Distinguishing filling curve types via special metrics

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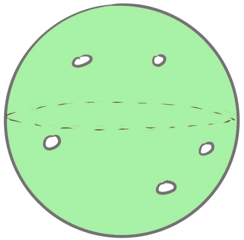
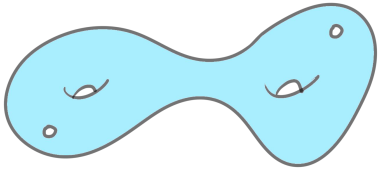
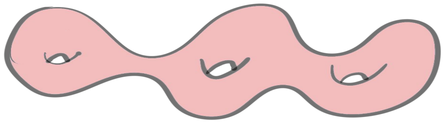
BACKGROUND

Surfaces

Finite type surfaces with negative Euler characteristic
(Topologically, genus g surfaces with n points removed)

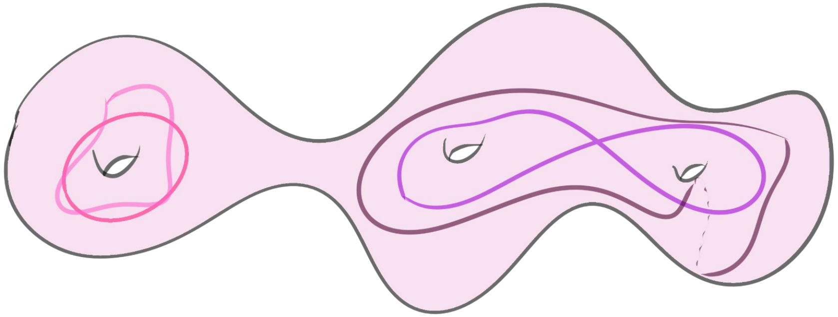






Curves

- We will consider all curves up to homotopy class
- In each homotopy class there is a unique geodesic (shortest one)
- Primitive, essential



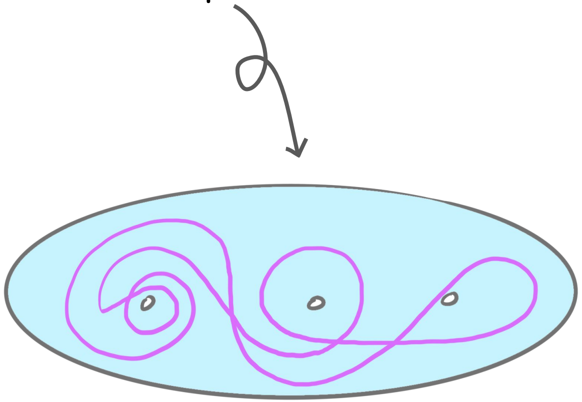
Filling Curves

Definition: A closed curve on a surface is said to be filling if it intersects every essential simple, non-peripheral closed curve on the surface.

Alternative definition:

Complement of a filling curve is a union of discs and annuli

Example:

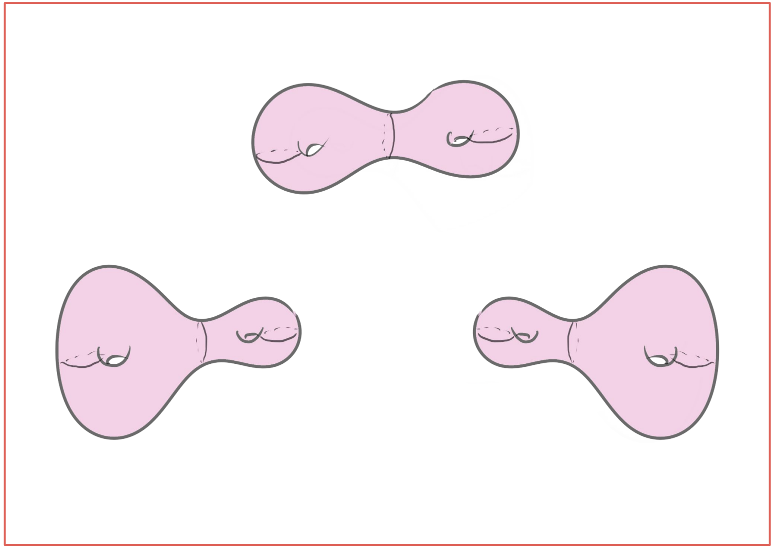


Teichmüller Space

Set of isotopy classes of marked hyperbolic structures.

