Number of simple and non-simple closed geodesics for random hyperbolic surfaces

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IHES

Surfaces, Manifolds, and Related Topics
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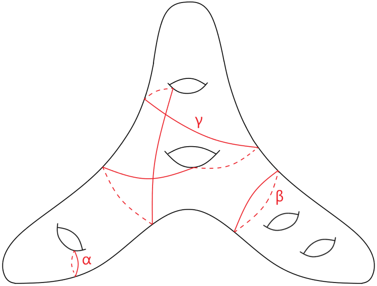
Closed Hyperbolic Surface

Let X_g be a closed hyperbolic surface of genus g (curvature -1).
Aim to count closed geodesics on X_g of different types.

Simple closed geodesic: no self-intersection point.

Non-simple closed geodesic: has self-intersection points.



Remark: A closed geodesic has self-intersection if and only if its homotopy class has self-intersection.

Counting of Closed Geodesics

Geodesics counting:
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The number of all closed geodesics is of exponential growth.

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$$\#\{\gamma \in \mathcal{P}(X_g) \mid \ \ell(\gamma) \leq L\} \sim \frac{e^L}{L} \ as \ L \to \infty.$$

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$$\#\{\gamma \in \mathcal{P}(X_g) \mid \ell(\gamma) \leq L\} \sim \frac{e^L}{L} \text{ as } L \to \infty.$$

The number of simple closed geodesics is of polynomial growth.

Theorem (Mirzakhani 2008)

$$\#\{\gamma \in \mathcal{P}(X_g) \mid \gamma \text{ simple and } \ell(\gamma) \leq L\} \sim C(X_g) L^{6g-6}$$

as $L \to \infty$, where $C(X_g)$ is a constant depending on X_g.

Counting of Closed Geodesics

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$$\#\{\gamma \in \mathcal{P}(X_g) \mid \ \ell(\gamma) \leq L\} \sim \frac{e^L}{L} \textit{ as } L \to \infty.$$

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as $L \to \infty$, *where* $C(X_g)$ *is a constant depending on* X_g.

So most geodesics will eventually be non-simple as $L \to \infty$.

Question: How large L (related to X_g) should be?

Answer

Theorem (Wu-X. 2022)

For random hyperbolic surfaces,

- *most closed geodesics of length* $\ll \sqrt{g}$ *are simple and non-separating,*
- *most closed geodesics of length* $\gg \sqrt{g}$ *are non-simple.*

Here $L(g) \ll \sqrt{g}$ *means* $\lim_{g\to\infty} \frac{L(g)}{\sqrt{g}} = 0$.

This is conjectured by Lipnowski and Wright 2021.
Later, a weaker but broader version of the second part is also proved by Dozier and Sapir 2023 in a totally different method.

Answer

$N(X_g, L)$: all closed geodesics. $N^{ns}(X_g, L)$: non-simple closed geodesics. $N^s_{nsep}(X_g, L)$: simple non-separating closed geodesics.

Theorem (Wu-X. 2022)

- *If* $\lim_{g\to\infty} L(g) = \infty$ *and* $\lim_{g\to\infty} \frac{L(g)}{\sqrt{g}} = 0$, *then there exists a function* $\delta(g) > 0$ *satisfying* $\lim_{g\to\infty} \delta(g) = 0$ *such that*

$$\lim_{g\to\infty} \mathrm{Prob}^g_{\mathrm{WP}}\left(X_g \in \mathcal{M}_g \mid \left|1 - \frac{N^s_{nsep}(X_g, L(g))}{N(X_g, L(g))}\right| < \delta(g)\right) = 1.$$

- *If* $\lim_{g\to\infty} \frac{L(g)}{\sqrt{g}} = \infty$, *then*

$$\lim_{g\to\infty} \mathrm{Prob}^g_{\mathrm{WP}}\left(X_g \in \mathcal{M}_g \mid \left|1 - \frac{N^{ns}(X_g, L(g))}{N(X_g, L(g))}\right| < \frac{g}{L(g)^2}\right) = 1.$$

Answer

Theorem (Wu-X. 2022)

For random hyperbolic surfaces,

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Need to explain more about:

- What is random hyperbolic surfaces
- Counting closed geodesics
- Counting simple closed geodesics

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Random Hyperbolic Surfaces of Weil-Petersson Model

Let $\mathcal{M}_g$ be the moduli space of closed hyperbolic surfaces of genus g. $\mathcal{M}_g$ is a $3g-3$ dimensional complex orbifold, and it admits a Kähler metric of **finite volume** called the Weil-Petersson metric.

