Z_j = x_j + y_j i, \quad \text{where } x_j, y_j \in \mathbb{R} \text{ and } y_j > 0 \quad (j = 1, 2). \tag{4.7}$$

Then the four tuples $(x_1, x_2, y_1, y_2) \in \mathbb{R}^2 \times (\mathbb{R}^+)^2$ form a system of coordinates of elements of $\mathbb{H}^2 \times \mathbb{H}^2$.

If $Z = (x_1 + y_1 i, x_2 + y_2 i)$, we use the following notation

$$\|cZ + d\| = ((cx_1 + d)^2 + c^2y_1^2)((c'x_2 + d')^2 + c'^2y_2^2).$$

Remark 4.2.2. *We point out that in this chapter points in the space $\mathbb{H}^2 \times \mathbb{H}^2$ are denoted by capital letters, whereas in Chapter 2, capital letters denote half-spaces of $\mathbb{X}$. In fact during the whole thesis, points of topological spaces $\mathbb{X}$ have been denoted by small letters. To stay consistent with this notation, in this chapter, we denote the part of Z contained in one copy of hyperbolic 2-space by small letters. Therefore we denote the point in $\mathbb{H}^2 \times \mathbb{H}^2$, which consists of coordinates in each copy of $\mathbb{H}^2$, by a capital letter.*

We introduce another system of coordinates $(s_1, s_2, r, h) \in \mathbb{R}^2 \times (\mathbb{R}^+)^2$ by setting

$$x_1 = s_1 + s_2\omega, \quad x_2 = s_1 + s_2\omega', \quad y_1^2 = \frac{h}{r}, \quad y_2^2 = rh \tag{4.8}$$

or equivalently

$$s_1 = \frac{\omega' x_1 - \omega x_2}{\omega' - \omega}, \quad s_2 = \frac{x_1 - x_2}{\omega - \omega'}, \quad r = \frac{y_1}{y_2}, \quad h = y_1 y_2. \tag{4.9}$$

So, each element Z of $\mathbb{H}^2 \times \mathbb{H}^2$ is represented by either $(x_1, x_2, y_1, y_2) \in \mathbb{R}^2 \times (\mathbb{R}^+)^2$, or $(s_1, s_2, r, h) \in \mathbb{R}^2 \times (\mathbb{R}^+)^2$, or $(x_1, x_2, r, h) \in \mathbb{R}^2 \times (\mathbb{R}^+)^2$. Then the norm, ratio and height of Z are defined respectively by

$$\|Z\| = (x_1^2 + y_1^2)(x_2^2 + y_2^2), \quad r(Z) = r = \frac{y_1}{y_2}, \quad h(Z) = h = y_1 y_2.$$

Using (4.5) and (4.6), we get

$$h(\gamma(Z)) = \frac{h(Z)}{\|cZ + d\|}. \tag{4.10}$$

Let

$$\Gamma = \mathrm{PSL}_2(R), \tag{4.11}$$

the Hilbert modular group, and we consider it as a discontinuous group of isometries of $\mathbb{H}^2 \times \mathbb{H}^2$. Let Γ_∞ denote the stabilizer of ∞ by the action of Γ (on $\widehat{\mathbb{C}}$). The elements of Γ_∞ are represented by the matrices $\begin{pmatrix} \epsilon_0^m & b \\ 0 & \epsilon_0^{-m} \end{pmatrix}$ with $m \in \mathbb{Z}$ and $b \in R$. As in the chapters before, we make an abuse of notations and thus if $\gamma \in \Gamma$ is represented by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then we simply write $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Hence, (4.5), (4.6) and (4.10) imply that

$$\text{if } \gamma = \begin{pmatrix} \epsilon_0^m & b \\ 0 & \epsilon_0^{-m} \end{pmatrix} \in \Gamma_\infty \text{ then } h(\gamma(Z)) = h(Z) \text{ and } r(\gamma(Z)) = \epsilon_0^{4m} r(Z). \tag{4.12}$$

The second rows of the elements of Γ form the following set

$$\mathcal{S} = \{(c, d) \in R^2 \mid cR + dR = R\}.$$

4.3 Fundamental Domain

In this section we compute a fundamental domain for the group $\Gamma = \mathrm{PSL}_2(R)$ (4.11) via methods analogue to those used for computing a Ford fundamental domain for a discrete group acting on the hyperbolic 2-space (see Section 1.2.3). This part of our work is based on the ideas of H. Cohn in [Coh65a, Coh65b].

We start introducing the following subsets of $\mathbb{H}^2 \times \mathbb{H}^2$, expressed in the (s_1, s_2, r, h)-coordinates:

$$\begin{aligned} V_i^{+,\geq} &= \left\{(s_1, s_2, r, h) \in \mathbb{H}^2 \times \mathbb{H}^2 \mid s_i \geq \frac{1}{2}\right\}, \\ V_i^{-,\geq} &= \left\{(s_1, s_2, r, h) \in \mathbb{H}^2 \times \mathbb{H}^2 \mid s_i \geq -\frac{1}{2}\right\}, \end{aligned}$$

for $i = 1, 2$ and

$$\begin{aligned} V_3^{+,\geq} &= \left\{(s_1, s_2, r, h) \in \mathbb{H}^2 \times \mathbb{H}^2 \mid r \geq \epsilon_0^2\right\}, \\ V_3^{-,\geq} &= \left\{(s_1, s_2, r, h) \in \mathbb{H}^2 \times \mathbb{H}^2 \mid r \geq \epsilon_0^{-2}\right\}. \end{aligned}$$

Moreover, for every $c, d \in \mathcal{S}$ with $c \neq 0$, let

$$V_{c,d}^{\geq} = \{Z \in \mathbb{H}^2 \times \mathbb{H}^2 \mid \|cZ + d\| \geq 1\}.$$

We also define the sets $V_i^{\pm,\leq}$ and $V_{c,d}^{\leq}$ (respectively, $V_i^{\pm}$ and $V_{c,d}$) by replacing $\geq$ by $\leq$ (respectively, $=$) in the previous definitions.

For every $(c, d) \in \mathcal{S}$ with $c \neq 0$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ we have that

$$\begin{aligned}
\| - c\gamma(Z) + a\| &= | - c\gamma(Z_1) + a|^2 | - c'\gamma'(Z_2) + a'|^2 \\
&= | - c\frac{aZ_1 + b}{cZ_1 + d} + a|^2 | - c'\frac{a'Z_2 + b'}{c'Z_2 + d'} + a'|^2 \\
&= |\frac{1}{cZ_1 + d}|^2 |\frac{1}{c'Z_2 + d'}|^2 \\
&= \frac{1}{|cZ_1 + d|^2 |c'Z_2 + d'|^2} \\
&= \frac{1}{\|cZ + d\|},
\end{aligned} \tag{4.13}$$

and therefore

$$\gamma\left(V_{c,d}^{\geq}\right) = V_{-c,a}^{\leq}.$$

We define

$$\begin{aligned}
\mathcal{F}_\infty &= V_1^{+,\leq} \cap V_1^{-,\geq} \cap V_2^{+,\leq} \cap V_2^{-,\geq} \cap V_3^{+,\leq} \cap V_3^{-,\geq} \\
&= \{(s_1, s_2, r, h) \in \mathbb{R}^2 \times (\mathbb{R}^+)^2 \mid |s_1|, |s_2| \leq \frac{1}{2}, \epsilon_0^{-2} \leq r \leq \epsilon_0^2\}; \\
\mathcal{F}_0 &= \bigcap_{(c,d)\in\mathcal{S}} V_{c,d}^{\geq} = \{Z \in \mathbb{H}^2 \times \mathbb{H}^2 \mid \|cZ + d\| \geq 1 \text{ for all } (c, d) \in \mathcal{S}\}; \\
\mathcal{F} &= \mathcal{F}_\infty \cap \mathcal{F}_0.
\end{aligned}$$

Lemma 4.3.1. *$\mathcal{F}_\infty$ is a fundamental domain for Γ_∞.*

Proof. We first prove that if $1 \neq \gamma \in \Gamma_\infty$ and $Z = (Z_1, Z_2) = (x_1 + y_1 i, x_2 + y_2 i) = (s_1, s_2, r, h)$ then $\{Z, \gamma(Z)\}$ cannot be contained in the interior of $\mathcal{F}_\infty$. Let

$$\gamma = \begin{pmatrix} \epsilon_0^m & b \\ 0 & \epsilon_0^{-m} \end{pmatrix}.$$

By (4.12), $r(\gamma(Z)) = \epsilon_0^{4m} r$. If Z and $\gamma(Z)$ belong to the interior of Γ_∞ then $\epsilon_0^{-2} < r, \epsilon_0^{4m} r < \epsilon_0^2$ and therefore $m = 0$. Therefore the transformation γ is simply a translation by the parameter (b, b') with $b \in R$. Now $b = b_1 + b_2\omega$, with $b_1, b_2 \in \mathbb{Z}$. As Z and $\gamma(Z)$ belong to the interior of $\mathcal{F}_\infty$, then $|s_1|, |s_2|, |s_1 + b_1|, |s_2 + b_2| < \frac{1}{2}$. Thus $b = 0$ and hence $\gamma = 1$, as desired.

Let $Z = (x_1 + y_1 i, x_2 + y_2 i) = (s_1, s_2, r, h)$ be an arbitrary point in $\mathbb{H}^2 \times \mathbb{H}^2$, with $x_i, y_i \in \mathbb{R}$. We will show that there exists $\gamma \in \Gamma_\infty$ such that $\gamma(Z) \in \mathcal{F}_\infty$. As $\epsilon_0 > 1$, $\lim_{n\to+\infty} \epsilon_0^n = \infty$ and $\lim_{n\to-\infty} \epsilon_0^n = 0$ and hence there exists $n \in \mathbb{Z}$ such that

$$\epsilon_0^{4n-2} \leq r \leq \epsilon_0^{4n+2}.$$

Let $\gamma = \begin{pmatrix} \epsilon_0^{-n} & 0 \\ 0 & \epsilon_0^n \end{pmatrix}$. By (4.12), $\epsilon_0^{-2} \leq r(\gamma(Z)) = \epsilon_0^{-4n} r \leq \epsilon_0^2$. So we may assume that $\epsilon_0^{-2} \leq r \leq \epsilon_0^2$. Now consider $\gamma = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, with $b = b_1 + b_2\omega$, $b_1, b_2 \in \mathbb{Z}$ and $|s_i + b_i| \leq \frac{1}{2}$. By (4.9) and (4.12), the (s_1, s_2, r, h)-coordinates of $\gamma(Z)$ are $(s_1 + b_1, s_2 + b_2, r, h)$ and hence $\gamma(Z) \in \mathcal{F}_\infty$, as desired.

The boundary of $\mathcal{F}_\infty$ is included in $\bigcup_{i=1}^3 V_i^+ \cup \bigcup_{i=1}^3 V_i^-$. This clearly has Lebesgue measure 0. The connectedness of $\mathcal{F}$ is evident. ∎

For a subset C of $\mathbb{H}^2 \times \mathbb{H}^2$ and $\mu > 0$ let

$$V_{C,\mu} = \{(c, d) \in R \times R \mid \|cZ + d\| \leq \mu \text{ for some } Z \in C\}. \tag{4.14}$$

We say that a subset C of $\mathbb{H}^2 \times \mathbb{H}^2$ is *hyperbolically bounded* if it is bounded in the metric of $\mathbb{H}^2 \times \mathbb{H}^2$ or equivalently if it is bounded in the Euclidean metric and there is a positive number ϵ such that $y_1, y_2 > \epsilon$ for every $(x_1 + y_1 i, x_2 + y_2 i) \in C$. For the next lemma we need the following proposition from [Neu99].

Proposition 4.3.2. *[Neu99, Lemma 7.2] Let K be a number field and $\mathcal{O}_K$ the ring of integers of K. Let N denote the norm map of K over $\mathbb{Q}$. Then, for a given $\lambda \in \mathbb{Z}$, up to multiplication by units, there are only finitely many $a \in \mathcal{O}_k$ such that $N(a) = \lambda$.*

Lemma 4.3.3. *Let C be a hyperbolically bounded subset of $\mathbb{H}^2 \times \mathbb{H}^2$ and let $\mu > 0$. Then there are finitely many elements $(c_1, d_1), \ldots, (c_k, d_k)$ of $R \times R$ such that $V_{C,\mu} = \{(uc_i, ud_i) \mid i = 1, \ldots, k, u \in \mathcal{U}(R)\}$.*

Proof. As, by Proposition 4.3.2, $\{d \in R \mid N(d) \leq \mu\}$ is finite up to units, it is clear that $V_{C,\mu}$ has finitely many elements of the form $(0, d)$ with $d \in R$, up to units. As C is hyperbolically bounded there is $s > 0$ such that $h(Z) \geq s$ for every $Z \in C$. By Proposition 4.3.2, there are $c_1, \ldots, c_k \in R \setminus \{0\}$ such that if $c \in R \setminus \{0\}$ with $N(c)^2 \leq \frac{\mu}{s^2}$ then $c = uc_i$ for some $i = 1, \ldots, k$ and some $u \in \mathcal{U}(R)$. Moreover, for every $i = 1, \ldots, k$, the set

$$C_i = \{(x, y) \in \mathbb{R}^2 \mid (c_i x_1 + x)^2 c_i'^2 y_2^2 + (c_i' x_2 + y)^2 c_i^2 y_1^2 \leq \mu \\ \text{for some } (x_1 + y_1 i, x_2 + y_2 i) \in C\}$$

is a bounded subset of $\mathbb{R}^2$. As $\{(d, d') \mid d \in R\}$ is a discrete subset of $\mathbb{R}^2$, there are $e_{i1}, \ldots, e_{ij_i} \in R$ such that if $d \in R$ and $(d, d') \in C_i$ then $d = e_{il_i}$ for some $1 \leq l_i \leq j_i$. Assume that $\|cZ + d\| \leq \mu$ for some $Z = (x_1 + y_1 i, x_2 + y_2 i) \in C$. Then $N(c)^2 s^2 \leq$

$N(c)^2 h(Z)^2 = c^2 c'^2 y_1^2 y_2^2 \leq ((cx_1+d)^2+c^2y_1^2)((c'x_2+d')^2+c'^2y_2^2) = \|cZ+d\| \leq \mu$. Hence $c = uc_i$ for some $i = 1, \ldots, k$ and $u \in \mathcal{U}(R)$. Moreover, as $N(u) = uu' = \pm 1$, we have

$$\begin{aligned}(c_ix_1 + u^{-1}d)^2 c_i'^2 y_2^2 + (c_i'x_2 + (u^{-1}d)')^2 c_i^2 y_1^2 & \\ &= (cx_1+d)^2 c'^2 y_2^2 + (c'x_2+d')^2 c^2 y_1^2 \\ &\leq \|cZ+d\|,\end{aligned}$$

and by assumption, the latter is smaller than μ. Therefore $(u^{-1}d, (u^{-1}d)') \in C_i$. Thus $u^{-1}d = e_{il_i}$ for some $1 \leq l_i \leq j_i$. This proves that there are finitely many elements $(c_1, d_1), \ldots, (c_k, d_k) \in (R \setminus \{0\}) \times R$ such that $V_{C,\mu}$ is contained in $\{u(c_i, d_i) \mid i = 1, \ldots, k, u \in \mathcal{U}(R)\}$. Note that if $u \in \mathcal{U}(R)$ then $\|uc_iZ + ud_i\| = \|c_iZ + d_i\|$. Hence, the result follows. ∎

The following lemma is a kind of generalization of Lemma 1.2.46. Its proof is very similar to the proof of Lemma 1.2.46, but for completeness' sake we give it here.

Lemma 4.3.4. *$\mathcal{F}_0 = \{Z \in \mathbb{H}^2 \times \mathbb{H}^2 \mid Z$ has maximal height in its Γ-orbit$\}$.*

Proof. We first claim that for a fixed $Z \in \mathbb{H}^2 \times \mathbb{H}^2$, the set $\{\|cZ + d\| \mid (c, d) \in \mathcal{S}\}$ has a minimum. Indeed, let $\pi = \|Z\|$. Clearly $V_{\{Z\},\pi} \cap \mathcal{S} \neq \emptyset$ as it contains $(1, 0)$. By Lemma 4.3.3, $V_{\{Z\},\pi} \cap \mathcal{S} = \{u(c_i, d_i) \mid i = 1, \ldots, k, u \in \mathcal{U}(R)\}$ for some $(c_1, d_1), \ldots, (c_k, d_k) \in \mathcal{S}$. Let $0 \neq m = \min\{\|c_iZ + d_i\| \mid i = 1, \ldots, k\}$. If $\|cZ + d\| < m \leq \pi$, with $(c, d) \in \mathcal{S}$ then $(c, d) = u(c_i, d_i)$ for some i and some $u \in \mathcal{U}(R)$. Then $m > \|cZ + d\| = N(u)^2\|c_iZ + d_i\| \geq m$, a contradiction. Hence the claim follows. Consequently, by (4.10), the Γ-orbit of Z has an element of maximum height.

Consider a Γ-orbit and let Z be an element in this orbit with maximal height. Hence for every $\gamma \in \Gamma$, $h(\gamma(Z)) \leq h(Z)$ and hence by (4.10), $\|cZ + d\| \geq 1$, for every $(c, d) \in \mathcal{S}$. Thus $Z \in \mathcal{F}_0$.

To prove the other inclusion, let $Z \in \mathcal{F}_0$. Then $\|cZ + d\| \geq 1$, for every $(c, d) \in \mathcal{S}$ and hence for every $\gamma \in \Gamma$, $h(\gamma(Z)) \leq h(Z)$. Thus every element of $\mathcal{F}_0$ reaches the maximum height in its orbit. ∎

The next lemma is the generalization of Lemma 1.2.47 to $\mathbb{H}^2 \times \mathbb{H}^2$.

Lemma 4.3.5. *[Coh65b, Coh65a] $\mathcal{F}$ is fundamental region for Γ.*

Proof. We first show that $\mathcal{F}$ contains a point of every Γ-orbit. To prove this, consider an orbit and let Z be an element in this orbit with maximal height. By Lemma 4.3.1, there is $W \in \mathcal{F}_\infty$ and $\tau \in \Gamma_\infty$ such that $W = \tau(Z)$. By (4.12), $h(Z) = h(W)$ and hence we may assume that $Z \in \mathcal{F}_\infty$. Thus, by Lemma 4.3.4, $Z \in \mathcal{F}_0$ and so $Z \in \mathcal{F}$.

Now we will show that if two points of the same orbit are in $\mathcal{F}$ then they necessarily are on the border of $\mathcal{F}$. Suppose that $Z, \gamma(Z) \in \mathcal{F}$, for some $1 \neq \gamma \in \Gamma$. If $\gamma \in \Gamma_\infty$ then, by Lemma 4.3.1, Z belongs to the border of $\mathcal{F}_\infty$ and hence it belongs to the border of $\mathcal{F}$. Otherwise $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \neq 0$ and, by (4.10) we have

$$h(Z) = h(\gamma(Z)) = \frac{h(Z)}{\|cZ + d\|}.$$

Therefore $\|cZ+d\| = 1$ and thus Z lies on the border of $\mathcal{F}_0$ and thus on the border of $\mathcal{F}$.

Finally the boundary of $\mathcal{F}$ is contained in $\bigcup_{i=1}^{3} V_i^+ \cup \bigcup_{i=1}^{3} V_i^- \cup \bigcup_{(c,d)\in\mathcal{S}} V_{c,d}$ with $c \neq 0$. As $\mathcal{S}$ is countable, the boundary of $\mathcal{F}$ has measure 0. ∎

Define

$$B = \left\{(s_1, s_2, r) \mid |s_1|, |s_2| \leq \frac{1}{2},\ \epsilon_0^{-2} \leq r \leq \epsilon_0^2\right\}.$$

Observe that

$$\mathcal{F}_\infty = B \times \mathbb{R}^+.$$

We can think of $\mathcal{F}$ as the region above a "floor", which is given by the sets $V_{c,d}$ for $(c,d) \in \mathcal{S}$ and limited by "six walls", which are given by the sets $V_i^\pm$ for $i = 1, 2, 3$.

We now give an alternative description of $\mathcal{F}$ on which the "floor" is given by the graph of a function h_0 defined on B.

For each $c, d \in R$ we define the function $f_{c,d} : \mathbb{R}^4 \to \mathbb{R}$ by

$$f_{c,d}(x_1, x_2, y_1, y_2) = \left[(cx_1 + d)^2 + c^2 y_1^2\right]\left[(c'x_2 + d')^2 + c'^2 y_2^2\right]. \tag{4.15}$$

Observe that if $Z \in \mathbb{H}^2 \times \mathbb{H}^2$ then $f_{c,d}(Z) = \|cZ + d\|$ and if $d \in \mathcal{U}(R)$ then $f_{0,d}(Z) = 1$. Thus

$$\mathcal{F}_0 = \{Z \in \mathbb{H}^2 \times \mathbb{H}^2 \mid f_{c,d}(Z) \geq 1, \text{ for all } (c,d) \in \mathcal{S}\}. \tag{4.16}$$

We use the mixed coordinate system described in Section 4.2 and write

$$\begin{aligned} &f_{c,d}(x_1, x_2, r, h) \\ &\quad = [(cx_1+d)(c'x_2+d')]^2 + \left[(cx_1+d)^2 c'^2 r + (c'x_2+d')^2 \frac{c^2}{r}\right] h + N(c)^2 h^2. \end{aligned} \tag{4.17}$$

If $c, d \in R$, $(x_1, x_2, r) \in \mathbb{R}^2 \times \mathbb{R}^+$ and $v \in \mathbb{R}$, then we define

$$\begin{aligned} &f_{c,d,x_1,x_2,r}(v) \\ &= [(cx_1+d)(c'x_2+d')]^2 + \left[(cx_1+d)^2 c'^2 r + (c'x_2+d')^2 \frac{c^2}{r}\right] v + N(c)^2 v^2. \end{aligned}$$

and if $c \neq 0$ then we set

$$\begin{aligned} h_1(c,d,x_1,x_2,r) = &\sqrt{\frac{1}{N(c)^2} + \frac{1}{4}\left[\left(x_1 + \frac{d}{c}\right)^2 r - \left(x_2 + \frac{d'}{c'}\right)^2 \frac{1}{r}\right]^2} - \\ &\frac{1}{2}\left[\left(x_1 + \frac{d}{c}\right)^2 r + \left(x_2 + \frac{d'}{c'}\right)^2 \frac{1}{r}\right]. \end{aligned}$$

If $u \in \mathcal{U}(R)$ then the following statements hold (throughout this chapter we will use these without explicit reference):

$$\begin{aligned} f_{uc,ud} &= f_{c,d}, & (4.18)\\ h_1(uc, ud, x_1, x_2, r) &= h_1(c, d, x_1, x_2, r), & (4.19)\\ (c,d) \in \mathcal{S} &\Leftrightarrow (uc, ud) \in \mathcal{S} & (4.20)\\ V^{\geq}_{uc,ud} &= V^{\geq}_{c,d}, & (4.21)\\ V_{uc,ud} &= V_{c,d}. & (4.22)\end{aligned}$$

Set $\mathcal{C} = \{(x_1, x_2) \in \mathbb{R}^2 \mid -1 < (cx_1 + d)(c'x_2 + d') < 1\}$. Moreover, if $f = f_{c,d,x_1,x_2,r}$ then

$$f'(v) = \left[(cx_1 + d)^2 c'^2 r + (c'x_2 + d')^2 \frac{c^2}{r}\right] + 2N(c)^2 v. \quad (4.23)$$

and hence f is strictly increasing on $[\min(0, h_1), \infty)$.

For $(s_1, s_2, r) \in B$ let

$$h_0(s_1, s_2, r) = \sup\{h_1 > 0 \mid f_{c,d,x_1,x_2,r}(h_1) = 1, \text{ for some } (c, d) \in \mathcal{S}, \text{ with } c \neq 0\},$$

where we understand that the supremum of the empty set is 0. If $(c, d) \in \mathcal{S}$, $c \neq 0$, $h_1 > 0$ and $f_{c,d,x_1,x_2,r}(h_1) = 1$ then $h_1^2 \leq \frac{1}{N(c)^2} \leq 1$. Hence the supremum defining $h_0(s_1, s_2, r)$ exists and

$$h_0(s_1, s_2, r) \leq 1. \quad (4.24)$$

By (4.16) and the monotonicity $f_{c,d,x_1,x_2,r}$ we have

$$\begin{aligned} \mathcal{F}_0 &= \{(s_1, s_2, r, h) \in \mathbb{R}^2 \times (\mathbb{R}^+)^2 \mid h \geq h_0(s_1, s_2, r)\}, & (4.25)\\ \mathcal{F} &= \{(s_1, s_2, r, h) \in B \times \mathbb{R}^+ \mid h \geq h_0(s_1, s_2, r)\}. & (4.26)\end{aligned}$$

From (4.24) and (4.26) the following lemmas easily follow.

Lemma 4.3.6. *If $Z = (s_1, s_2, r, h) \in \mathbb{H}^2 \times \mathbb{H}^2$ with $|s_1| \leq \frac{1}{2}$, $|s_2| \leq \frac{1}{2}$, $\epsilon_0^{-2} \leq r \leq \epsilon_0^2$ and $h \geq 1$, then $Z \in \mathcal{F}$. Moreover, if the inequalities are strict then $Z \in \mathcal{F}^\circ$.*

The following lemma then follows easily.

Lemma 4.3.7. *$\mathcal{F}$ is path-connected.*

Proof. Let Z and W be two points of $\mathcal{F}$. We prove that we may construct a path joining both points. Set $Z = (s_1, s_2, r, h)$ and $W = (s_1', s_2', r', h')$. As $Z \in \mathcal{F} = \mathcal{F}_0 \cap \mathcal{F}_\infty$, by (4.25) and Lemma 4.3.1, $|s_1| \leq \frac{1}{2}$, $|s_2| \leq \frac{1}{2}$, $\epsilon_0^{-1} \leq r \leq \epsilon_0$ and $h > h_0(s_1, s_2, r)$. Thus, again by (4.25), it is easy to see that every point $Z' = (s_1, s_2, r, \tilde{h})$ with $\tilde{h} \geq h$ is also contained in $\mathcal{F}_0 \cap F_\infty$. Set $h^* = \max\{h, h', 1\}$ and set $Z_1 = (s_1, s_2, r, 2h^*)$. Then by the previous there exists clearly a path in $\mathcal{F}$ joining Z to Z_1. Equivalently, if $W_1 = (s_1', s_2', r', 2h^*)$, there exists a path in $\mathcal{F}$ joining W to W_1. By Lemma 4.3.6, every point $Y = (s_1'', s_2'', r'', 2h^*)$ with $|s_1''| \leq \frac{1}{2}$, $|s_2''| \leq \frac{1}{2}$ and $\epsilon_0^{-1} \leq r'' \leq \epsilon_0$ is in $\mathcal{F}$. Hence is is easy to see that there exists a path in $\mathcal{F}$ joining Z_1 to W_1. This concludes the proof. ∎

The following corollary is an immediate consequence of the previous lemma.

Corollary 4.3.8. *$\mathcal{F}$ is a fundamental domain for Γ.*

In order to have a finite procedure to compute the fundamental domain $\mathcal{F}$, we need to replace $\mathcal{S}$ in the definition of $\mathcal{F}$ by a suitable finite set. In our next result we obtain this for R a principal ideal domain (PID, for short). This result is based on [Coh65b, Section 5]. Define, for $(c, d) \in R \times R$,

$$G(c,d) = (cx_1 + d)^2 + c^2 y_1^2 + (c'x_2 + d')^2 + c'^2 y_2^2, \tag{4.27}$$

where x_1, x_2, y_1, y_2 are fixed real numbers. If we set $c = c_0 + c_1\omega$ and $d = d_0 + d_1\omega$, $G(c, d)$ is a quadratic form in four variables with determinant $y_1^2 y_2^2 (\omega - \omega')^4 = \left(\frac{4k}{k_0^2} y_1 y_2\right)^2$. Hence, by [KZ72], there exists an integral quadruple, or a pair $(c, d) \in R \times R$ respectively, such that

$$G(c,d) \leq \sqrt{2\frac{4k}{k_0^2} y_1 y_2}.$$

Now fix $(Z_1, Z_2) \in \mathbb{H}^2 \times \mathbb{H}^2$. As

$$\|cZ + d\| \leq \frac{G(c,d)^2}{4},$$

for every $(c, d) \in R \times R$, there exists $(c, d) \in R \times R$, such that

$$\|cZ + d\| \leq \frac{2k}{k_0^2} y_1 y_2.$$

We may suppose that $(c, d) \in \mathcal{S}$. Indeed, suppose on the contrary that $(c, d) \notin \mathcal{S}$. As R is a PID, this means that c and d have a common division, say $\xi \in R$. Then $c = \tilde{c}\xi$ and $d = \tilde{d}\xi$ and $\|cz + d\| = |N(\xi)|^2 \|\tilde{c}Z + \tilde{d}\|$ and thus $\|\tilde{c}Z + \tilde{d}\| \leq \frac{2k}{|N(\xi)|^2 k_0^2} y_1 y_2 \leq \frac{2k}{k_0^2} y_1 y_2$. Hence, there exists $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, with $a, b \in \mathbb{R}$, such that

$$h(\gamma(Z)) = \frac{h(Z)}{\|cZ + d\|} \geq \frac{k_0^2}{2k}.$$

This gives the following lemma.

