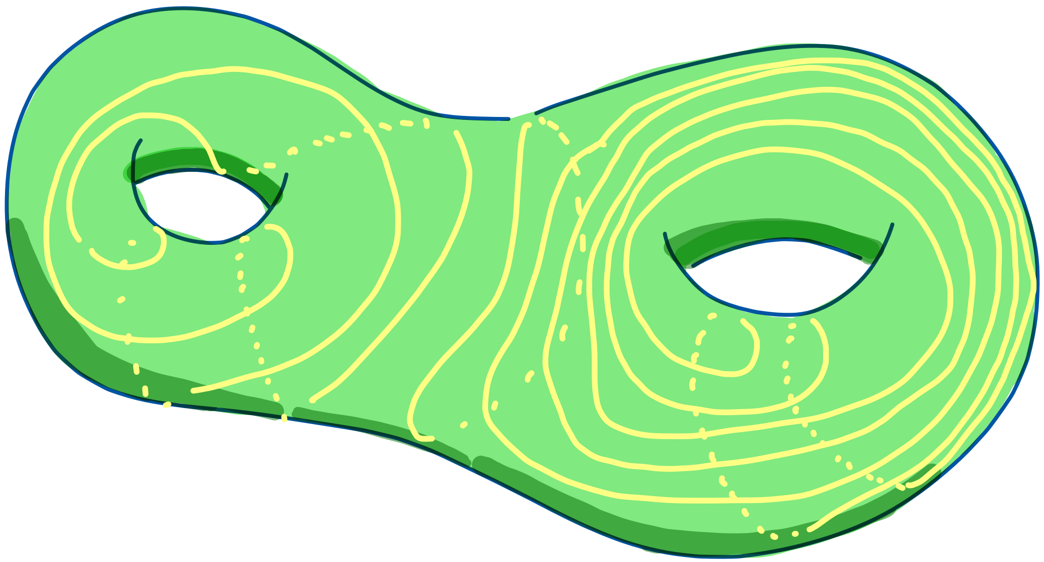
length partition of random multi-geodesics on LARGE genus hyperbolic surfaces



joint work with

Vincent Delecroix

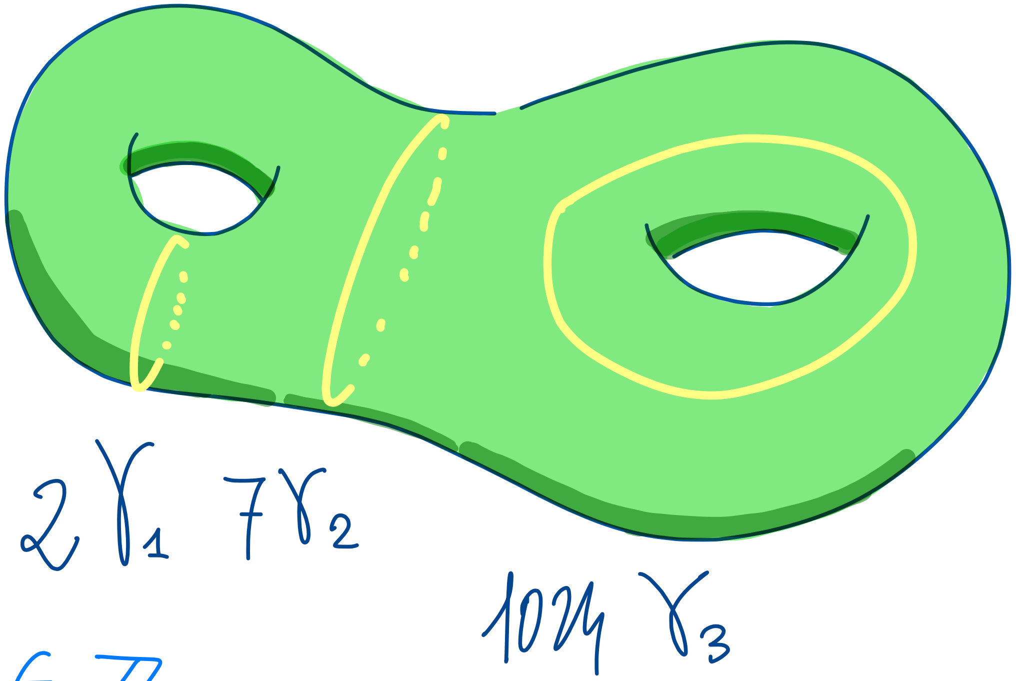
X hyperbolic surface of genus $g \geq 2$

(Complete. Connected. oriented..)

