# Tiling billiards in the wind-tree model

Magali Jay

Université d'Aix-Marseille

Conference SMaRT, Luxembourg
18 March 2024

# Plan

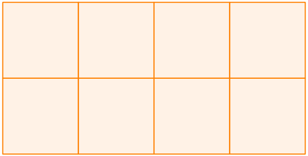
# Some context

# Tiling billiards: Definition

## Definition

*A tiling billiard is a dynamical system whose trajectories are:*

- *in a polygonal tiling,*

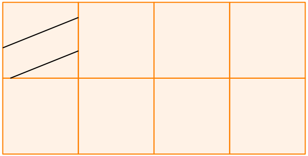


# Tiling billiards: Definition

## Definition

*A tiling billiard is a dynamical system whose trajectories are:*

- *in a polygonal tiling,*
- *in straight line in each tile,*

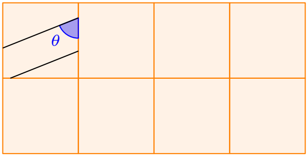


# Tiling billiards: Definition

## Definition

*A tiling billiard is a dynamical system whose trajectories are:*

- *in a polygonal tiling,*
- *in straight line in each tile,*
- *refracting when crossing a side.*

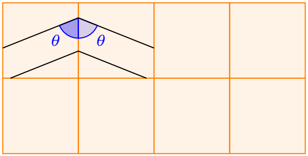


# Tiling billiards: Definition

**Definition**

*A tiling billiard is a dynamical system whose trajectories are:*

- *in a polygonal tiling,*
- *in straight line in each tile,*
- *refracting when crossing a side.*

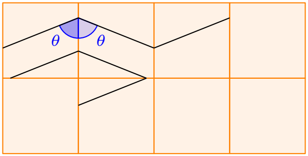


# Tiling billiards: Definition

**Definition**

*A tiling billiard is a dynamical system whose trajectories are:*

- *in a polygonal tiling,*
- *in straight line in each tile,*
- *refracting when crossing a side.*

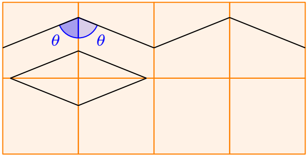


# Tiling billiards: Definition

## Definition

*A tiling billiard is a dynamical system whose trajectories are:*

- *in a polygonal tiling,*
- *in straight line in each tile,*
- *refracting when crossing a side.*



# Tiling billiards: Definition

**Definition**

*A tiling billiard is a dynamical system whose trajectories are:*

- *in a polygonal tiling,*
- *in straight line in each tile,*
- *refracting when crossing a side.*

**Fig. 1.** Photograph of the left-handed metamaterial (LHM) sample. The LHM sample consists of square copper split ring resonators and copper wire strips on fiber glass circuit board material. The rings and wires are on opposite sides of the boards, and the boards have been cut and assembled into an interlocking lattice.

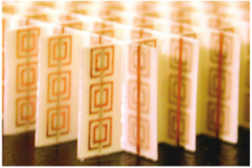


Figure: Metamaterial - From : *Experimental Verification of a Negative Index of Refraction*, R. A. Shelby, D. R. Smith, S. Schultz

# An example: the triangle tiling billiards

**Theorem (Baird-Smith,Davis,Fromm,Iyer - 2018 - and Hubert,Paris-Romaskevich - 2019)**

*For any triangle, for almost every initial direction, the trajectory is either periodic or at bounded distance from a line.*