Each point in Teichmüller space of Σ, can be denoted as (X,f) where X is a surface with complete, finite area hyperbolic structure with geodesic boundary and f is a diffeomorphism from Σ to X.

$(X, f) \sim (Y, g)$ if $f \circ g^{-1}$ is isotopic to an isometry.



3 different points in Teichmüller space

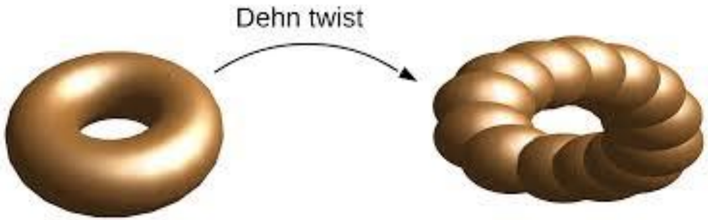
Mapping Class Group

Group of orientation preserving isometries up to isotopy

MCG := $\mathrm{Diffeo}^{+}(\Sigma) / \sim$

$f \sim g$ if $f \circ g^{-1}$ is isotopic to identity

Easy example: Dehn twists



The MCG acts naturally on the Teichmüller space.

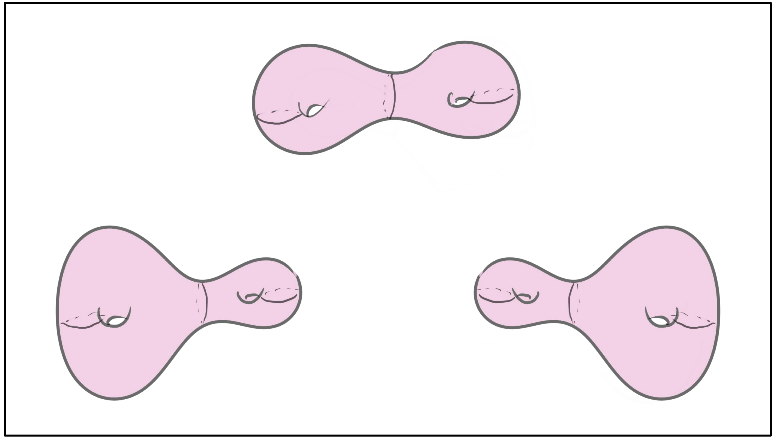
$$g \circ (X,f) \longrightarrow (X, f \circ g^{-1})$$

Acts by "unmarking"

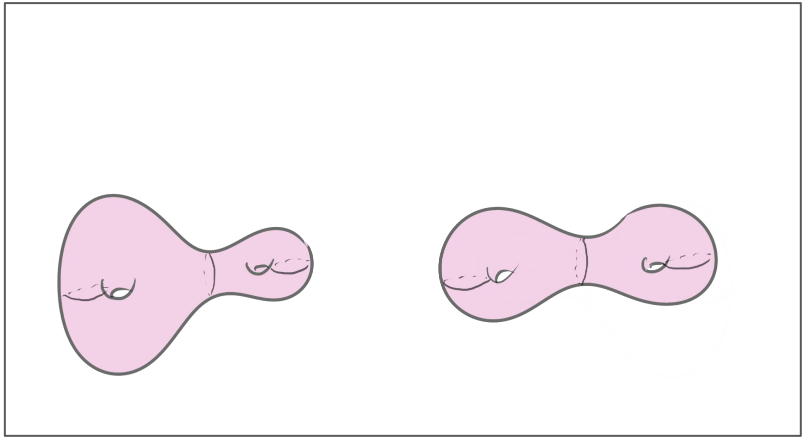
Moduli Space

Moduli space Mg is the quotient of Teichmüller space under the action of MCG.