A **multi-geodesic** is a multiset of disjoint simple closed geodesics

$\{a, a, b\} \neq \{a, b\}$

simple: no self-intersection



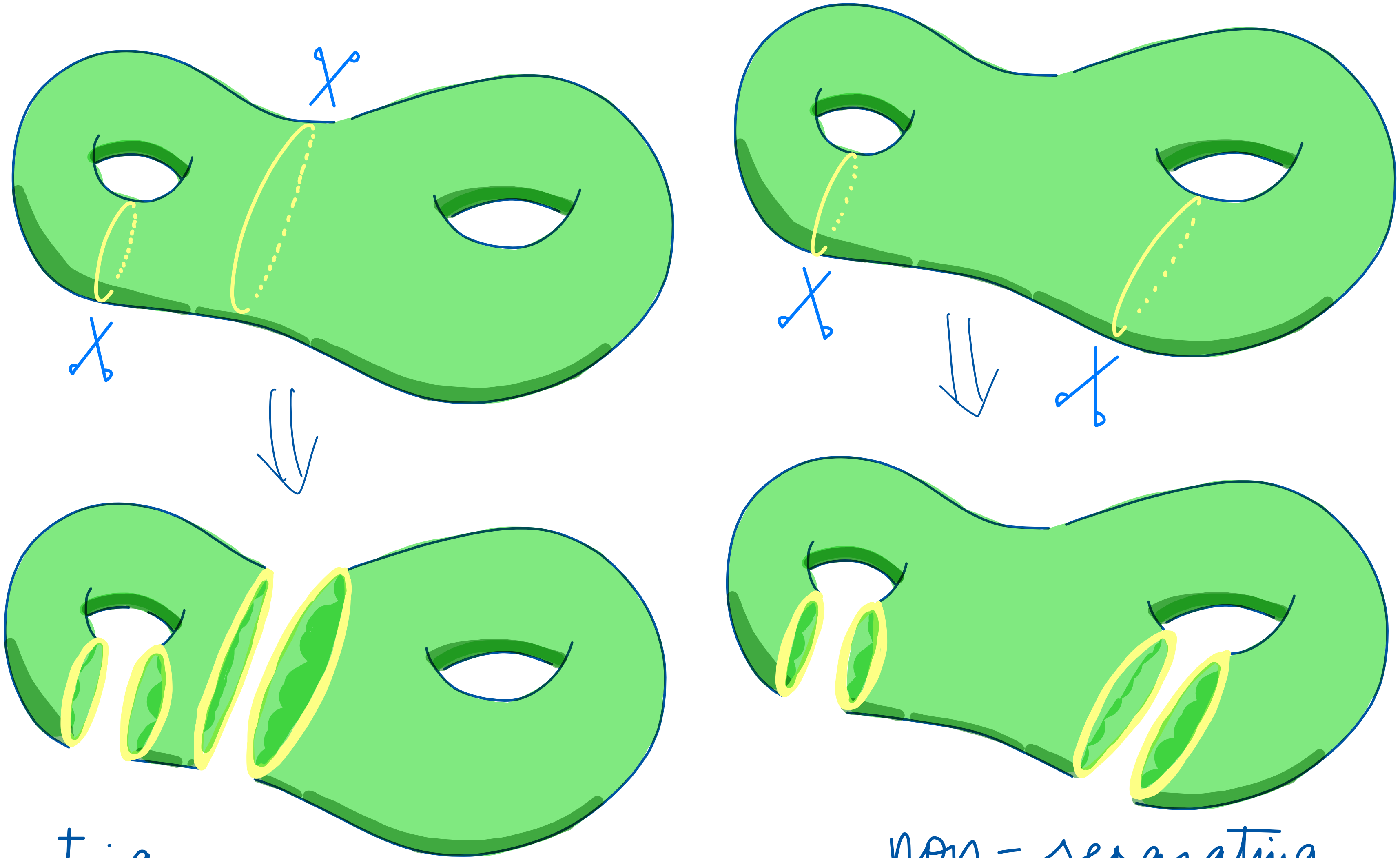
Ex $\gamma = 2\gamma_1 + 7\gamma_2 + 1024\,\gamma_3$