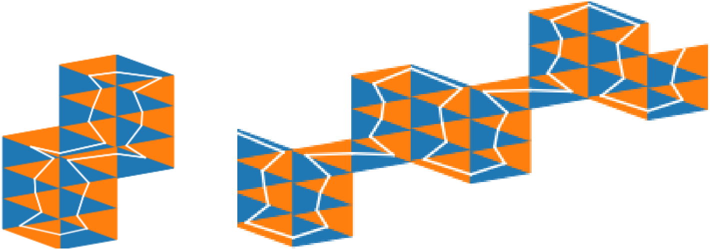


Figure: The two generic types of trajectories

# Wind-tree model: Definition

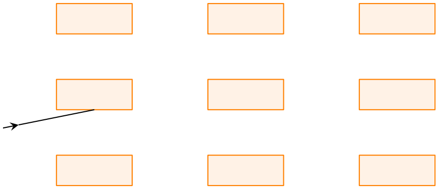


Figure: The wind-tree model

# Wind-tree model: Definition

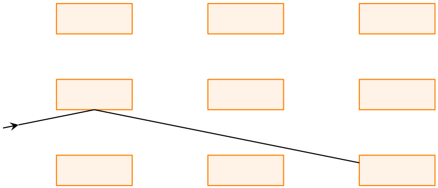


Figure: The wind-tree model

# Wind-tree model: Definition

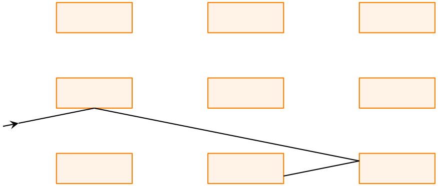


Figure: The wind-tree model

# Wind-tree model: Definition

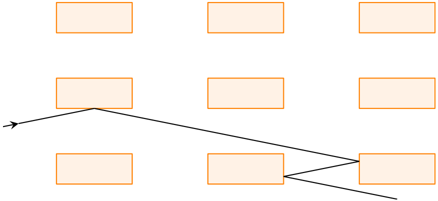


Figure: The wind-tree model

# Wind-tree model: Definition

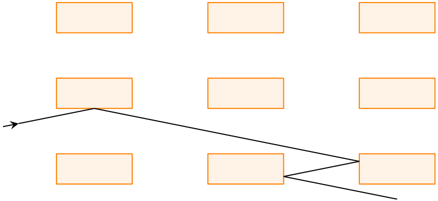


Figure: The wind-tree model

Denote $T(a, b)$ this arrangement: each rectangle, of size $a \times b$, is centered in a point of $\mathbb{Z}^2$. Denote $\varphi_t$ the flow, i.e. $\varphi_t(x, \theta)$ is the point, at time $t$, of trajectory that begins in $x$ with angle $\theta$.

# Wind-tree model: Recurrence and diffusion rate

**Theorem (Avila, Hubert - 2020)**

*For every $(a, b) \in (0, 1)^2$ , for almost every initial direction $\theta$, for every initial point $x$, the trajectory $t \mapsto \varphi_t(x, \theta)$ is recurrent.*

# Wind-tree model: Recurrence and diffusion rate

**Theorem (Avila, Hubert - 2020)**

*For every $(a, b) \in (0, 1)^2$ , for almost every initial direction $\theta$, for every initial point $x$, the trajectory $t \mapsto \varphi_t(x, \theta)$ is recurrent.*

**Theorem (Delecroix, Hubert, Lelièvre - 2017)**

*For every $(a, b) \in (0, 1)^2$ , for almost every initial direction $\theta$, for every initial point $x$ having infinite future orbit, the following holds:*

$$\limsup_{t \to \infty} \frac{\log d(x, \varphi_t(x, \theta))}{\log t} = \frac{2}{3}.$$

# Wind-tree model: Recurrence and diffusion rate

**Theorem (Avila, Hubert - 2020)**

*For every $(a,b) \in (0,1)^2$ , for almost every initial direction $\theta$, for every initial point $x$, the trajectory $t \mapsto \varphi_t(x,\theta)$ is recurrent.*

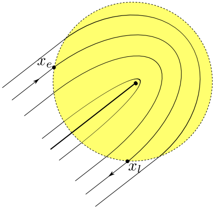
**Theorem (Barazer - 2024)**

*For every $(a,b) \in (0,1)^2$ , for almost every initial direction $\theta$, for every initial point $x$ having infinite future orbit, the following holds:*

$$\lim_{t\to\infty} \frac{\log \frac{1}{t}\int_0^t d(x,\varphi_t(x,\theta))}{\log t} = \frac{2}{3}.$$

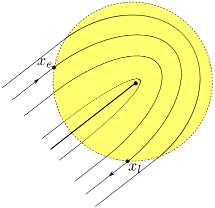
# Eaton Lenses: Definition

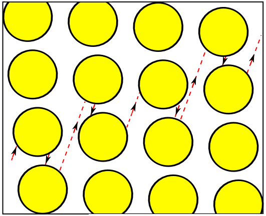
Let $R > 0$.



# Eaton Lenses: Definition

Let $R > 0$. Let $\Lambda$ be a lattice of $\mathbb{R}^2$.