Random Hyperbolic Surfaces of Weil-Petersson Model

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The Weil-Petersson metric induces a probability measure $\mathrm{Prob}_{\mathrm{WP}}^{g}$ on $\mathcal{M}_g$.

$$\mathrm{Prob}_{\mathrm{WP}}^{g}(\mathcal{A}) := \frac{1}{\mathrm{Vol}(\mathcal{M}_g)} \int_{\mathcal{M}_g} \mathbf{1}_{\mathcal{A}}(X)\, d\,\mathrm{Vol}_{\mathrm{WP}}(X) = \frac{\mathrm{Vol}(\mathcal{A})}{\mathrm{Vol}(\mathcal{M}_g)}.$$

And the expectation is defined by

$$\mathbb{E}_{\mathrm{WP}}^{g}[f] := \frac{1}{\mathrm{Vol}(\mathcal{M}_g)} \int_{\mathcal{M}_g} f(X)\, d\,\mathrm{Vol}_{\mathrm{WP}}(X).$$

Random Hyperbolic Surfaces of Weil-Petersson Model

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$$\mathbb{E}_{\mathrm{WP}}^{g}[f] := \frac{1}{\mathrm{Vol}(\mathcal{M}_g)} \int_{\mathcal{M}_g} f(X)\, d\,\mathrm{Vol}_{\mathrm{WP}}(X).$$

Say a property P holds **for random hyperbolic surfaces**, or equivalently say P happens **with high probability**, if

$$\lim_{g\to\infty} \mathrm{Prob}_{\mathrm{WP}}^{g}(X_g \in \mathcal{M}_g \mid P \text{ holds for } X_g) = 1.$$

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$$\lim_{g\to\infty} \mathrm{Prob}_{\mathrm{WP}}^{g}(X_g \in \mathcal{M}_g |\ P \text{ holds for } X_g) = 1.$$

e.g. P is:

- most closed geodesics of length $\ll \sqrt{g}$ are simple and non-separating,
- most closed geodesics of length $\gg \sqrt{g}$ are non-simple.

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Prime Geodesic Theorem

Let

$$0 = \lambda_0(X_g) < \lambda_1(X_g) < \lambda_2(X_g) < \cdots \to \infty$$

be the eigenvalues of the Laplacian operator on X_g.

Theorem (Randol, Huber)

For a fixed X_g,

$$\#\{\gamma \mid \ell(\gamma) \leq L\} = \mathrm{Li}(e^L) + \sum_{0<\lambda_j(X_g)<\frac{1}{4}} \mathrm{Li}(e^{s_j L}) + O_{X_g}(\frac{e^{\frac{3}{4}L}}{L})$$

where $\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}$ *as* $x \to \infty$, *and* $s_j = \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_j}$.

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Problem: $O_{X_g}(\cdot)$ depends on X_g. But we want $L = L(g)$ growing along with X_g changes.

Prime Geodesic Theorem

How the remainder depends on X_g?

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Theorem (Wu-X. 2022)

$$\#\{\gamma \mid \ell(\gamma) \leq L\} = \mathrm{Li}(e^L) + \sum_{0<\lambda_j(X_g)<\frac{1}{4}} \mathrm{Li}(e^{s_j L})$$

$$+O\left(g\frac{e^{\frac{5}{6}L}}{L} + g\frac{e^{\frac{2}{3}L}}{L}\max\left\{0, \log\left(\frac{1}{\mathrm{sys}(X_g)}\right)\right\}\right)$$

where $\mathrm{sys}(X_g)$ *is the length of the shortest closed geodesic in* X_g*.*

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Corollary (Wu-X. 2022)

If $e^{e^{\frac{1}{3}L}} > \frac{1}{\mathrm{sys}(X_g)}$*, then there exists a universal constant* $c > 0$ *such that*

$$\#\{\gamma \mid \ \ell(\gamma) \leq L\} \leq c\ g\ \frac{e^L}{L}.$$

Prime Geodesic Theorem for Random Hyperbolic Surfaces

Theorem (Wu-X. 2022)

For any $\epsilon > 0$, for random hyperbolic surfaces,

$$\#\{\gamma \mid \ \ell(\gamma) \leq L\} = \mathrm{Li}(e^L) + O\left(g \cdot e^{(\frac{3}{4}+\epsilon)L}\right).$$