In general $\gamma = m_1 \gamma_1 + \dots + m_k \gamma_k$, $\quad m_i \in \mathbb{Z}_{\geq 1}$

$k \leq 3g - 3$

$= 3g-3 \Rightarrow$ pants decomposition

Topology : separating ?

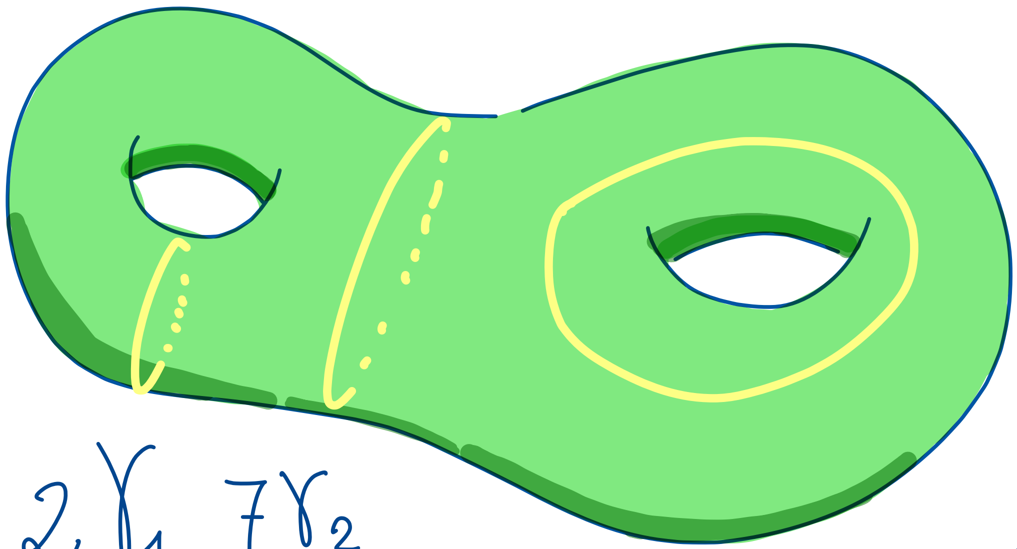


separating

non-separating

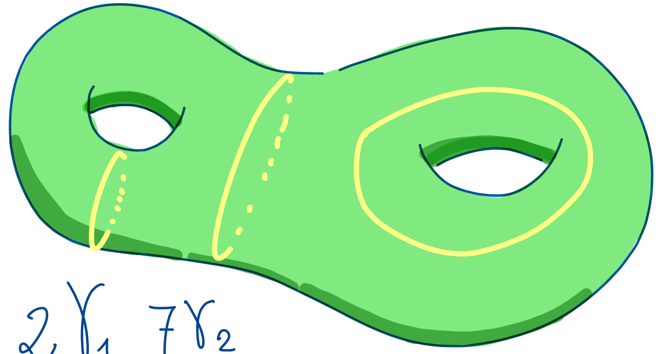
Topology

- separating ?
- number of components ?



$$\#\text{ of components}\left(2\gamma_1 \quad 7\gamma_2 \quad 1024\gamma_3 \right) = 3$$

$$\gamma = m_1\gamma_1 + \dots + m_k\gamma_k$$



Geometry

hyperbolic metric

• (total) length $\ell_X(\gamma) = m_1\ell_X(\gamma_1) + \dots + m_k\ell_X(\gamma_k)$