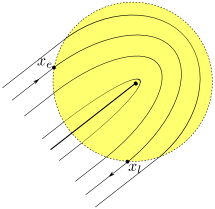


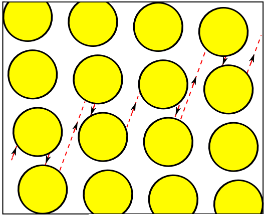



Denote $L(\Lambda, R)$ this system of Eaton lenses: each lens, of radius $R$, is centered in a lattice point.

# Eaton Lenses: Definition

Let $R > 0$. Let $\Lambda$ be a lattice of $\mathbb{R}^2$.



Denote $L(\Lambda, R)$ this system of Eaton lenses: each lens, of radius $R$, is centered in a lattice point.
Say that the pair $(\Lambda, R)$ is *admissible* when the lenses are pairwise disjoint.

# Eaton Lenses: trapped trajectories

**Theorem (Frączek, Schmoll - 2014)**

*For almost every admissible pair* $(\Lambda, R)$ *there exist constants* $C = C(\Lambda, R) > 0$ *and* $\Theta = \Theta(\Lambda, R) \in \mathbb{S}^1$*, such that every vertical light ray in* $L(\Lambda, R)$ *is trapped in an infinite band of width* $C > 0$ *in direction* $\Theta$*.*

# Eaton Lenses: trapped trajectories

**Theorem (Frączek, Schmoll - 2014)**

*For **almost every** admissible pair $(\Lambda, R)$ there exist constants $C = C(\Lambda, R) > 0$ and $\Theta = \Theta(\Lambda, R) \in \mathbb{S}^1$, such that every **vertical** light ray in $L(\Lambda, R)$ is trapped in an infinite band of width $C > 0$ in direction $\Theta$.*

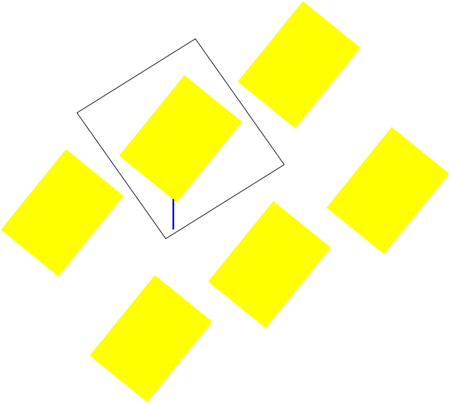
**Theorem (Frączek, Shi, Ulcigrai - 2018)**

*For **every** admissible pair $(\Lambda, R)$, for **almost every direction** $\eta$, there exist constants $C = C(\Lambda, R, \eta) > 0$ and $\Theta = \Theta(\Lambda, R, \eta) \in \mathbb{S}^1$, such that every light ray in direction $\eta$ in $L(\Lambda, R)$ is trapped in an infinite band of width $C > 0$ in direction $\Theta$.*

# Tiling billiard in the wind-tree model

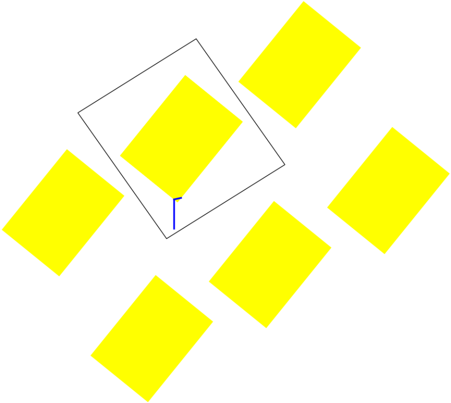
# Setting

Let $\Lambda \subset \mathbb{R}^2$ be a lattice. Let $a, b > 0$. Let $\alpha \in [0, \pi)$



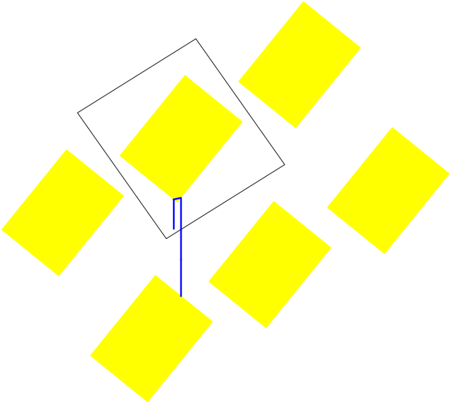
# Setting

Let $\Lambda \subset \mathbb{R}^2$ be a lattice. Let $a, b > 0$. Let $\alpha \in [0, \pi)$



