A new perspective on Teichmüller theory

joint work with Jean-Marc Schlenker

Mingkun LIU

University of Luxembourg

December 20, 2022

Classical Teichmüller theory

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The *classical Teichmüller theory* studies *nice* maps

$$\pi_1(S) \longrightarrow \mathrm{PSL}(2,\mathbb{R})$$

where $\pi_1(S)$ is the fundamental group of S, and

$$\mathrm{PSL}(2,\mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R},\ ad-bc=1 \right\} \Big/ \{\pm \mathrm{Id}\}$$

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Edited by:

Athanase Papadopoulos: *Institut de Recherche Mathématique Avancée, Strasbourg, France*

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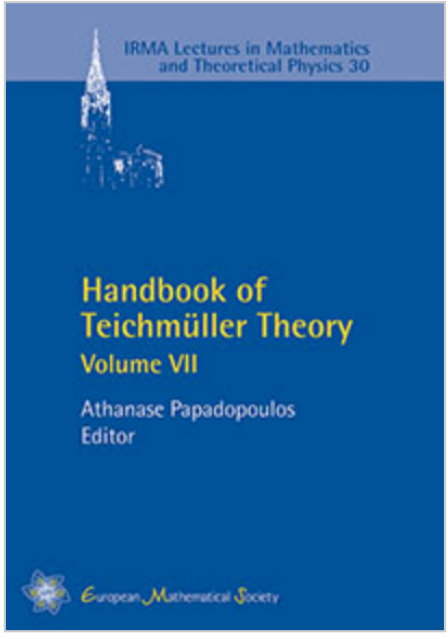

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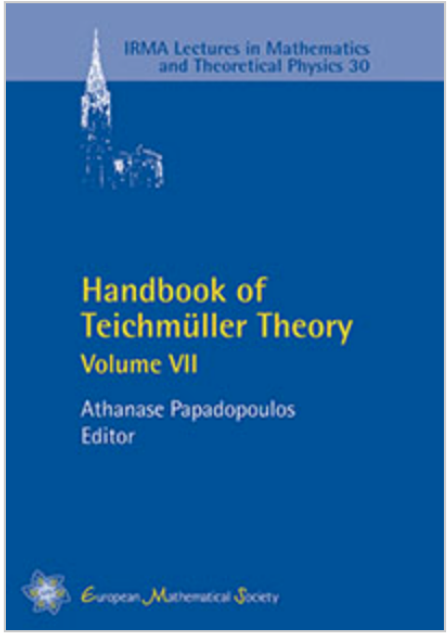

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Contents
Current events
Random article
About Wikipedia
Contact us
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Handbook

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This article is about the type of reference work. For the subnotebook computer, see Gateway Handbook.
"Pocket reference" redirects here. For the all-purpose reference work by Thomas J. Glover, see Pocket Ref.
"Vademecum" redirects here. For other uses, see Vademecum (disambiguation).

A **handbook** is a type of reference work, or other collection of instructions, that is intended to provide ready reference. The term originally applied to a small or portable book containing information useful for its owner, but the *Oxford English Dictionary* defines the current sense as "any book ... giving information such as facts on a particular subject, guidance in some art or occupation, instructions for operating a machine, or information for tourists."[1]

A handbook is sometimes referred to as a ***vade mecum*** (Latin, "go with me") or **pocket reference**. It may also be referred to as an enchiridion.

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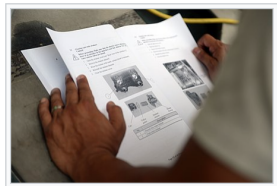

A technician referring to an instructional handbook for the operation of a machine

See also [edit]

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Main page
Contents
Current events
Random article
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Handbook

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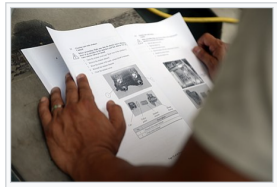

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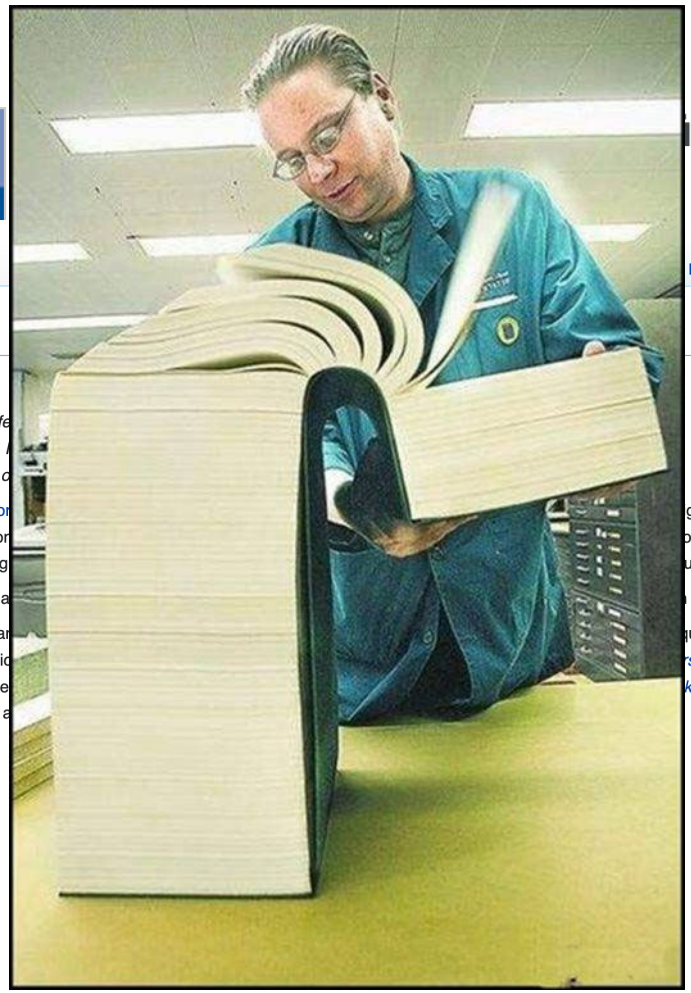

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Contents
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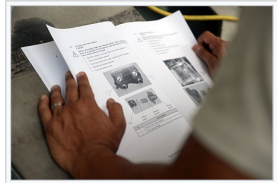

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> Generality is an idol before whom mathematicians torture themselves.
>
> Arthur Eddington

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Lower Teichmüller theory

Lower Teichmüller theory, or Schlenker theory

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that we shall call *Schlenkerian representations*.

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Therefore, we obtain the following complete classification of Schlenkerian representations.

Theorem (Riemann–Schlenker uniformization theorem)

All Schlenkerian representations are isomorphic.

Work in progress

Work in progress: Lowest Teichmüller theory

Lowest Teichmüller theory, or Schlenkest theory/program

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> And if you gaze long into an abyss, the abyss also gazes into you.
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> Friedrich Nietzsche

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Study *good* representations in the form

$$\pi_1(S) \longrightarrow \mathrm{PSL}(☺, \mathbb{R}).$$

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Voilà voilà ☺