Lemma 4.3.9. *If R is a PID and $(s_1, s_2, r, h) \in \mathcal{F}_0$ then $h > \frac{k_0^2}{2k}$.*

Regrouping the above results, we obtain the following theorem.

Theorem 4.3.10. *Let k be a square-free integer greater than 1. Let k_0 and ω be as in (4.4), $R = \mathbb{Z}[\omega]$, $\Gamma = \mathrm{PSL}_2(R)$ and $\mathcal{F}$ the fundamental domain for Γ given in Lemma 4.3.5 and Corollary 4.3.8. Let $\mathcal{S}_1$ be a set of representatives, up to multiplication by units in R, of the couples $(c, d) \in R^2$ satisfying the following conditions:*

$$\begin{aligned} &c \neq 0, \quad cR + dR = R, \quad |N(c)| \leq \frac{2k}{k_0^2}, \\ &\left|\frac{d}{c}\right| < \epsilon_0 \sqrt{\frac{2k}{N(c)^2 k_0^2} - \frac{k_0^2}{2k}} + \frac{1+\omega}{2} \quad \text{and} \quad \left|\frac{d'}{c'}\right| < \epsilon_0 \sqrt{\frac{2k}{N(c)^2 k_0^2} - \frac{k_0^2}{2k}} + \frac{1-\omega'}{2}. \end{aligned} \tag{4.28}$$

Then $\mathcal{S}_1$ is finite and if R is a principal ideal domain then

$$\mathcal{F} = \mathcal{F}_\infty \bigcap \bigcap_{(c,d)\in\mathcal{S}_1} V_{c,d}^{\geq}.$$

In particular, $V_{c,d} \cap \mathcal{F} \neq \emptyset$ if and only if $(c,d) \in \mathcal{S}_1$.

Proof. That $\mathcal{S}_1$ is finite is a consequence of the well known fact that the set of elements of R with a given norm is finite modulo multiplication by units (see Proposition 4.3.2) and that the image of R by the map $x \mapsto (x, x')$ is a discrete additive subgroup of $\mathbb{R}^2$, so that it intersects every compact subset in finitely many elements.

Assume now that R is a PID. Then, by Lemma 4.3.9, $h_0(s_1, s_2, r) \geq \frac{k_0^2}{2k} > 0$. Therefore the set

$$\mathcal{S}_{s_1,s_2,r} = \left\{ (c,d) \in \mathcal{S} \mid c \neq 0 \text{ and } h_1(c,d,s_1,s_2,r) \geq \frac{k_0^2}{2k} \right\}$$

is not empty and

$$h_0(s_1,s_2,r) = \sup\{h_1(c,d,s_1,s_2,r) \mid (c,d) \in \mathcal{S}_{s_1,s_2,r}\}. \tag{4.29}$$

We claim that if $(s_1,s_2,r) \in B$ and $(c,d) \in \mathcal{S}_{s_1,s_2,r}$ then (c,d) satisfies the conditions of (4.28). Indeed, clearly (c,d) satisfies the first two conditions. It satisfies the third condition since

$$N(c)^2 \left(\frac{k_0^2}{2k}\right)^2 \leq N(c)^2 h_1^2 \leq f_{c,d,x_1,x_2,r}(h_1) = 1.$$

To prove that it satisfies the last two conditions of (4.28) let $x_1 = s_1 + s_2\omega$, $x_2 = s_1 + s_2\omega'$ and $h_1 = h_1(c,d,s_1,s_2,r)$ Recall that $|s_1|, |s_2| \leq \frac{1}{2}$ and hence

$$\frac{-1-\omega}{2} \leq x_1 \leq \frac{1+\omega}{2} \quad \text{and} \quad \frac{-1+\omega'}{2} \leq x_2 \leq \frac{1-\omega'}{2}. \tag{4.30}$$

Furthermore

$$\begin{aligned}
& N(c)^2 \frac{k_0^2}{2k}\left(\frac{k_0^2}{2k} + \left(x_1 + \frac{d}{c}\right)^2 r + \left(x_2 + \frac{d'}{c'}\right)^2 \frac{1}{r}\right) \\
= \;& N(c)^2 \left(\frac{k_0^2}{2k}\right)^2 + \left((cx_1+d)^2 c'^2 r + (c'x_2+d')^2 c^2 \frac{1}{r}\right)\frac{k_0^2}{2k} \\
\leq \;& N(c)^2 h_1^2 + \left((cx_1+d)^2 c'^2 r + (c'x_2+d')^2 c^2 \frac{1}{r}\right) h_1 + (cx_1+d)^2(c'x_2+d')^2 \\
= \;& f_{c,d,x_1,x_2,r}(h_1) = 1,
\end{aligned}$$

This, together with $\epsilon_0^{-2} \leq r \leq \epsilon_0^2$ implies

$$\begin{cases}
|x_1 + \frac{d}{c}| < \frac{1}{\sqrt{r}}\sqrt{\frac{2k}{N(c)^2 k_0^2} - \frac{k_0^2}{2k}} \leq \epsilon_0 \sqrt{\frac{2k}{N(c)^2 k_0^2} - \frac{k_0^2}{2k}}, \\
|x_2 + \frac{d'}{c'}| < \sqrt{r}\sqrt{\frac{2k}{N(c)^2 k_0^2} - \frac{k_0^2}{2k}} \leq \epsilon_0 \sqrt{\frac{2k}{N(c)^2 k_0^2} - \frac{k_0^2}{2k}}.
\end{cases}$$

Combining this with (4.30) we also have

$$\begin{cases} |\frac{d}{c}| < \epsilon_0\sqrt{\frac{2k}{N(c)^2k_0^2} - \frac{k_0^2}{2k}} + \frac{1+\omega}{2}, \\ |\frac{d'}{c'}| < \epsilon_0\sqrt{\frac{2k}{N(c)^2k_0^2} - \frac{k_0^2}{2k}} + \frac{1-\omega'}{2}. \end{cases}$$

This proves the claim.

Suppose now that R is a PID. Combining the claim with (4.19) and (4.29) we deduce that if $(s_1, s_2, r) \in B$ then $h_0(s_1, s_2, r) = \sup\{h_1(c, d, s_1, s_2, r) \mid (c, d) \in \mathcal{S}_1 \cap \mathcal{S}_{s_1,s_2,r}\}$. As $\mathcal{S}_1$ is finite, this implies that $h_0(s_1, s_2, r) = h_1(c, d, s_1, s_2, r)$ for some $(c_0, d_0) \in \mathcal{S}_1$. Then $(s_1, s_2, r, h) \in \mathcal{F}$ if and only if $(s_1, s_2, r, h) \in V^{\geq}_{c_0,d_0}$ if and only if $(s_1, s_2, r, h) \in V^{\geq}_{c,d}$ for every $(c, d) \in \mathcal{S}$. Therefore, in case R is a PID, $\mathcal{F} = \mathcal{F}_\infty \cap \bigcap_{(c,d)\in\mathcal{S}_1} V^{\geq}_{c,d}$ as desired.

To get the second part of the theorem, notice that $\mathcal{F} \cap V_{c,d} \subseteq \partial\mathcal{F}_0$ for $(c, d) \in \mathcal{S}$, $c \neq 0$. Moreover, similarly as in (4.25),

$$\partial\mathcal{F}_0 = \{(s_1, s_2, r, h) \in \mathbb{R}^2 \times (\mathbb{R}^+)^2 \mid h = h_0(s_1, s_2, r)\}.$$

Thus by the same reasoning as above, $\mathcal{F}_0 \cap V_{c,d} \neq \emptyset$ if and only if $(c, d) \in \mathcal{S}_1$ and hence the result follows. ∎

Recall the definition of a locally finite fundamental domain from Definition 1.2.9.

Lemma 4.3.11. *The fundamental domains $\mathcal{F}$ for Γ and $\mathcal{F}_\infty$ for Γ_∞ are locally finite.*

Proof. Let C be a compact subset of $\mathbb{H}^2\times\mathbb{H}^2$ and let $\gamma \in \Gamma_\infty$ be such that $C\cap\gamma(\mathcal{F}_\infty) \neq \emptyset$. Let $Z = (x_1, x_2, y_1, y_2) \in C \cap \gamma(\mathcal{F}_\infty)$. As C is compact, the coordinates of Z are bounded. As $\gamma^{-1} \in \Gamma_\infty$, $\gamma^{-1} = \begin{pmatrix} \epsilon_0^m & b \\ 0 & \epsilon_0^{-m} \end{pmatrix}$ for some $m \in \mathbb{Z}$ and $b \in R$, and hence $\gamma^{-1}(Z) = (\epsilon^{2m}x_1 + b, \epsilon^{-2m}x_2 + b', \epsilon^{2m}y_1, \epsilon^{-2m}y_2)$. As y_1 and y_2 are bounded, there are only finitely many $m \in \mathbb{Z}$ such that $\gamma^{-1}(Z) \in \mathcal{F}_\infty$, or equivalently $Z \in \gamma(\mathcal{F}_\infty)$, and this for every $Z \in C$. Moreover as the first two coordinates of Z are bounded and $\{(b, b') \mid b \in R\}$ is a discrete subset of R^2 for each m only finitely many b's in R satisfy that $\epsilon^{2m}x_1+b$ and $\epsilon^{-2m}x_2+b'$ satisfy the conditions imposed on x_1 and x_2 for Z to be in $\mathcal{F}_\infty$. Thus there are only finitely many $\gamma \in \Gamma_\infty$ such that $C \cap \gamma(\mathcal{F}_\infty) \neq \emptyset$ and therefore $\mathcal{F}_\infty$ is locally finite.

Now suppose $C\cap\gamma(\mathcal{F}) \neq \emptyset$ for C a compact subset of $\mathbb{H}^2 \times \mathbb{H}^2$ and $\gamma \in \Gamma$. If $\gamma \in \Gamma_\infty$, then we are done by the first part. So we may suppose that $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \neq 0$. If $Z \in C \cap \gamma(\mathcal{F})$ and $c \neq 0$ then, by (4.13), $\frac{1}{\|-cZ+a\|} = \|c\gamma^{-1}(Z) + d\| \geq 1$ and therefore $\| - cZ + a\| \leq 1$, in other words $(-c, a) \in V_{C,1}$, where $V_{C,1}$ is defined as in Lemma 4.3.3. Using Lemma 4.3.3 we deduce that $(-c, a)$ belongs to a finite subset, up to units in R. So suppose $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\gamma_u = \begin{pmatrix} ua & u^{-1}b \\ uc & u^{-1}d \end{pmatrix}$ are such that $C \cap \gamma(\mathcal{F}) \neq \emptyset$ and

$C \cap \gamma_u(\mathcal{F}) \neq \emptyset$ respectively for some $u \in \mathcal{U}(R)$. Then

$$\gamma^{-1}\gamma_u = \begin{pmatrix} u & * \\ 0 & u^{-1} \end{pmatrix},$$

where $*$ denotes some element of R. Denote the latter matrix by U. Then $\gamma_u = \gamma U$ and $U \in \Gamma_\infty$. Hence we also have $C \cap \gamma U(\mathcal{F}) \neq \emptyset$ or equivalently $\gamma(C) \cap U(\mathcal{F}) \neq \emptyset$. As $\gamma(C)$ is still a compact subset of $\mathbb{H}^2 \times \mathbb{H}^2$ and $U(\mathcal{F}) \subseteq U(\mathcal{F}_\infty)$, by the first part of the proof there are only finitely many units u such that γ_u satisfies $C \cap \gamma_u(\mathcal{F}) \neq \emptyset$ for every fixed $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For every $(-c, a) \in V_{C,1}$, fix a matrix $\gamma_{-c,a} = \begin{pmatrix} a & b_{-c,a} \\ c & d_{-c,a} \end{pmatrix} \in \Gamma$. Now consider an arbitrary matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and set $U = \gamma_{-c,a}^{-1}\gamma$. Then $U \in \Gamma_\infty$ and $U^{-1}(\gamma_{-c,a}^{-1}(C)) \cap \mathcal{F}_\infty \neq \emptyset$ if and only if $\gamma_{-c,a}^{-1}(C) \cap U(\mathcal{F}_\infty) \neq \emptyset$. As $\gamma_{-c,a}^{-1}(C)$ is compact, by the first part of the proof there are only finitely many $U \in \Gamma_\infty$ that satisfy this and hence there are also only finitely many $\gamma \in \Gamma$ satisfying $C \cap \gamma(\mathcal{F}) \neq \emptyset$. ∎

Corollary 4.3.12. *For every $Z \in \mathbb{H}^2 \times \mathbb{H}^2$, there exists $\lambda > 0$ such that if $B(Z, \lambda) \cap \gamma(\mathcal{F}) \neq \emptyset$, then $Z \in \gamma(\mathcal{F})$ for $\gamma \in \Gamma$.*

Proof. Let $Z \in \mathbb{H}^2 \times \mathbb{H}^2$. Take some $\lambda' > 0$ randomly. As the closed ball $\overline{B(Z, \lambda')}$ is compact, $\overline{B(Z, \lambda')} \cap \gamma(\mathcal{F}) \neq \emptyset$ for only finitely many $\gamma \in \Gamma$. Hence also $B(Z, \lambda') \cap \gamma(\mathcal{F}) \neq \emptyset$ for only finitely many $\gamma \in \Gamma$. Set $\lambda_0 = \min\{d(Z, \gamma(\mathcal{F})) \mid \gamma \in \Gamma \text{ such that } B(Z, \lambda') \cap \gamma(\mathcal{F}) \neq \emptyset \text{ and } Z \notin \gamma(\mathcal{F})\}$, where $d(Z, \gamma(\mathcal{F}))$ denotes the Euclidean distance from Z to the the set $\gamma(\mathcal{F})$. Take $\lambda = \frac{\lambda_0}{2}$ and the corollary is proven. ∎

The following corollary is now easy to prove.

Corollary 4.3.13. *$\partial\mathcal{F} = \bigcup_{\gamma \neq 1} \mathcal{F} \cap \gamma(\mathcal{F})$.*

Proof. If $1 \neq \gamma \in \Gamma$, we have that $\mathcal{F}^\circ \cap \gamma(\mathcal{F}) = \emptyset$ and hence $\mathcal{F} \cap \gamma(\mathcal{F}) \subseteq \partial\mathcal{F}$. This proves that $\bigcup_{\gamma \neq 1} \mathcal{F} \cap \gamma(\mathcal{F}) \subseteq \partial\mathcal{F}$. If $Z \in \mathcal{F} \setminus \bigcup_{\gamma \neq 1} \gamma(\mathcal{F})$, then by Corollary 4.3.12 there is a neighbourhood U of Z such that $U \cap \gamma(\mathcal{F}) = \emptyset$ for every $1 \neq \gamma \in \Gamma$. Thus $U \subset \mathcal{F}^\circ$ and hence $Z \in \mathcal{F}^\circ$. This proves the other inclusion. ∎

4.4 Some Topological and Geometrical Properties of the Fundamental Domain

In order to determine a presentation of Γ, we need to obtain more information on the fundamental domain $\mathcal{F}$ for Γ constructed in the previous section. Thus in this section we will study some properties of $\mathcal{F}$. First we recall some notions from real algebraic geometry (for more details see [BCR98]).

Definition 4.4.1. *A real algebraic set is the set of zeros in $\mathbb{R}^n$ of some subset of $\mathbb{R}[X_1, \ldots, X_n]$. A real algebraic variety is an irreducible algebraic set, i.e. one which is not the union of two proper real algebraic subsets.*

Definition 4.4.2. *A real semi-algebraic set is a set of the form*

$$\bigcup_{i=1}^{s} \bigcap_{j=1}^{n_i} \{x \in \mathbb{R}^n \mid f_{i,j}(x) *_{i,j} 0\},$$

where $f_{i,j}(x) \in \mathbb{R}[X_1, \ldots, X_n]$ and $*_{i,j}$ is either $=$ or $<$, for $i = 1, \ldots, s$ and $j = 1, \ldots, n_i$.

Definition 4.4.3. *Let A be a real semi-algebraic set. Set*

$$\mathcal{P}(A) = \mathbb{R}[X_1, \ldots, X_n]/\mathcal{I},$$

where $\mathcal{I}$ is the set of polynomial functions defining A. The dimension of A, denoted $dim(A)$, is the dimension of the ring $\mathcal{P}(A)$, i.e. the maximal length of chains of prime ideals of $\mathcal{P}(A)$.

Definition 4.4.4. *Let A be a semi-algebraic set and let x be a point of A. The local dimension of A at x, denoted $dim(A_x)$, is $dim(U)$, where U is a semi-algebraic neighbourhood of x in A such that for any other semi-algebraic neighbourhood U' of x in A with $U' \subseteq U$, $dim(U') = dim(U)$.*

Note that the existence of a semi-algebraic neighbourhood U satisfying the given properties in Definition 4.4.4 is guaranteed by [BCR98, Proposition 2.8.10].

For example the sets $V_i^{\pm,\geq}$ and $V_{c,d}^{\geq}$ are real semi-algebraic sets and the sets $V_i^{\pm}$ and $V_{c,d}$ are real algebraic sets and in fact, we will prove that they are real algebraic varieties.

The following lemma describes when two sets V_{c_1,d_1} and V_{c_2,d_2} are equal.

Lemma 4.4.5. *Let $(c_1, d_1), (c_2, d_2) \in \mathcal{S}$. Then $V_{c_1,d_1} = V_{c_2,d_3}$ if and only if $(c_2, d_2) = (uc_1, ud_1)$ for some $u \in \mathcal{U}(R)$. Moreover if $c_1d_2 = c_2d_1$ and $N(c_1)^2 \neq N(c_2)^2$ then $V_{c_1,d_1} \cap V_{c_2,d_2} = \emptyset$.*

Proof. One implication has already been given in (4.22). For the other implication, suppose that $V_{c_1,d_1} = V_{c_2,d_2}$. The case $c_1c_2 = 0$ follows from the fact that if $(c, d) \in \mathcal{S}$ then $c = 0$ if and only if $V_{c,d} = \mathbb{H}^2 \times \mathbb{H}^2$. So assume $c_1c_2 \neq 0$. We can then rewrite $f_{c,d}$ as

$$f_{c,d}(x_1, x_2, y_1, y_2) = N(c)^2 \left[\left(x_1 + \frac{d}{c}\right)^2 + y_1^2\right]\left[\left(x_2 + \frac{d'}{c'}\right)^2 + y_2^2\right].$$

In particular, if $c \neq 0$ then the intersection of $V_{c,d}$ with $A = \{(x, 0, y, 1) \mid x \in \mathbb{R}, y \in \mathbb{R}^+\}$ is formed by the points $(x, 0, y, 1)$ such that (x, y) belongs to the ball with centre $\left(\frac{d}{c}, 0\right)$ and radius $\frac{1}{|N(c)|}\sqrt{1 + \left(\frac{d'}{c'}\right)^2}$. Therefore, if $V_{c_1,d_1} = V_{c_2,d_2}$ then $\frac{d_1}{c_1} = \frac{d_2}{c_2}$, or equivalently, $c_1d_2 = c_2d_1$, and $N(c_1)^2 = N(c_2)^2$. As $(c_1, d_1), (c_2, d_2) \in \mathcal{S}$, $g_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \Gamma$ and

$g_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in \Gamma$ and $g_2 g_1^{-1} \in \Gamma_\infty$. Thus $g_2 = \begin{pmatrix} u^{-1} & b \\ 0 & u \end{pmatrix} g_1$ for some $u \in \mathcal{U}(R)$ and $b \in R$ and so $(c_2, d_2) = (uc_1, ud_1)$.

The second part follows easily. ∎

Let

$$\begin{aligned} \mathcal{V}_\infty &= \{V_i^+, V_i^- \mid 1 \leq i \leq 3\}, \\ \mathcal{V} &= \{V_{c,d} \mid (c,d) \in \mathcal{S}, c \neq 0\} \text{ and} \\ \mathcal{M} &= \{\gamma(M) \mid \gamma \in \Gamma \text{ and } M \in \mathcal{V} \cup \mathcal{V}_\infty\}. \end{aligned}$$

Observe that

$$\partial \mathcal{F}_0 \subseteq \bigcup_{V \in \mathcal{V}} V, \quad \partial \mathcal{F}_\infty \subseteq \bigcup_{V \in \mathcal{V}_\infty} V \quad \text{and} \quad \partial \mathcal{F} \subseteq \bigcup_{V \in \mathcal{V} \cup \mathcal{V}_\infty} V. \tag{4.31}$$

Let $V \in \mathcal{V} \cup \mathcal{V}^\infty$. If $V = V_i^\pm$ (respectively, $V = V_{c,d}$) then let $V^\geq = V_i^{\pm,\geq}$ (respectively, $V^\geq = V_{c,d}^\geq$). Define $V^\leq$ in a similar way. Clearly, each $V^\geq$ and $V^\geq$ are real semi-algebraic sets, V is the intersection with $\mathbb{H}^2 \times \mathbb{H}^2$ of a real algebraic set and it is the boundary of $V^\leq$ and $V^\geq$ and its intersection. Thus $\mathcal{F}_\infty, \mathcal{F}_0$ and $\mathcal{F}$ and their boundaries are real semi-algebraic sets.

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \setminus \{1\}$ such that if $c = 0$ then $a > 0$ and if moreover $a = 1$ then $b = s_1 + s_2\omega$ with $s_1, s_2 \in \mathbb{Z}$. Then let

$$E_\gamma = \begin{cases} V_{c,d}^\geq, & \text{if } c \neq 0; \\ V_3^{+,\leq}, & \text{if } c = 0 \text{ and } a < 1; \\ V_3^{-,\geq}, & \text{if } c = 0 \text{ and } a > 1; \\ V_1^{+,\leq}, & \text{if } c = 0, a = 1 \text{ and } b = s_1 + s_2\omega \text{ with } s_1 < 0; \\ V_1^{-,\geq}, & \text{if } c = 0, a = 1 \text{ and } b = s_1 + s_2\omega \text{ with } s_1 > 0; \\ V_2^{+,\leq}, & \text{if } c = 0, a = 1 \text{ and } b = s_2\omega \text{ with } s_2 < 0; \\ V_2^{-,\leq}, & \text{if } c = 0, a = 1 \text{ and } b = s_2\omega \text{ with } s_2 > 0. \end{cases} \tag{4.32}$$

Let E'_γ be the set obtained by interchanging in the definition of E_γ the roles of $\leq$ and $\geq$ and let

$$V_\gamma = \partial E_\gamma = E_\gamma \cap E'_\gamma.$$

Remark 4.4.6. *Observe the resemblance between E_γ and E'_γ defined above and (1.25) and (1.26) respectively. Of course, geometrically speaking, the two objects are not at all the same. However the nice thing is that they carry similar algebraic structures, which allow to generalize properties of the classical Ford fundamental domain.*

The following lemma follows by straightforward calculations.

Lemma 4.4.7. *If $\gamma \in \Gamma \setminus \{1\}$ then $V_\gamma \in \mathcal{V}_\infty \cup \mathcal{V}$, $\mathcal{F} \subseteq E_\gamma$, $\gamma^{-1}(\mathcal{F}) \subseteq E'_\gamma$ and hence $\mathcal{F} \cap \gamma^{-1}(\mathcal{F}) \subseteq V_\gamma$.*

By Lemma 4.3.3 and (4.22) it is easy to prove the following lemma.

Lemma 4.4.8. *The set $\mathcal{V} \cup \mathcal{V}_\infty$ is locally finite.*

We first calculate the local dimensions of $\mathcal{F}$ and $\partial\mathcal{F}$.

Lemma 4.4.9. *The fundamental domain $\mathcal{F}$ has local dimension 4 at every point and its boundary $\partial\mathcal{F}$ has local dimension 3 at every point.*

Proof. Let $Z \in \mathcal{F}$. We have to prove that for every $\lambda > 0$, $B(Z,\lambda) \cap \mathcal{F}$ is of dimension 4. This is obvious for $Z \in \mathcal{F}^\circ$. So suppose $Z \in \partial\mathcal{F}$. If $Z \notin \partial\mathcal{F}_0$, then $Z \in V$ for some $V \in \mathcal{V}_\infty$. As $\partial\mathcal{F}_0$ is closed, there is $\lambda_0 > 0$ such that $B(Z,\lambda_0) \cap \partial F_0 = \emptyset$. We may assume without loss of generality that λ_0 is smaller than 1 and $2\epsilon_0$. If $Z = (s_1, s_2, r = s_3, h)$ then let $Z' = (s_1', s_2', r' = s_3', h)$ where for $i = 1,2,3$ we have

$$s_i' = \begin{cases} s_i - \frac{\lambda_0}{\sqrt{3}}, & \text{if } Z \in V_i^+; \\ s_i + \frac{\lambda_0}{\sqrt{3}}, & \text{if } Z \in V_i^-; \\ s_i, & \text{otherwise.} \end{cases}$$

Then $Z' \in B(Z,\lambda_0) \cap \mathcal{F}_\infty^\circ \cap \mathcal{F}_0^\circ$ and thus there exists $\lambda' > 0$ such that $B(Z',\lambda') \subseteq \mathcal{F}^\circ \cap B(Z,\lambda_0)$. Hence $B(Z,\lambda_0) \cap \mathcal{F}$ is of dimension 4. By the choice of λ_0, this is true for any $\lambda < \lambda_0$ and of course also for any $\lambda > \lambda_0$. Finally suppose $Z \in \partial\mathcal{F} \cap \partial\mathcal{F}_0$. By (4.25) and (4.26) Z is of the form $(s_1, s_2, r, h_0(s_1, s_2, r))$ with $(s_1, s_2, r) \in B$. Let $\lambda > 0$. By (4.25) the point $Z' = (s_1, s_2, r, h_0(s_1, s_2, r) + \frac{\lambda}{2})$ is in $B(Z,\lambda) \cap \mathcal{F}_0^\circ$. If $Z' \in \mathcal{F}^\circ$, then there exists $\lambda' > 0$ such that $B(Z',\lambda') \subseteq B(Z,\lambda) \cap \mathcal{F}^\circ$ and hence $B(Z,\lambda) \cap \mathcal{F}$ has dimension 4. If not, then $Z' \in \partial\mathcal{F} \cap \mathcal{F}_0^\circ$ and hence by the above there exists $\lambda' > 0$ such that $B(Z',\lambda') \subseteq B(Z,\lambda)$ and $B(Z',\lambda') \cap \mathcal{F}$ has dimension 4. Thus also $B(Z,\lambda) \cap \mathcal{F}$ has dimension 4.

To prove the second part, take $Z \in \partial\mathcal{F}$, $\lambda > 0$ and set $\overline{B} = \overline{B(Z,\lambda)}$, $U_1 = \overline{B} \cap \mathcal{F}$ and $U_2 = \overline{B} \cap (\mathcal{F}^c \cup \partial\mathcal{F})$, where $\mathcal{F}^c$ denotes the complementary of $\mathcal{F}$ in $\mathbb{H}^2 \times \mathbb{H}^2$. Then $\overline{B}$, U_1 and U_2 satisfy the conditions of Lemma A.1.2 and hence $B(Z,\lambda) \cap \partial\mathcal{F} = U_1 \cap U_2$ has dimension 3. ∎

The next lemmas give more details on the elements of $\mathcal{V} \cup \mathcal{V}_\infty$. Their proofs are based on long and tedious calculations. The interested reader may find all the details in Appendix A.2.

Lemma 4.4.10. *The elements of $\mathcal{V} \cup \mathcal{V}_\infty$ are path-connected.*

Lemma 4.4.11. *The elements of $\mathcal{V} \cup \mathcal{V}_\infty$ are non-singular irreducible real algebraic varieties of dimension 3. Moreover if two different varieties M_1 and M_2 intersect non-trivially, with $M_1, M_2 \in \mathcal{V} \cup \mathcal{V}_\infty$, then their intersection has local dimension 2 at every point.*

Lemma 4.4.12. *The elements of $\mathcal{M}$ are non-singular irreducible real algebraic varieties of dimension 3 and the intersection of two different elements of $\mathcal{M}$ is of dimension at most 2.*

Lemma 4.4.13. *Let M_1, M_2, M_3 be pairwise different elements of $\mathcal{V} \cup \mathcal{V}_\infty$ with $M_1 \cap M_2 \cap M_3 \neq \emptyset$. Then $M_1 \cap M_2 \cap M_3$ is a real algebraic set of local dimension* 1 *at every point and with at most one singular point.*

In order to determine the border of $\mathcal{F}$, we need the following definition, which generalizes the essential hyperplanes introduced in Chapter 2.