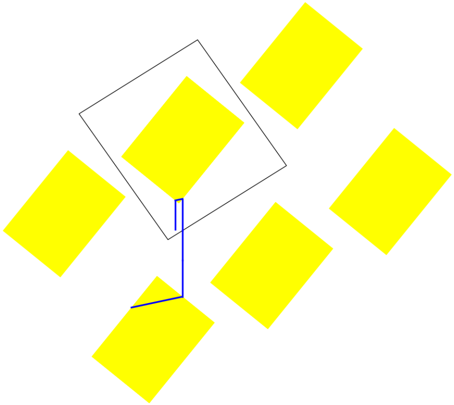
# Setting

Let $\Lambda \subset \mathbb{R}^2$ be a lattice. Let $a, b > 0$. Let $\alpha \in [0, \pi)$



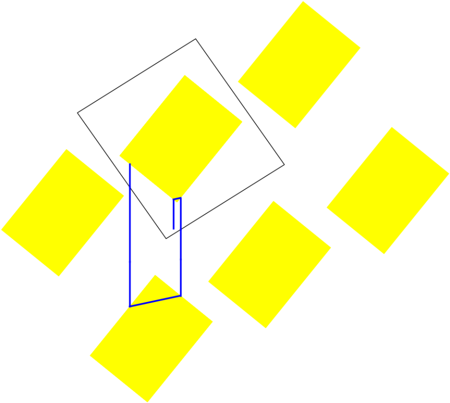
# Setting

Let $\Lambda \subset \mathbb{R}^2$ be a lattice. Let $a, b > 0$. Let $\alpha \in [0, \pi)$



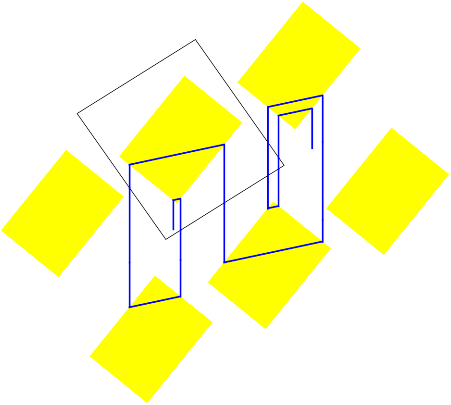
# Setting

Let $\Lambda \subset \mathbb{R}^2$ be a lattice. Let $a, b > 0$. Let $\alpha \in [0, \pi)$



# Setting

Let $\Lambda \subset \mathbb{R}^2$ be a lattice. Let $a, b > 0$. Let $\alpha \in [0, \pi)$



# The result

Denote $W(\Lambda, a, b, \alpha)$ this system: each rectangle, of size $a \times b$, is centered in a lattice point, making an angle $\alpha$ with the horizontal.
Say that the tuple $(\Lambda, a, b, \alpha)$ is *admissible* when the rectangles are pairwise disjoint.

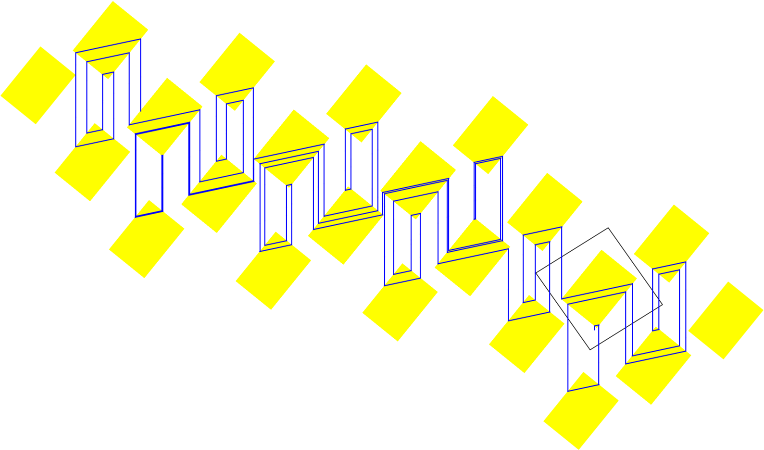
# The result

Denote $W(\Lambda, a, b, \alpha)$ this system: each rectangle, of size $a \times b$, is centered in a lattice point, making an angle $\alpha$ with the horizontal.
Say that the tuple $(\Lambda, a, b, \alpha)$ is *admissible* when the rectangles are pairwise disjoint.