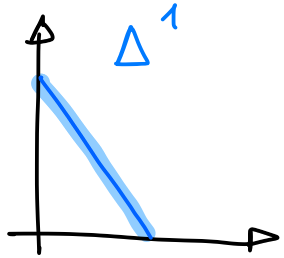
decreasingly ordered

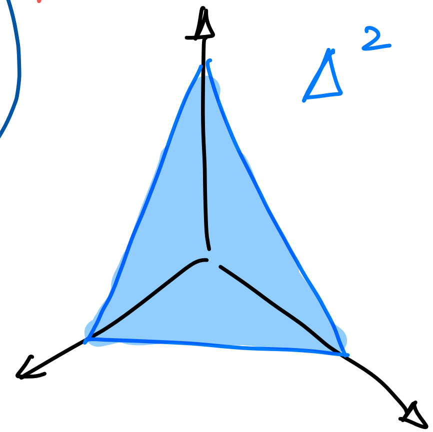
normalized

• length vector

$$\Delta^{k-1} = \{(x_1,\dots,x_k)\in\mathbb{R}^k_{\geq 0} \mid x_1+\dots+x_k = 1\}$$

$$\hat{\ell}^{\downarrow}_X(\gamma) = \frac{1}{\ell_X(\gamma)}\big(m_1\ell_X(\gamma_1),\dots,m_k\ell_X(\gamma_k)\big)^{\downarrow}$$





Ex $(10\%, 30\%, 4\%, 2\%, 50\%, 2\%)^{\downarrow}$

$= (50\%, 30\%, 10\%, 4\%, 2\%, 2\%)$

Random multi-geodesics

$R \gg 0$

$$\mathcal{S}_{X,R} = \left\{ \alpha \;\middle|\; \begin{array}{l} \alpha \text{ multi-geodesic on } X \\ \ell_X(\alpha) \leq R \end{array} \right\}$$

$\# \mathcal{S}_X = \infty \Rightarrow$ no uniform proba measure.

$\# \mathcal{S}_{X,R} < \infty \Rightarrow$ we equip it with a uniform proba measure!

$\Rightarrow$ functions defined on $\mathcal{S}_{X,R}$ become random variables

[V. Delecroix, É. Goujard, P. Zograf & A. Zorich]

Large genus asymptotic geometry of random square-tiled surfaces and of random multicurves

- $\mathbb{P}$ (a random multi-geodesic is separating) $\xrightarrow{\text{“} g \to \infty \text{”}} 0$

- Average number of components $\underset{g \to \infty}{\sim} \dfrac{\log g}{2}$

Topology

$$\{(x_1,x_2,\dots)\in\mathbb{R}^{\infty}_{\geq 0} \mid x_1+x_2+\dots=1\}$$

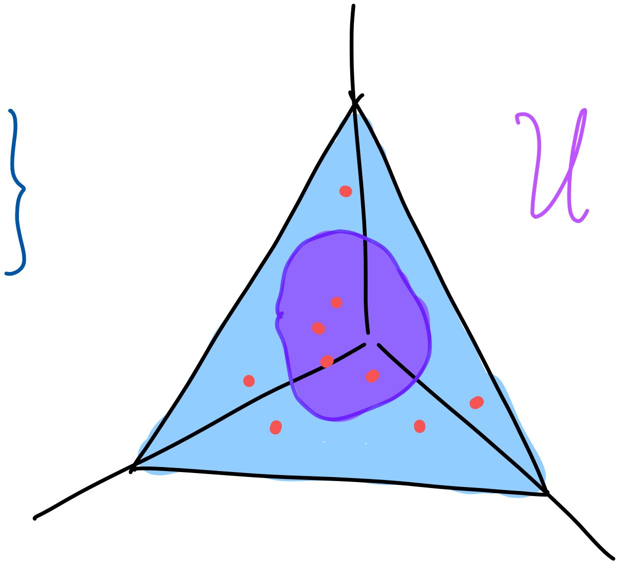
$$\hat{\ell}^{\downarrow}_{X,R} : \mathcal{S}_{X,R} \longrightarrow \Delta^{3g-4} \hookrightarrow \Delta^{\infty} = \{(x_1,x_2,\dots)\in\mathbb{R}^{\infty}_{\geq 0} \mid x_1+x_2+\dots=1\}$$