Definition 4.4.14. *An essential hypersurface of $\mathcal{F}$ is an element $M \in \mathcal{V} \cup \mathcal{V}_\infty$ such that $M \cap \mathcal{F}$ is of dimension* 3. *Let $\mathcal{V}_e$ denote the set of essential hypersurfaces of $\mathcal{F}$.*

Following the notation of [BCR98], for A a real semi-algebraic set, we denote by $A^{(d)}$ the set of points of A with local dimension d, i.e.

$$A^{(d)} \quad = \quad \{Z \in A \mid \forall \; \lambda > 0, \; dim(B(Z, \lambda) \cap A) = d\}. \tag{4.33}$$

Lemma 4.4.15.

$$\partial \mathcal{F} = \bigcup_{M \in \mathcal{V}_e} (M \cap \mathcal{F}) = \bigcup_{M \in \mathcal{V}_e} (M \cap \mathcal{F})^{(3)},$$

Proof. Clearly $\bigcup_{M \in \mathcal{V}_e} (M \cap \mathcal{F})^{(3)} \subseteq \bigcup_{M \in \mathcal{V}_e} (M \cap \mathcal{F}) \subseteq \partial \mathcal{F}$. Let $Z \in \partial \mathcal{F}$ and $\mathcal{V}_Z = \{M \in \mathcal{V} \cup \mathcal{V}_\infty \mid \; Z \in M\}$. We have to prove that $M \cap \mathcal{F}$ has local dimension 3 at Z for some $M \in \mathcal{V}_Z$. By (4.31) and Lemma 4.4.8, $\mathcal{V}_Z$ is a non-empty finite set. Thus there is an open ball $B = B(Z, \lambda_0)$ such that for every $M \in \mathcal{V} \cup \mathcal{V}_\infty$, $B \cap M \neq \emptyset$ if and only if $M \in \mathcal{V}_Z$. Thus, if $0 < \lambda \leq \lambda_0$, then $\partial \mathcal{F} \cap B = \cup_{M \in \mathcal{V}_Z} \mathcal{F} \cap B \cap M$ and by Lemma 4.4.11, $\partial \mathcal{F} \cap B$ has dimension 3. Thus $\mathcal{F} \cap B \cap M$ has dimension 3 for some $M \in \mathcal{V}_Z$. Therefore $M \cap \mathcal{F}$ has local dimension 3 at Z. ∎

Lemma 4.4.16. *If $V \in \mathcal{V}_\infty$, then $V \cap F$ has local dimension 3 at every point, i.e. $(V \cap \mathcal{F})^{(3)} = V \cap \mathcal{F}$.*

Proof. This follows by arguments similar to those used in the proof of Lemma 4.4.9. ∎

The following proposition gives information about essential hypersurfaces and will be important in the next two sections.

Proposition 4.4.17. *Assume $Z \in \partial \mathcal{F}$ and Z is contained in at most two elements M and M' of $\mathcal{V} \cup \mathcal{V}_\infty$. Then M and M' are essential and for every $\lambda > 0$, the intersections $B(Z, \lambda) \cap M \cap \mathcal{F}$ and $B(Z, \lambda) \cap M' \cap \mathcal{F}$ are of dimension* 3.

Proof. If Z is contained in a single element M of $\mathcal{V} \cup \mathcal{V}_\infty$, then the statement is obvious. So suppose that $Z \in M \cap M'$ for some $M, M' \in \mathcal{V} \cup \mathcal{V}_\infty$ with $M \neq M'$ and Z is contained in no other element of $\mathcal{V} \cup \mathcal{V}_\infty$. By Lemma 4.4.11, $M \cap M'$ is of dimension 2. Moreover $M = (M \cap M'^{<}) \cup (M \cap M') \cup (M \cap M'^{>})$ and, as by the proof of Lemma 4.4.11, both varieties M and M' are not tangent in Z, the first and the third intersections are of dimension 3. Thus $B(Z, \lambda) \cap M \cap M'^{>}$ is of dimension 3 and is contained in the boundary of $\mathcal{F}$. Hence M is an essential hypersurface and $B(Z, \lambda) \cap M \cap \mathcal{F}$ is of dimension 3. Inverting the role of M and M', we get the same result for M'. ∎

Finally, if R is a PID, we can prove compactness of the intersection of $\mathcal{F}$ with the elements of $\mathcal{V}$.

Lemma 4.4.18. *Assume R is a PID and let $V_{c,d}$, for $(c,d) \in \mathcal{S}$ with $c \neq 0$. Then $V_{c,d} \cap \mathcal{F}$ is compact.*

Proof. Let $Z = (s_1, s_2, r, h) \in V_{c,d} \cap \mathcal{F}$. As $V_{c,d} \cap \mathcal{F} \subseteq \mathcal{F}_\infty$, $|s_1| \leq \frac{1}{2}$, $|s_2| \leq \frac{1}{2}$ and $\epsilon_0^{-2} \leq r \leq \epsilon_0^2$. Moreover $V_{c,d} \cap \mathcal{F} \subseteq \mathcal{F}_0$ and hence, by Lemma 4.3.9, $h > \frac{k_0^2}{2k}$. As $Z \in \partial\mathcal{F}$, Lemma 4.3.6 yields that $h \leq 1$. Thus $V_{c,d} \cap \mathcal{F}$ is hyperbolically bounded and closed in $\mathbb{H}^2 \times \mathbb{H}^2$ and hence it is compact. ∎

4.5 Generators of Γ

In the next two sections, $\mathcal{T}$ denotes the tessellation of $\mathbb{H}^2 \times \mathbb{H}^2$ given by Γ and $\mathcal{F}$, i.e $\mathcal{T} = \{\gamma(\mathcal{F}) \mid \gamma \in \Gamma\}$. In order to get generators for Γ, we have to analyse the intersections between elements of $\mathcal{T}$. The next lemma is crucial.

Lemma 4.5.1. *Let T_1, T_2 and T_3 be three different elements of $\mathcal{T}$. Then $T_1 \cap T_2 \cap T_3 \subseteq M_1 \cap M_2$ for M_1, M_2 two different elements of $\mathcal{M}$. In particular the intersection of three different tiles has dimension at most 2.*

Proof. For every $i = 1, 2, 3$, let $\gamma_i \in \Gamma$ with $T_i = \gamma_i(\mathcal{F})$. Let $\mathcal{N} = \{M \in \mathcal{M} \mid T_i \cap T_j \subseteq M$ for some $1 \leq i \leq j \leq 3\}$. By Lemma 4.4.7, for every $1 \leq i < j \leq 3$, we have $T_i \cap T_j \subseteq \gamma_i(V_{\gamma_j^{-1}\gamma_i})$. Thus $\mathcal{N} \neq \emptyset$ and it is enough to show that $\mathcal{N}$ has at least two different elements. By means of contradiction, assume that $\mathcal{N} = \{M = \gamma(V)\}$ for some $V \in \mathcal{V}_\infty \cap \mathcal{V}$ and $\gamma \in \Gamma$. Let $M^{\geq} = \gamma(V^{\geq})$ and $M^{\leq} = \gamma(V^{\leq})$. Then $M = \gamma_i\left(V_{\gamma_j^{-1}\gamma_i}\right)$ and, by Lemma 4.4.7, for each $1 \leq i < j \leq 3$ either $T_i \subseteq M^{\geq}$ and $T_j \subseteq M^{\leq}$ or viceversa. By symmetry one may assume that $T_1 \subseteq M^{\geq}$. Then $T_2 \subseteq M^{\leq}$ and hence $T_3 \subseteq M^{\leq} \cap M^{\geq} = M$, a contradiction because $\dim(T_3) = 4$ and $\dim(M) = 3$. ∎

Our final goal is to generalize Theorem 2.4.7 to the case under consideration, i.e. the Hilbert Modular group. This theorem is based on generators and relations associated to sides and edges of a fundamental polyhedron. One problem is that the fundamental domain $\mathcal{F}$ constructed in Theorem 4.3.5, is not a polyhedron, i.e. it is not the intersection of convex half-spaces. So the classical notion of side and edge does not make sense in this case. Hence, in the following definition, we redefine the notion of side.

Definition 4.5.2. *For each $\gamma \in \Gamma \setminus \{1\}$, set*

$$S_\gamma = \mathcal{F} \cap \gamma^{-1}(\mathcal{F}).$$

A side *of $\mathcal{F}$ with respect to Γ is a set of the form S_γ that has dimension 3.*

Similar to Definition 2.4.1, in the case S_γ has dimension 3, we say that γ is a *side-pairing transformation* of Γ with respect to $\mathcal{F}$, or simply a *pairing transformation*. Observe that if γ is a pairing transformation then so is γ^{-1} and $\gamma(S_\gamma) = S_{\gamma^{-1}}$. Hence the pairing transformations "pair" the sides of $\mathcal{F}$. More generally, a *side* of $\mathcal{T}$ is a side of $\gamma(\mathcal{F})$ for some $\gamma \in \Gamma$. Equivalently, the sides of $\mathcal{T}$ are the sets of dimension 3 of the form $\gamma(\mathcal{F}) \cap \phi(\mathcal{F})$, with $\gamma, \phi \in \Gamma$.

Thanks to Definition 4.5.2 and the lemmas we have proven before, we can now generalize the generating part of Theorem 2.4.7. The proof is very similar to the one in Chapter 2. For completeness' sake and to convince the reader, we nevertheless give it here.

Theorem 4.5.3. *Let $\mathcal{F}$ be the fundamental domain for $\Gamma = \mathrm{PSL}_2(R)$ described in Theorem 4.3.5. Then Γ is generated by the pairing transformations of $\mathcal{F}$ with respect to Γ.*

Proof. Let $\mathcal{L} = \{\bigcap_{i\in I} \gamma_i(\mathcal{F}) \mid dim(\bigcap_{i\in I} \gamma_i(\mathcal{F})) \leq 2\}$ and consider the set

$$\Omega = \mathbb{H}^2 \times \mathbb{H}^2 \setminus \bigcup_{Y\in\mathcal{L}} Y.$$

This is a set of elements $Z \in \mathbb{H}^2 \times \mathbb{H}^2$ that belong either to the interior of a tile of $\mathcal{F}$ or to a unique side of $\mathcal{T}$. Indeed, let $Z \in \Omega$ and suppose Z is not contained in the interior of a tile of $\mathcal{F}$. Then $Z \in \gamma_1(\mathcal{F}) \cap \gamma_2(\mathcal{F})$ for at least two distinct elements $\gamma_1 \neq \gamma_2 \in \Gamma$. Because of Lemma 4.5.1, $\gamma_1(\mathcal{F})$ and $\gamma_2(\mathcal{F})$ are the only tiles containing Z. Thus, $Z \in \gamma_1(\mathcal{F}) \cap \gamma_2(\mathcal{F})$ and $Z \notin \gamma(\mathcal{F})$ for $\gamma \in \Gamma \setminus \{\gamma_1, \gamma_2\}$. By the definition of Ω, $dim(\gamma_1(\mathcal{F}) \cap \gamma_2(\mathcal{F})) = 3$ and hence $\gamma_1(\mathcal{F}) \cap \gamma_2(\mathcal{F})$ is the unique side containing Z.

Note that $\mathbb{X} = \mathbb{H}^2 \times \mathbb{H}^2$ and $\mathcal{L}$ satisfy the hypotheses of Lemma A.1.1 and hence Ω is path-connected. Let $\gamma \in \Gamma$ and $Z \in \mathcal{F}^\circ$. Put $W = \gamma(Z) \in \gamma(\mathcal{F})^\circ$. There exists a path p in Ω joining Z and W. Let $A = \{h \in \Gamma \mid p \cap h(\mathcal{F}) \neq \emptyset\}$. As p is compact and as $\mathcal{F}$ is locally finite by Lemma 4.3.11, A is finite. We define recursively a sequence of subsets of A by setting $A_0 = \{1\}$ and if $i \geq 1$ then $A_i = \{h \in A \mid h(\mathcal{F}) \cap k(\mathcal{F})$ is a side for some $k \in A_{i-1} \setminus \bigcup_{j<i-1} A_j\}$. Let $B = \bigcup_{i\geq 0} A_i$. We claim that $B = A$. Otherwise let $\alpha : [0,1] \to p \subseteq \Omega$ be a continuous function with image p and such that $\alpha(0) = Z$ and $\alpha(1) = \gamma(Z)$ and let $a = \min\{t \in [0,1] \mid \alpha(t) \in h(\mathcal{F})$ for some $h \in A \setminus B\}$. This minimum exists because $A \setminus B$ is non-empty and $\bigcup_{k\in A\setminus B} k(\mathcal{F})$ is closed. Moreover $a > 0$ because $\alpha(0) = Z \in \mathcal{F}^\circ$ and $1 \in B$, so that $Z \notin h(\mathcal{F})$ for each $h \in A \setminus B$. Then $\alpha([0,a)) \subseteq \bigcup_{h\in B} h(\mathcal{F})$ and as this union is closed, $\alpha(a) \in h(\mathcal{F}) \cap k(\mathcal{F})$ for some $h \in B$ and $k \in A \setminus B$. As $\alpha(a) \in \Omega$, $h(\mathcal{F}) \cap k(\mathcal{F})$ has dimension 3 and hence it is a side. This contradicts the definition of B. Hence $A = B$ and in particular $\gamma \in B$. By using the sets A_i, we create a sequence $\gamma_0 = 1, \gamma_1, \dots, \gamma_k = g$ such that for every $1 \leq j$, $\gamma_{j-1}(\mathcal{F}) \cap \gamma_j(\mathcal{F})$ is a side. Hence $\mathcal{F} \cap \gamma_{j-1}^{-1}\gamma_j(\mathcal{F})$ is a side of $\mathcal{F}$ and thus $\gamma_{j-1}^{-1}\gamma_j$ is one of the proposed generators. As $\gamma = (\gamma_0^{-1}\gamma_{i_1})(\gamma_{i_1}^{-1}\gamma_{i_2})\dots(\gamma_{i_k-1}^{-1}\gamma_{i_k})$, the result follows. ∎

Recall that the border of $\mathcal{F}$ is covered by the essential hypersurfaces of $\mathcal{F}$. In fact it is also covered by the sides of $\mathcal{F}$.

Lemma 4.5.4. *The border of $\mathcal{F}$ is the union of the sides of $\mathcal{F}$.*

Proof. Since the sides of $\mathcal{F}$ are contained in $\partial\mathcal{F}$, we only have to prove that if $Z \in \partial\mathcal{F}$, then Z belongs to a side of $\mathcal{F}$. As Z is in $\partial\mathcal{F}$ and $\mathcal{F}$ is a locally finite fundamental domain, $Z \in \gamma(\mathcal{F})$ for only finitely many $\gamma \in \Gamma$, say $\gamma_1 = 1, \dots, \gamma_k$. Choose $\lambda_0 > 0$ such that the closed ball $\overline{B(Z,\lambda_0)}$ intersects $\gamma_i(\mathcal{F})$ if and only if $Z \in \gamma_i(\mathcal{F})$ (this is possible by Corollary 4.3.12). Then

$$Z \in \overline{B(Z,\lambda_0)} = \left(\overline{B(Z,\lambda_0)} \cap \mathcal{F}\right) \cup \left(\overline{B(Z,\lambda_0)} \cap \cup_{i=2}^{k} \gamma_i(\mathcal{F})\right).$$

Thus by Lemma A.1.2, $\mathcal{F} \cap \gamma_i(\mathcal{F})$ is of dimension 3, for at least one $1 \le i \le k$ and hence Z is contained in a side of $\mathcal{F}$. ∎

In order to give more precise information on the generators of Γ described in Theorem 4.5.3, we need to analyse the relationship between pairing transformations and essential hypersurfaces. By Lemma 4.4.15, $\partial\mathcal{F}$ is the union of the sets of the form $(\mathcal{F} \cap V)^{(3)}$ with V running through the essential hypersurfaces and by Lemma 4.5.4, $\partial\mathcal{F}$ also is the union of the sides of $\mathcal{F}$ with respect to Γ. Moreover, if γ is a pairing transformation then $S_\gamma \subset \mathcal{F} \cap V_\gamma$, by Lemma 4.4.7. Hence V_γ is an essential hypersurface of $\mathcal{F}$ and it is the unique essential hypersurface of $\mathcal{F}$ containing S_γ. Conversely, let V be an essential hypersurface and let

$$\Gamma_V = \{\gamma \mid \gamma \text{ is a pairing transformations such that } S_\gamma \subseteq V\}.$$

Clearly, $\gamma \in \Gamma_V$ if and only if $\dim S_\gamma = 3$ and $V = V_\gamma$. Moreover,

$$(V \cap \mathcal{F})^{(3)} \subseteq \bigcup_{\gamma \in \Gamma_V} S_\gamma.$$

Each pairing transformation belongs to Γ_V for some essential hypersurface V. We will show that, in order to generate Γ it is enough to take one element of Γ_V for each essential hypersurface.

We start dealing with the elements of $\mathcal{V}_\infty$. Clearly Γ_∞ is generated by the following elements:

$$P_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}, \quad P_3 = \begin{pmatrix} \epsilon_0 & 0 \\ 0 & \epsilon_0^{-1} \end{pmatrix}.$$

Using (4.24) and (4.26) it is easy to see that the six sets $V_i^{\pm}$, with $i = 1, 2, 3$, are essential hypersurfaces. Moreover, a straightforward calculation shows that if $i = 1, 2$ then

$$\Gamma_{V_i^{\pm}} = \{P_i^{\mp 1}\} \quad \text{(with opposite signs on both sides).} \tag{4.34}$$

Thus, for $i = 1, 2$, P_i (respectively, P_i^{-1}) is a pairing transformation and its side covers the part of the boundary given by $V_i^{+} \cap \mathcal{F}$ (respectively, $V_i^{-} \cap F$).

However to cover $\mathcal{F} \cap V_3^{\pm}$ with sides, we may need more than one side.

Lemma 4.5.5. *If $g \in \Gamma_{V_3^{\pm}}$ then $g = QP_3^{\mp}$ (with opposite signs on both sides) for some $Q \in \langle P_1, P_2 \rangle$ and the inversion map is a bijection $\Gamma_{V_3^+} \to \Gamma_{V_3^-}$. Moreover $\Gamma_{V_3^+}$ and $\Gamma_{V_3^-}$ are finite.*

Proof. The first statement follows from Lemma 4.4.7, Lemma 4.4.11, (4.32) and some easy computations. If $\gamma \in \Gamma_{V_3^+}$ then $\gamma = \begin{pmatrix} \epsilon_0^{-1} & b \\ 0 & \epsilon_0 \end{pmatrix}$ with $b \in R$. Moreover, by (4.5) and (4.6), $\mathcal{F} \cap \gamma(\mathcal{F})$ is a non-empty subset formed by elements of the form (x_1, x_2, y_1, y_2) such that (x_1, x_2) and $(\widehat{x_1}, \widehat{x_2}) = (\epsilon_0^{-2} x_1 + \epsilon_0^{-1} b, \epsilon_0^2 x_2 + \epsilon_0 b')$ belong to a compact set. As $\{(b, b') \mid b \in R\}$ is discrete, we deduce that b belongs to a finite subset of R. Thus $\Gamma_{V_3^+}$ is finite. Hence $\Gamma_{V_3^-}$ is finite too. ∎

We now deal with the essential hypersurfaces of the form $V_{c,d}$ with $(c, d) \in \mathcal{S}$ (and necessarily $c \neq 0$).

Lemma 4.5.6. *Let $(c, d) \in \mathcal{S}$ with $c \neq 0$. Then the second row of every element of $\Gamma_{V_{c,d}}$ is of the form (uc, ud) for some $u \in \mathcal{U}(R)$. Equivalently, if the second row of $\gamma \in \Gamma$ is (c, d) then $\Gamma_{V_{c,d}} \subseteq \Gamma_\infty \gamma$.*

Proof. Let $\gamma \in \Gamma_{V_{c,d}}$ and let $v \in R^2$ be the second row of γ. Then $S_\gamma \subseteq V_\gamma \cap V_{c,d}$, by Lemma 4.4.7, and hence $V_\gamma = V_{c,d}$, by Lemma 4.4.12. Then $v = (uc, ud)$ for some $u \in \mathcal{U}(R)$, by Lemma 4.4.5. ∎

Let $\mathcal{S}_e$ denote the set of $(c, d) \in \mathcal{S}$ such that $V_{c,d}$ is an essential hypersurface of $\mathcal{F}$.

Corollary 4.5.7. *For every $(c, d) \in \mathcal{S}_e$ choose $a, b \in R$ with $P_{c,d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Then $\Gamma = \langle P_1, P_2, P_3, P_{c,d} \mid (c, d) \in \mathcal{S}_e \rangle$.*

Proof. By Theorem 4.5.3, Γ is generated by $\cup_{V \in \mathcal{V}_e} \Gamma_V$ and hence it is enough to show that $\langle P_1, P_2, P_3, \gamma_{c,d} \mid (c, d) \in \mathcal{S} \rangle$ contains Γ_V for each $V \in \mathcal{V}_e$. This is a consequence of (4.34), Lemma 4.5.5, Lemma 4.5.6 and the fact that Γ_∞ is generated by P_1, P_2 and P_3. ∎

In case R is a PID we can combine Theorem 4.3.10 and Corollary 4.5.7 to get the following

Corollary 4.5.8. *Suppose that R is a PID and let $\mathcal{S}_1$ be as in Theorem 4.3.10. For each $(c, d) \in \mathcal{S}_1$ choose $a, b \in R$ such that $P_{c,d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Then $\Gamma = \langle P_1, P_2, P_3, P_{c,d} \mid (c, d) \in \mathcal{S}_1 \rangle$.*

4.6 Defining Relations of Γ

In this section we generalize the relation part of Theorem 2.4.7 (see the last proof of Chapter 2). As in the classical case, there are two types of relations. These will be called again the *pairing relations* and the *cycle relations*. The pairing relations are quite obvious to establish.

Notation 4.6.1. *Given a side* S *of* $\mathcal{F}$*, let* γ_S *denote the unique element of* Γ *such that* $S = \mathcal{F} \cap \gamma_S^{-1}(\mathcal{F})$ *and let* $S^* = \mathcal{F} \cap \gamma_S(\mathcal{F})$*.*

Remark 4.6.2. *Note that the definition of* γ_S *is different from the definition given in Chapter 2 in Definition 2.4.1. As, in this chapter, we work with a concrete fundamental domain, which is based on the construction of a Ford domain, this new notation makes more sense in this context. Indeed, for Ford domains and Dirichlet domains, the sides of the domain are included in the isometric spheres* I_γ*, defined in Subsection 1.2.3 or in the bisectors* Σ_γ*, defined in (3.1). In Subsection 1.2.3, we have seen that for* $\gamma \in \Gamma$ *and* $\mathcal{F}$ *a Ford domain or a Dirichlet domain,* $\mathcal{F} \cap \gamma^{-1}(\mathcal{F}) \subseteq I_\gamma$*, respectively* $\mathcal{F} \cap \gamma^{-1}(\mathcal{F}) \subseteq \Sigma_\gamma$*.*

Observe that if S is a side then $\gamma_S(S) = S^*$. Moreover $S \mapsto \gamma_S$ and $\gamma \mapsto S_\gamma$ define mutually inverse bijections between the sides of $\mathcal{F}$ and the pairing transformations such that $\gamma_S^{-1} = \gamma_{S^*}$, or equivalently $S_\gamma^* = S_{\gamma^{-1}}$. The pairing relation given by S is then simply $\gamma_S \gamma_{S^*} = 1$. Note that in case $S = S^*$, we get as pairing relation $\gamma_S^2 = 1$.

We will now turn to the cycle relations. For this we need to introduce the notion of edge. Similar, as in Chapter 2, we give a more general concept and introduce the notion of cell. However, we cannot just retake Definition 2.3.1, because we do not have the notion of relative interior in this context. Hence we change the definition a bit.

Definition 4.6.3. *A* cell C *of* $\mathcal{T}$ *is a non-empty intersection of tiles of* $\mathcal{F}$ *satisfying the following property: for every* $\gamma \in \Gamma$*, either* $C \subseteq \gamma(\mathcal{F})$ *or* $dim(C \cap \gamma(\mathcal{F})) \leq dim(C) - 1$*. Clearly, the cells of dimension 4 are the tiles. By Lemma 4.5.1, the sides of* $\mathcal{T}$ *are the cells of dimension 3. A cell of dimension* 2 *is called an* edge*. If a cell or edge is contained in a tile* T*, then it is called a cell or an edge of* T*.*

Observe that the inequality $dim(C \cap \gamma(\mathcal{F})) \leq dim(C) - 1$ "plays the role" of the relation $C^r \cap T = \emptyset$ in Definition 2.3.1. A cell is always a finite intersection of tiles. Indeed, consider $Z \in C$. As $\{Z\}$ is compact, Z is contained in only finitely many tiles of $\mathcal{F}$, and hence so is C. The following proposition generalizes in some sense the notion of relative interior of a cell.

Proposition 4.6.4. *Let* C *be a cell of* $\mathcal{T}$*. Then there exists* $Z \in C$*, such that, for every* $\gamma \in \Gamma$*,* $Z \in \gamma(\mathcal{F})$ *if and only if* $C \subseteq \gamma(\mathcal{F})$*.*

Proof. Let $\gamma_1, \ldots, \gamma_n$ be the elements $\gamma \in \Gamma$ such that $C \subseteq \gamma(\mathcal{F})$. Suppose, by contradiction, that there do not exist $Z \in C$ that satisfy the statement of the proposition. Then, for every $Z \in C$, there exists $\gamma_Z \in (\Gamma \setminus \{\gamma_i \mid 1 \leq i \leq n\})$ such that $Z \in \gamma_Z(\mathcal{F})$. Put $\Gamma^* = \{\gamma_Z \mid Z \in C\}$. This is a countable set because Γ is countable, and clearly $C \subseteq \cup_{\gamma \in \Gamma^*} \gamma(\mathcal{F})$. Thus

$$C = \cup_{\gamma \in \Gamma^*} \left(\gamma(\mathcal{F}) \cap C\right).$$

As $C \not\subseteq \gamma(\mathcal{F})$ for $\gamma \in \Gamma^*$ and because C is a cell, $dim(C \cap \gamma(\mathcal{F})) \leq dim(C) - 1$ and hence $C = \cup_{\gamma \in \Gamma^*} (\gamma(\mathcal{F}) \cap C)$ has dimension at most $dim(C) - 1$, a contradiction. ∎

The next lemma is an obvious consequence of the definition of an edge.

Lemma 4.6.5. *Let $\gamma \in \Gamma$. If E is an edge then $\gamma(E)$ is an edge. In particular, if E is an edge of $\mathcal{F}$ contained in some side $S_\gamma = \mathcal{F} \cap \gamma^{-1}(\mathcal{F})$, with $\gamma \in \Gamma$, then $\gamma(E)$ is an edge of $\mathcal{F}$.*

In order to prove more results on the edges of $\mathcal{F}$, we first have to analyse the sides a bit more in detail.

Lemma 4.6.6. *Let $(c,d) \in \mathcal{S}$ with $c \neq 0$. If $t \in \Gamma_\infty$ then there exists $(c_0, d_0) \in \mathcal{S}$, $c_0 \neq 0$, such that $t(V_{c,d}) = V_{c_0,d_0}$.*

Proof. Let $Z \in V_{c,d}$. Fix $\alpha, \beta \in R$ such that $\alpha c + \beta d = 1$. Write $t = \begin{pmatrix} \epsilon_0^m & b \\ 0 & \epsilon_0^{-m} \end{pmatrix}$, for some $m \in \mathbb{Z}$. Let $c_0 = \epsilon_0^{-2m} c \in R$ and $d_0 = d - \epsilon_0^{-m} bc \in R$. Then $(c_0, d_0) \in \mathcal{S}$, because $\alpha' c_0 + \beta' d_0 = 1$ for $\alpha' = \epsilon_0^m \beta b + \epsilon_0^{2m} \alpha \in R$ and $\beta' = \beta \in R$. Moreover, for every $Z \in \mathbb{H}^2 \times \mathbb{H}^2$, we have

$$\|c_0 t(Z) + d_0\| = \|\epsilon_0^{-2m} c(\epsilon_0^{2m} Z + \epsilon_0^m b) + d - \epsilon_0^{-m} bc\| = \|cZ + d\|$$

Hence $t(V_{c,d}) = V_{c_0,d_0}$. ∎

Similarly to Lemma 4.5.1, one may analyse the intersection of three sides.

Lemma 4.6.7. *The intersection of three distinct sides of $\mathcal{F}$ is of dimension at most* 1.