**Theorem (J.+)**

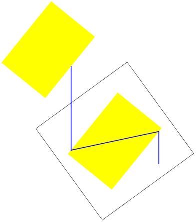
*For almost every admissible tuple $(\Lambda, a, b, \alpha)$, there exist constants $C = C(\Lambda, R, a, b, \alpha) > 0$ and $\Theta = \Theta(\Lambda, R, a, b, \alpha) \in \mathbb{S}^1$, such that every vertical trajectory in $W(\Lambda, a, b, \alpha)$ is trapped in an infinite band of width $C > 0$ in direction $\Theta$.*

# Illustration of the result

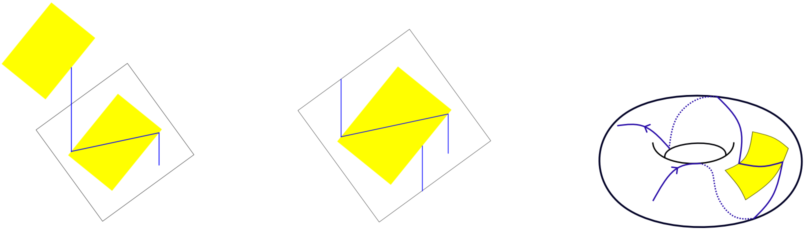


# Sketch of the proof

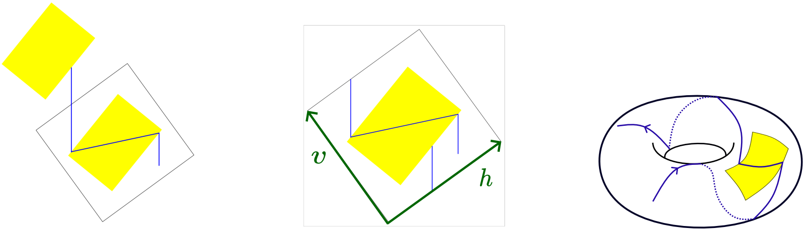
# A corresponding surface



# A corresponding surface

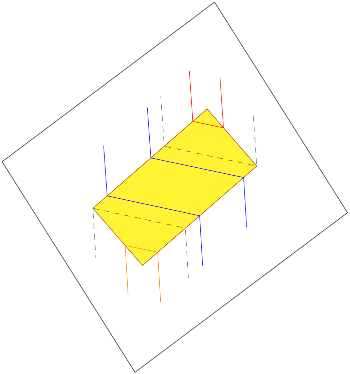


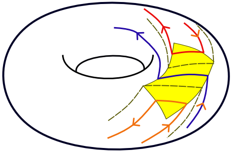
# A corresponding surface



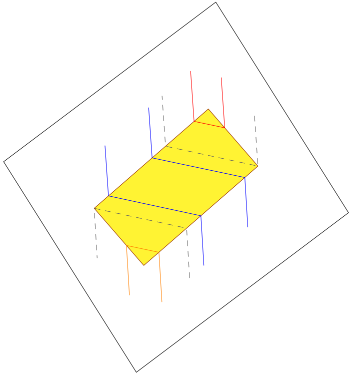

The question is now: how many times does the trajectory intersect the curves h and v?

