Distinguishing filling curve types via special metrics

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BACKGROUND
Surfaces

Finite type surfaces with negative Euler characteristic
(Topologically, genus $g$ surfaces with $n$ points removed)
Curves

- We will consider all curves up to homotopy class.
- In each homotopy class there is a unique geodesic (shortest one).
- Primitive, essential.
Filling Curves

Definition: A closed curve on a surface is said to be filling if it intersects every essential simple, non-peripheral closed curve on the surface.

Alternative definition:

Complement of a filling curve is a union of discs and annuli.
Teichmüller Space

Set of isotopy classes of marked hyperbolic structures.

Each point in Teichmüller space of $\Sigma$, can be denoted as $(X, f)$ where $X$ is a surface with complete, finite area hyperbolic structure with geodesic boundary and $f$ is a diffeomorphism from $\Sigma$ to $X$.

$(X, f) \sim (Y, g)$ if $f \circ g^{-1}$ is isotopic to an isometry.
Mapping Class Group

Group of orientation preserving isometries up to isotopy

\[ \text{MCG} := \text{Diffeo}^+(\Sigma) / \sim \]

\[ f \sim g \text{ if } f \circ g^{-1} \text{ is isotopic to identity} \]

The MCG acts naturally on the Teichmüller space.

\[ g \circ (X, f) \rightarrow (X, f \circ g^{-1}) \]

Acts by "unmarking"
Moduli Space

Moduli space $M_g$ is the quotient of Teichmüller space under the action of MCG.

Two points $(X, f)$ and $(X, g)$ that map to the same point in moduli space differ by the action of the mapping class $g^{-1} \circ f$. 

3 different points in Teichmüller space

2 different points in Moduli space
Topological Types

Two curves are said to be of the same topological type if there is a mapping class group element taking one to the other.

Different

Same
Fix a topological surface $\Sigma$ and let $\text{Teich}(\Sigma)$ denote its Teichmüller space. Consider a non-simple closed curve $\gamma$ in $\Sigma$.

For $(\phi, X) \in \Sigma$. Let $l_\gamma(X)$ denote the ‘$X$-length’ of the geodesic in the free homotopy class of $\phi(\gamma)$.

We define the length infimum of $\gamma$ as follows:

$$m_\gamma = \inf \{ l_\gamma(X) : (\phi, X) \in \text{Teich}(\Sigma) \}$$
Properties:

- Invariant under action on Mapping Class Group on Teichmüller space.
- The infimum is attained. *(Mumford compactness theorem)*
- The infimum is unique. *(Convexity of Weil-Petersson geodesics)*
Other invariants

Self intersection number

If $\gamma$ non-simple closed curve in $\Sigma$, then the self intersection number of $\gamma$ denoted by $i(\gamma, \gamma)$ is the minimum number of self-intersection points of a curve in its free homotopy class in general position.
RELATION BETWEEN INVARIANTS
Motivation

Self intersection number vs Length

Want to buy: Self intersections

- To achieve more intersections we would need longer curves.
- However, we can get more intersection for less length in thin parts of the surface.

Currency: Length
Questions:

1. Relations between these invariants?
2. Is the inf invariant a complete invariant? Are there curves with same inf invariant?
3. Can the inf invariant distinguish curves that the self intersection number can’t?
Known Results:

- Length bounds on shortest k-geodesics. (A. Basmajian, 2013)
- Explicit inf length values for a certain of curves (uniform filling curves). (E. Girondo, G. González-Diez, R.A. Hidalgo, 2023)
- For geodesic currents… (J. Sapir, S. Hensel, 2021, 2023)
- Length minima for an infinite family of filling closed curves on a one-holed torus. (Z. Wang and Y. Zhang, 2022)
CONSTRUCTING FILLING CURVES
Minimal filling curve

**Definition:** Filling curve with the minimum self intersection number on a given surface.

- For a closed surface of genus $g$, this curve is unique and has intersection number $2g - 1$. (C. Arettines, 2015).
  - The complement is one disk!
  - $m_y = (4g - 2) \arccosh(2 \cos [\frac{\pi}{4g-2}] + 1)$

- We extend his construction to surfaces with boundaries (genus $g$ and $n$ boundary components).
  - Intersection number = $2g - 1 + \max \{0, (n - 1)\}$
  - Complement consists of annuli.
Family 1: Separating Curve Case

Start with a (minimal) filling curve $\gamma_0$.

Cut at an intersection point with a separating curve.

Given a pair of positive integers $(m, n)$, let $\gamma$ be the curve $\eta^m \ast \gamma_0^n$ (based at the intersection point).

A $(2,2)$ curve.
Family 2: Punctured Surface Case

Start with a (minimal) filling curve $\gamma_0$, with a subloop homotopic to a puncture.