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Theorem (Wu-X. 2022)

For any $L > 12 \log g$,

$$\mathbb{E}_{\mathrm{WP}}^{g}[\#\{\gamma \mid \ell(\gamma) \leq L\}] = \mathrm{Li}(e^L) \cdot \left(1 + O(\frac{1}{g^{0.99}})\right).$$

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Conjecture

For any $\epsilon > 0$*, for random hyperbolic surfaces,*

$$\#\{\gamma \mid \ \ell(\gamma) \leq L\} = \mathrm{Li}(e^L) + O\left(g \cdot e^{(\frac{1}{2}+\epsilon)L}\right).$$

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Simple and Non-simple Geodesics

Recall that as $L \to \infty$,

$$\#\{\gamma \mid \ \ell(\gamma) \leq L\} \sim \frac{e^L}{L},$$

$$\#\{\gamma \mid \ \text{simple and } \ell(\gamma) \leq L\} \sim C(X_g)L^{6g-6}.$$

Question: How large L should be so that most geodesics are non-simple?

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Question: How large L should be so that most geodesics are non-simple?

Surprisingly, if $0 \ll L = o(\sqrt{g})$, as $g \to \infty$,

$$\mathbb{E}_{\mathrm{WP}}^{g}\left[\#\left\{\gamma \,\middle|\, \begin{array}{c} \ell(\gamma) \leq L \\ \text{simple non-separating} \end{array} \right\}\right] = \mathrm{Li}(e^L) \cdot (1 + o(1))$$

$$\sim \ \#\{\gamma \in \mathcal{P}(X_g) \mid \ \ell(\gamma) \leq L\}.$$

Mirzakhani's Integration Formula

The key tool in the study of random hyperbolic surfaces of Weil-Petersson model is **Mirzakhani's integration formula**.

Let γ be a simple closed curve. Consider the orbit under Mod_g-action:

$$\mathcal{O}_\gamma = \{h \cdot \gamma |\ h \in \mathrm{Mod}_g\}.$$

Theorem (Mirzakhani 2007)

For a function $F : \mathbb{R}_{\geq 0} \to \mathbb{R}$

$$\int_{\mathcal{M}_g} \sum_{\alpha \in \mathcal{O}_\gamma} F(\ell_\alpha(X))\, d\,\mathrm{Vol}_{\mathrm{WP}}(X) = C_\gamma \int_{\mathbb{R}_{\geq 0}} F(t) V_g(\gamma; t) t\ dt$$

where the constant $C_\gamma \in (0, 1]$ *only depends on* γ*.*

Remark: In general, Mirzakhani's integration formula holds for simple closed multi-curves $\Gamma = (\gamma_1, \cdots, \gamma_k)$.

Mirzakhani's Integration Formula

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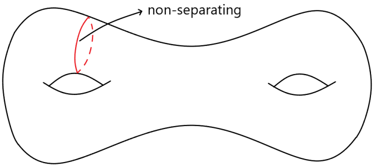
where the constant $C_\gamma \in (0, 1]$ *only depends on* γ.

For example, if γ is a non-separating curve, then

$$C_\gamma = \frac{1}{2}, \qquad V_g(\gamma; t) = V_{g-1,2}(t, t).$$



Simple Geodesics

Take $F(x) = \mathbf{1}_{[0,L]}(x)$, γ oriented primitive simple non-separating, then

$$\mathbb{E}^g_{\mathrm{WP}}\left[\#\left\{\gamma \,\middle|\, \begin{array}{c}\ell(\gamma)\le L\\ \text{simple non-separating}\end{array}\right\}\right] = \int_0^L x\frac{V_{g-1,2}(x,x)}{V_g}dx.$$

where $V_g =$ Weil-Petersson volume of $\mathcal{M}_g$,
$V_{g-1,2}(x,x) =$ Weil-Petersson volume of $\mathcal{M}_{g-1,2}(x,x)$.

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where V_g = Weil-Petersson volume of $\mathcal{M}_g$,
$V_{g-1,2}(x,x)$ = Weil-Petersson volume of $\mathcal{M}_{g-1,2}(x,x)$.