Proof. Let S_1, S_2, S_3 be three distinct sides of $\mathcal{F}$ and let $\gamma_i = \gamma_{S_i}$. By Lemma 4.4.7, each $S_i \subseteq V_{\gamma_i}$ and hence V_{γ_i} is an essential hypersurface of $\mathcal{F}$. If $V_{\gamma_i} \neq V_{\gamma_j}$, for every $1 \leq i < j \leq 3$, then by Lemma 4.4.13, $S_1 \cap S_2 \cap S_3$ is of dimension at most 1. Assume now that $V_{\gamma_1} = V_{\gamma_2} = V_{\gamma_3} = V_{c,d}$, for some $(c,d) \in \mathcal{S}$, $c \neq 0$. Then, by Lemma 4.5.6 and Lemma 4.4.7, $\gamma_1(S_1 \cap S_2 \cap S_3) = \mathcal{F} \cap \gamma_1(\mathcal{F}) \cap t_2^{-1}(\mathcal{F}) \cap t_3^{-1}(\mathcal{F}) \subseteq V_{-c,a} \cap V_{t_2} \cap V_{t_3}$, for some $a \in R$. If $V_{t_2} \neq V_{t_3}$, then by Lemma 4.4.13, the last intersection is of dimension at most 1 as desired. Suppose that $V_{t_2} = V_{t_3} = V_i^{\pm}$. Using that $t_2 \neq t_3$ it is easy to prove that $i = 3$ and hence $t_2, t_3 \notin \langle P_1, P_2 \rangle$. By Lemma 4.5.5, $t_3 = Qt_2$ for $Q \in \langle P_1, P_2 \rangle$. Thus $t_2\gamma_1(S_1 \cap S_2 \cap S_3) \subseteq t_2(V_{-c,a}) \cap \mathcal{F} \cap Q^{-1}(\mathcal{F}) \cap t_2(\mathcal{F})$. By Lemma 4.6.6, $t_2(V_{-c,a}) = V_{c',d'}$ for some $(c', d') \in \mathcal{S}$, $c' \neq 0$ and by Lemma 4.4.7 and the fact that $t_2 \notin \langle P_1, P_2, \rangle$, $t_2(\mathcal{F}) \cap \mathcal{F} \subseteq \mathcal{V}_3^{\pm}$ and $Q^{-1}(\mathcal{F}_\infty) \cap \mathcal{F}_\infty \subseteq V_i^{\pm}$ with $i = 1$ or 2. Hence Lemma 4.4.13 allows to conclude.

Next assume that $V_{\gamma_1} = V_{\gamma_2} = V_{\gamma_3} = V_i^{\pm}$ with $i = 1, 2$ or 3. By Lemma 4.4.7 and the fact that the γ_i's are different we conclude that $i = 3$ and hence $\gamma_1, \gamma_2, \gamma_3 \in \Gamma_\infty \setminus \langle P_1, P_2 \rangle$. Then, again by the same argument as above, $\gamma_2 = Q_2\gamma_1$ and $\gamma_3 = Q_3\gamma_1$ for $Q_2, Q_3 \in \langle P_1, P_2 \rangle$. Thus $\gamma_1(S_1 \cap S_2 \cap S_3) \subseteq \mathcal{F} \cap \gamma_1(\mathcal{F}) \cap Q_2^{-1}(\mathcal{F}) \cap Q_3^{-1}(\mathcal{F}) \subseteq V_3^{\pm} \cap V_i^{\pm} \cap V_j^{\pm}$. with $i, j \in \{1, 2\}$. However, as $Q_2 \neq Q_3$, $V_i^{\pm} \neq V_j^{\pm}$ and again Lemma 4.4.13 allows to conclude.

So up to permutations of the sides, we are left to deal with the case $V_{\gamma_1} = V_{\gamma_2} \neq V_{\gamma_3}$ and four possible subcases:

1. $V_{\gamma_1} = V_{c_1,d_1}$ and $V_{\gamma_3} = V_{c_3,d_3}$ with $(c_1, d_1), (c_3, d_3) \in \mathcal{S}$ and c_1 and c_3 different from 0;

2. $V_{\gamma_1} = V_{c_1,d_1}$ and $V_{\gamma_3} \in \mathcal{V}_\infty$ with $(c_1, d_1) \in \mathcal{S}$ and $c_1 \neq 0$;

3. $V_{\gamma_1} \in \mathcal{V}_\infty$ and $V_{\gamma_3} = V_{c_3,d_3}$ for $(c_3, d_3) \in \mathcal{S}$ and $c_3 \neq 0$.

4. $V_{\gamma_1}, V_{\gamma_3} \in \mathcal{V}_\infty$.

We deal with each case separately.

In the first case, by Lemma 4.5.6, $\gamma_2 = t\gamma_1$, for some $1 \neq t \in \Gamma_\infty$. Thus $\gamma_1(S_1 \cap S_2 \cap S_3) = \gamma_1(\mathcal{F}) \cap \mathcal{F} \cap t^{-1}(\mathcal{F}) \cap \gamma_1\gamma_3^{-1}(\mathcal{F}) \subseteq V_{\gamma_1^{-1}} \cap V_t \cap V_{\gamma_3\gamma_1^{-1}}$. We claim that $V_{\gamma_1^{-1}}, V_t$ and $V_{\gamma_3\gamma_1^{-1}}$ are pairwise different. On the one side $V_t \in \mathcal{V}_\infty$ while $V_{\gamma_1^{-1}} \in \mathcal{V}$ and hence $V_{\gamma_1} \neq V_t$. On the other side, since $V_{\gamma_1} = V_{c_1,d_1}$, the last row of γ_1 is of the form (uc_1, ud_1) with $u \in \mathcal{U}(R)$, by Lemma 4.4.5. If $\gamma_3\gamma_1^{-1} \in \Gamma_\infty$ then the last row of γ_3 is of the same form and hence $V_{c_1,d_1} = V_{c_3,d_3}$ contradicting the hypothesis. Thus $\gamma_1\gamma_3^{-1} \notin \Gamma_\infty$ and therefore $V_{\gamma_3\gamma_1^{-1}} \neq V_t$. Finally, if $V_{\gamma_1^{-1}} = V_{\gamma_3\gamma_1^{-1}}$ then the last rows of γ_1^{-1} and $\gamma_3\gamma_1^{-1}$ differs in a unit and hence $\gamma_3 \in \Gamma_\infty$, contradicting the fact that $V_{\gamma_3} = V_{c_3,d_3}$. This proves the desired result.

In the second case, again by the same reasoning, $\gamma_2 = t\gamma_1$ for some $t \in \Gamma_\infty$ and $\gamma_3 \in \Gamma_\infty$. Thus $S_1 \cap S_2 \cap S_3 = \mathcal{F} \cap \gamma_1^{-1}(\mathcal{F}) \cap \gamma_1^{-1}t^{-1}(\mathcal{F}) \cap \gamma_3^{-1}(\mathcal{F}) \subseteq V_{\gamma_1} \cap V \cap \left(\gamma_1^{-1}(\mathcal{F}) \cap \gamma_1^{-1}t^{-1}(\mathcal{F})\right)$ with V en element of $\mathcal{V}_\infty$. Observe that

$$\gamma_1\left(\gamma_1^{-1}(\mathcal{F}) \cap \gamma_1^{-1}t^{-1}(\mathcal{F})\right) = \mathcal{F} \cap t^{-1}(\mathcal{F}) \subseteq V',$$

for some element $V' \in \mathcal{V}_\infty$. Hence $S_1 \cap S_2 \cap S_3 \subseteq V_{\gamma_1} \cap V \cap \gamma_1^{-1}(V')$. By applying the implicit function theorem, as in the proof of Lemma 4.4.11 and distinguishing between the different cases for V and V', it is now a matter of a straightforward tedious calculation to prove that $V_{\gamma_1} \cap V \cap \gamma_1^{-1}(V')$ has dimension at most 1. We leave the Details to Appendix A.3.

In the third case $V_{\gamma_1} = V_3^\pm$ by (4.34). and $\gamma_2 = Q\gamma_1 \in \langle P_1, P_2\rangle P_3^\pm$ for some $Q \in \langle P_1, P_2\rangle$. Then $\gamma_1(S_1 \cap S_2 \cap S_3) = \mathcal{F} \cap \gamma_1(\mathcal{F}) \cap Q^{-1}(\mathcal{F}) \cap \gamma_1\gamma_3^{-1}(\mathcal{F}) \subseteq V_{\gamma_1^{-1}} \cap V_Q \cap V_{\gamma_3\gamma_1^{-1}}$. Since $\gamma_3\gamma_1^{-1} \notin \Gamma_\infty$ and $\gamma_1^{-1} \notin \langle P_1, P_2\rangle$ we deduce that $V_{\gamma_1^{-1}}$, V_Q and $V_{\gamma_3\gamma_1^{-1}}$ are different and hence the result follows from Lemma 4.4.13.

In the fourth case, again $V_{\gamma_1} = V_{\gamma_2} = V_3^\pm$ and $\gamma_2 = Q\gamma_1$, as above. Moreover $S_1 \cap S_2 \cap S_3 = \mathcal{F} \cap \gamma_1^{-1}(\mathcal{F}) \cap \gamma_1^{-1}Q^{-1}(\mathcal{F}) \cap \gamma_3^{-1}(\mathcal{F}) \subseteq V_{\gamma_1} \cap V_{\gamma_3} \cap \left(\gamma_1^{-1}(\mathcal{F}) \cap \gamma_1^{-1}Q^{-1}(\mathcal{F})\right)$, with V_{γ_1} and V_{γ_3} two different elements of $\mathcal{V}_\infty$. As in case 2, $\left(\gamma_1^{-1}(\mathcal{F}) \cap \gamma_1^{-1}Q^{-1}(\mathcal{F})\right) = \gamma_1^{-1}(V')$ for some $V' \in \{V_1^\pm, V_2^\pm\}$. Again the details are given in Appendix A.3 ∎

We now describe $\mathcal{F} \cap V$ for $V \in \mathcal{V} \cup \mathcal{V}_\infty$ in terms of intersection of tiles.

Lemma 4.6.8. *Let $V \in \mathcal{V} \cup \mathcal{V}_\infty$ and set $\Gamma_V^* = \{\gamma \in \Gamma \mid \mathcal{F} \cap \gamma(\mathcal{F}) \subseteq V \cap \mathcal{F}\}$. Then*

$$V \cap \mathcal{F} = \bigcup_{\gamma \in \Gamma_V^*} (\mathcal{F} \cap \gamma(\mathcal{F})).$$

Proof. One inclusion is obvious. To prove the other one, take $Z \in V \cap \mathcal{F}$. Suppose $V \in \mathcal{V}$, i.e. $V = V_{c,d}$ for some $(c,d) \in \mathcal{S}$. Take $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ for some $a, b \in R$. By (4.10), $h(\gamma(Z)) = h(Z)$ and hence by Lemma 4.3.4, $\gamma(Z) \in \mathcal{F}_0$. As $\mathcal{F}_\infty$ is a fundamental domain for Γ_∞, there exists $\tau \in \Gamma_\infty$ such that $\tau\gamma(Z) \in \mathcal{F}_\infty$. As, by (4.12), $h(\tau\gamma(Z)) = h(\gamma(Z)) = h(Z)$, we have $\tau\gamma(Z) \in \mathcal{F}$ and thus $Z \in \mathcal{F} \cap \gamma^{-1}\tau^{-1}(\mathcal{F})$. By Lemma 4.4.7, $\mathcal{F} \cap \gamma^{-1}\tau^{-1}(\mathcal{F}) \subseteq V_{c,d} \cap \mathcal{F}$.

If $V \in \mathcal{V}_\infty$ a similar reasoning may be applied. ∎

Next we show that every edge is contained in precisely two sides. To prove this we will make use of the following lemma.

Lemma 4.6.9. *Let $Z \in \mathcal{F}$ and let C denote the intersection of the tiles of $\mathcal{T}$ containing Z (i.e. $C = \bigcap_{Z \in \gamma(\mathcal{F})} \gamma(\mathcal{F})$). If $\dim(C) = 2$, then the number of sides of $\mathcal{F}$ containing C is exactly 2.*

Proof. By Lemma 4.6.7, C is not contained in three different sides. So we only have to show that C is contained in two different sides.

Clearly, every intersection of tiles containing Z also contains C. We claim that a similar property holds for the elements $V \in \mathcal{V} \cup \mathcal{V}_\infty$. Indeed, if $Z \in V$ then by Lemma 4.6.8, there is $\gamma_1 \in \Gamma$ such that $Z \in \mathcal{F} \cap \gamma_1(\mathcal{F}) \subseteq V$. Hence $C \subseteq \mathcal{F} \cap \gamma_1(\mathcal{F}) \subseteq V$, as desired.

By Lemma 4.4.15, Z is contained in at least one essential hypersurface W. Since C has dimension 2, Z cannot be contained in more than two elements of $\mathcal{V} \cup \mathcal{V}_\infty$, by Lemma 4.4.13. Then, the elements of $\mathcal{V} \cup \mathcal{V}_\infty$ containing Z are essential hypersurfaces of $\mathcal{F}$ by Proposition 4.4.17. Take $\lambda > 0$ such that $B(Z,\lambda) \cap \gamma(\mathcal{F}) = \emptyset$ for every γ with $Z \notin \gamma(\mathcal{F})$ and $B(Z,\lambda) \cap W = \emptyset$ for every essential hypersurface W of $\mathcal{F}$ not containing Z. Again by Proposition 4.4.17, if W is an essential hypersurface of $\mathcal{F}$ containing Z then $B(Z,\lambda) \cap W \cap \mathcal{F}$ is of dimension 3 and $B(Z,\lambda) \cap W \cap \mathcal{F} \subseteq \bigcup_{1 \neq \gamma \in \Gamma, Z \in \gamma(\mathcal{F})} (\mathcal{F} \cap \gamma(\mathcal{F}) \cap W)$ and thus one of these intersections is of dimension 3. Therefore, if W is an essential hypersurface of $\mathcal{F}$ containing C then it contains a side containing Z. Thus, if C is contained in exactly two elements of $\mathcal{V} \cup \mathcal{V}_\infty$ then each contains one side containing Z and these two sides have to be different by Lemma 4.4.11.

Hence we may assume that Z is contained in exactly one element W of $\mathcal{V} \cup \mathcal{V}_\infty$. Then C is contained in at least one side S_1 and $S_1 \subseteq W$. Let $\gamma_1 \in \Gamma_{S_1}$ (i.e. $S_1 = \mathcal{F} \cap \gamma_1^{-1}(\mathcal{F})$). Then, by Lemma 4.4.7, $W = V_{\gamma_1}$. As $\dim(C) = 2$, $\dim(S_{\gamma_1}) = 3$ and $C \subseteq \mathcal{F} \cap S_1$ there is $\gamma_2 \in \Gamma \setminus \{1, \gamma_1\}$ with $Z \in \gamma_2^{-1}(\mathcal{F})$. Thus $\gamma_2 = t\gamma_1$ with $t \in \Gamma_\infty \setminus \{1\}$ and if $\gamma_1 \in \Gamma_\infty$ then $t \in \langle P_1, P_2 \rangle$, by Lemma 4.5.5 and Lemma 4.5.6. We consider separately two cases.

First assume that $W = V_i^{\pm}$ with $i = 1$ or 2 and without loss of generality, we may assume that $W = V_1^{+}$. Then $\gamma_1 = P_1^{-1}$ by (4.34) and $\gamma_2 = P_1^{\alpha} P_2^{\beta}$ for some integers α and β. Moreover $Z = (\frac{1}{2}, s_2, r, h)$ and $(\alpha + \frac{1}{2}, s_2 + \beta, r, h) = \gamma_2(Z) \in \mathcal{F}$. Thus $\alpha \in \{0, -1\}$. However, if $\alpha = 0$ then $V_{\gamma_2} = V_2^{\pm} \neq W$. Thus $\alpha = -1$ and hence $\beta = \pm 1$, because $\gamma_1 \neq \gamma_2$. Then $Z \in V_2^{\pm}$, a contradiction.

Assume now that either $W \in \mathcal{V}$ or $W = V_3^{\pm}$. In both cases $V_t \neq V_{\gamma_1}$, because in the first case $V_{\gamma_1} \in \mathcal{V}$ and $V_t \in \mathcal{V}_\infty$ and in the second case $V_t = V_i^{\pm}$ with $i = 1$ or 2.

Moreover $\gamma_1(C) \subseteq \mathcal{F} \cap \gamma_1(\mathcal{F}) \cap t^{-1}(\mathcal{F}) \subseteq V_{\gamma_1^{-1}} \cap V_t$. As $\mathcal{F} \subseteq E_t$, $\gamma_1(S_{\gamma_1}) = S_{\gamma_1^{-1}} \subseteq E_t$. Let $M = \gamma_1^{-1}(V_t)$, $M^{\geq} = \gamma_1^{-1}(E_t)$ and $M^{\leq} = \gamma_1^{-1}(E_t')$. Then, by the previous, $S_{\gamma_1} \subseteq M^{\geq}$. By the choice of λ and Lemma 4.4.11, $B(Z, \lambda) \cap W \cap \mathcal{F} \cap M^{\leq}$ has dimension 3. Thus, by Lemma 4.5.4, $Z \in S$ for some side $S \subseteq W$ and different from S_{γ_1}. Hence also C is contained in two different sides. ∎

The following is a consequence of Proposition 4.6.4 and Lemma 4.6.9.

Corollary 4.6.10. *If E is an edge of the fundamental domain $\mathcal{F}$, then there are precisely two sides that contain E.*

The following lemma is obvious.

Lemma 4.6.11. *Let E_1 and E_2 be two different edges of some tile. Then the intersection $E_1 \cap E_2$ is of dimension at most 1.*

Finally, in order to be able to describe the relations, we need two more lemmas.

Lemma 4.6.12. *Let $\gamma_1, \gamma_2 \in \Gamma$. Assume $\gamma_1(\mathcal{F}) \cap \gamma_2(\mathcal{F})$ has dimension 2. Then,*

1. *there exists $Z_0 \in \gamma_1(\mathcal{F}) \cap \gamma_2(\mathcal{F})$ such that $\bigcap_{Z_0 \in \gamma(\mathcal{F})} \gamma(\mathcal{F})$ is of dimension 2,*
2. *for every such Z_0, the set $\bigcap_{Z_0 \in \gamma(\mathcal{F})} \gamma(\mathcal{F})$ is an edge contained in $\gamma_1(\mathcal{F}) \cap \gamma_2(\mathcal{F})$.*

Proof. 1. We first show the existence of a point $Z_0 \in \gamma_1(\mathcal{F}) \cap \gamma_2(\mathcal{F})$ such that $\bigcap_{Z_0 \in \gamma(\mathcal{F})} \gamma(\mathcal{F})$ is of dimension 2. For every $Z \in \gamma_1(\mathcal{F}) \cap \gamma_2(\mathcal{F})$, let $\Gamma_Z = \{\gamma \in \Gamma \setminus \{\gamma_1, \gamma_2\} \mid Z \in \gamma(\mathcal{F})\}$. We claim that $\Gamma_Z \neq \emptyset$ for every $Z \in \gamma_1(\mathcal{F}) \cap \gamma_2(\mathcal{F})$. Otherwise there is a $\lambda > 0$ such that $B(Z, \lambda) \cap \gamma(\mathcal{F}) = \emptyset$ for every $\gamma \in \Gamma \setminus \{\gamma_1, \gamma_2\}$. Then $B(Z, \lambda) \subseteq \gamma_1(\mathcal{F}) \cup \gamma_2(\mathcal{F})$, which is in contradiction with Lemma A.1.2. This proves the claim. Thus $\gamma_1(\mathcal{F}) \cap \gamma_2(\mathcal{F}) = \bigcup_{Z \in \gamma_1(\mathcal{F}) \cap \gamma_2(\mathcal{F})} \left(\bigcap_{\gamma \in \Gamma_Z} \gamma(\mathcal{F}) \cap \gamma_1(\mathcal{F}) \cap \gamma_2(\mathcal{F}) \right)$. As, by assumption, $\gamma_1(\mathcal{F}) \cap \gamma_2(\mathcal{F})$ has dimension 2 and Γ is countable, it follows that there exists $Z_0 \in \gamma_1(\mathcal{F}) \cap \gamma_2(\mathcal{F})$ with $\bigcap_{Z_0 \in \gamma(\mathcal{F})} \gamma(\mathcal{F})$ of dimension 2.

2. Since Z_0 belongs to only finitely many tiles (by Lemma 4.3.11), say $\gamma_1(\mathcal{F})$, $\gamma_2(\mathcal{F})$, ..., $\gamma_n(\mathcal{F})$, we have that $Z_0 \in \bigcap_{i=1}^n \gamma_i(\mathcal{F})$ and $\bigcap_{i=1}^n \gamma_i(\mathcal{F})$ has dimension 2. We want to prove that this intersection is an edge. Let $\gamma_0 \in \Gamma \setminus \{\gamma_1, \ldots, \gamma_n\}$. As $Z_0 \notin \gamma_0(\mathcal{F})$, it is clear that $\bigcap_{i=1}^n \gamma_i(\mathcal{F}) \not\subseteq \gamma_0(\mathcal{F})$. Hence it remains to prove that $\bigcap_{i=1}^n \gamma_i(\mathcal{F})$ intersects $\gamma_0(\mathcal{F})$ in dimension at most 1. Suppose this is not the case, i.e. $\bigcap_{i=0}^n \gamma_i(\mathcal{F})$ is of dimension 2. As in the first part of the proof, there exists $Z_1 \in \bigcap_{i=0}^n \gamma_i(\mathcal{F})$ such that $\bigcap_{Z_1 \in \gamma(\mathcal{F})} \gamma(\mathcal{F})$ is of dimension 2. Let $\gamma_0, \ldots, \gamma_m$ be all elements of Γ (with $m \geq n$) such that $Z_1 \in \gamma_i(\mathcal{F})$. So $\bigcap_{i=0}^m \gamma_i(\mathcal{F})$ is of dimension 2. By Corollary 4.3.12, let $\lambda_1 > 0$ and $B = \overline{B(Z_1, \lambda_1)}$ be such that $B \cap \gamma(\mathcal{F}) \neq \emptyset$ if and only if $Z_1 \in \gamma(\mathcal{F})$. Then $B = (B \cap \bigcup_{i=1}^n \gamma_i(\mathcal{F})) \cup (B \cap (\bigcup_{i=n+1}^m \gamma_i(\mathcal{F}) \cup \gamma_0(\mathcal{F})))$, where both factors are closed sets of dimension 4 (by Lemma 4.4.9). Hence by Lemma A.1.2, $(B \cap \bigcup_{i=1}^n \gamma_i(\mathcal{F})) \cap (B \cap (\bigcup_{i=n+1}^m \gamma_i(\mathcal{F}) \cup \gamma_0(\mathcal{F})))$ is of dimension 3. Thus there exists $1 \leq j_1 \leq n$ and $n+1 \leq j_2 \leq m$ or $j_2 = 0$ such that $Z_1 \in \gamma_{j_1}(\mathcal{F}) \cap \gamma_{j_2}(\mathcal{F})$ and the latter intersection is of dimension 3. Hence $\gamma_{j_1}(\mathcal{F}) \cap \gamma_{j_2}(\mathcal{F})$ is a side of the tile $\gamma_{j_1}(\mathcal{F})$. We now come back to Z_0. Let $\lambda_0 > 0$

and $B' = \overline{B(Z_0, \lambda_0)}$ be such that $B \cap \gamma(\mathcal{F}) \neq \emptyset$ if and only if $Z_0 \in \gamma(\mathcal{F})$. Thus $B' = (B' \cap \gamma_{j_1}(\mathcal{F})) \cup \bigcup_{i=1, i\neq j_1}^{n} (B' \cap \gamma_i(\mathcal{F}))$ and again by Lemma A.1.2, there exists $1 \leq j_3 \leq n$ and $j_3 \neq j_1$ such that $\gamma_{j_1}(\mathcal{F}) \cap \gamma_{j_3}(\mathcal{F})$ is of dimension 3. Thus it is a side of the tile $\gamma_{j_1}(\mathcal{F})$. Hence $Z_0 \in \partial\gamma_{j_1}(\mathcal{F})$, such that $\bigcap_{Z_0 \in \gamma(\mathcal{F})} \gamma(\mathcal{F})$ is of dimension 2. Moreover the latter is contained in $\gamma_{j_1}(\mathcal{F}) \cap \gamma_{j_3}(\mathcal{F})$, which is a side. Hence by Lemma 4.6.9, there exists a second side of $\gamma_{j_1}(\mathcal{F})$, which contains $\bigcap_{Z_0 \in \gamma(\mathcal{F})} \gamma(\mathcal{F})$. Thus this side also contains Z_0 and hence there exists $1 \leq j_4 \leq n$ with $j_4 \neq j_1$ and $j_4 \neq j_3$ such that $\gamma_{j_1}(\mathcal{F}) \cap \gamma_{j_4}(\mathcal{F})$ is a side. Thus the tile $\gamma_{j_1}(\mathcal{F})$ contains three sides, $\gamma_{j_1}(\mathcal{F}) \cap \gamma_{j_2}(\mathcal{F})$, $\gamma_{j_1}(\mathcal{F}) \cap \gamma_{j_3}(\mathcal{F})$ and $\gamma_{j_1}(\mathcal{F}) \cap \gamma_{j_4}(\mathcal{F})$ and their intersection contains $\bigcap_{i=0}^{m} \gamma_i(\mathcal{F})$, which is of dimension 2. This contradicts Lemma 4.6.7. Hence $\bigcap_{i=1}^{n} \gamma_i(\mathcal{F})$ is an edge and it contains Z_0. ∎

Lemma 4.6.13. *Let E be an edge. The finitely many elements $\gamma \in \Gamma$ with $E \subseteq \gamma(\mathcal{F})$ can be ordered, say as $\gamma_0, \gamma_1, \ldots, \gamma_m = \gamma_0$, such that $\gamma_{j-1}(\mathcal{F}) \cap \gamma_j(\mathcal{F})$ is a side (containing E) for every $1 \leq j \leq m$. Moreover, up to cyclic permutations and reversing the ordering, there is only one possible ordering with this property.*

Proof. Recall that there are only finitely many $\gamma \in \Gamma$ with $E \subseteq \gamma(\mathcal{F})$. Let γ_0 be such an element. Then E is an edge of $\gamma_0(\mathcal{F})$ and hence, by Corollary 4.6.10, there exists two sides, say $\gamma_0(\mathcal{F}) \cap \gamma_1\mathcal{F}$ and $\gamma_0(\mathcal{F}) \cap \gamma_{m-1}(\mathcal{F})$ of $\gamma_0(\mathcal{F})$ containing E. Now E also is an edge of $\gamma_1(\mathcal{F})$ and $\gamma_0(\mathcal{F}) \cap \gamma_1(\mathcal{F})$ is one of the two sides of $\gamma_1(\mathcal{F})$ containing E. Hence there exists a third tile, say $\gamma_2(\mathcal{F})$, such that $\gamma_1(\mathcal{F}) \cap \gamma_0(\mathcal{F})$ and $\gamma_1(\mathcal{F}) \cap \gamma_2(\mathcal{F})$ are the two different sides of $\gamma_1(\mathcal{F})$ containing E. So $\gamma_2 \notin \{\gamma_0, \gamma_1\}$. One may continue this process and have a sequence $\gamma_{-1}, \gamma_0, \gamma_1, \gamma_2, \ldots$ of elements of Γ such that $\gamma_{i-1}(\mathcal{F}) \cap \gamma_i(\mathcal{F})$ and $\gamma_i(\mathcal{F}) \cap \gamma_{i+1}(\mathcal{F})$ are the two sides of $\gamma_i(\mathcal{F})$ containing E for every $i \geq 0$. In particular every three consecutive elements of the list of γ_i's are different. As there are only finitely many tiles containing E, after finitely many steps we obtain $\gamma_j \in \Gamma$ with $\gamma_i = \gamma_j$ and $0 \leq i < j$. Let j be minimal with this property. We claim that $i = 0$. Indeed, if $0 < i$, then by construction, $\gamma_i(\mathcal{F}) \cap \gamma_{i-1}(\mathcal{F})$, $\gamma_i(\mathcal{F}) \cap \gamma_{i+1}(\mathcal{F})$, $\gamma_j(\mathcal{F}) \cap \gamma_{j-1}(\mathcal{F}) = \gamma_i(\mathcal{F}) \cap \gamma_{j-1}(\mathcal{F})$ and $\gamma_j(\mathcal{F}) \cap \gamma_{j+1}(\mathcal{F}) = \gamma_i(\mathcal{F}) \cap \gamma_{j+1}(\mathcal{F})$ are sides of $\gamma_i(\mathcal{F})$, all containing E. By Corollary 4.6.10, $\gamma_{i-1} = \gamma_{j-1}$ and $\gamma_{i+1} = \gamma_{j+1}$ or $\gamma_{i+1} = \gamma_{j-1}$ and $\gamma_{i-1} = \gamma_{j+1}$. Both cases contradict the minimality of i. It remains to prove that $\{\gamma_0, \ldots, \gamma_{j-1}\} = \{\gamma \in \Gamma \mid E \subseteq \gamma(\mathcal{F})\}$. This may be easily done by arguments similar as the arguments used in the end of the proof of Lemma 4.6.12.