$$\gamma \longmapsto \ell^{\downarrow}_{X}(\gamma)$$

set of multi-geod on X of length $\leq R$

uniform proba measure

$$\mathbb{P}\Big(\hat{\ell}^{\downarrow}_{X,R} \in \mathcal{U}\Big) = \frac{\#\{\bullet \mid \bullet \in \mathcal{U}\}}{\#\{\bullet\}}, \qquad \mathcal{U} \subset \Delta^{\infty}$$



$$\left\{ (x_1, x_2, \dots) \in \mathbb{R}^{\infty}_{\geqslant 0} \;\middle|\; x_1 + x_2 + \dots = 1 \right\}$$

$$\|$$

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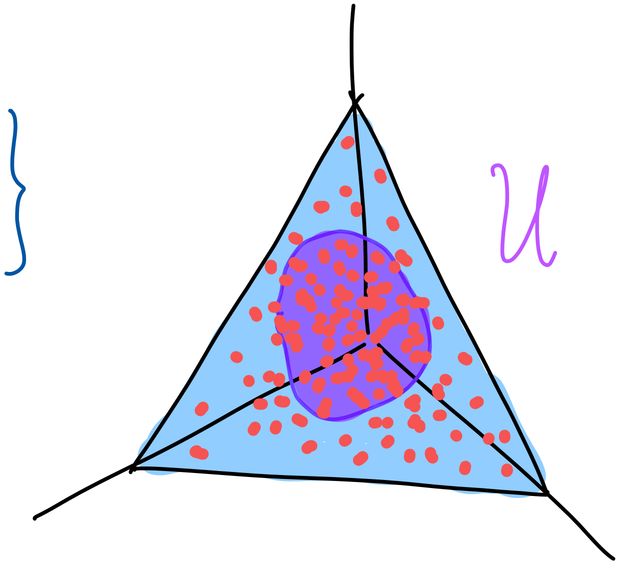

$$\{(x_1, x_2, \dots) \in \mathbb{R}^\infty_{\geq 0} \mid x_1 + x_2 + \dots = 1\}$$

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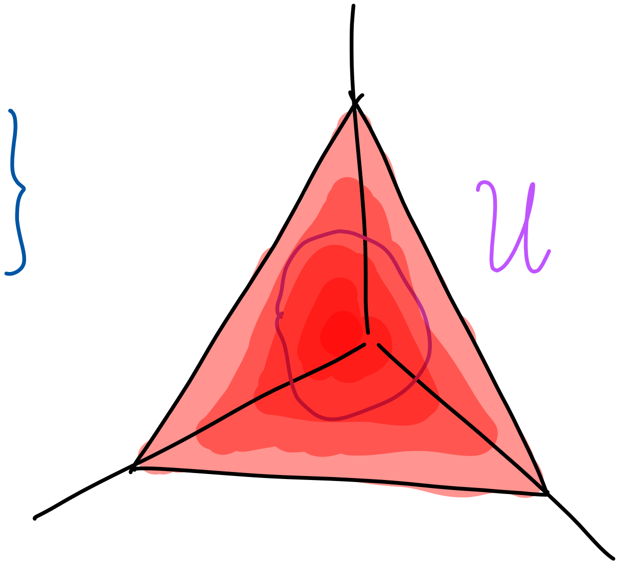
$$\mathcal{U} \cap \Delta^\infty$$



Theorem (Delecroix - L)