For $0 \ll x = o(\sqrt{g})$,

$$\frac{V_{g-1,2}(x,x)}{V_g} \sim \left(\frac{\sinh(x/2)}{x/2}\right)^2 \sim \frac{e^x}{x^2}.$$

Simple Geodesics of $L \ll \sqrt{g}$

Then for $0 \ll L = o(\sqrt{g})$,

$$\mathbb{E}_{\mathrm{WP}}^{g}\left[\#\left\{\gamma \,\middle|\, \begin{array}{c} \ell(\gamma) \leq L \\ \text{simple non-separating} \end{array}\right\}\right] = \mathrm{Li}(e^{L}) \cdot (1 + o(1)) \,.$$

Recall that for $L > 12 \log g$,

$$\mathbb{E}_{\mathrm{WP}}^{g}\left[\#\left\{\gamma \mid \ell(\gamma) \leq L\right\}\right] = \mathrm{Li}(e^{L}) \cdot \left(1 + O\left(\frac{1}{g^{0.99}}\right)\right) .$$

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$$\Rightarrow \ \mathbb{E}_{\mathrm{WP}}^{g}\left[\#\left\{\begin{array}{c}\text{non-simple} \\ \text{or separating}\end{array} \leq L\right\}\right] \ll \ \mathrm{Li}(e^{L}) \ \approx \ \#\left\{\gamma \mid \ell(\gamma) \leq L\right\}$$

$\Rightarrow$ On random hyperbolic surfaces, if length $L \ll \sqrt{g}$, then most closed geodesics are simple non-separating.

Simple Geodesics of $L \gg \sqrt{g}$

If $\frac{L}{\sqrt{g}} \to \infty$, aim to show

$$\mathbb{E}_{\mathrm{WP}}^{g}\left[\#\left\{\begin{array}{c}\text{simple}\\\text{non-separating}\end{array} \leq L\right\} + \#\left\{\begin{array}{c}\text{simple}\\\text{separating}\end{array} \leq L\right\}\right] \ll \mathrm{Li}(e^{L}).$$

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Recall:

$$\mathbb{E}^g_{\mathrm{WP}}\left[\#\left\{\begin{array}{c}\text{simple}\\ \text{non-separating}\end{array} \le L\right\}\right] = \int_0^L x\frac{V_{g-1,2}(x,x)}{V_g}dx.$$

Need an estimation on the Weil-Petersson volume of large boundary length.

Weil-Petersson Volume of Large Boundary Length

It is known that

$$\left(1 - O(\frac{x^2}{g})\right)\left(\frac{\sinh(x/2)}{x/2}\right)^2 \leq \frac{V_{g,2}(x,x)}{V_{g,2}} \leq \left(\frac{\sinh(x/2)}{x/2}\right)^2.$$

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Theorem (Wu-X. 2022)

If $x^2 > g$, then for any integer $k \geq 1$ exist $c(k) > 0$ such that

$$\frac{V_{g,2}(x,x)}{V_{g,2}} \leq c(k)(\frac{g}{x^2})^k \left(\frac{\sinh(x/2)}{x/2}\right)^2.$$

Remark: In general, similar result for $V_{g,n}(x_1, \cdots, x_n)$.

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Remark: In general, similar result for $V_{g,n}(x_1, \cdots, x_n)$.
Then for $L \gg \sqrt{g}$,

$$\begin{aligned}\mathbb{E}^g_{\rm WP}\left[\#\left\{\begin{matrix}\text{simple}\\ \text{non-separating}\end{matrix} \leq L\right\}\right] &\leq c(k)(\frac{g}{L^2})^k \,\mathrm{Li}(e^L)\\ &\ll \mathbb{E}^g_{\rm WP}\left[\#\left\{\ell(\gamma) \leq L\right\}\right].\end{aligned}$$

Weil-Petersson Volume and Intersection Number

$$V_{g,2}(x,x) = \sum_{d_1+d_2=0}^{3g-1} [\tau_{d_1}\tau_{d_2}]_{g,2} \frac{(x/2)^{2(d_1+d_2)}}{(2d_1+1)!(2d_2+1)!}.$$

Essential to study the intersection numbers $[\tau_{d_1}\tau_{d_2}]_{g,2}$.

$$\text{Mirzakhani:} \quad 1 - O(\tfrac{(d_1+d_2)^2}{g}) \leq \frac{[\tau_{d_1}\tau_{d_2}]_{g,2}}{V_{g,2}} \leq 1.$$

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Theorem (Wu-X. 2022)

If $(d_1+d_2)^2 > g$, then for any integer $k \geq 1$ exist $c(k) > 0$ such that

$$\frac{[\tau_{d_1}\tau_{d_2}]_{g,2}}{V_{g,2}} \leq c(k)\left(\frac{g}{(d_1+d_2)^2}\right)^k.$$

Remark: In general, similar result for $[\tau_{d_1}\cdots\tau_{d_n}]_{g,n}$.