To prove the last part, notice that instead of starting with the chosen element γ_0 one could have started with any of the finitely many elements $\gamma \in \Gamma$ such that $E \subseteq \gamma(\mathcal{F})$. Second, note that once the element γ_0 is fixed, there exists two unique tiles $\gamma_{m-1}(\mathcal{F})$ and $\gamma_1(\mathcal{F})$ such that $\gamma_0(\mathcal{F}) \cap \gamma_{m-1}(\mathcal{F})$ and $\gamma_0(\mathcal{F}) \cap \gamma_1(\mathcal{F})$ are sides. Hence, up to a choice of the first element γ_0, thus up to a cyclic permutations, and up to a choice of a second element, thus up to reversing ordering, there is only one possible ordering. ∎

Based on the previous lemma, we give the following new definition.

Definition 4.6.14. *Let E be an edge. We call an ordering $(\gamma_0, \ldots, \gamma_{m-1}, \gamma_0)$ of the elements $\gamma \in \Gamma$ such that $E \subseteq \gamma(\mathcal{F})$ as in Lemma 4.6.13 an* edge loop *of E.*

Note that the definition of edge loop in Chapter 2 is slightly different (see (2.2). However the only difference is that in (2.2), we give the definition in terms of tiles and here we give the definition in terms of the elements $\gamma \in \Gamma$ associated to the tiles. We now come to a description of the relations of Γ. We first define what is called a cycle. To do so, we fix some notations. This is the exact equivalent of (2.3).

Definition 4.6.15. *Let E be an edge of $\mathcal{F}$ and S a side of $\mathcal{F}$ containing E. Then define recursively the sequence $(E_1, S_1, E_2, S_2, \ldots)$ as follows:*

1. *$E_1 = E$ and $S_1 = S$,*
2. *$E_{i+1} = \gamma_{S_i}(E_i)$,*
3. *S_{i+1} is the only side of $\mathcal{F}$ different from S_i^* that contains E_{i+1}.*

Note that parts 2 and 3 are justified by Lemma 4.6.5 and Corollary 4.6.10. For every i, clearly $E_i \subseteq S_i$ and $E_{i+1} \subseteq S_i^*$, as $\gamma_{S_i}(S_i) = S_i^*$. Moreover, each pair (E_i, S_i) determines the pairs (E_{i-1}, S_{i-1}) and (E_{i+1}, S_{i+1}). This is clear for the subsequent pair (E_{i+1}, S_{i+1}) but also for the previous one (E_{i-1}, S_{i-1}) because if $i \geq 2$, then S_{i-1}^* is the only side of $\mathcal{F}$ containing E_i and different from S_i and $E_{i-1} = \gamma_{S_{i-1}}^{-1}(E_i) = \gamma_{S_{i-1}^*}(E_i)$. Thus each pair (E_i, S_i) determines the sequence $(E_1, S_1, \ldots)$.

Now we relate the sequence $(E_1, S_1, E_2, S_2, \ldots)$ with an edge loop of E. Let $L_{\mathcal{F}} = \{\gamma \in \Gamma \mid E \subseteq \gamma(\mathcal{F})\}$. Clearly $1, \gamma_{S_1}^{-1} \in L_{\mathcal{F}}$ and as S_1 is a side of $\mathcal{F}$, there is a unique edge loop of E of the form $(\gamma_0 = 1, \gamma_1 = \gamma_{S_1}, \gamma_2, \ldots, \gamma_{m-1}, \gamma_0)$. As $S_1^* = S_{\gamma_1}$ and S_2 are the two sides of $\mathcal{F}$ containing E_2, $S_1 = \gamma_1^{-1}(S_1)^*$ and $\gamma_1^{-1}(S_2)$ are the two sides of $\gamma_1^{-1}(\mathcal{F})$ containing $\gamma_1^{-1}(E_2) = E$. Therefore $\gamma_1^{-1}(S_2) = \gamma_1^{-1}(\mathcal{F}) \cap \gamma_2(\mathcal{F})$ and hence $S_2 = \mathcal{F} \cap \gamma_1\gamma_2(\mathcal{F})$. A similar argument shows that $S_i = \mathcal{F} \cap \gamma_1\gamma_2 \ldots \gamma_i(\mathcal{F})$ for every i. In particular, the sequence $(E_1, S_1, E_2, S_2, \ldots)$ is periodic, i.e. there is $n > 0$ such that $(E_{i+n}, S_{i+n}) = (E_i, S_i)$ for every i. If n is minimal with this property, then $(E_i, S_i) \neq (E_j, S_j)$ for $1 \leq i < j \leq n$. We call $(E_1, S_1, \ldots E_n, S_n)$ the *cycle* determined by (E_1, S_1). Hence this proves the following lemma.

Lemma 4.6.16. *Let E_1 be an edge of $\mathcal{F}$ and S_1 a side of $\mathcal{F}$ containing the edge E_1. The cycle starting with $(E_1, S_1, \ldots)$ is a finite cycle.*

Note that S_1 and S_n^* are the two different sides of $\mathcal{F}$ containing E_1 and thus there are two cycles starting with the edge E_1, namely $(E_1, S_1, \ldots E_n, S_n)$ and $(E_1, S_n^*, E_n, S_{n-1}^*, \ldots, E_2, S_1^*)$.

It is now also clear that if E is an edge and S is a side containing E, then all the cycles containing E are cyclic permutations of the cycle starting with (E, S) and the cycles obtained by replacing in those the sides by their paired sides and reversing the order. In particular, if $E_i = E_j$ with $1 \leq i < j \leq n$, then $S_i \neq S_j$ and hence

$$\begin{aligned} & (E_i, S_i, E_{i+1}, S_{i+1}, \ldots, E_n, S_n, E_1, S_1, \ldots, E_{i-1}, S_{i-1}) \\ = \; & (E_j, S_{j-1}^*, E_{j-1}, S_{j-2}^*, \ldots, S_2^*, E_1, S_n^*, E_n, S_{n-1}^*, \ldots, E_{j+1}, S_j^*). \end{aligned}$$

Lemma 4.6.17. *If $(E_1, S_1, E_2, S_2, \ldots, E_n, S_n)$ is a cycle of $\mathcal{F}$ then $\gamma_{S_n}\gamma_{S_{n-1}} \ldots \gamma_{S_1}$ has finite order.*

Proof. Let $\gamma = \gamma_{S_n}\gamma_{S_{n-1}}, \ldots, \gamma_{S_1}$. Clearly $\gamma(E_1) = E_1$ and thus $\gamma^k(E_1) = E_1$ for all non-negative integers k. Hence $E_1 \subseteq \gamma^k(\mathcal{F})$ and because every edge is contained in only finitely many tiles, γ has finite order. ∎

Because of Theorem 4.5.3, we thus obtain a natural group epimorphism

$$\varphi : \Delta \to \Gamma : [\gamma_S] \mapsto \gamma_S \tag{4.35}$$

where Δ is the group given by the following presentation

- *Generators*: a generator $[\gamma_S]$ for each side S of $\mathcal{F}$
- *Relations*: $[\gamma_S]\,[\gamma_{S}*] = 1$ if the sides S and S^* are paired and

  $$([\gamma_{S_n}] \ldots [\gamma_{S_1}])^m = 1$$

  if $(E_1, S_1, E_2, S_2, \ldots, E_n, S_n)$ is a cycle and m is the order of $\gamma_{S_n} \ldots \gamma_{S_1}$.

Our next aim is to show that φ is also injective and thus we obtain a presentation of Γ. To that end we introduce the following definition, which is the equivalent of Definition 2.4.3.

Definition 4.6.18. *A* loop of tiles *is a finite list of tiles*

$$(h_0(\mathcal{F}), h_1(\mathcal{F}), \ldots, h_n(\mathcal{F}))$$

with $h_i \in \Gamma$ such that $h_0(\mathcal{F}) = h_n(\mathcal{F})$ and $h_{i-1}(\mathcal{F}) \cap h_i(\mathcal{F})$ is of dimension 3, for every $i = 1, \ldots n$.

The last condition means that $\mathcal{F} \cap h_i^{-1}h_{i-1}(\mathcal{F})$ is of dimension 3, which is equivalent to $\gamma_i = h_{i-1}^{-1}h_i$ being a side-paring transformation. Moreover we get the relation $\gamma_1\gamma_2 \ldots \gamma_n = 1$ which is called a *loop relation*. In Theorem 4.6.25 we will show that these relations form a complete set of relations of Γ.

It is easy to see that the pairing and cycle relations are determined by loop relations. Indeed, let S and S^* be two paired sides of $\mathcal{F}$. Then

$$(\mathcal{F}, \gamma_S(\mathcal{F}), \gamma_S\gamma_{S^*}(\mathcal{F}) = \mathcal{F})$$

is a loop of tiles which gives as loop relation the pairing relation. Let $(E_1,\ S_1,\ E_2,\ S_2, \ldots,\ E_n,\ S_n)$ be a cycle. We have seen that there exists a positive integer m such that $(\gamma_{S_n}\gamma_{S_{n-1}} \ldots \gamma_{S_1})^m = 1$. Set $h_0 = 1$ and $h_i = h_{i-1}\gamma_{S_j}$ where $j \equiv i \mod (n)$ and $i \in \{1, \ldots, mn\}$. Consider $(h_0(\mathcal{F}), \ldots, h_{mn}(\mathcal{F}))$. Clearly $h_{mn} = 1$ and hence $h_{mn}(\mathcal{F}) = \mathcal{F} = h_0(\mathcal{F})$. Also, for every $i \in \{1, \ldots, mn\}$, $h_{i-1}(\mathcal{F}) \cap h_i(\mathcal{F}) = h_{i-1}(\mathcal{F} \cap \gamma_{S_j}(\mathcal{F}))$, where γ_{S_j} is a side-paring transformation and hence $\mathcal{F} \cap \gamma_{S_j}(\mathcal{F})$ and thus also $h_{i-1}(\mathcal{F}) \cap h_i(\mathcal{F})$

is of dimension 3. So, by definition, $(h_0(\mathcal{F}), \ldots, h_{mn}(\mathcal{F}))$ is a loop of tiles and the associated loop relation is the cycle relation.

Consider the union of intersections of varieties of $\mathcal{M}$ such that these intersections have dimension at most 1. Let Ω denote the complement of this set in $\mathbb{H}^2 \times \mathbb{H}^2$. Note that by Lemma A.1.1, Ω is path-connected.

To finish the proof of the generalization of Theorem 2.4.7, we follow a very similar line as in Chapter 2. To convince the reader that everything works fine, we give nevertheless all the details. In the remainder we fix α to be a continuous map

$$\alpha : [0,1] \to \Omega,$$

such that

$$\alpha(0) \in g(\mathcal{F})^\circ \text{ and } \alpha(1) \in g'(\mathcal{F})^\circ,$$

for some $g, g' \in \Gamma$ and such that α is made up of a finite number of line segments, which are parametrized by polynomials of degree at most one. Moreover, for each line segment forming α, we suppose that at least one of its end-points does not belong to any element in $\mathcal{M}$. Note that such a map exists. Indeed, let $\mathcal{M}_\alpha$ denote the set consisting of the elements of $\mathcal{M}$ that have non-empty intersection with (the image of) α. Then it is easy to see that $\Omega \setminus \bigcup_{M \in \mathcal{M}_\alpha} M$ is dense in Ω. Hence, by Lemma A.1.3, such a map α indeed exists.

Lemma 4.6.19. *The set* $\{t \in [0,1] \mid \alpha(t) \in \partial g(\mathcal{F}) \text{ for some } g \in \Gamma\}$ *is finite.*

Proof. We know from Lemma 4.3.11 that the compact set $\alpha([0,1])$ only intersects finitely many tiles and thus also only finitely many sides. By Lemma 4.4.7, for every side S of some tile $g(\mathcal{F})$, $g(\mathcal{F}) \cap S$ is contained in a precise variety $M \in \mathcal{M}$. As the elements of $\mathcal{M}$ are real semi-algebraic varieties, which are given by polynomials, a line segment is either contained in such a variety or it intersects it in finitely many points. The path α consists of finitely many line segments, such that at least one of the end-points of these line segments does not belong to any element in $\mathcal{M}$. If such a line segment l is parametrized by a polynomial of first degree then there are only finitely many $t \in [0,1]$ such that $l(t) \in M$ for some $M \in \mathcal{M}$. Moreover, if a line segment l is parametrized by a polynomial of degree 0, i.e. the image of l is just a point, then by definition this point is not contained in any element of $\mathcal{M}$ and $l \cap M = \emptyset$ for every $M \in \mathcal{M}$. Thus $\alpha(t)$ belongs to a side for only finitely many t. Hence the lemma follows by Lemma 4.5.4. ∎

The following lemma defines a partition of α and shows that it exists and is unique. This is the equivalent of the α-adapted list given in Definition 2.4.11. Note that both definitions are not the same. As the fundamental domain $\mathcal{F}$ given in this chapter is a lot less nicer than the fundamental polyhedron defined in Chapter 2, a lot of arguments used in Chapter 2 are not possible here (for example, we do not have convexity of $\mathcal{F}$ in this case). This is why we change the definition a bit. The advantage is that we get uniqueness with our new definition, in the contrary to Definition 2.4.11.

Lemma 4.6.20. *Let* α *be a continuous map* $[0,1] \to \Omega$ *such that*

1. $\alpha(0) \in g(\mathcal{F})^\circ$ and $\alpha(1) \in g'(\mathcal{F})^\circ$ for some $g, g' \in \Gamma$,

2. α is made up of a finite number of line segments,

3. for each of these line segments at least one of its two end-points does not belong to any element in $\mathcal{M}$.

Then there exists a unique ordered list $\mathcal{L} = (a_0, g_1, a_1, g_2, a_2 \ldots, g_n, a_n)$, where $g_i \in \Gamma$, $0 = a_0 < a_1 < \ldots < a_n = 1$ and for every $1 \le i \le n$,

(i) $g_{i-1} \neq g_i$,

(ii) $\alpha([a_{i-1}, a_i]) \subseteq g_i(\mathcal{F})$ and

(iii) there exists $\epsilon_0 > 0$, such that $\alpha((a_i, a_i + \epsilon)) \cap g_i(\mathcal{F}) = \emptyset$.

We call $\mathcal{L}$ the partition of α*.*

Proof. As $\alpha(0) \in g(\mathcal{F})^\circ$, we set $g_1 = g$ and $a_0 = 0$. By Lemma 4.6.19, there are only finitely many $t \in [0, 1]$ such that $\alpha(t) \in \partial g(\mathcal{F})$ for some $g \in \Gamma$, say $t_1 < t_2 < \ldots < t_m$. Put $t_0 = 0$ and $t_{m+1} = 1$. For each $1 \le i \le m + 1$, the set $\alpha(t_{i-1}, t_i)$ is contained in the interior of only one tile, say $h_i(\mathcal{F})$ with $h_i \in \Gamma$. We now construct recursively $(a_0 = 0 = t_0, g_1 = g, a_1, g_1, \ldots, g_n, a_n)$. Let $a_1 = t_i$ with t_i maximal such that $\alpha([0, t_i]) \subseteq g_1(\mathcal{F})$. Then $h_1 = \ldots = h_i = g_1$, $\alpha([t_i, t_{i+1}]) \subseteq h_{i+1}(\mathcal{F})$ and we set $g_2 = h_{i+1}$. Assume we have constructed $(a_0, g_1, a_1, \ldots, a_k, g_k)$ satisfying conditions (i)-(iii) and such that $a_k = t_i$ for some i and $g_k = h_{i+1}$. Then let $a_{k+1} = t_j$ with j maximal with $\alpha([t_i, t_j]) \subseteq g_k(\mathcal{F})$ and $g_{k+1} = h_{j+1}$. After finitely many steps, we obtain an ordered list $(a_0 = 0 = t_0, g_1 = g, a_1, g_1, \ldots, g_n, a_n)$ satisfying conditions (i)-(iii). Clearly such a sequence is unique. ∎

Remark 4.6.21. *As, by assumption, $\alpha(0) \in g(\mathcal{F})^\circ$ and $\alpha(1) \in g'(\mathcal{F})^\circ$ for some $g, g' \in \Gamma$, the first element g_1 of the partition of α equals g and the last element g_n equals g'.*

The following definition is exactly the same as the one given in Chapter 2. However, as we have uniqueness in the previous lemma, we do not need all the lemmas proved in Chapter 2. Let $g, h \in \Gamma$. Let C be a cell of $\mathcal{T}$ of dimension $m \ge 2$ and that is contained in $g(\mathcal{F}) \cap h(\mathcal{F})$. We define $\kappa_C(g, h) \in \Delta$ as follows.

- If $m = 4$ then $\kappa_C(g, h) = 1$.
- If $m = 3$ then $\kappa_C(g, h) = [g^{-1}h]$.
- If $m = 2$ then C is an edge contained in $g(\mathcal{F}) \cap h(\mathcal{F})$ and thus, by Lemma 4.6.13, g and h belong to an edge loop of C. Up to a cyclic permutation, we can write the edge loop of C as $(g = k_0, \ldots, k_t = h, k_{t+1}, \ldots, k_m = g)$ (or the equivalent edge loop $(g = k_m, k_{m-1}, \ldots, k_t = h, k_{t-1}, \ldots, k_1, k_0 = g)$) and we set

$$\kappa_C(g, h) = [k_0^{-1}k_1][k_1^{-1}k_2] \cdots [k_{t-1}^{-1}k_t] = [k_m^{-1}k_{m-1}] \cdots [k_{t+1}^{-1}k_t].$$

Observe that $\kappa_C(g,g) = 1$ in the three cases. The following lemma is exactly the same as Lemma 2.4.9. Therefore we omit its proof.

Lemma 4.6.22. *Let $g, h \in \Gamma$ and let C be a cell of $\mathcal{T}$ of dimension $m \geq 2$ and that is contained in $g(\mathcal{F}) \cap h(\mathcal{F})$. The following properties hold.*

1. $\kappa_C(g,h) = \kappa_C(h,g)^{-1}$.
2. *If D is cell of $\mathcal{T}$ contained in C and of dimension at least 2 then $\kappa_D(g,h) = \kappa_C(g,h)$.*
3. *If $g_1, \dots, g_n \in \Gamma$ and $C \subseteq \bigcap_{i=1}^n g_i(\mathcal{F})$ then $\kappa_C(g_1, g_n) = \kappa_C(g_1, g_2)\ \kappa_C(g_2, g_3) \dots \kappa_C(g_{n-1}, g_n)$.*

Let $Z \in \Omega$, $g, h \in \Gamma$ and $Z \in C \subseteq g(\mathcal{F}) \cap h(\mathcal{F})$ for some cell C. Then, by the definition of Ω, the dimension of C is at least 2 and we define

$$\kappa_Z(g,h) = \kappa_C(g,h).$$

Note that the same reasoning that showed that (2.4) is well-defined, applies here. By parts 1 and 3 of Lemma 4.6.22 we have $\kappa_Z(g,h) = \kappa_Z(h,g)^{-1}$ and if $Z \in \cap_{i=1}^n g_i(\mathcal{F})$ with $g_1, \dots, g_n \in \Gamma$ then

$$\kappa_Z(g_1, g_n) = \kappa_Z(g_1, g_2) \cdots \kappa_Z(g_{n-1}, g_n).$$

Definition 4.6.23. *Let α be a continuous map $[0,1] \to \Omega$ as in Lemma 4.6.20 and let $\mathcal{L} = (a_0, g_1, a_1, g_2,\ \ a_2 \dots, g_n, a_n)$ be the partition of α. We define*

$$\Phi(\mathcal{L}) = \kappa_{\alpha(a_1)}(g_1, g_2)\ \kappa_{\alpha(a_2)}(g_2, g_3) \cdots \kappa_{\alpha(a_{n-1})}(g_{n-1}, g_n),$$

if $n \neq 1$. If $n = 1$, we set $\Phi(\mathcal{L}) = 1$.

Observe that if $i \in \{1, \dots, n\}$ then $\alpha(a_i) \in g_i(\mathcal{F}) \cap g_{i+1}(\mathcal{F})$. By the definition of Ω, $g_i(\mathcal{F}) \cap g_{i+1}(\mathcal{F})$ has dimension 2 or 3. If it has dimension 3, then $g_i(\mathcal{F}) \cap g_{i+1}(\mathcal{F})$ is a side containing $\alpha(a_i)$. If $g_i(\mathcal{F}) \cap g_{i+1}(\mathcal{F})$ has dimension 2, then $\bigcap_{g,\ \alpha(a_i) \in g(\mathcal{F})} g(\mathcal{F})$ has dimension 2 by the definition of Ω and thus, by Lemma 4.6.12, $g_i(\mathcal{F}) \cap g_{i+1}(\mathcal{F})$ contains an edge that contains $\alpha(a_i)$. Hence $\alpha(a_i) \in C \subseteq g_i(\mathcal{F}) \cap g_{i+1}(\mathcal{F})$, for some cell C and thus $\kappa_{\alpha(a_i)}(g_i, g_{i+1})$ is well defined.

Let $Z, W \in \Omega$ and let $\mathbb{P}$ be the set of all the continuous maps $\alpha : [0,1] \to \Omega$ with $\alpha(0) = Z$ and $\alpha(1) = W$, and such that α verifies the conditions of Lemma 4.6.20. The set $\mathbb{P}$ may be considered as a metric space with the metric determined by the infinite norm: $\|\alpha\|_\infty = \max\{|\alpha(t)| \mid t \in [0,1]\}$. We define the map

$$\Phi : \mathbb{P} \to \Delta$$

by $\Phi(\alpha) = \Phi(\mathcal{L})$, where $\mathcal{L}$ is the partition of α. This is well defined as by Lemma 4.6.20 the partition of α exists and is unique. The next lemma will be a crucial part in the proof of the injectivity of the map φ defined in (4.35). Observe that its proof has quite some overlap with the proof of Lemma 2.4.16. Nevertheless, as it is one of the main lemmas leading to the proof of Theorem 4.6.25, we give the complete proof here.

Lemma 4.6.24. *If both Z and W belong to the interior of some tile then $\Phi : \mathbb{P} \to \Delta$ is constant.*

Proof. We claim that it is sufficient to show that Φ is locally constant. Indeed, assume this is the case and let $\alpha, \beta \in \mathbb{P}$. By Lemma A.1.4, α and β are homotopic in Ω and by Lemma A.1.7, there is a homotopy $H(t, -)$ in $\mathbb{P}$ from α to β. In fact Lemma A.1.4 and Lemma A.1.7 replace the concept of strongly homotopic used in Chapter 2. Let c denote the supremum of the $s \in [0, 1]$ for which $\Phi(H(s, -)) = \Phi(\alpha)$. Since, by assumption, Φ is constant in a neighbourhood of $H(x, -)$, it easily follows that $c = 1$ and thus $\Phi(\alpha) = \Phi(\beta)$.

To prove that Φ is locally constant, let $\alpha, \beta \in \mathbb{P}$ and let $\mathcal{L}_1 = (a_0, g_1, a_1, \dots , g_n, a_n)$ and $\mathcal{L}_2$ be the partition of α and β respectively. Moreover we denote by $d(-, -)$ the Euclidean distance. Let $\{0 < d_1 < d_2 < \ldots < d_m < 1\}$ be the sets of elements $d \in [0, 1]$ such that $\alpha(d) \in \partial g(\mathcal{F})$ for some $g \in \Gamma$ for every $1 \leq i \leq m$. Lemma 4.6.19 ensures that this set is finite. Denote by $k_1, \dots , k_{m+1}$ the elements in Γ, such that $\alpha(d_{i-1}, d_i) \subseteq k_i(\mathcal{F})$ for $1 \leq i \leq m + 1$, where we set $d_0 = a_0 = 0$ and $d_{m+1} = a_n = 1$. Observe that $k_1 = g_1$, $k_{m+1} = g_n$ and $\{k_1, \dots , k_{m+1}\} = \{g_1, \dots , g_n\}$. Since $\mathcal{F}$ is locally finite, there is $\delta_1 > 0$ such that for every $i \in \{0, 1, \dots , m + 1\}$ and every $g \in \Gamma$, if $B(\alpha(d_i), 2\delta_1)) \cap g(\mathcal{F}) \neq \emptyset$ then $\alpha(d_i) \in g(\mathcal{F})$. Since α is continuous there is $\epsilon < \min\left\{\frac{d_i - d_{i-1}}{2} \mid i \in \{1, \dots , m + 1\}\right\}$ such that, for every $i \in \{0, 1, \dots , m + 1\}$, $d(\alpha(t), \alpha(d_i)) < \delta_1$ for every t with $|t - d_i| < \epsilon$. For every $i \in \{1, \dots , m\}$, let $d'_i = d_i - \epsilon$ and $d''_i = d_i + \epsilon$. We also set $d'_{m+1} = 1$ and $d''_0 = 0$. Observe that $d''_i \leq d'_{i+1}$ for every $i \in \{0, \dots , m + 1\}$. Each $\alpha([d''_{i-1}, d'_i])$ is compact and it is contained in $k_i(\mathcal{F})^\circ$. Again using that $\mathcal{F}$ is locally finite we obtain a positive number δ_2 such that for every $i \in \{0, 1, \dots , m+1\}$, $d(\alpha(t), g(\mathcal{F})) > \delta_2$ for every $t \in [d''_{i-1}, d'_i]$ and every $g \in \Gamma$ with $g \neq k_i$. Let $\delta = \min\{\delta_1, \delta_2\}$.