Two points (X, f) and (X, g) that map to the same point in moduli space differ by the action of the mapping class $g^{-1} \circ f$



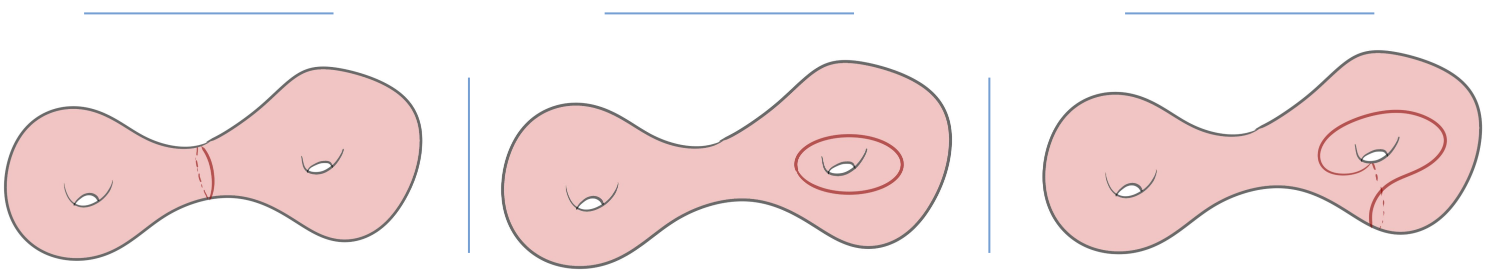
3 different points in Teichmüller space



2 different points in Moduli space

Topological Types

Two curves are said to be of the same topological type if there is a mapping class group element taking one to the other.



Different

Same

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INVARIANTS

Inf invariant

Fix a topological surface Σ and let Teich(Σ) denote its Teichmuller space. Consider a non-simple closed curve γ in Σ.

For (ϕ,X) in(Σ) . Let $\ell_\gamma(X)$ denote the 'X-length' of the geodesic in the free homotopy class of $\phi(\gamma)$.

We define the length infimum of γ as follows:

$$m_\gamma = \inf \{ \ell_\gamma(X) : (\phi, X) \text{ in Teich}(\Sigma) \}$$

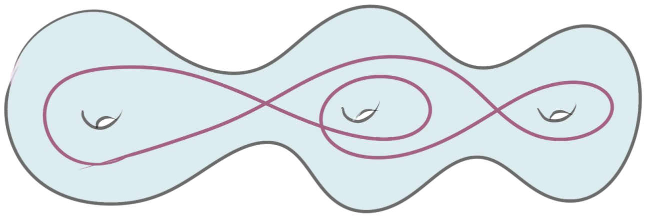
Properties:

- Invariant under action on Mapping Class Group on Teichmuller space.
- The infimum is attained. (Mumford compactness theorem)
- The infimum is unique. (Convexity of Weil-Petersson geodesics)

Other invariants

Self intersection number

If γ non-simple closed curve in **Σ** , then the self intersection number of γ denoted by i(γ ,γ) is the minimum number of self-intersection points of a curve in its free homotopy class in general position.



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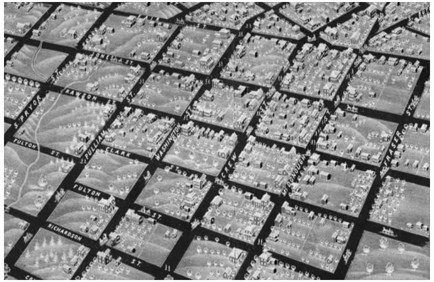
RELATION BETWEEN INVARIANTS

Motivation

Self intersection number vs Length



Want to buy : Self intersections

Currency: Length

- To achieve more intersections we would need longer curves.
- However, we can get more intersection for less length in thin parts of the surface.

Questions:

- Relations between these invariants?
- Is the inf invariant a complete invariant? Are there curves with same inf invariant?
- Can the inf invariant distinguish curves that the self intersection number can't?