This is why $\sqrt{g}$ is the critical point.

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Further Questions

Question

What happen for random hyperbolic surfaces with cusps and boundaries?

Need to study (for large number of cusps/boundaries):

prime geodesic theorem

Weil-Petersson volume

$\cdots$

Further Questions

Question

How many simple closed geodesics for $L \gg \sqrt{g}$*?*

How does the counting decrease from $\frac{e^L}{L}$ *to polynomial growth?*

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How many simple closed geodesics for $L \gg \sqrt{g}$?

How does the counting decrease from $\frac{e^L}{L}$ to polynomial growth?

Known:

$$\mathbb{E}_{\mathrm{WP}}^{g}\left[\#\left\{\begin{matrix}\text{simple}\\ \text{non-separating}\end{matrix} \leq L\right\}\right] \sim \frac{e^L}{L} \qquad \text{for } L \ll \sqrt{g}$$

$$\mathbb{E}_{\mathrm{WP}}^{g}\left[\#\left\{\begin{matrix}\text{simple}\\ \text{non-separating}\end{matrix} \leq L\right\}\right] \leq c(k)\left(\frac{g}{L^2}\right)^k \frac{e^L}{L} \qquad \text{for } L \gg \sqrt{g}$$

$$e^{0.034L} \leq \mathbb{E}_{\mathrm{WP}}^{g}\left[\#\left\{\begin{matrix}\text{simple}\\ \text{non-separating}\end{matrix} \leq L\right\}\right] \leq e^{0.037L} \qquad \text{for } L = 1000g.$$

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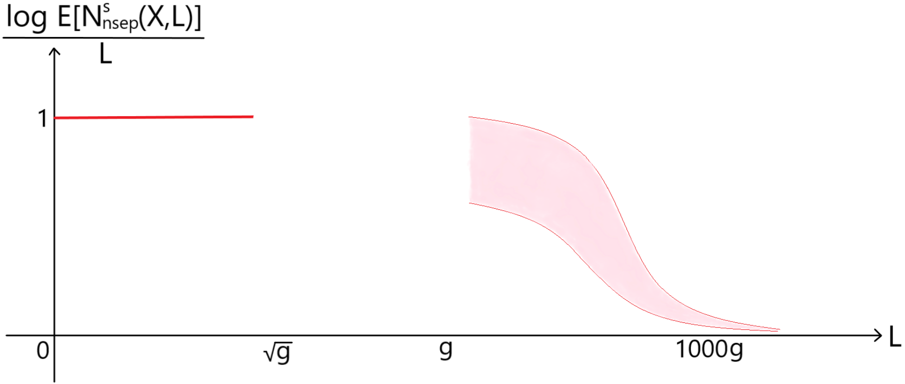

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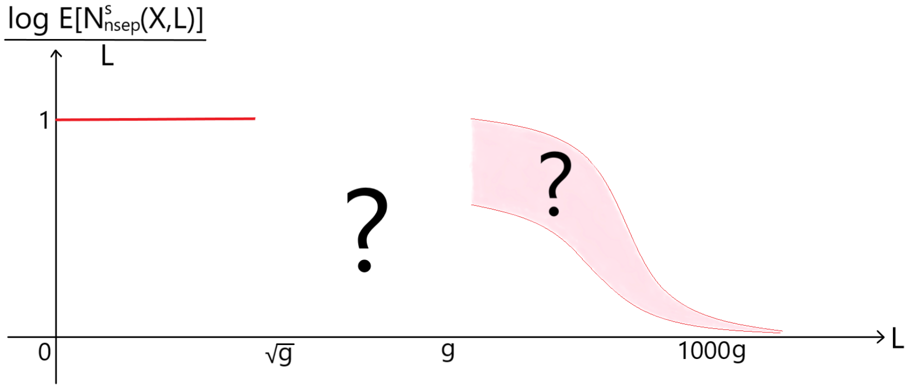

Thank You!

1. Yunhui Wu, Yuhao Xue: **Prime geodesic theorem and closed geodesics for large genus**, 62 pages, arXiv:2209.10415.