We will prove that if $\beta \in B_{\|\ \|_\infty}(\alpha, \delta)$, then $\Phi(\alpha) = \Phi(\beta)$. So assume $\beta \in \mathbb{P}$ with $\|\alpha - \beta\|_\infty < \delta$. Then $d(\alpha(t), \beta(t)) < \delta$ for every $t \in [0, 1]$. In particular, as $d(\beta(t), \alpha(d_i)) < 2\delta_1$,

$$\text{if } t \in (d'_i, d''_i) \text{ and } \beta(t) \in g(\mathcal{F}) \text{ then } \alpha(d_i) \in g(\mathcal{F}). \tag{4.36}$$

Moreover, since $d(\alpha(t), \beta(t)) < \delta_2$,

$$\text{if } t \in [d''_{i-1}, d'_i] \text{ and } \beta(t) \in g(\mathcal{F}) \text{ then } g = k_i. \tag{4.37}$$

The interval $[0, 1]$ may be written as

$$[0, d'_1] \cup [d'_1, d''_1] \cup [d''_1, d'_2] \cup \ldots [d'_m, d''_m] \cup [d''_m, d_{m+1}] .$$

Based on this information, we construct $\mathcal{L}_2$ and prove that $\Phi(\mathcal{L}_1) = \Phi(\mathcal{L}_2)$. By (4.37), the elements $k_1, \ldots k_{m+1}$ appear in that order in $\mathcal{L}_2$. Between those elements may appear other elements $h \in \Gamma$. For each $1 \leq i \leq m+1$, there are two possibilities: k_i and k_{i+1} are equal or they are different. If $k_i = k_{i+1}$ and k_i and k_{i+1} are two consecutive elements in $\mathcal{L}_2$, then $\beta((d''_{i-1}, d'_{i+1})) \subseteq k_i(\mathcal{F}) = k_{i+1}(\mathcal{F})$. According to the definition of the partition of β, k_i and k_{i+1} are represented by just one element in $\mathcal{L}_2$. Hence we may suppose, without loss of generality, that if k_i and k_{i+1} are two consecutive elements of $\mathcal{L}_2$, then they are different. Thus

$$\mathcal{L}_2 = (0, k_1, b_1, *_1, k_2, b_2, *_2, \dots, k_m, b_m, *_m, k_{m+1}, 1),$$

where $d'_i < b_i < d''_i$ and $*_i$ represents a, possibly empty, sequence (h_{i1}, b_{i1}, h_{i2}, g_{i2}, ..., h_{in_i}, b_{in_i}) with $h_{ij} \in \Gamma$ and $d'_i < b_{ij} < d''_i$ for every $1 \leq j \leq n_i$. As stated above, if $*_i$ is empty, then $k_i \neq k_{i+1}$. In that case $k_i = g_j$ and $k_{i+1} = g_{j+1}$ are two consecutive elements in $\mathcal{L}_1$ and $b_i = a_j$. Moreover

$$\begin{aligned}\beta([d''_{i-1}, d'_i]) &\subseteq k_i(\mathcal{F}),\\ \beta([d''_i, d'_{i+1}]) &\subseteq k_{i+1}(\mathcal{F}),\\ \beta(b_i) &\in k_i(\mathcal{F}) \cap k_{i+1}(\mathcal{F}).\end{aligned}$$

By (4.36) and the fact that $b_i \in [d'_i, d''_i]$, for every $g \in \Gamma$ such that $\beta(b_i) \in g(\mathcal{F})$ we have that $\alpha(d_i) = \alpha(a_j) \in g(\mathcal{F})$. Hence, by Lemma 4.6.12,

$$\bigcap_{g,\, \alpha(a_j) \in g(\mathcal{F})} g(\mathcal{F})$$

is a cell, which is contained in every cell containing $\beta(b_i)$. Thus both $\alpha(a_j)$ and $\beta(b_i)$ are included in a same cell C contained in $g_j(\mathcal{F}) \cap g_{j+1}(\mathcal{F})$. Thus,

$$\kappa_{\alpha(a_j)}(g_j, g_{j+1}) = \kappa_C(g_j, g_{j+1}) = \kappa_C(k_i, k_{i+1}) = \kappa_{\beta(b_i)}(k_i, k_{i+1}).$$

Suppose now that $*_i$ is not empty. We will analyse the subsequence

$$(k_i, b_i, h_{i1}, b_{i1}, h_{i2}, b_{i2} \dots, h_{in_i}, b_{in_i}, k_{i+1}, b_{i+1}).$$

By lemma 4.6.12, $\alpha(d_i) \in C$, $\beta(b_i) \in C_i$ and $\beta(b_{ij}) \in C_{ij}$ for C, C_i and C_{ij} cells for $1 \leq j \leq n_i$. As $b_i \in (d'_i, d''_i)$ and also $b_{ij} \in (d'_i, d''_i)$ for every $1 \leq j \leq n_i$, by (4.36) we have that $\alpha(d_i) \in k_i(\mathcal{F}) \cap k_{i+1}(\mathcal{F}) \cap \bigcap_{j=1}^{n_i} h_{ij}(\mathcal{F})$. Hence $C \subseteq C_i \cap \bigcap_{i=1}^{n_i} C_{ij}$. If $k_i \neq k_{i+1}$, then $k_i = g_l$, $k_{i+1} = g_{l+1}$ and $d_i = a_l$ for some $1 \leq l \leq n$ and by parts 2 and 3 of Lemma 4.6.22

$$\begin{aligned}\kappa_{\alpha(a_l)}(g_l, g_{l+1}) &= \kappa_C(k_i, k_{i+1})\\ &= \kappa_C(k_i, h_{i1})\kappa_C(h_{i1}, h_{i2}) \dots \kappa_C(h_{in_i} k_{i+1})\\ &= \kappa_{C_i}(k_i, h_{i1})\kappa_{C_{i1}}(h_{i1}, h_{i2}) \dots \kappa_{C_{in_i}}(h_{in_i} k_{i+1})\\ &= \kappa_{\beta(b_i)}(k_i, h_{i1})\kappa_{\beta(b_{i1})}(h_{i1}, h_{i2}) \dots \kappa_{\beta(b_{in_i})}(h_{in_i} k_{i+1})\end{aligned}$$

If $k_i = k_{i+1}$, then $k_i = k_{i+1} = g_l$ for some $1 \leq l \leq n$. Then in $\Phi(\alpha)$, there is no κ-term corresponding to the subsequence above and by parts 2 and 3 of Lemma 4.6.22, we have

$$\begin{aligned}&\kappa_{\beta(b_i)}(k_i, h_{i1})\kappa_{\beta(b_{i1})}(h_{i1}, h_{i2}) \dots \kappa_{\beta(b_{in_i})}(h_{in_i} k_{i+1})\\ =\ &\kappa_{C_i}(k_i, h_{i1})\kappa_{C_{i1}}(h_{i1}, h_{i2}) \dots \kappa_{C_{in_i}}(h_{in_i} k_{i+1})\\ =\ &\kappa_C(k_i, h_{i1})\kappa_C(h_{i1}, h_{i2}) \dots \kappa_C(h_{in_i} k_{i+1})\\ =\ &\kappa_C(k_i, k_{i+1})\\ =\ &\kappa_C(k_i, k_i)\\ =\ &1.\end{aligned}$$

This shows that $\Phi(\mathcal{L}_1) = \Phi(\mathcal{L}_2)$. ∎

We are now ready to prove that $\varphi : \Delta \to \Gamma$ (see (4.35)) is injective. Proving the injectivity is equivalent to proving that if $g_1 \dots g_n = 1$ with each $g_i \in \Gamma$ and $\mathcal{F} \cap g_i(\mathcal{F})$ a side, then $[g_1] \dots [g_n] = 1$ in Δ. So, suppose that $g_1 \dots g_n = 1$ with each $g_i \in \Gamma$ and $\mathcal{F} \cap g_i(\mathcal{F})$ a side. For every $i = 0, 1, \dots, n$, put $h_i = g_1 \dots g_i$ and put $h_0 = 1$. Choose $Z \in \mathcal{F}^\circ$ and, for every $i = 1, \dots, n$, choose a path from $h_{i-1}(Z)$ to $h_i(Z)$ which is only contained in $h_{i-1}(\mathcal{F}) \cup h_i(\mathcal{F})$ and which is made up of a finite number of line segments, whose end-points do not belong to any element in $\mathcal{M}$. This is possible by Lemma 4.3.7 and Lemma A.1.3. Let α be the path obtained by juxtaposing those paths. So $\alpha(0) = Z = \alpha(1)$ and α is contained in the space $\mathbb{P}$. Let $\mathcal{L}_1 = (a_0 = 0, h_0, a_1, \dots, h_n, a_{n+1} = 1)$ be the partition of α. Let β be the constant path $\beta(t) = Z$ for every $t \in [0,1]$ and let $\mathcal{L}_2 = (0,1,1)$ be the partition of β. Clearly β is also contained in $\mathbb{P}$. Now we have that

$$\begin{aligned} [g_1] \dots [g_n] &= [h_0^{-1}h_1][h_1^{-1}h_2] \dots [h_{n-1}^{-1}h_n] \\ &= \kappa_{a_1}(h_0,h_1)\kappa_{a_2}(h_1,h_2) \dots \kappa_{a_n}(h_{n-1},h_n) \\ &= \Phi(\mathcal{L}_1) \\ &= \Phi(\alpha). \end{aligned}$$

By Lemma 4.6.24, $\Phi(\alpha) = \Phi(\beta)$. However $\Phi(\beta) = \Phi(\mathcal{L}_2) = \kappa_1(1,1) = 1$ and hence $[g_1] \dots [g_n] = 1$.

To sum up we have proven the following theorem.

Theorem 4.6.25. *Let $\mathcal{F}$ be the fundamental domain for Γ as defined above. Then the following is a presentation of Γ:*

- *Generators: the pairing transformations of $\mathcal{F}$,*
- *Relations: the pairing relations and the cycle relations.*

It is well known that Γ is finitely presented. However it is not clear whether the presentation given in Theorem 4.6.25 is finite. The following proposition implies that at least the presentation is finite in case R is a PID.

Lemma 4.6.26. *If R is a PID then every tile of $\mathcal{F}$ intersects only finitely many other tiles. In particular the fundamental domain $\mathcal{F}$ has finitely many cells and thus finitely many sides and edges.*

Proof. It is enough to prove that $\mathcal{F}$ intersects only finitely many other tiles. Moreover, if $1 \neq \gamma \in \Gamma$ then $\mathcal{F} \cap \gamma(\mathcal{F}) \subseteq \cup_{V \in \mathcal{V}_e} \mathcal{F} \cap V$ and $\mathcal{V}_e$ is finite by Theorem 4.3.10. Thus it is enough to prove that if $V \in \mathcal{V} \cap \mathcal{V}_\infty$ then $X_V = \{\gamma \in \Gamma \mid V \cap \mathcal{F} \cap \gamma(\mathcal{F}) \neq \emptyset\}$ is finite. If $V \in \mathcal{V}$ then $V \cap F$ is compact, by Lemma 4.4.18. Since $\mathcal{F}$ is locally finite (Lemma 4.3.11), X_V is finite. So assume that $V \in \mathcal{V}_\infty$. By (4.24), (4.26) and Lemma 4.3.9, there are positive real numbers $a < b$ such that $B \times (b, +\infty) \subseteq \mathcal{F} \subseteq B \times (a, +\infty)$ (in the (s_0, s_1, r, h)-coordinates). Let $K = B \times [a, b]$. If $\gamma \in X_V$ then either $K \cap \gamma(\mathcal{F}) \neq \emptyset$ or $V \cap (B \times (b, +\infty)) \cap \gamma(\mathcal{F}) \neq \emptyset$. As K is compact and $\mathcal{T}$ is locally finite, only finitely many $\gamma \in \Gamma$ satisfy the first condition. On the other hand it is easy to see that $\mathbb{R}^2 \times \mathbb{R}^+ \times (b, \infty) = \cup_{\gamma \in \Gamma_\infty} \gamma(B \times (b, +\infty))$ (in the (s_1, s_2, r, h)-coordinates). Thus if γ satisfies the second condition then $\gamma \in \Gamma_\infty$ and $\mathcal{F} \cap \gamma(\mathcal{F}) \neq \emptyset$. It remains to prove that only finitely many elements of Γ_∞ satisfy the last condition. Fix $h_0 > 0$. Assume that $Z_1 = \gamma(Z_2) \in \mathcal{F}_\infty \cap \gamma(\mathcal{F}_\infty)$ with $\gamma \in \Gamma_\infty$. Let Z_1' and Z_2' be the elements of $\mathbb{H}^2 \times \mathbb{H}^2$ obtained by replacing the last coordinate of Z_1 and Z_2 by h_0. Then $Z_1' = \gamma(Z_2') \in (B \times \{h_0\}) \cap \mathcal{F}_\infty \cap \gamma(\mathcal{F}_\infty)$. Thus $(B \times \{h_0\}) \cap \gamma(\mathcal{F}_\infty) \neq \emptyset$. As $B \times \{h_0\}$ is compact and $\mathcal{F}_\infty$ is a locally finite fundamental domain for Γ_∞, only finitely many elements satisfies the last condition. This finishes the proof. ∎

4.7 Conclusions and Further Work

First we stress the benefit of the progress made on the Hilbert modular group: the fact that we were able to establish a theoretical method for finding a presentation of $\mathrm{SL}_2(R)$ where R is the ring of integers of some number field K, which acts discontinuously on a direct product of two copies of hyperbolic 2-space, gives hope for the investigation on some exceptional components of type (1). In fact the results obtained for actions on $\mathbb{H}^2 \times \mathbb{H}^2$ are an important first step in the good direction.

The following step to be taken in this direction would be to determine a finite presentation of the group $\mathrm{SL}_2(R)$ with R the ring of integers of $\mathbb{Q}(\sqrt{d})$ with d a positive integer, but R need not to be a principal ideal domain. In fact Theorem 4.6.25 gives a presentation of $\mathrm{SL}_2(R)$, but this presentation is only guaranteed to be finite in the case R is a principal ideal domain (see Corollary 4.5.8 and Lemma 4.6.26). The PID-condition allows to use the result of [KZ72] and hence guarantees a minimal height for points contained in the fundamental domain (see Lemma 4.3.9). This then implies that the fundamental domain is finite-sided. So, the problem is to find another argument (without making use of PID) to prove that the fundamental domain is finite-sided.

The next step is to generalize the above techniques to arbitrary discrete subgroups of the group $\mathrm{SL}_2(\mathbb{R}) \times \overset{(r)}{\cdots} \times \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{C}) \times \overset{(s)}{\cdots} \times \mathrm{SL}_2(\mathbb{C})$. These subgroups will have a discontinuous action on $\mathbb{H}^2 \times \overset{(r)}{\cdots} \times \mathbb{H}^2 \times \mathbb{H}^3 \times \overset{(s)}{\cdots} \times \mathbb{H}^3$. The main difficulty is to generalize the above result to more copies of hyperbolic spaces and to also allow copies that are 3-spaces (and not only 2-spaces). It seems plausible that the work done so far can be extended to more than two copies of hyperbolic 2-spaces. To also allow copies of the form of hyperbolic 3-spaces, one has to do more work. In fact the construction of the fundamental domain $\mathcal{F}$ of Theorem 4.3.5 is based on the construction of a fundamental domain for Γ_∞ (see Theorem 4.3.1). In case of hyperbolic 3-space, several more symmetries are showing up and thus the group Γ_∞ will be more complicated than in dimension 2.

For the applications to units of integral group rings we also have to deal with subgroups of direct products of special linear groups. In first instance, one needs to deal with subgroups of finite index. The construction of the fundamental domain probably should be similar. Again, one of the main issues will be to prove that the proposed fundamental domain has finitely many sides.

One of our original aims is to deal with the unit group of $\mathbb{Z}(Q_8 \times C_7)$. For this, one also should handle subgroups of direct products which are not necessarily of finite index in direct products of special linear groups over ring of integers. Recall that the study of the unit group of $\mathbb{Z}(Q_8 \times C_7)$ is reduced to that of $\mathrm{SL}_1(\mathcal{H}(\mathbb{Z}[\xi_7]))$, which has a discrete embedding in $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$.

Another idea for finding generators for the concrete example $\mathrm{SL}_1(\mathcal{H}(\mathbb{Z}[\xi_7]))$ is to start with the work done by Macbeath in [Mac64]. More precisely, the idea is to use Theorem 1.2.10, instead of Theorem 2.4.7. We could construct algorithmically a finite-sided fundamental domain based on the construction explained above. A problem here is that our algorithm has to handle quadratic equations in 9 variables. Then [Mac64, Corollary of Theorem 2] gives a method to find the generators of $\mathrm{SL}_1(\mathcal{H}(\mathbb{Z}[\xi_7]))$ based on the fundamental domain constructed. The latter theorem also gives a method to compute the relations between those generators. However, to do this in practice, one needs a lot of information on the structure of the fundamental domain, and hence the program explained above will be necessary.

Finally, also exceptional components of type (3) should be studied. One idea is to realize this via Clifford Möbius transformations. These are generalizations of classical Möbius transformations to higher dimensions. Roughly speaking, Clifford Möbius transformations are 2-by-2 matrices over Clifford algebras. The Clifford algebra $\mathcal{C}_n$ is the associative algebra over the real numbers generated by n elements $i_1, \ldots, i_n$ that anti-commute and square to -1. For $n = 0$ and $n = 1$, this is simply $\mathbb{R}$ and $\mathbb{C}$. For $n = 2$, $\mathcal{C}_n$ may be identified with the Hamiltonian quaternions $\mathcal{H}$. The idea is to generalize the theory of classical Möbius transformations to Clifford Möbius transformation and get hence actions on hyperbolic space of higher dimension than 3. More details on this may be found in [Par07]. The quaternionic Möbius transformations, i.e. 2-by-2 matrix transformations over quaternion algebras, play a special role in this generalization. In fact, for $n \geq 3$, the algebra $\mathcal{C}_n$ contains zero divisors, which complicate the generalization a lot. So $\mathcal{C}_2$ is in fact the largest case without zero-divisors. The whole theory of classical Möbius transformations can be generalized to this case, except that one has to be careful about commutativity. For example, when generalizing the group $\mathrm{SL}_2(\mathbb{C})$ to $\mathrm{SL}_2(\mathcal{H})$, one needs to replace the classical determinant by the Dieudonné determinant. Quaternionic Möbius transformations give a natural action on $\mathbb{H}^4$ and by a similar manipulation as the Poincaré extension, one also gets an action on $\mathbb{H}^5$. Applications of quaternionic Möbius transformations to hyperbolic geometry were for example studied in [Kel01, Kel03]. It is now not complicated to see that this theory could serve as a good starting point for the investigation on exceptional components of type (3).

A

Appendices

A.1 Topological Lemmas

Lemma A.1.1. *Let $\mathbb{X}$ be a path-connected open subset of $\mathbb{R}^n$, such that for every two points $Z_0, Z_1 \in \mathbb{X}$, there exists $\epsilon > 0$ such that the open cylinder with axis $[Z_0, Z_1]$ and radius ϵ is in $\mathbb{X}$. If $\mathcal{L}$ is a locally finite collection of semi-algebraic varieties of dimension at most $n-2$ in $\mathbb{X}$ then $\mathbb{X} \setminus \cup_{L\in\mathcal{L}} L$ is path-connected.*

Proof. Let Z_0 and Z_1 be two points in $\mathbb{X} \setminus \cup_{L\in\mathcal{L}} L$. Consider the Euclidean geodesic segment $[Z_0, Z_1]$. If $[Z_0, Z_1] \cap L = \emptyset$ for every $L \in \mathcal{L}$, we are done. Otherwise consider the cylinder C with axis $[Z_0, Z_1]$ and radius ϵ. Note that $C \subseteq \mathbb{X}$ by definition. Moreover take ϵ small enough such that for $i = 0, 1$, $B(Z_i, \epsilon) \cap W_i$ does not intersect $\mathcal{L}$, where W_i is the hyperplane containing Z_i and perpendicular to $[Z_0, Z_1]$. Let $\varphi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the linear projection which maps $[Z_0, Z_1]$ to a point S. Then the Zariski-closure of $\varphi(\cup_{L\in\mathcal{L}} L)$ has dimension at most $n-2$ and hence $\varphi(C) \setminus \varphi(\cup_{L\in\mathcal{L}} L)$ is dense in $\varphi(C)$. Hence there exists a point $S_0 \in \varphi(C) \setminus \varphi(\cup_{L\in\mathcal{L}} L)$. Thus $\gamma_0 = \varphi^{-1}(S_0)$ is a Euclidean geodesic in C parallel to $[Z_0, Z_1]$ and $\gamma_0 \cap L = \emptyset$ for every $L \in \mathcal{L}$. Denote by γ_1 a path in W_0 with endpoints Z_0 and $\gamma_0 \cap W_0$ and by γ_2 a path in W_2 with endpoints Z_1 and $\gamma_0 \cap W_2$. Let γ be the concatenation of the paths γ_1, γ_0 and γ_2. By construction γ is a path joining Z_0 and Z_1 and being disjoint from every element in $\mathcal{L}$. ∎

Lemma A.1.2. *Let $\overline{B}$ be a Euclidean closed ball in $\mathbb{R}^n$. If $\overline{B} = U_1 \cup U_2$, where U_1 and U_2 are closed semi-algebraic varieties, with $U_1 \not\subseteq U_2$ and $U_2 \not\subseteq U_1$, then $dim(U_1 \cap U_2) \geq n-1$.*

Proof. We prove this by contradiction. So, suppose that $\overline{B} = U_1 \cup U_2$, where U_1 and U_2 are closed semi-algebraic varieties and suppose that $dim(U_1 \cap U_2) \leq n-2$. Consider $\overline{B'} = \overline{B} \setminus (U_1 \cap U_2)$. By Lemma A.1.1, this is still a connected space. Set $U_i' = U_i \setminus (U_1 \cap U_2)$ for $i = 1, 2$. Obviously U_1' and U_2' are disjoint closed non-empty sets in $\overline{B'}$, which contradicts the connectedness of $\overline{B'}$. ∎

Lemma A.1.3. *Let X be a connected open subset of $\mathbb{R}^n$ and let Z, W be two points in X. Then there exists a path $\gamma : [0,1] \to X$ such that $\alpha(0) = Z$ and $\alpha(1) = W$ and which is made up of a finite number of line segments, which are parametrized by equations of degree 1. Moreover, if $Y \subseteq X$ is a subset with dense complement, then the end-points of the line segments can be chosen to avoid Y.*

Proof. As X is path-connected, there exists a path $\gamma : [0,1] \to X$ such that $\gamma(0) = Z$ and $\gamma(1) = W$. Since $\gamma([0,1])$ is compact, there exists a finite number of open euclidean balls $B_1, \ldots, B_m$, such that $\gamma([0,1]) \subseteq \cup_{i=1}^m B_i$. We claim that there exists a finite sequence $0 = t_0 < t_1 < t_2 < \ldots < t_n = 1$ such that for each i there is a ball in the above covering which contains the entire path segment $\gamma([t_i, t_{i+1}])$. Indeed for each j decompose $\gamma^{-1}(B_j) \in [0,1]$ into connected components $I_{j,k}$ where k ranges over some (possibly infinite) index set K. As the $I_{j,k}$ are connected open subsets of $[0,1]$, they are intervals. The $\{I_{j,k}\}_{j,k}$ form an open covering of $[0,1]$ and hence we can choose a finite sub-covering. Now we just have to place finitely many points t_i on $[0,1]$ (the first being 0 and the last being 1) such that for each i there is an interval $I_{j,k}$ in our finite sub-covering containing both t_i and t_{i+1}. This proves the claim.

For each t_i choose a $y_i \in X$ such that $y_i \in B_j$ for every B_j such that $\gamma(t_i) \in B_j$ and $y_i \notin Y$. This is possible, because there are only finitely many balls B_j and hence $\bigcap_{B_j, \gamma(t_i) \in B_j} B_j$ contains an open neighbourhood of $\gamma(t_i)$. As Y has dense complement in X, this open neighbourhood cannot be completely contained in Y. We now define α piecewise. Let $\alpha|_{[t_i, t_{i+1}]}$ be given by the straight line connecting y_i and y_{i+1}:

$$\alpha(x) := y_i + \frac{x - t_i}{t_{i+1} - t_i}(y_{i+1} - y_i) \text{ for } x \in [t_i, t_{i+1}].$$

By construction, there exists a ball B_j containing both y_i and y_{i+1}. As a ball is a convex set, $\alpha|_{[t_i, t_{i+1}]} \subseteq B_j$. As all the balls are contained in X, the path α is contained in X. ∎

Lemma A.1.4. *Let X be a simply connected connected open subset of $\mathbb{R}^n$ for some $n \geq 3$. Let $\{T_i\}_i$ be a finite family of algebraic varieties in X of codimension bigger or equal than 3. Then $X \setminus \cup_i T_i$ is simply connected.*

Proof. Let γ and γ' be two loops in $X \setminus \cup_i T_i$. We can assume without loss that the loops are made up of line segments and the homotopy between them (in X) is made up of pieces of finitely many two-dimensional planes. Since $\gamma([0,1])$ and $\gamma'([0,1])$ both have positive distance from $\cup_i T_i$ and ∂X there is some $1 > \epsilon > 0$ such that for each $v \in \mathbb{R}^n$ with $|v| < \epsilon$ and each homotopy H between γ and γ' we can define a homotopy H_v (in $\mathbb{X}$) with agrees with a scaled version of $H + v$ on $[\mu, 1-\mu] \times [\mu, 1-\mu]$, where $0 < \mu < \frac{1}{2}$, and which agrees with H on the rim of $[0,1] \times [0,1]$. The points on the rim of $[0,1] \times [0,1]$ are connected with the corresponding points on the rim of $[\mu, 1-\mu] \times [\mu, 1-\mu]$ by a straight line. Now the image of H_v intersects with $\cup_i T_i$ if and only if the image of H shifted by v does. Now assume W is one of the two dimensional planes that make up H and let $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-2}$ be a (linear) projection which maps W to a point w. Then for $v \in \mathbb{R}^n$ the shifted plane $W + v$ intersects $\cup_i T_i$ if and only if $w + \varphi(v) \in \varphi(\cup_i T_i)$. However the Zariski-closure of $\varphi(\cup_i T_i)$ has codimension at least 1 in $\mathbb{R}^{n-2}$, and thus its pre-image under φ has codimension at least 1 in $\mathbb{R}^n$. This implies that the set of $v \in \mathbb{R}^n$ which do not lie in a pre-image contain the complement of an algebraic set of codimension at least 1. Now we consider the set of points $v \in \mathbb{R}^n$ such that none of the 2-planes that make up H intersects with $\cup_i T_i$ when shifted by v. By the above, these are the complement of the union of a finite number of algebraic sets of codimension at least 1, and therefore we find an element v in it of norm strictly smaller than ϵ. Thus H_v is a homotopy between γ and γ' in $X \setminus \cup_i T_i$. ∎

Lemma A.1.5. *Let X be a connected open subset of $\mathbb{R}^n$. Let $[\gamma] \in \pi_1(X)$ be a homotopy class. Then there is a closed path $\gamma' \in [\gamma]$ starting and ending at x_0 which is made up of a finite number of line segments, which are parametrized by equations of degree 1. Moreover, if we are given a subset $Y \subseteq X$ with dense complement, then we can choose the end-points of the line segments to avoid Y.*

Proof. By Lemma A.1.3, there exists $\gamma' : [0,1] \to X$ starting and ending at x_0 which is made up of a finite number of line segments, whose end-points avoid Y. We use the straight line homotopy between γ and γ', i.e.

$$H(x,t) := x \cdot \gamma(t) + (1-x) \cdot \gamma'(t),$$

both loops are homotopic. Hence we only need to show that this homotopy stays in X. Indeed, by the construction in Lemma A.1.3, for $t_i \le t \le t_{i+1}$ the values $H(x,t)$ are convex combinations of elements in the ball which contains the path segment $\gamma([t_i, t_{i+1}])$ as well as y_i and y_{i+1}, and they hence do not leave this ball and therefore they do not leave X. ∎

Lemma A.1.6. *Let X be a connected open subset of $\mathbb{R}^n$. Let γ and γ' be two paths starting at Z and ending at W which are both made up of line segments, that are parametrized by equations of degree 1. Assume γ and γ' are homotopic. Then there is an integer N and a homotopy*

$$H : [0,1] \times [0,1] \longrightarrow X$$

such that each path $H(x,-)$ starts at Z and ends at W for $x \in [0,1]$ and is made up of $\le N$ line segments, which are parametrized by equations of degree 1.

Proof. In what follows, we equip both $\mathbb{R}^n$ and $[0,1] \times [0,1]$ with the euclidean norm $|\cdot|$. Let $H'(x,t)$ be an arbitrary homotopy between γ and γ' in X with $H'(0,-) = \gamma(-)$ and $H'(1,-) = \gamma'(-)$. If $\partial X \neq \emptyset$ then we denote by ϵ half the distance between $H'([0,1] \times [0,1])$ and ∂X. This distance indeed exists and is positive because $H'([0,1] \times [0,1])$ is compact. If $\partial X = \emptyset$, then let $\epsilon > 0$ be arbitrarily. As H' is a continuous function on the compact set $[0,1] \times [0,1]$, it is uniformly continuous. Hence there exists $\delta > 0$ such that $|(x,t) - (x',t')| < \delta$ implies $|H'(x,t) - H'(x',t')| < \epsilon$. Let $M \in \mathbb{N}$ be such that

$1/M < \delta/2$. Let $0 = t_1 < t_2 < \ldots < t_N = 1$ be such that for each i, $|t_i - t_{i+1}| < \delta/2$ and all the endpoints of the line segments that make up γ and γ' are among the t_i (i.e. both γ and γ' become straight lines when restricted to an interval of the form $[t_i, t_{i+1}]$). Then we define the homotopy $H: [0,1] \times [0,1] \longrightarrow X$ piecewise, with the definition on the rectangle $[j/M, (j+1)/M] \times [t_i, t_{i+1}]$ being given by the plane

$$\begin{aligned} H(x,t) \quad := \quad & H'(j/M, t_i) + \tfrac{t-t_i}{t_{i+1}-t_i} \cdot [H'(j/M, t_{i+1}) - H'(j/M, t_i)] \\ & +(Mx-j) \cdot \Big[H'((j+1)/M, t_i) + \tfrac{t-t_i}{t_{i+1}-t_i} \cdot (H'((j+1)/M, t_{i+1}) \\ & -H'((j+1)/M, t_i)) - H'(j/M, t_i) \\ & - \tfrac{t-t_i}{t_{i+1}-t_i} \cdot (H'(j/M, t_{i+1}) - H'(j/M, t_i)) \Big]. \end{aligned} \tag{A.1}$$

Observe that the functions H and H' coincide on the edges of the rectangle $[j/M, (j+1)/M] \times [t_i, t_{i+1}]$. It is easy to see that $H(0,-) = \gamma(-)$, $H(1,-) = \gamma'(-)$, $H(-,0) = Z$ and $H(-,1) = W$ and, by construction, $H(x,-)$ is a line segment on every interval $[t_i, t_{i+1}]$ for $1 \le i \le N-1$. So we only have to prove that the image of the homotopy $H(x,t)$ is contained in X. The rectangle $[j/M, (j+1)/M] \times [\tau_i, \tau_{i+1}]$ lies in an δ-ball around $(j/M, \tau_i)$, and the values are convex combinations of the values of H' on the vertices of the rectangle. Thus all four of those values lie in an ϵ-ball around $H'(j/M, \tau_i)$. Hence the distance between the points of $H([j/M, (j+1)/M] \times [\tau_i, \tau_{i+1}])$ and $H'(j/M, \tau_i)$ is smaller than ϵ_+, which keeps them away from ∂X. This proves the result. ∎

Lemma A.1.7. *Let X, γ, γ', Z and W be given exactly as in Lemma A.1.6. Assume we are additionally given a subset $Y \subseteq X$ with dense complement and the property that a line which is not wholly contained in Y has finite intersection with it. Assume also that the starting and ending point Z and W do not lie in Y. Then we can choose H satisfying all the properties we imposed in Lemma A.1.6, with the addition that there exist $0 = \tau_1 < \tau_2 < \ldots < \tau_N = 1$ such that $H(x,-)|_{[\tau_i, \tau_{i+1}]}$ is a line segment for each $x \in (0,1)$, which is parametrized by equations of degree 1, and at least one of the two points $H(x, \tau_i)$ and $H(x, \tau_{i+1})$ lies outside Y.*

Proof. Let H be the homotopy constructed in Lemma A.1.6. Choose $\epsilon > 0$ smaller than the distance between $H([0,1] \times [0,1])$ and ∂X. We can assume without loss that each $H(j/M, \tau_i)$ for $0 < j < M$ and $1 < i < N$ lies outside Y. Indeed we can replace each $H'(j/M, \tau_i)$ by an arbitrary point in an ϵ-ball around it and we define a new homotopy $\widetilde{H}$ via the same formula (A.1). Clearly we will still have a valid homotopy and now $\widetilde{H}(j/M, \tau_i)$ lies outside Y for $0 < j < M$ and $1 < i < N$. Observe that we have not modified H on $\partial([0,1] \times [0,1])$. Due to the assumptions on Y, we have that each path $H(-, \tau_i)$ intersects Y in at most finitely many points. Therefore we can choose a continuous functions $\varphi_i: [0,1] \longrightarrow [0,1]$ such that

1. Each φ_i is strictly increasing.
2. For all natural numbers $0 \le j \le M$ we have $\varphi_i(j/M) = j/M$ (for each i).
3. For each $x \in (0,1)$ at most one of the values $H(\varphi_i(x), \tau_i)$, for $1 \le i \le N$, lies in Y.