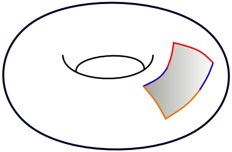
# A corresponding surface



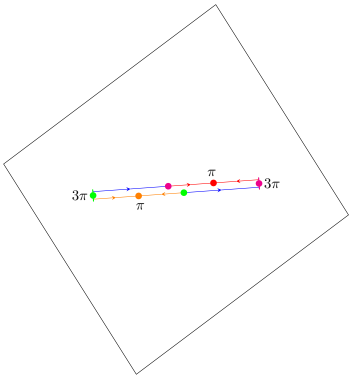


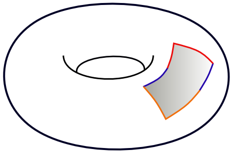
# A corresponding surface



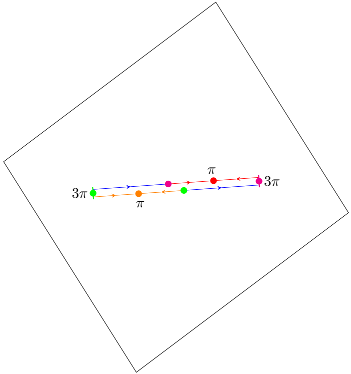


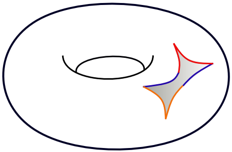
# A corresponding surface



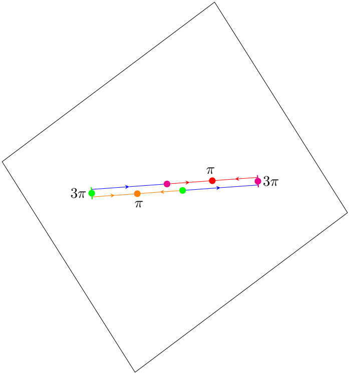



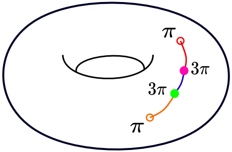
# A corresponding surface



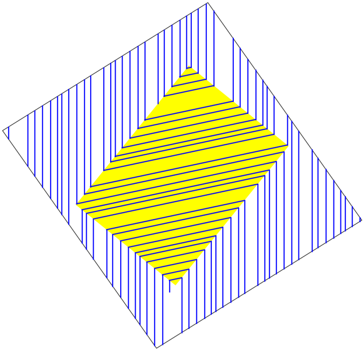
# A corresponding surface

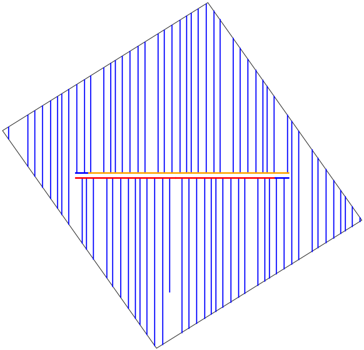


# Trajectories on both surfaces

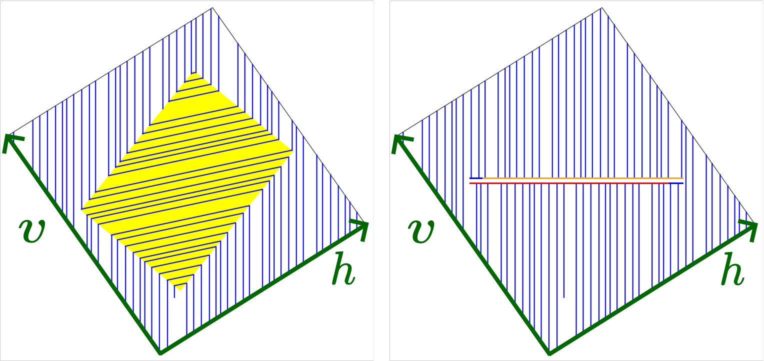
We get the same trajectory on both surfaces.





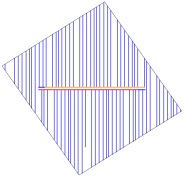
# Trajectories on both surfaces

We get the same trajectory on both surfaces.
**Goal**: Compute the number of intersections between the curve and $h$ (resp. $v$).



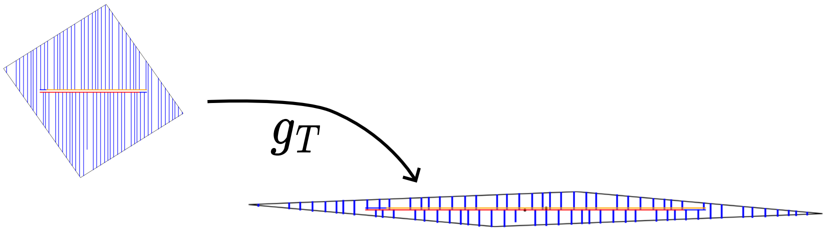
# Teichmüller flow

We renormalize the surface via the Teichmüller flow to get a smaller curve.