Known Results:

- Length equivalent curves. (C. Leininger, 2003)
- Universal length bounds for non-simple closed geodesics on hyperbolic surfaces. (A. Basmajian, H.Parlier, and J.Souto., 2017)
- Length bounds on shortest k-geodesics. (A. Basmajian, 2013)
- Explicit inf length values for a certain of curves (uniform filling curves). (E. Girondo, G. González-Diez, R.A. Hidalgo, 2023)
- For geodesic currents… (J. Sapir, S. Hensel, 2021, 2023)
- Length minima for an infinite family of filling closed curves on a one-holed torus. (Z.Wang and Y.Zhang, 2022)

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CONSTRUCTING FILLING CURVES

Minimal filling curve

Definition: Filling curve with the minimum self intersection number on a given surface.

- For a closed surface of genus g, this curve is unique and has intersection number 2g - 1. (C. Arettines , 2015).
  - The complement is one disk!
  - $m\,\gamma = (4g - 2)\,\mathrm{arccosh}\,(2 \cos[\,\pi/4g{-}2\,] + 1)$
- We extend his construction to surfaces with boundaries (genus g and n boundary components).
  - Intersection number = $2g - 1 + \max\{0\,,, (n - 1)\}$
  - Complement consists of annuli.

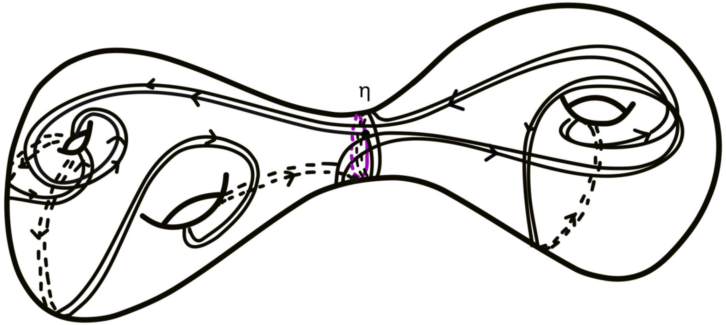
Family 1: Separating Curve Case

Start with a (minimal) filling curve γ_0

Cut at an intersection point with a separating curve.

Given a pair of positive integers (m, n) , let γ be the curve $\eta^m * \gamma_0^n$ (based at the intersection point)

A (2,2) curve.



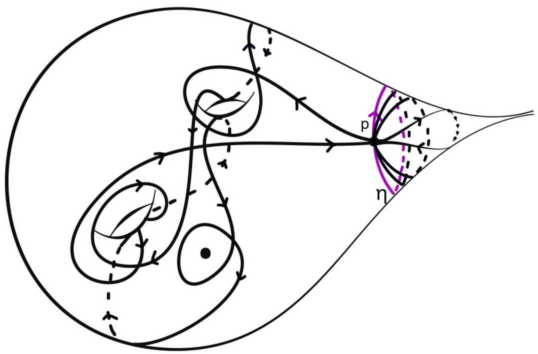
Family 2: Punctured Surface Case

Start with a (minimal) filling curve γ_o, with a subloop homotopic to a puncture.

Start at an intersection point of the subloop.

Given (m, n) , let γ be the curve $\eta^m * \gamma_o^n$ that start at p and goes around γ_o, n times, then around η, m times.

A (2,1) curve.



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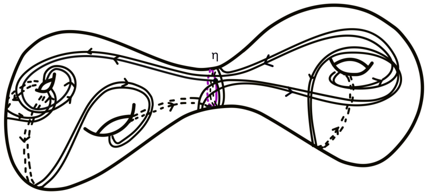
LENGTHS AND INTERSECTION NUMBERS

Self Intersection numbers:

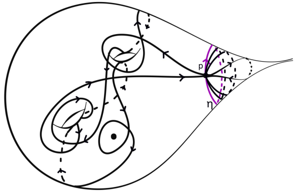
Separating Curve Case:

$$i(\gamma, \gamma) = i(\gamma_0, \gamma_0)n^2 + (i(\gamma_0, \eta)n - 1)m$$



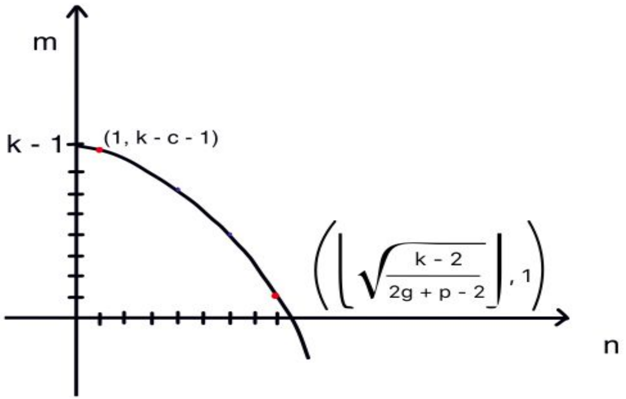
Punctured Surface Case:

$$i(\gamma, \gamma) = i(\gamma_0, \gamma_0)n^2 + m \pm 1$$

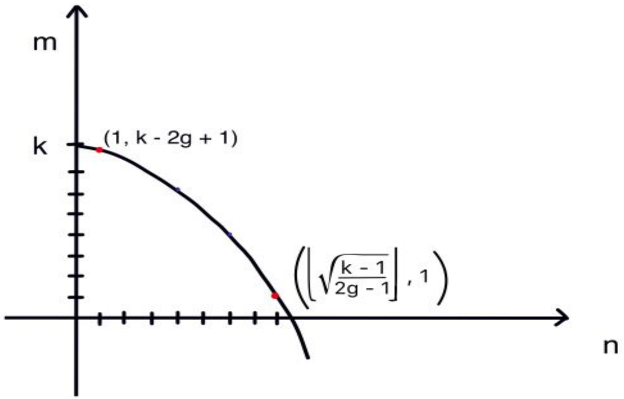


Admissible Pairs

For a fixed k = i(γ, γ) where k >> 2g − 1, there are several choices of curves (pairs of integers(m,n)) in both curve family with k self-intersections. We call these *admissible pairs*.



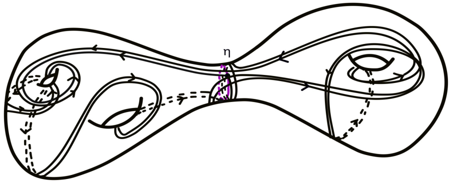
Separating Curve Case



Punctured Surface Case

Coarse length bounds

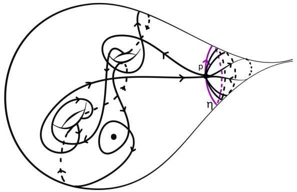
Separating Curve Case:



$$2n\,[d + r(\ell_\eta(X)/2] + m\ell_\eta(X) - C \le \ell_\gamma\,(X) \le 2n\,[D + 2\,r(\ell_\eta(X)/2)) + K\ell_\eta(X)] + m\ell_\eta(X)$$

Constants d, D, and K are independent of n and m, and only depend on γ_0

Punctured Surface Case:



$$nd + 2\log m - c_2 \le \ell_\gamma(X) \le 2\log m + n\ell_{\gamma_0}(X) + c_1$$

where c_1, c_2, d are positive constants that depend only on γ_0

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RESULTS

Theorem 1 (Separating Curve case)

Suppose $\chi(\Sigma) \leq -2$. There exists an infinite set of positive integer K and a collection of curve pairs $\{(\alpha_k, \beta_k)\}$, $k \in K$ such that

(1) α_k and β_k are each filling curves

(2) $i(\alpha_k, \alpha_k) = i(\beta_k, \beta_k) = k$

(3) $m_{\alpha_k} \lesssim \log k < \sqrt{k} \lesssim m_{\beta_k}$

(4) the optimal metrics $\{X_{\beta_k}\}$ are contained in a compact subspace of moduli space.