Now define a continuous bijective function $\varphi : [0,1] \times [0,1] \to [0,1] \times [0,1]$ in a piecewise manner, with the values on $[0,1] \times [\tau_i, \tau_{i+1}]$ being given by

$$\varphi(x,t) := \left(\varphi_i(x) + \frac{t - \tau_i}{\tau_{i+1} - \tau_i} \cdot (\varphi_{i+1}(x) - \varphi_i(x)), t\right)$$

Then $H \circ \varphi$ is the desired homotopy. ▮

In the case, we will need the last lemma, we are given a countable number of algebraic surfaces in $\mathbb{R}^n$. The complement of the union of those surfaces is dense in $\mathbb{R}^n$. Therefore we can find, for each $Z \in \mathbb{R}^n$, an $Z' \in \mathbb{R}^n$ arbitrarily close to Z which lies on none of these surfaces. Assume we are given two homotopic paths γ and γ' which are made up from a finite number of line segments with the end-points avoiding the algebraic surfaces (by the preceding considerations we will always be able to find such representatives). Then we can find a homotopy $H(x,t)$ between γ and γ' such that at each point x the path $H(x,-)$ is made up of line segments which all have at most finite intersection with each of the algebraic surfaces. Note that a line segment is either completely contained in an algebraic surface, or it has only finite intersection with it. So only having one of the end points of the line segment lying outside of an algebraic surface guarantees finite intersection.

A.2 Appendix for Section 4.4

Lemma A.2.1. *The elements of $\mathcal{V} \cup \mathcal{V}_\infty$ are path-connected.*

Proof. It is easy to see that the elements of $\mathcal{V}_\infty$ are path-connected. So consider an element $V_{c,d} \in \mathcal{V}$ for $(c,d) \in \mathcal{S}$. Then

$$V_{c,d} = \{(x_1, x_2, r, h) \in \mathbb{R}^2 \times \left(\mathbb{R}^+\right)^2 \mid f_{c,d}(x_1, x_2, r, h) = 1\},$$

where $f_{c,d}(x_1, x_2, r, h)$ is given by (4.17). It is easy to see that for every (x_1, x_2, r) in $\mathbb{R}^2 \times \mathbb{R}^+$, the equation $f_{c,d}(x_1, x_2, r, h) = 1$ has a solution in h which is given by

$$\begin{aligned} h_1(c,d,x_1,x_2,r) = &\sqrt{\frac{1}{N(c)^2} + \frac{1}{4}\left[\left(x_1 + \frac{d}{c}\right)^2 r - \left(x_2 + \frac{d'}{c'}\right)^2 \frac{1}{r}\right]^2} \\ &- \frac{1}{2}\left[\left(x_1 + \frac{d}{c}\right)^2 r + \left(x_2 + \frac{d'}{c'}\right)^2 \frac{1}{r}\right]. \end{aligned}$$

This solution is positive if and only if

$$\begin{aligned} &\sqrt{\frac{1}{N(c)^2} + \frac{1}{4}\left[\left(x_1 + \frac{d}{c}\right)^2 r - \left(x_2 + \frac{d'}{c'}\right)^2 \frac{1}{r}\right]^2} \\ &> \frac{1}{2}\left[\left(x_1 + \frac{d}{c}\right)^2 r + \left(x_2 + \frac{d'}{c'}\right)^2 \frac{1}{r}\right], \end{aligned}$$

which is equivalent with

$$-1 < (cx_1 + d)(c'x_2 + d') < 1.$$

So set

$$\mathcal{C} = \{(x_1, x_2) \in \mathbb{R}^2 \mid -1 < (cx_1 + d)(c'x_2 + d') < 1\}. \tag{A.2}$$

Then we have proved that for every $(x_1, x_2, r) \in \mathcal{C} \times \mathbb{R}^+$, the equation $f_{c,d}(x_1, x_2, r, h) = 1$ has a positive solution in h. Inversely, it is also easy to see that if $(x_1, x_2) \notin \mathcal{C}$, then $f_{c,d}(x_1, x_2, r, h) \geq 1$ and hence $f_{c,d}(x_1, x_2, r, h) = 1$ has no positive solution in h. Thus we can describe $V_{c,d}$ as

$$V_{c,d} = \{(x_1, x_2, r, h_1(c, d, x_1, x_2, r)) \mid (x_1, x_2, r) \in \mathcal{C} \times \mathbb{R}^+\}.$$

As $\mathcal{C}$ is path-connected and h_1 is continuous, the result follows. ∎

In the following lemmas we will make use of a proposition from [BCR98]. For the convenience of the reader, we state the proposition.

Proposition A.2.2. *[BCR98, Proposition 3.3.10] Let $V \subseteq \mathbb{R}^n$ be an algebraic set and x a point of V. The following properties are equivalent.*

(i) The point x is non-singular in dimension d.

(ii) There exists an irreducible component V' of V, with $dim(V') = d$, such that V' is the only irreducible component of V containing x and x is a non-singular point of V'.

(iii) There exists $n - d$ polynomials $f_1, \ldots, f_{n-d}$ among the polynomials defining V and an open neighbourhood U of $x \in \mathbb{R}^n$ for the Euclidean topology, such that $V \cap U = \{x \in \mathbb{R}^n \mid P_i(x) = 0$ for $1 \leq i \leq n - d\}$ and the rank of the Jacobian matrix $\left[\frac{\partial f_j}{\partial x_i}\right]$ is equal to $n - d$.

Lemma A.2.3. *The elements of $\mathcal{V} \cup \mathcal{V}_\infty$ are non-singular irreducible real algebraic varieties of dimension 3. Moreover if two different varieties M_1 and M_2 intersect non-trivially, with $M_1, M_2 \in \mathcal{V} \cup \mathcal{V}_\infty$, then their intersection has local dimension 2 at every point.*

Proof. Consider first the set $V_{c,d}$, for $(c, d) \in \mathcal{S}$ and $c \neq 0$. In fact the set $V_{c,d}$ can be rewritten as $\{(x_1, x_2, r, h) \in \mathbb{H}^2 \times \mathbb{H}^2 \mid f_{c,d}(x_1, x_2, r, h) = 1\}$ where $f_{c,d}$ is the function on 4 variables which has an analytical expression as in (4.17). By (4.23) it is easy to see that $\frac{\partial f_{c,d}}{\partial h}$ only takes strictly positive values on $\mathbb{H}^2 \times \mathbb{H}^2$. Hence by Proposition A.2.2, every point of $V_{c,d}$ is non-singular in dimension 3. Moreover, by the same proposition every point of $V_{c,d}$ is contained in a unique component of $V_{c,d}$. As $V_{c,d}$ is path-connected, by Lemma A.2.1, it is irreducible. It is evident that the elements of $\mathcal{V}_\infty$ are irreducible real algebraic varieties of dimension 3.

For the second part, we use the expression of the functions $f_{c,d}$ in the variables x_1, x_2, y_1, y_2 from (4.15). Now define the map

$$F : \mathbb{R}^4 \to \mathbb{R}^2 : (x_1, x_2, y_1, y_2) \mapsto (f_{c_1,d_1}(x_1, x_2, y_1, y_2), f_{c_2,d_2}(x_1, x_2, y_1, y_2)),$$

for $(c_1, d_1), (c_2, d_2) \in \mathcal{S}$. The intersection of V_{c_1,d_1} and V_{c_2,d_2} may be expressed as the pre-image of the point $(1, 1) \in \mathbb{R}^2$ by the function F. We have four partial derivatives:

$$\begin{aligned}
\frac{\partial f_{c_i,d_i}}{\partial x_1} &= 2c_i(c_i x_1 + d_i)\left[({c_i}' x_2 + d_i')^2 + {c_i}'^2 y_2^2\right], \\
\frac{\partial f_{c_i,d_i}}{\partial x_2} &= 2c_i'({c_i}' x_2 + d_i')\left[(c_i x_1 + d_i)^2 + c_i^2 y_1^2\right], \\
\frac{\partial f_{c_i,d_i}}{\partial y_1} &= 2c_i^2 y_1\left[(c_i' x_2 + d_i')^2 + {c_i}'^2 y_2^2\right], \\
\frac{\partial f_{c_i,d_i}}{\partial y_2} &= 2{c_i}'^2 y_2\left[(c_i x_1 + d_i)^2 + c_i^2 y_1^2\right].
\end{aligned}$$

The sub-matrix $\begin{pmatrix} \frac{\partial f_{c_1,d_1}}{\partial x_1} & \frac{\partial f_{c_1,d_1}}{\partial y_1} \\ \frac{\partial f_{c_2,d_2}}{\partial x_1} & \frac{\partial f_{c_2,d_2}}{\partial y_1} \end{pmatrix}$ of the Jacobian matrix of F has determinant

$$4c_1c_2y_1(c_2d_1 - c_1d_2)\left[(c_1'x_2 + d_1')^2 + {c_1}'^2 y_2^2\right]\left[(c_2'x_2 + d_2')^2 + {c_2}'^2 y_2^2\right],$$

which is 0 if and only if $\frac{d_1}{c_1} = \frac{d_2}{c_2}$. By Lemma 4.4.5, in that case V_{c_1,d_1} equals V_{c_2,d_2}. Thus for different manifolds V_{c_1,d_1} and V_{c_2,d_2}, the determinant of the submatrix above does not vanish in any point of $V_{c_1,d_1} \cap V_{c_2,d_2}$ and hence the Jacobian matrix of the map F is of rank 2 at every point of $V_{c_1,d_1} \cap V_{c_2,d_2}$. Again, by Proposition A.2.2, every point of $V_{c_1,d_1} \cap V_{c_2,d_2}$ is non-singular in dimension 2.

Consider now the case of $M_1 \cap M_2$ with $M_1 = V_{c,d} \in \mathcal{V}$ and $M_2 = V \in \mathcal{V}_\infty$. Then

$$V_{c,d} \cap V = \{(x_1, x_2, r, h) \in \mathbb{R}^2 \times \left(\mathbb{R}^+\right)^2 \mid F_{c,d}(x_1, x_2, r, h) = 1\},$$

where $F_{c,d} : \mathbb{R}^3 \to \mathbb{R}$ with $F_{c,d}(x_1, x_2, r, h) = f_{c,d}(x_1, x_2, r, h)$ with r fixed or x_2 depending on x_1. Again, by (4.23), $\frac{\partial F_{c,d}}{\partial h} \neq 0$ and thus by proposition A.2.2, $V_{c,d} \cap V$ has dimension 2 in every point.

Finally, it is easy to see that $M_1 \cap M_2$ has dimension 2 in every point if $M_1, M_2 \in \mathcal{V}_\infty$.

∎

Lemma A.2.4. *The elements of $\mathcal{M}$ are non-singular irreducible real algebraic varieties of dimension 3 and the intersection of two different elements of $\mathcal{M}$ is of dimension at most 2.*

Proof. The first part follows easily from Lemma A.2.3 and [BCR98, Theorem 2.8.8] and the fact that the action of $\mathrm{PSL}_2(R)$ on $\mathbb{H}^2 \times \mathbb{H}^2$ is a bijective semi-algebraic map. The second part follows trivially from the irreducibility, by [Sha94, Theorem I.6.1]. ∎

Lemma A.2.5. *If $V \in \{V_3^+, V_3^-\}$ and $V_{c,d} \in \mathcal{V}$, then $V \cap V_{c,d}$ is an irreducible variety of dimension 2.*

Proof. We first prove that $V_{c,d} \cap V$ is path-connected. Observe that for every couple $(x_1, x_2) \in \mathcal{C}$, with $\mathcal{C}$ as in (A.2), there exists $h > 0$ such that $f_{c,d}(x_1, x_2, \epsilon_0^{\pm 2}, h) = 1$, with $f_{c,d}$ defined as in (4.17). As $\mathcal{C}$ is path-connected, the result follows. We then prove that $V_{c,d} \cap V$ is a smooth (without singular points) variety of dimension 2. Indeed

$$V_{c,d} \cap V = \{Z = (x_1, x_2, \epsilon_0^{\pm 2}, h) \in \mathbb{H}^2 \times \mathbb{H}^2 \mid f_{c,d}(x_1, x_2, \epsilon_0^{\pm 2}, h) = 1\}.$$

Again, by (4.23), $\frac{\partial f_{c,d}}{\partial h} \neq 0$ and hence by [BCR98, Proposition 3.310], every point of $V_{c,d} \cap V$ is non-singular in dimension 2. Moreover it follows from [BCR98, Proposition 3.3.10], that every point of $V_{c,d} \cap V$ is contained in a unique irreducible component of $V_{c,d} \cap V$. As the variety is path-connected, it has to be irreducible. ∎

Lemma A.2.6. *Let M_1, M_2, M_3 be pairwise different elements of $\mathcal{V} \cup \mathcal{V}_\infty$ with $M_1 \cap M_2 \cap M_3 \neq \emptyset$. Then $M_1 \cap M_2 \cap M_3$ is a real algebraic set of local dimension 1 at every point and with at most one singular point.*

Proof. Suppose that $M_i = V_{c_i,d_i}$ for $i = 1, 2, 3$. Similar, as in the proof of Lemma A.2.3, we apply the implicit function theorem to the function $F : \mathbb{R}^4 \to \mathbb{R}^3$ defined by

$$(x_1, x_2, y_1, y_2) \mapsto (f_{c_1,d_1}(x_1, x_2, y_1, y_2), f_{c_2,d_2}(x_1, x_2, y_1, y_2), f_{c_3,d_3}(x_1, x_2, y_1, y_2)),$$

where f_{c_i,d_i} is defined as in (4.15). So $V_{c_1,d_1} \cap V_{c_2,d_2} \cap V_{c_3,d_3} = F^{-1}(1, 1, 1)$. It is sufficient to prove that the Jacobian matrix of F is of rank 3 except possibly for at most one point. The Jacobian matrix of F has the following form

$$2 \begin{pmatrix} c_1(c_1x_1 + d_1)\alpha_1 & c_2(c_2x_1 + d_2)\alpha_2 & c_3(c_3x_1 + d_3)\alpha_3 \\ c_1'(c_1'x_2 + d_1')\beta_1 & c_2'(c_2'x_2 + d_2')\beta_2 & c_3'(c_3'x_2 + d_3')\beta_3 \\ c_1^2 y_1\alpha_1 & c_2^2 y_1 \alpha_2 & c_3^2 y_1 \alpha_3 \\ c_1'^2 y_2 \beta_1 & c_2'^2 y_2 \beta_2 & c_3'^2 y_2 \beta_3 \end{pmatrix}, \tag{A.3}$$

where $\alpha_i = (c_i'x_2 + d_i')^2 + c_i'^2 y_2^2$ and $\beta_i = (c_i x_1 + d_i)^2 + c_i^2 y_1^2$. Note that $\beta_i = \alpha_i^{-1}$ because $(x_1, x_2, y_1, y_2) \in V_{c_1,d_1} \cap V_{c_2,d_2} \cap V_{c_3,d_3}$. In particular α_i is non-zero for every $i = 1, \ldots, 3$. Let det_{ijk} denote the determinant of (A.3) given by the rows i, j and k. Then

$$det_{123} \quad = \quad (\gamma_1(c_1'x_2 + d_1') + \gamma_2(c_2'x_2 + d_2') + \gamma_3(c_3'x_2 + d_3'))y_1,$$

with $\gamma_i = c_i' c_{i+1} c_{i+2} \alpha_i^{-1} \alpha_{i+1} \alpha_{i+2} (c_{i+1} d_{i+2} - c_{i+2} d_{i+1})$, where the indices are to be interpreted modulo 3. Note that each $\gamma_i \neq 0$ because of Lemma 4.4.5 and the assumption that the three varieties considered are distinct. Also

$$det_{134} = (-\gamma_1 c_1' - \gamma_2 c_2' - \gamma_3 c_3')y_1 y_2.$$

If both determinants are 0, then (because $y_1 \neq 0$ and $y_2 \neq 0$)

$$\begin{cases} \gamma_1 c_1' + \gamma_2 c_2' + \gamma_3 c_3' = 0 \\ \gamma_1 d_1' + \gamma_2 d_2' + \gamma_3 d_3' = 0. \end{cases}$$

Hence the vector $(\gamma_1, \gamma_2, \gamma_3)$ is perpendicular to the vectors (c_1', c_2', c_3') and (d_1', d_2', d_3') and thus

$$(\gamma_1, \gamma_2, \gamma_3) = t((c_1', c_2', c_3') \times (d_1', d_2', d_3'))$$

for some $t \in \mathbb{R}$. So

$$\begin{pmatrix} c_1' c_2 c_3 \alpha_1^{-1} \alpha_2 \alpha_3 (c_2 d_3 - c_3 d_3) \\ c_2' c_3 c_1 \alpha_2^{-1} \alpha_3 \alpha_1 (c_3 d_1 - c_1 d_3) \\ c_3' c_1 c_2 \alpha_3^{-1} \alpha_1 \alpha_2 (c_1 d_2 - c_2 d_1) \end{pmatrix} = t \begin{pmatrix} c_2' d_3' - c_3' d_2' \\ c_3' d_1' - c_1' d_3' \\ c_1' d_2' - c_2' d_1' \end{pmatrix}.$$

Dividing the first coordinate by the second and the third, we get that

$$\alpha_1^{-2} = \mu_2 \alpha_2^{-2} \tag{A.4}$$
$$\alpha_1^{-2} = \mu_3 \alpha_3^{-2}, \tag{A.5}$$

for some μ_2 and μ_3 depending only on c_i and d_i for $1 \leq i \leq 3$. Because α_i^{-1} can be interpreted as c_i^2 times the square of the Euclidean distance from the point (x_1, y_1) to the point $(-\frac{d_i}{c_i}, 0)$, equations (A.4) and (A.5) take the form

$$d\left((x_1, y_1), \left(-\frac{d_1}{c_1}, 0\right)\right) = \lambda_i d\left((x_1, y_1), \left(-\frac{d_i}{c_i}, 0\right)\right)$$

with $\lambda_i = \frac{c_i}{c_1} \sqrt[4]{\mu_i}$ for $i = 2, 3$. Hence, modulo a translation (x_1, y_1) is a solution for (x, y) of the system

$$\begin{cases} x^2 + y^2 = \lambda_2^2((x-a)^2 + y^2) \\ x^2 + y^2 = \lambda_3^2((x-b)^2 + y^2), \end{cases}$$

when a and b are different non-zero real numbers. If $\lambda_i = 1$ for $i = 2$ or $i = 3$, then the corresponding equation represents the bisector (which is a line) of $(0, 0)$ and $(a, 0)$ or of $(0, 0)$ and $(b, 0)$. Otherwise, the equation represents the circle $\mathcal{C}_i$ with centre $\left(-\frac{\lambda_2^2 a}{1-\lambda_2^2}, 0\right)$ and radius $\frac{\lambda_2 a}{1-\lambda_2^2}$, or the same formulas with λ_3 and b (because $y > 0$ and $a \neq b$, the case $\lambda_2 = 1 = \lambda_3$ is impossible). Such two different lines, two different circles or a circle and a line intersect in at most one point having $y > 0$. So, if there are more than two points in $\mathbb{H}^2 \times \mathbb{H}^2$ satisfying the two equations then $\lambda_i \neq 1$ for $i \neq 2, 3$ and the equations represent the same circles. In this case

$$\begin{cases} \frac{\lambda_2^2 a}{1-\lambda_2^2} = \frac{\lambda_3^2 b}{1-\lambda_3^2} \\ \frac{\lambda_2 a}{1-\lambda_2^2} = \frac{\lambda_3 b}{1-\lambda_3^2}. \end{cases}$$

Dividing the first equation by the second we conclude that $\lambda_2 = \lambda_3$ thus $a = b$, a contradiction.

So we have shown that there exists at most one possible couple (x_1, y_1) which can be completed to a point (x_1, x_2, y_1, y_2) in the intersection and such that $det_1 = det_2 = 0$. Therefore, the condition that the rank of the Jacobian matrix in (x_1, x_2, y_1, y_2) is less than 3 determines the coordinates x_1 and y_1. By symmetry, it also determines the coordinates x_2 and y_2. This means that the intersection $V_{c_1,d_1} \cap V_{c_2,d_2} \cap V_{c_3,d_3}$ is a real algebraic variety of dimension at most 1.

Now suppose that at least one of the M_1, M_2 and M_3 is in $\mathcal{V}_\infty$. If M_1, M_2, $M_3 \in \mathcal{V}_\infty$, then it is evident that the dimension is at most 1. If two of the three are in $\mathcal{V}_\infty$, then a similar reasoning as in the proof of Lemma A.2.3 may be applied. Indeed suppose $M_1 = V_{c,d}$ for some $(c, d) \in \mathcal{S}$ and $M_2, M_3 \in \mathcal{V}_\infty$. Then $M_1 \cap M_2 \cap M_3 = F^{-1}(1)$, where $F : \mathbb{R}^2 \to \mathbb{R}$ is given by $f_{c,d}(x_1, x_2, r, h)$ where either r is fixed and x_2 depends on x_1 or x_1 and x_2 are fixed. Exactly as in the proof of Lemma A.2.3, it is easy to see that $\frac{\partial F}{\partial h}$ only takes strictly positive values and hence the result follows by Proposition A.2.2.

So we are left with the case that $M_1, M_2 \in \mathcal{V}$ and $M_3 \in \mathcal{V}_\infty$. First suppose that $M_3 \in \{V_1^\pm, V_2^\pm\}$. Then we proceed in exactly the same way as in the case $M_1, M_2, M_3 \in \mathcal{V}$. Let $F : \mathbb{R}^4 \to \mathbb{R}^3$ be the function defined by

$$(x_1, x_2, y_1, y_2) \mapsto (f_{c_1,d_1}(x_1, x_2, y_1, y_2), f_{c_2,d_2}(x_1, x_2, y_1, y_2), g(x_1, x_2, y_1, y_2)),$$

where $g : \mathbb{R}^4 \to \mathbb{R}$ is the function defining M_3. Now the Jacobian of F is the following

$$\begin{pmatrix} 2c_1(c_1x_1 + d_1)\alpha_1 & 2c_2(c_2x_1 + d_2)\alpha_2 & \theta' \\ 2c_1'(c_1'x_2 + d_1')\beta_1 & 2c_2'(c_2'x_2 + d_2')\beta_2 & -\theta \\ 2c_1^2 y_1\alpha_1 & 2c_2^2 y_1\alpha_2 & 0 \\ 2c_1'^2 y_2\beta_1 & 2c_2'^2 y_2\beta_2 & 0 \end{pmatrix}, \tag{A.6}$$

with $\theta = 1$ or $\theta = \omega$ depending on $M_3 = V_2^\pm$ or $M_3 = V_1^\pm$ respectively. Then, computing the subdeterminants of the Jacobian, we get

$$\begin{aligned} det_{134} &= 4\theta' y_1 y_2 (c_1^2 c_2'^2 \alpha_1\beta_2 - c_1'^2 c_2^2 \alpha_2\beta_1) \\ det_{123} &= 4y_1 \Big[\theta'(c_1' c_2^2 (c_1'x_2 + d_1')\alpha_2\beta_1 - c_1^2 c_2'(c_2'x_2 + d_2')\alpha_1\beta_2) \\ &\qquad +\theta(c_1c_2^2(c_1x_1 + d_1)\alpha_1\alpha_2 - c_2c_1^2(c_2x_1 + d_2)\alpha_1\alpha_2)\Big] \end{aligned}$$

Both subdeterminants are 0 if and only if

$$\begin{aligned} c_1^2 c_2'^2 \alpha_1\beta_2 - c_1'^2 c_2^2 \alpha_2\beta_1 &= 0 \\ \theta' c_1' c_2^2 d_1' \beta_1\alpha_2 - \theta' c_1^2 c_2' d_2' \beta_2\alpha_1 + \theta c_1c_2(c_2d_1 - c_1d_2)\alpha_1\alpha_2 &= 0 \end{aligned}$$

If we set

$$\begin{aligned} \delta_1 &= c_1' c_2^2 \alpha_2\beta_1 \\ \delta_2 &= c_1^2 c_2' \alpha_1\beta_2 \\ \delta_3 &= c_1c_2(c_2d_1 - c_1d_2)\alpha_1\alpha_2 \end{aligned}$$

then $det_{134} = 0 = det_{123}$ if and only if

$$\begin{aligned} c_2'\delta_2 - c_1'\delta_1 + 0 \cdot \delta_3 &= 0 \\ \theta' d_1'\delta_1 - \theta' d_2'\delta_2 + \theta\delta_3 &= 0 \end{aligned}$$

In the same way as in the beginning of the proof we get that

$$(\gamma_1, \gamma_2, \gamma_3) = t((-c_1', c_2', 0) \times (d_1', -d_2', 1))$$

for some constant $t \in \mathbb{R}$ and thus

$$\begin{pmatrix} c_1' c_2^2 \alpha_1 \beta_2 \\ c_1^2 c_2' \alpha_2 \beta_1 \\ c_1 c_2 (c_2 d_1 - c_1 d_2) \beta_1 \beta_2 \end{pmatrix} = t \begin{pmatrix} c_2' \\ c_1' \\ c_1' d_2' - c_2' d_1' \end{pmatrix}.$$

Dividing the third row by the first and the second (observe that this is possible because none of the coefficients is 0), we get

$$\begin{aligned} \beta_1^2 &= \mu_1, \\ \beta_2^2 &= \mu_2, \end{aligned}$$

for some μ_1 and μ_2 constants depending only on c_i and d_i for $1 \leq i \leq 3$ and θ. Hence the point (x_1, y_1) has to be at distance $\frac{\sqrt[4]{\mu_i}}{|c_i|}$ from the point $(\frac{-d_i}{c_i}, 0)$ for $i = 1, 2$. As $\frac{d_1}{c_1} \neq \frac{d_2}{c_2}$ (otherwise $M_1 = M_2$ or $M_1 \cap M_2 = \emptyset$ by Lemma 4.4.5) and as $y_1 > 0$, we get only one possible point (x_1, y_1) satisfying this condition. By symmetry there is also just one possible couple (x_2, y_2) such that the subdeterminants might be 0. This proves the result.

Finally suppose $M_1, M_2 \in \mathcal{V}$ and $M_3 = V_3^{\pm}$. By Lemma A.2.5 $M_i \cap M_3$ is an irreducible variety of dimension 2 for $i = 1, 2$. Hence $M_1 \cap M_2 \cap M_3$ has dimension strictly lower than 2 or $M_1 \cap M_2 \cap M_3 = M_1 \cap M_3$. Suppose by contradiction that the latter is the case. Then, as $M_1 \cap M_3 \subseteq M_2 \cap M_3$ and $M_2 \cap M_3$ is irreducible, $M_1 \cap M_3 = M_2 \cap M_3$. We have that $M_i \cap M_3 = g_{c_i,d_i}^{-1}(1)$, where $g_{c_i,d_i} : \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$g_{c_i,d_i}(x_1, x_2, h) = f_{c_i,d_i}(x_1, x_2, r, h),$$

with r fixed (because of M_3) and f_{c_i,d_i} defined as in (4.17). Define

$$\mathcal{C}_i = \{(x_1, x_2) \in \mathbb{R}^2 \mid -1 < (c_i x_1 + d_i)(c_i' x_2 + d_i') < 1\}.$$

For every $(x_1, x_2) \in \mathcal{C}_i$, there exists a precise $h > 0$ such that $(x_1, x_2, r, h) \in M_1 \cap M_3$. Thus $M_1 \cap M_3 = M_2 \cap M_3$ implies that $\mathcal{C}_1 = \mathcal{C}_2$. This also implies that the closures of $\mathcal{C}_1$ and $\mathcal{C}_2$ are equal. Hence, if we prove that the system

$$\begin{aligned} (c_1 x_1 + d_1)(c_1' x_2 + d_1') &= \mu_1 \\ (c_2 x_1 + d_2)(c_2' x_2 + d_2') &= \mu_2 \end{aligned}$$

with $\mu_i \in \{1, -1\}$, only has finitely many solutions, we have a contradiction. This system is equivalent with

$$\begin{aligned} x_1 &= \frac{\mu_1}{c_1(c_1' x_2 + d_1')} - \frac{d_1}{c_1}, \\ x_1 &= \frac{\mu_2}{c_2(c_2' x_2 + d_2')} - \frac{d_2}{c_2}, \end{aligned}$$

which is then equivalent with

$$\frac{\mu_1}{c_1(c_1' x_2 + d_1')} - \frac{d_1}{c_1} = \frac{\mu_2}{c_2(c_2' x_2 + d_2')} - \frac{d_2}{c_2}.$$

This is again equivalent with an equation of second degree beginning as follows

$$(d_2 c_1 c_1' c_2' - d_1 c_2 c_1' c_2') x^2 + \ldots$$

By Lemma 4.4.5, the coefficient of x^2 is different from 0 and hence the equation has at most 2 solutions for x_2. Hence $\mathcal{C}_1 \neq \mathcal{C}_2$ and thus the result follows. ∎

A.3 Appendix for Lemma 4.6.7

In this section, we prove the second and the fourth case of Lemma 4.6.7. The proof is not very elegant, but is based on rather tedious and brute force computations. So we suggest that only the really motivated and interested reader continues reading this.