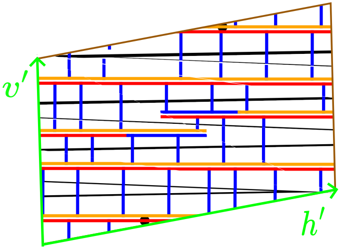
# Teichmüller flow

We renormalize the surface via the Teichmüller flow to get a smaller curve.



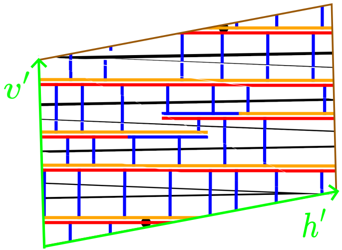
# Teichmüller flow

After cutting and pasting, we get new curves, $h'$ and $v'$, that intersect our trajectory only a few times.



# Teichmüller flow

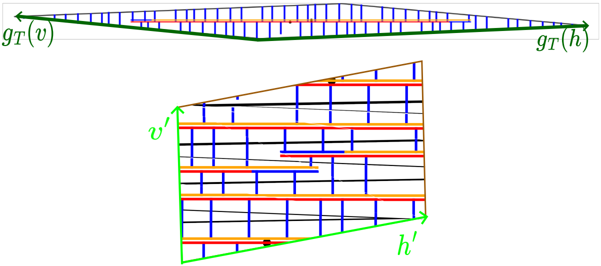
After cutting and pasting, we get new curves, $h'$ and $v'$, that intersect our trajectory only a few times.




**Question**: What is the relation between $h'$, $v'$ and $g_T(h)$, $g_T(v)$?

# Kontsevich-Zorich cocycle

Cutting and pasting corresponds to a change of basis of the homology $H_1(g_T(\Sigma))$ of the surface $g_T(\Sigma)$ from "old" basis $(g_T(h), g_T(v))$ to the "new" basis $(h', v')$.



# Kontsevich-Zorich cocycle

Cutting and pasting corresponds to a change of basis of the homology $H_1(g_T(\Sigma))$ of the surface $g_T(\Sigma)$ from "old" basis $(g_T(h), g_T(v))$ to the "new" basis $(h', v')$. This is given by a discrete version of the Kontsevich-Zorich cocycle.

# Kontsevich-Zorich cocycle

Cutting and pasting corresponds to a change of basis of the homology $H_1(g_T(\Sigma))$ of the surface $g_T(\Sigma)$ from "old" basis $(g_T(h), g_T(v))$ to the "new" basis $(h', v')$. This is given by a discrete version of the Kontsevich-Zorich cocycle.

We denote $A_T$ the transition matrix from the basis $(g_T(h), g_T(v))$ to the basis $(h', v')$, i.e.

$$\begin{cases} g_T(h) & = kh' + mv' \\ g_T(v) & = lh' + nv' \end{cases},$$

where

$$A_T^{-1} = \begin{pmatrix} k & l \\ m & n \end{pmatrix}.$$

We are interested in the growth of $A_T^{-1}$ as $T$ goes to infinity.

# A contracted direction

We can apply Oseledets' theorem to the Kontsevich-Zorich cocycle.

For any generic surface (which is the case for almost every admissible parameters $(\Lambda, a, b, \alpha)$), there exist an homology class $w = xh + yv$ such that $A_T^{-1}$ contracts $\begin{pmatrix} x \\ y \end{pmatrix}$.

# A contracted direction

We can apply Oseledets' theorem to the Kontsevich-Zorich cocycle.

For any generic surface (which is the case for almost every admissible parameters $(\Lambda, a, b, \alpha)$), there exist an homology class $w = xh + yv$ such that $A_T^{-1}$ contracts $\begin{pmatrix} x \\ y \end{pmatrix}$.

The curve $\gamma$ can intersect the curve $xh + yv$ only a bounded number of times.

# A contracted direction

We can apply Oseledets' theorem to the Kontsevich-Zorich cocycle.

For any generic surface (which is the case for almost every admissible parameters $(\Lambda, a, b, \alpha)$), there exist an homology class $w = xh + yv$ such that $A_T^{-1}$ contracts $\begin{pmatrix} x \\ y \end{pmatrix}$.

The curve $\gamma$ can intersect the curve $xh + yv$ only a bounded number of times.

The vector $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ gives the direction of the strip in which the trajectory is trapped.

# Thank you!