(5) The metrics $\{X_{\alpha_k}\}$ limit to a stratum S in $\partial\mathcal{M}(\Sigma)$ which correspond corresponds to η being pinched.

Theorem 2 (Punctured Surface case)

Suppose Σ has negative Euler characteristic with genus g and $n \geq 1$ punctures. There exists an infinite set of positive integer K and a collection of curve pairs $\{(\alpha_k, \beta_k)\}$, $k \in K$ so that

(1) α_k and β_k are each filling curves

(2) $i(\alpha_k, \alpha_k) = i(\beta_k, \beta_k) = k$,

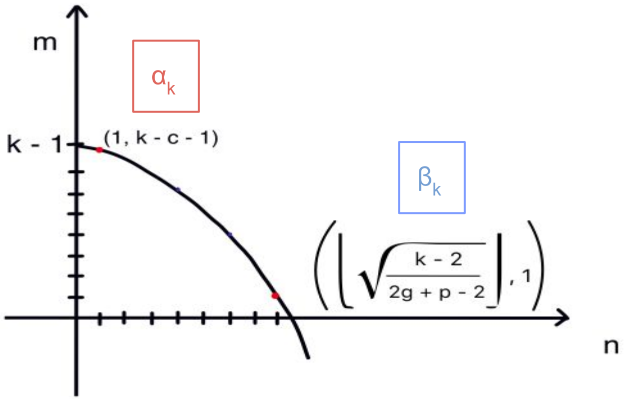
(3) $m_{\alpha_k} \neq m_{\beta_k}$

(4) $m_{\alpha_k} \lesssim 2 \log k < c\sqrt{k} \lesssim m_{\beta_k}$

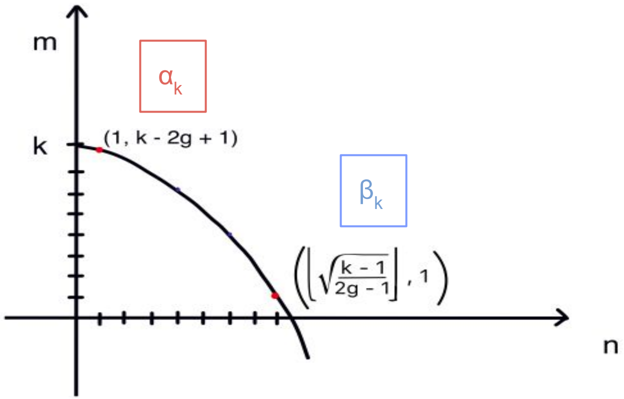
(5) the metrics X_{α_k} and X_{β_k} stay within a compact subspace of $\mathcal{M}(\Sigma)$ for all k.

Admissible Pairs

For a fixed k = i(γ, γ) where k >> 2g − 1, there are several choices of curves (pairs of integers(m,n)) in both curve family with k self-intersections. We call these *admissible pairs*.



Separating Curve Case



Punctured Surface Case

Proof Sketch (Separating Curve case)

$m_{\alpha_k} \leq 2D + 4 \log (m / \log m) + K (\log m / m) + \log m$, where $m = (k - 2g + 1)$

$m_{\beta_k} \geq 2d \lfloor \sqrt{(k-1)/(2g-1)} \rfloor - C$

Thus, $m_{\alpha_k} \lesssim \log k < \sqrt{k} \lesssim m_{\beta_k}$

Proof Sketch (Punctured Surface case)

$m_{\alpha_k} \leq 2\log(k - c - 1) + \ell_{\gamma_0}(X) + c_2$

$m_{\beta_k} \geq d\sqrt{(k-2)/(2g+p-2)} - c_1$

So, $m_{\alpha_k} \lesssim 2\log k < c\sqrt{k} \lesssim m_{\beta_k}$

Questions

- What about the other curves with k self intersections? (In progress)
- Examples of filling curves with same self intersection number and same inf?
- Understanding the inf length spectrum . . .

THANKS!