Start at an intersection point of the subloop.

Given $(m, n)$, let $\gamma$ be the curve $\eta^m \ast \gamma_0^n$ that start at $p$ and goes around $\gamma_0$, $n$ times, then around $\eta$, $m$ times.

A (2,1) curve.
LENGTHS AND INTERSECTION NUMBERS
Self Intersection numbers:

Separating Curve Case:

\[ i(\gamma, \gamma) = i(\gamma_0, \gamma_0)n^2 + (i(\gamma_0, \eta)n - 1)m \]

Punctured Surface Case:

\[ i(\gamma, \gamma) = i(\gamma_0, \gamma_0)n^2 + m \pm 1 \]
For a fixed $k = i(\gamma, \gamma)$ where $k >> 2g - 1$, there are several choices of curves (pairs of integers $(m, n)$) in both curve family with $k$ self-intersections. We call these *admissible pairs*.
Coarse length bounds

Separating Curve Case:

\[2n [d + r(\ell(X)/2)] + m\ell(X) - C \leq \ell(X) \leq 2n [D + 2 r(\ell(X)/2)) + K\ell(X)] + m\ell(X)\]

Constants d, D, and K are independent of n and m, and only depend on \(\gamma_0\).

Punctured Surface Case:

\[nd + 2 \log m - c_2 \leq \ell(X) \leq 2 \log m + n\ell_0(X) + c_1\]

where \(c_1, c_2, d\) are positive constants that depend only on \(\gamma_0\).
RESULTS
Theorem 1 (Separating Curve case)

Suppose $\chi(\Sigma) \leq -2$. There exists an infinite set of positive integer $K$ and a collection of curve pairs $\{(\alpha_k, \beta_k)\}, k \in K$ such that

1. $\alpha_k$ and $\beta_k$ are each filling curves
2. $i(\alpha_k, \alpha_k) = i(\beta_k, \beta_k) = k$
3. $m_{\alpha_k} \leq \log k < \sqrt{k} \leq m_{\beta_k}$
4. the optimal metrics $\{X_{\beta_k}\}$ are contained in a compact subspace of moduli space.
5. The metrics $\{X_{\alpha_k}\}$ limit to a stratum $S$ in $\partial M(\Sigma)$ which correspond corresponds to $\eta$ being pinched.
Suppose $\Sigma$ has negative Euler characteristic with genus $g$ and $n \geq 1$ punctures. There exists an infinite set of positive integer $K$ and a collection of curve pairs $\{(\alpha_k, \beta_k)\}, k \in K$ so that

1. $\alpha_k$ and $\beta_k$ are each filling curves
2. $i(\alpha_k, \alpha_k) = i(\beta_k, \beta_k) = k,$
3. $m_{\alpha_k} \neq m_{\beta_k}$
4. $m_{\alpha_k} \leq 2 \log k < c \sqrt{k} \leq m_{\beta_k}$
5. the metrics $X_{\alpha_k}$ and $X_{\beta_k}$ stay within a compact subspace of $\mathcal{M}(\Sigma)$ for all $k.$
For a fixed $k = i(γ, γ)$ where $k >> 2g - 1$, there are several choices of curves (pairs of integers $(m,n)$) in both curve family with $k$ self-intersections. We call these *admissible pairs*.

**Admissible Pairs**

- **Separating Curve Case**
  - $(1, k - c - 1)$
  - $\beta_k$ at $\left(\frac{\sqrt{k - 2}}{2g + p - 2}, 1\right)$

- **Punctured Surface Case**
  - $(1, k - 2g + 1)$
  - $\beta_k$ at $\left(\frac{k - 1}{2g - 1}, 1\right)$
Proof Sketch (Separating Curve case)

\[ m_\alpha_k \leq 2D + 4 \log \left( \frac{m}{\log m} \right) + K \left( \log m / m \right) + \log m, \text{ where } m = (k - 2g + 1) \]

\[ m_{\beta_k} \geq 2d \left\lfloor \sqrt{(k - 1)/(2g - 1)} \right\rfloor - C \]

Thus, \[ m_\alpha_k \leq \log k < \sqrt{k} \leq m_{\beta_k} \]
Proof Sketch  (Punctured Surface case)

\[ m^\alpha_k \leq 2 \log(k - c - 1) + \ell_{r_0} (X) + c_2 \]

\[ m^\beta_k \geq d \sqrt{(k - 2)/(2g + p - 2)} - c_1 \]

So, \( m^\alpha_k \approx 2 \log k < c \sqrt{k} \approx m^\beta_k \)
Questions

- What about the other curves with k self intersections? (In progress)
- Examples of filling curves with same self intersection number and same inf?
- Understanding the inf length spectrum . . .
THANKS!

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