A.3.1 Second Case

In the second case, we have to show that the intersection $V_{\gamma_1} \cap V \cap \gamma_1^{-1}(V')$ has dimension at most 1, with $\gamma_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $c \neq 0$ and $V, V' \in \mathcal{V}_\infty$. As $\mathcal{V}_\infty = \{V_1^\pm, V_2^\pm, V_3^\pm\}$, we have to distinguish basically 9 different cases:

(i) $V = V_1^\pm$ and $V' = V_1^\pm$,

(ii) $V = V_2^\pm$ and $V' = V_2^\pm$,

(iii) $V = V_3^\pm$ and $V' = V_3^\pm$,

(iv) $V = V_1^\pm$ and $V' = V_2^\pm$,

(v) $V = V_1^\pm$ and $V' = V_3^\pm$,

(vi) $V = V_2^\pm$ and $V' = V_1^\pm$,

(vii) $V = V_2^\pm$ and $V' = V_3^\pm$,

(viii) $V = V_3^\pm$ and $V' = V_1^\pm$ and

(ix) $V = V_3^\pm$ and $V' = V_2^\pm$.

Case (i)

We first compute $\gamma_1^{-1}(V')$.To simplify the computations, we set

$$\begin{aligned}\alpha_1 &= (cx_1+d)^2+c^2y_1^2, &\text{(A.7)}\\ \alpha_2 &= (c'x_2+d')^2+c'^2y_2^2, &\text{(A.8)}\\ \beta_1 &= (ax_1+b)(cx_1+d)+acy_1^2, &\text{(A.9)}\\ \beta_2 &= (a'x_2+b')(c'x_2+d')+a'c'y_2^2. &\text{(A.10)}\end{aligned}$$

We then have

$$\begin{aligned}\gamma_1^{-1}(V') &= \{\gamma_1^{-1}(Z) \mid Z \in \mathbb{H}^2\times\mathbb{H}^2 \text{ and } Z \in V'\}\\ &= \{Z \in \mathbb{H}^2\times\mathbb{H}^2 \mid \gamma_1(Z)\in V'\}\\ &= \{Z \in \mathbb{H}^2\times\mathbb{H}^2 \mid \frac{\omega'\frac{\beta_1}{\alpha_1}-\omega\frac{\beta_2}{\alpha_2}}{\omega'-\omega} = \pm\frac{1}{2}\},\end{aligned}$$

where the last equality is obtained by (4.9) and (4.5) and (4.6). Observe that here α_i and β_i are defined differently as in the proof of Lemma A.2.6. Under this settings, we get that

$$\gamma_1^{-1}(V') = \{Z \in \mathbb{H}^2\times\mathbb{H}^2 \mid \omega'\beta_1\alpha_2-\omega\beta_2\alpha_1 = \pm\frac{1}{2}(\omega'-\omega)\},$$

where we have used the fact that $\alpha_1\alpha_2=1$. Hence $Z \in S_1\cap S_2\cap S_3 \subseteq V_{\gamma_1}\cap V\cap\gamma_1^{-1}(V')$ if and only if Z satisfies the following three equations:

$$\begin{aligned}\alpha_1\alpha_2 &= 1 &\text{(A.11)}\\ \frac{\omega'x_1-\omega x_2}{\omega'-\omega} &= \pm\frac{1}{2} &\text{(A.12)}\\ \omega'\beta_1\alpha_2-\omega\beta_2\alpha_1 &= \pm\frac{1}{2}(\omega'-\omega), &\text{(A.13)}\end{aligned}$$

where the second one is obtained by (4.9).

To use the implicit function theorem, let $F:\mathbb{R}^4\to\mathbb{R}^3$ with $F(x_1,x_2,y_1,y_2)$ be equal to

$$(\left[(cx_1+d)^2+c^2y_1^2\right]\left[(c'x_2+d')^2+c'^2y_2^2\right], \omega'\beta_1\alpha_2-\omega\beta_2\alpha_1, \frac{\omega'x_1-\omega x_2}{\omega'-\omega}).$$

The Jacobian of F is the following matrix

$$\begin{pmatrix} 2c(cx_1+d)\alpha_2 & \omega'\left[a(cx_1+d)+c(ax_1+b)\right]\alpha_2-2c\omega(cx_1+d)\beta_2 & \frac{\omega'}{\omega'-\omega}\\ 2c'(c'x_2+d')\alpha_1 & 2c'\omega'(c'x_2+d')\beta_1-\omega\left[a'(c'x_2+d')+c'(a'x_2+b')\right]\alpha_1 & \frac{-\omega}{\omega'-\omega}\\ 2c^2y_1\alpha_2 & 2ac\omega'\alpha_2y_1-2c^2\omega\beta_2y_1 & 0\\ 2c'^2y_2\alpha_1 & 2c'^2\omega'\beta_1y_2-2a'c'\omega\alpha_1y_2 & 0\end{pmatrix}.$$

If det_{ijk} denotes the determinant of the submatrix formed by the rows i, j and k, then

$$det_{234} = \frac{-4cc'\omega y_1y_2}{\omega'-\omega}\left[c\omega\alpha_1(c'x_2+d')+c'\omega'\alpha_2(cx_1+d)\right]$$

and

$$det_{124} = \frac{2\omega c' y_2}{\omega' - \omega}\Big[-c'\omega'\left((cx_1+d)^2 - c^2y_1^2\right)\alpha_2 \\ -2c\omega\alpha_1(cx_1+d)(c'x_2+d') - c'\omega'\alpha_1^2\Big].$$

Suppose that $det_{234} = 0 = det_{124}$. Then we have the following

$$c\omega\alpha_1(c'x_2+d') + c'\omega'\alpha_2(cx_1+d) = 0, \qquad \text{(A.14)}$$

$$-c'\omega'\left((cx_1+d)^2 - c^2y_1^2\right)\alpha_2 - 2c\omega\alpha_1(cx_1+d)(c'x_2+d') - c'\omega'\alpha_1^2 = 0. \qquad \text{(A.15)}$$

Note that $\frac{-4cc'\omega y_1 y_2}{\omega'-\omega}$ and $\frac{2\omega c' y_2}{\omega'-\omega}$ are different from 0 as c, ω, y_1 and y_2 are different from 0. Putting (A.14) into (A.15), we get

$$-c'\omega'\left((cx_1+d)^2 - c^2y_1^2\right)\alpha_2 + 2c'\omega'\alpha_2(cx_1+d)^2 - c'\omega'\alpha_1^2 = 0.$$

By multiplying the latter by α_1 and taking (A.11) into account, we get

$$\begin{aligned} & -c'\omega'\left((cx_1+d)^2 - c^2y_1^2\right) + 2c'\omega'(cx_1+d)^2 - c'\omega'\alpha_1^3 = 0 \\ \Leftrightarrow\quad & (cx_1+d)^2 - c^2y_1^2 - 2(cx_1+d)^2 + \alpha_1^3 = 0 \\ \Leftrightarrow\quad & -\alpha_1 + \alpha_1^3 = 0 \\ \Leftrightarrow\quad & \alpha_1 = 1. \end{aligned}$$

By (A.11), $\alpha_2 = 1$ and hence, replacing α_1 and α_2 by 1 in the equations (A.12), (A.13) and (A.14), we are left with the following system

$$\begin{aligned} \alpha_1 &= 1 \\ \alpha_2 &= 1 \\ \frac{\omega' x_1 - \omega x_2}{\omega' - \omega} &= \pm\frac{1}{2}, \\ \omega'\beta_1 - \omega\beta_2 \mp \frac{1}{2}(\omega' - \omega) &= 0, \\ c\omega(c'x_2+d') + c'\omega'(cx_1+d) &= 0 \end{aligned}$$

It is now easy to see that this system has at most one unique solution in (x_1, x_2, y_1, y_2). Hence the rank of the Jacobian of F is 3, except at one possible point and thus, by Proposition A.2.2, $V_{\gamma_1} \cap V \cap \gamma_1^{-1}(V')$ has local dimension 1 at every point, except possibly one single point.

Case (ii)

This case is very similar to case (i). The equations (A.11), (A.12) and (A.13) become

$$\alpha_1\alpha_2 = 1 \qquad \text{(A.16)}$$

$$\frac{x_1 - x_2}{\omega - \omega'} = \pm\frac{1}{2} \qquad \text{(A.17)}$$

$$\beta_1\alpha_2 - \beta_2\alpha_1 = \pm\frac{1}{2}(\omega - \omega')\alpha_1\alpha_2, \qquad \text{(A.18)}$$

where α_i and β_i for $i = 1, 2$ are defined as before. The subdeterminants are also very similar, i.e.

$$det_{234} = \frac{4cc'y_1y_2}{\omega - \omega'} \left[c\alpha_1(c'x_2 + d') + c'\alpha_2(cx_1 + d)\right]$$
$$det_{124} = \frac{-2c'y_2}{\omega - \omega'} \left[-c'\left((cx_1 + d)^2 - c^2y_1^2\right)\alpha_2 - 2c\alpha_1(cx_1 + d)(c'x_2 + d') - c'\alpha_1^2\right].$$

Thus again $det_{234} = 0 = det_{124}$ is impossible and thus $V_{\gamma_1} \cap V \cap \gamma_1^{-1}(V')$ has local dimension at most 1 at every, except at most 1, point.

Case(iii)

For $\gamma_1^{-1}(V')$, we have

$$\begin{aligned} \gamma_1^{-1}(V') &= \{\gamma_1^{-1}(Z) \mid Z \in \mathbb{H}^2 \times \mathbb{H}^2 \text{ and } Z \in V'\} \\ &= \{Z \in \mathbb{H}^2 \times \mathbb{H}^2 \mid \gamma_1(Z) \in V'\} \\ &= \{Z \in \mathbb{H}^2 \times \mathbb{H}^2 \mid \frac{y_1\alpha_2}{y_2\alpha_1} = \epsilon_0^{\pm 2}\}, \end{aligned}$$

where the last equality is obtained by (4.9) and (4.5) and (4.6). Hence $Z \in S_1 \cap S_2 \cap S_3 \subseteq V_{\gamma_1} \cap V \cap \gamma_1^{-1}(V')$ if and only if Z satisfies the following three equations:

$$\alpha_1\alpha_2 = 1 \tag{A.19}$$
$$\alpha_2 y_1 - \epsilon_0^{\pm 2}\alpha_1 y_2 = 0 \tag{A.20}$$
$$\frac{y_1}{y_2} = \epsilon_0^{\pm 2}, \tag{A.21}$$

where α_1 and α_2 are again defined as in (A.7) and (A.8). Again we define a function $F : \mathbb{R}^4 \to \mathbb{R}^3$ by

$$F(x_1, x_2, y_1, y_2) = (\alpha_1\alpha_2, \alpha_2 y_1 - \epsilon_0^{\pm 2}\alpha_1 y_2, \frac{y_1}{y_2}).$$

Then the Jacobian of F equals

$$\begin{pmatrix} 2c(cx_1 + d)\alpha_2 & -2c\epsilon_0^{\pm 2}(cx_1 + d)y_2 & 0 \\ 2c'(c'x_2 + d')\alpha_1 & 2c'(c'x_2 + d')y_1 & 0 \\ 2c^2y_1\alpha_2 & \alpha_2 - 2c^2\epsilon_0^{\pm 2}y_1y_2 & \frac{1}{y_2} \\ 2c'^2y_2\alpha_1 & -\epsilon_0^{\pm 2}\alpha_1 + 2c'^2y_1y_2 & -\frac{y_1}{y_2^2} \end{pmatrix}.$$

Consider det_{123}. It is easy to compute that

$$det_{123} = \frac{4cc'(cx_1 + d)(c'x_2 + d')}{y_2}\left(\epsilon_0^{\pm 2}\alpha_1 y_2 + \alpha_2 y_1\right).$$

Suppose that $det_{123} = 0$. As $c \neq 0$ and $\epsilon_0^{\pm 2}\alpha_1 y_2 + \alpha_2 y_1 > 0$, this is equivalent with $(cx_1+d)(c'x_2+d') = 0$. Hence we get the following conditions on $Z \in V_{\gamma_1} \cap V \cap \gamma_1^{-1}(V')$.

$$\begin{aligned} (cx_1+d)(c'x_2+d') &= 0 && \text{(A.22)} \\ \alpha_1\alpha_2 &= 1 && \text{(A.23)} \\ \alpha_2 y_1 - \epsilon_0^{\pm 2}\alpha_1 y_2 &= 0 && \text{(A.24)} \\ \frac{y_1}{y_2} &= \epsilon_0^{\pm 2} && \text{(A.25)} \end{aligned}$$

Joining (A.25) and (A.24) gives $\alpha_2\epsilon_0^{\pm 2}y_2 - \epsilon_0^{\pm 2}\alpha_1 y_2 = 0$, which is equivalent with $\alpha_1 = \epsilon_0^i\alpha_2$, where $i \in \{0, 4, -4\}$. Togeher with equation (A.23), this gives a condition on the value of α_1 and α_2. Equation (A.22) implies that $x_1 = -\frac{d}{c}$ or $x_2 - \frac{d'}{c'}$. This together with the exact value of α_1 and α_2 and (A.25) gives at most one point which is solution to the equations above. Hence $V_{\gamma_1} \cap V \cap \gamma_1^{-1}(V')$ has dimension at most 1.

Case (iv)

The case is again similar to the cases (i) and (ii). The equations (A.11), (A.12) and (A.13) become

$$\begin{aligned} \alpha_1\alpha_2 &= 1 && \text{(A.26)} \\ \frac{\omega' x_1 - \omega x_2}{\omega' - \omega} &= \pm\frac{1}{2} && \text{(A.27)} \\ \beta_1\alpha_2 - \beta_2\alpha_1 &= \pm\frac{1}{2}(\omega - \omega')\alpha_1\alpha_2, && \text{(A.28)} \end{aligned}$$

where α_i and β_i for $i = 1, 2$ are defined as in (A.7)-(A.10). The subdeterminants are also very similar, i.e.

$$\begin{aligned} det_{234} &= \frac{4\omega cc' y_1 y_2}{\omega' - \omega}\left[c\alpha_1(c'x_2+d') + c'\alpha_2(cx_1+d)\right] \\ det_{124} &= \frac{-2c'y_2}{\omega - \omega'}\Big[-c'\omega\left((cx_1+d)^2 - c^2y_1^2\right)\alpha_2 \\ &\quad -2c\omega\alpha_1(cx_1+d)(c'x_2+d') - c'\omega'\alpha_1^2\Big]. \end{aligned}$$

However the reasoning now goes on a bit differently. Suppose that $det_{234} = 0 = det_{124}$. Taking both equations together and multiplying by α_1 as in case (i), we get that

$$\begin{aligned} & -c'\omega\left((cx_1+d)^2 - c^2y_1^2\right) + 2c'\omega(cx_1+d)^2 - c'\omega'\alpha_1^3 = 0 \\ \Leftrightarrow\quad & \omega(cx_1+d)^2 - \omega c^2y_1^2 - 2\omega(cx_1+d)^2 + \omega'\alpha_1^3 = 0 \\ \Leftrightarrow\quad & -\omega\alpha_1 + \omega'\alpha_1^3 = 0 \\ \Leftrightarrow\quad & \alpha_1(-\omega + \omega'\alpha_1^2) = 0. \end{aligned}$$

As $\omega > 0$ and $\omega' < 0$, the last equation has no solution. Hence $V_{\gamma_1} \cap V \cap \gamma_1^{-1}(V')$ has dimension at most 1 at every point.

Case (v)

The function $F : \mathbb{R}^4 \to \mathbb{R}^3$, describing the intersection in this case, is

$$F(x_1, x_2, y_1, y_2) = \left(\alpha_1\alpha_2, y_1\alpha_2 - \epsilon_0^{\pm 2} y_2\alpha_1, \frac{\omega' x_1 - \omega x_2}{\omega' - \omega}\right)$$

and the equations defining the intersection $V_{\gamma_1} \cap V \cap \gamma_1^{-1}(V')$ are the following

$$\begin{aligned} \alpha_1\alpha_2 &= 1 && \text{(A.29)} \\ y_1\alpha_2 - \epsilon_0^{\pm 2} y_2\alpha_1 &= 0 && \text{(A.30)} \\ \frac{\omega' x_1 - \omega x_2}{\omega' - \omega} &= \pm\frac{1}{2}. && \text{(A.31)} \end{aligned}$$

The Jacobian of F is

$$\begin{pmatrix} 2c(cx_1 + d)\alpha_2 & -2c\epsilon_0^{\pm 2}(cx_1 + d)y_2 & \frac{\omega'}{\omega' - \omega} \\ 2c'(c'x_2 + d')\alpha_1 & 2c'(c'x_2 + d')y_1 & \frac{-\omega}{\omega' - \omega} \\ 2c^2 y_1\alpha_2 & \alpha_2 - 2c^2\epsilon_0^{\pm 2} y_1 y_2 & 0 \\ 2c'^2 y_2\alpha_1 & -\epsilon_0^{\pm 2}\alpha_1 + 2c'^2 y_1 y_2 & 0 \end{pmatrix}.$$

By Lemma A.2.5, $V_{\gamma_1^{-1}} \cap V'$ is irreducible of dimension 2 and hence also $V_{\gamma_1} \cap \gamma_1^{-1}(V')$ is an irreducible algebraic set of dimension 2 in every point. As $V_{\gamma_1} \cap V \cap \gamma_1^{-1}(V')$ is an algebraic set contained in $V_{\gamma_1} \cap \gamma_1^{-1}(V')$, $V_{\gamma_1} \cap V \cap \gamma_1^{-1}(V')$ has dimension strictly lower or $V_{\gamma_1} \cap V \cap \gamma_1^{-1}(V') = V_{\gamma_1} \cap \gamma_1^{-1}(V')$. Thus it remains to prove that $V_{\gamma_1} \cap V \cap \gamma_1^{-1}(V') \neq V_{\gamma_1} \cap \gamma_1^{-1}(V')$. Therefore we show that there exists a point $Z \in V_{\gamma_1} \cap \gamma_1^{-1}(V')$ such that $Z \notin V$. Consider (A.29) and (A.30) with $x_1 = \frac{-d}{c}$ and $x_2 = \frac{-d'}{c'}$. Then the equations become

$$\begin{aligned} c^2 c'^2 y_1^2 y_2^2 &= 1, && \text{(A.32)} \\ y_1 c'^2 y_2^2 - \epsilon_0^{\pm 2} c^2 y_1^2 y_2 &= 0. && \text{(A.33)} \end{aligned}$$

(A.33) is equivalent to $c'^2 y_2 - \epsilon_0^{\pm 2} y_1 c^2 = 0$ and hence $y_1 = \frac{c'^2}{\epsilon_0^{\pm 2} c^2} y_2$. Putting this into (A.32), we get

$$\begin{aligned} & c^2 \frac{c'^4}{\epsilon_0^{\pm 4} c^4} y_2^2 c'^2 y_2^2 = 1 \\ \Rightarrow \quad & c'^6 y_2^4 = \epsilon_0^{\pm 4} c^2 \\ \Rightarrow \quad & y_2 = \frac{\epsilon_0^{\pm 1}\sqrt{|c|}}{\sqrt{|c'^3|}}. \end{aligned}$$

Hence the point $(\frac{-d}{c}, \frac{-d'}{c'}, y_1, y_2)$ with y_1 and y_2 given by the equations above, is contained in $V_{\gamma_1} \cap \gamma_1^{-1}(V')$. This point is a point of V if and only if $-\omega'\frac{d}{c} + \omega\frac{d'}{c'} = \pm\frac{1}{2}(\omega' - \omega)$.

So suppose $-\omega'\frac{d}{c} + \omega\frac{d'}{c'} = \pm\frac{1}{2}(\omega' - \omega)$, then consider again (A.29) and (A.30) with $x_1 = \frac{-d}{\lambda c}$ and $x_2 = \frac{-d'}{\lambda c'}$, with $\lambda > 1$ real such that $\frac{(\lambda-1)^4 d^2 d'^2}{\lambda^4} < 1$. Observe that this is possible as $\frac{(\lambda-1)^4}{\lambda^4}$ is continuous and increasing from 0 to 1 on $\lambda \geq 1$. As $-\omega' \frac{d}{c} + \omega \frac{d'}{c'} = \pm\frac{1}{2}(\omega' - \omega)$, $-\omega' \frac{d}{\lambda c} + \omega \frac{d'}{\lambda c'} \neq \pm\frac{1}{2}(\omega' - \omega)$ and hence $(\frac{-d}{\lambda c}, \frac{-d'}{\lambda c'}, y_1, y_2) \notin V$ for every $y_1, y_2 > 0$. Equations (A.29) and (A.30) become this time

$$\left(\frac{(\lambda-1)^2 d^2}{\lambda^2} + c^2 y_1^2\right)\left(\frac{(\lambda-1)^2 d'^2}{\lambda^2} + c'^2 y_2^2\right) = 1, \tag{A.34}$$

$$y_1\left(\frac{(\lambda-1)^2 d'^2}{\lambda^2} + c'^2 y_2^2\right) - \epsilon_0^{\pm 2} y_2 \left(\frac{(\lambda-1)^2 d^2}{\lambda^2} + c^2 y_1^2\right) = 0. \tag{A.35}$$

Multiplying (A.35) by $\left(\frac{(\lambda-1)^2 d^2}{\lambda^2} + c^2 y_1^2\right)$, we get the following equation

$$y_1 - \epsilon_0^{\pm 2} y_2 \left(\frac{(\lambda-1)^2 d^2}{\lambda^2} + c^2 y_1^2\right)^2 = 0$$
$$\Rightarrow \quad y_2 = \frac{y_1}{\epsilon_0^{\pm 2}\left(\frac{(\lambda-1)^2 d^2}{\lambda^2} + c^2 y_1^2\right)^2}.$$

Combining this with (A.34), we get an equation in y_1

$$\left(\frac{(\lambda-1)^2 d^2}{\lambda^2} + c^2 y_1^2\right)\left(\frac{(\lambda-1)^2 d'^2}{\lambda^2} + \frac{c'^2}{\epsilon_0^{\pm 4}\left(\frac{(\lambda-1)^2 d^2}{\lambda^2} + c^2 y_1^2\right)^4} y_1^2\right) = 1,$$

which is equivalent with

$$\left(\frac{(\lambda-1)^2 d^2}{\lambda^2} + c^2 y_1^2\right)\frac{(\lambda-1)^2 d'^2}{\lambda^2} + \frac{c'^2 y_1^2}{\epsilon_0^{\pm 4}\left(\frac{(\lambda-1)^2 d^2}{\lambda^2} + c^2 y_1^2\right)^3} = 1.$$

The left side of the equation is continuous and in the limit $y_1 = 0$ it equals $\frac{(\lambda-1)^4 d^2 d'^2}{\lambda^4}$, which is strictly smaller than 1 by the choice of λ. In the limit $y_2 \to +\infty$, it goes to $+\infty$. Hence there exists a value for y_1 such that the equation is satisfied. Hence the point $(\frac{-d}{\lambda c}, \frac{-d'}{\lambda c'}, y_1, y_2)$ with y_1 and y_2 given by the equations above, is contained in $V_{\gamma_1} \cap \gamma_1^{-1}(V')$, but not in V.

Case (vi)

This can be deduced from case (iv). Indeed consider $V_{\gamma_1} \cap V \cap \gamma_1^{-1}(V')$ with V and V' described as in case (vi). Then

$$\gamma_1(V_{\gamma_1} \cap V \cap \gamma_1^{-1}(V') = V_{\gamma_1^{-1}} \cap \gamma_1^{-1}(V) \cap V'$$

and the latter has dimension at most 1 by case (iv).

Case (vii)

This case can be treated in a similar way as case (v).

Case (viii)

This can be deduced from case (v) in the same way as case (vi) follows from case (iv).

Case (ix)

This can be deduced from case (vii) in the same way as case (vi) follows from case (iv).

A.3.2 Fourth Case

In the fourth case we still have to prove that $V_{\gamma_1} \cap V \cap \gamma_1^{-1}(V')$ has dimension at most 1 for $V_{\gamma_1} = V_3^{\pm}$, and $V, V' \in \{V_1^{\pm}, V_2^{\pm}\}$. Observe that $V \neq V_3^{\pm}$, because otherwise $V = V_{\gamma_1}$ or V and V_{γ_1} would be disjoint.

We first compute $\gamma_1^{-1}(V')$. As $V_{\gamma_1} = V_3^{\pm}$, $\gamma_1 = \begin{pmatrix} \epsilon_0^m & b \\ 0 & \epsilon_0^{-m} \end{pmatrix}$ for some $m \neq 0$ and $b \in R$. A point Z is a point of $\gamma_1^{-1}(V')$ if and only if

$$\begin{aligned} & \gamma_1(Z) \in V' \\ \Rightarrow \quad & E(x_1, x_2) = \pm\frac{1}{2}(\omega - \omega') \end{aligned}$$

where $E(x_1, x_2)$ is defined by

$$E(x_1, x_2) = \omega' \epsilon_0^{2m} x_1 - \omega \epsilon_0^{-2m} x_2 + \omega' \epsilon_0^m b - \omega \epsilon_0^{-m} b'$$

or

$$E(x_1, x_2) = \epsilon_0^{2m} x_1 - \epsilon_0^{-2m} x_2 + \epsilon_0^m b - \epsilon_0^{-m} b',$$

depending on whether $V' = V_1^{\pm}$ or $V' = V_2^{\pm}$.

Thus the equations defining if a point Z is in the intersection $V_{\gamma_1} \cap V \cap \gamma_1^{-1}(V')$ are the following

$$\frac{y_1}{y_2} = \epsilon_0^{\pm 2}, \tag{A.36}$$

$$G(x_1, x_2) = \pm\frac{1}{2}(\omega - \omega'), \tag{A.37}$$

$$E(x_1, x_2) = \pm\frac{1}{2}(\omega - \omega'), \tag{A.38}$$

where G is given by

$$G(x_1, x_2) = \omega' x_1 - \omega x_2$$

or

$$G(x_1, x_2) = x_1 - x_2,$$

depending on whether $V = V_1^{\pm}$ or $V = V_2^{\pm}$. If we set

$$F(x_1, x_2, y_1, y_2) = (\frac{y_1}{y_2}, G(x_1, x_2), E(x_1, x_2)),$$

then the Jacobian of F is given by

$$\begin{pmatrix} 0 & a'_V & a'_{V'}\epsilon_0^{2m} \\ 0 & -a_V & -a_{V'}\epsilon_0^{-2m} \\ \frac{1}{y_2} & 0 & 0 \\ \frac{-y_1}{y_2^2} & 0 & 0 \end{pmatrix},$$

where $a_V = \omega$ or $a_V = 1$ depending on whether $V = V_1^{\pm}$ or $V = V_2^{\pm}$. The parameter $a_{V'}$ is defined similarly depending on V'. Computing det_{123}, we get

$$det_{123} = \frac{1}{y_2}\left(-a'_V a_{V'}\epsilon_0^{-2m} + a_V a'_{V'}\epsilon_0^{2m}\right).$$

Depending on V and V', $det_{123} = 0$ leads to four different equations:

$$\begin{aligned} -\epsilon_0^{-2m} + \epsilon_0^{2m} &= 0, \\ -\omega\epsilon_0^{-2m} + \omega'\epsilon_0^{2m} &= 0, \\ -\omega'\epsilon_0^{-2m} + \omega\epsilon_0^{2m} &= 0, \\ -\omega\omega'\epsilon_0^{-2m} + \omega\omega'\epsilon_0^{2m} &= 0. \end{aligned}$$

The first and the last equations are verified if and only if $\epsilon_0^{2m} = \epsilon_0^{-2m}$, which is in contradiction with $m \neq 0$. The second and the third equations are impossible because ω and ω' have opposite signs and $\epsilon_0 > 0$. Hence the Jacobian always has rank 3 and thus $V_{\gamma_1} \cap V \cap \gamma_1^{-1}(V')$ has dimension 1 at every point.

This finally completes the proof of Lemma 4.6.7.

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Index