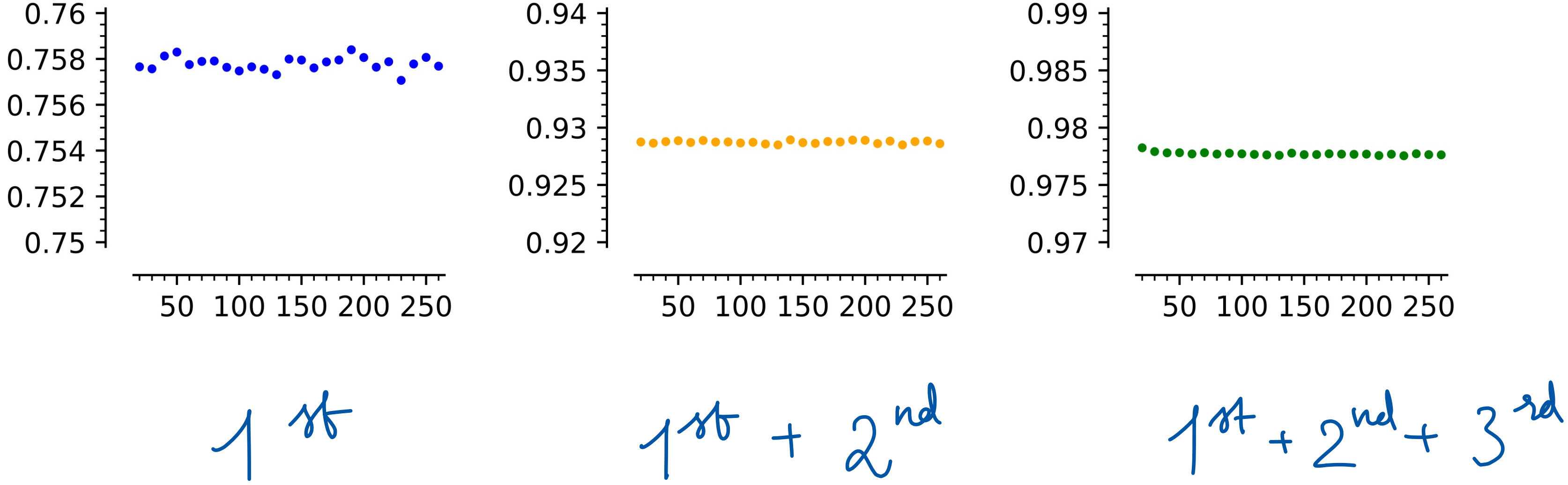
- $\hat{\ell}_{X,R}^{\downarrow} \xrightarrow[R \to \infty]{(d)} L^{(g)}$ (in distribution), where $L^{(g)}$:
  - admits an explicit density function
  - does not depend on X
- $L^{(g)} \xrightarrow[g \to \infty]{(d)}$ Poisson-Dirichlet distribution of parameter $1/2$

Corollary

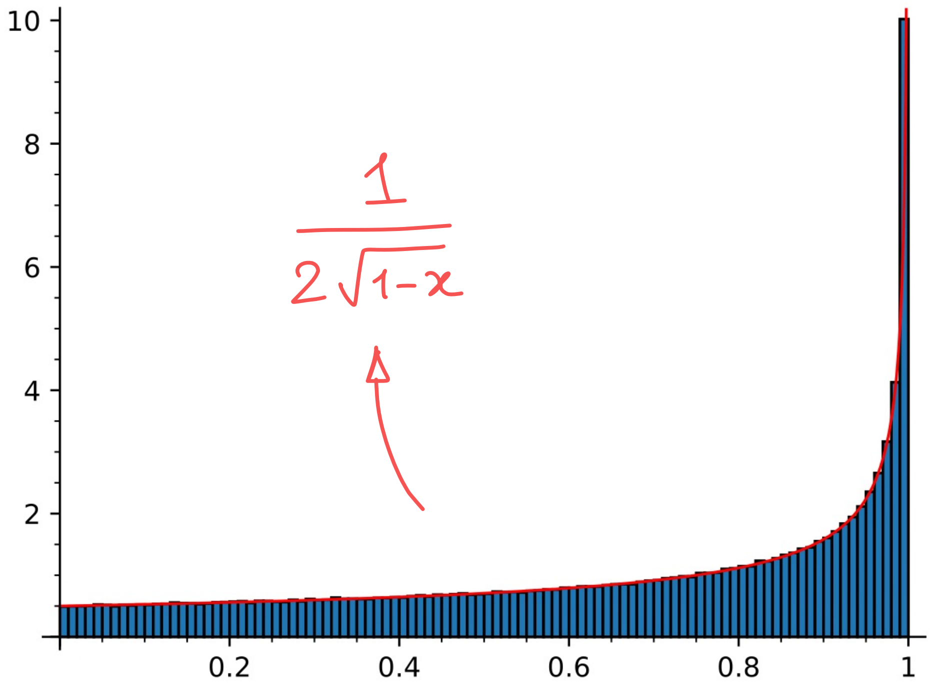
When g is large, in average:

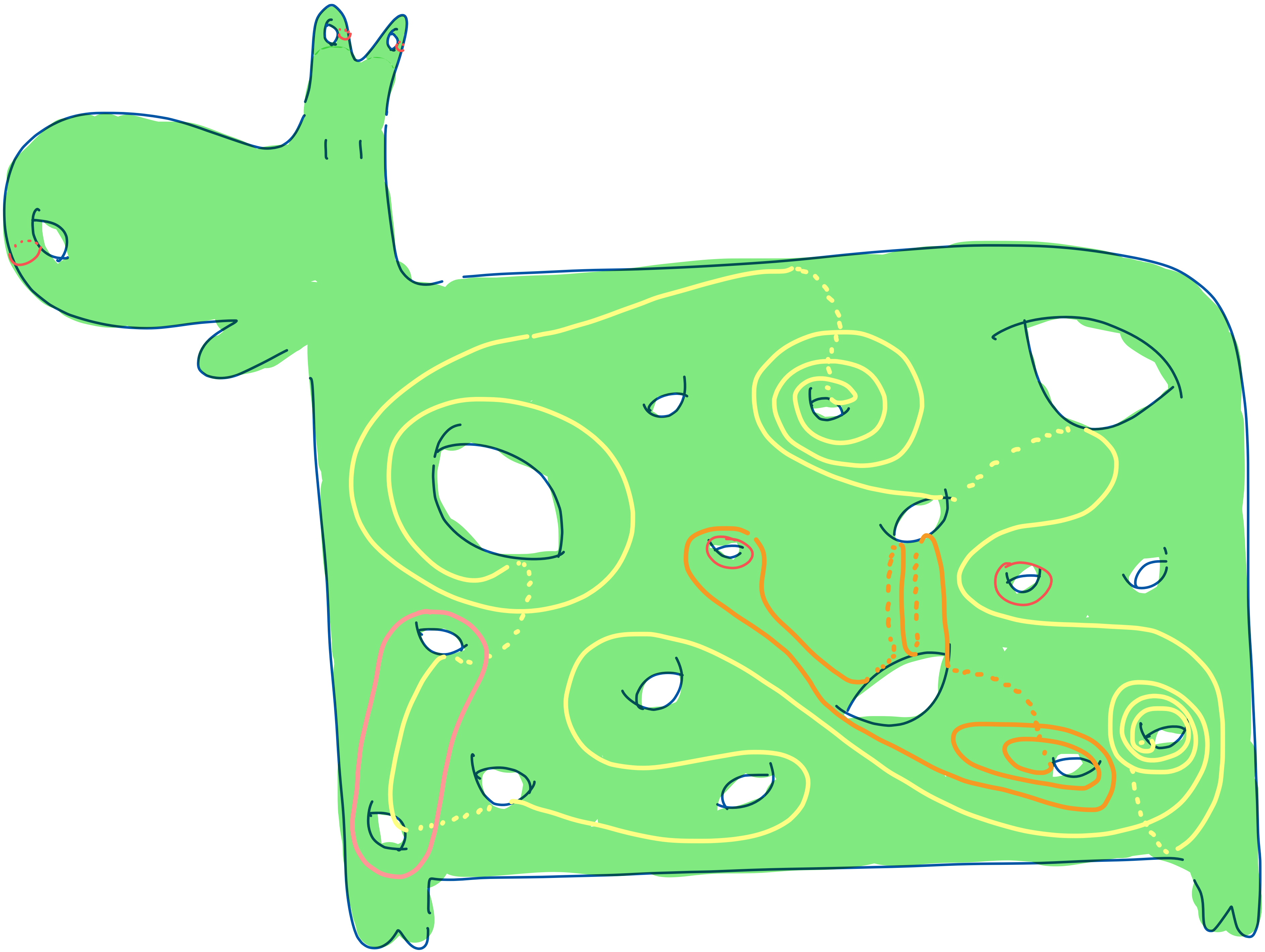
largest component : 75.8 %
2^{nd} : 17.1 %
3^{rd} : 4.9 %

97.8 %



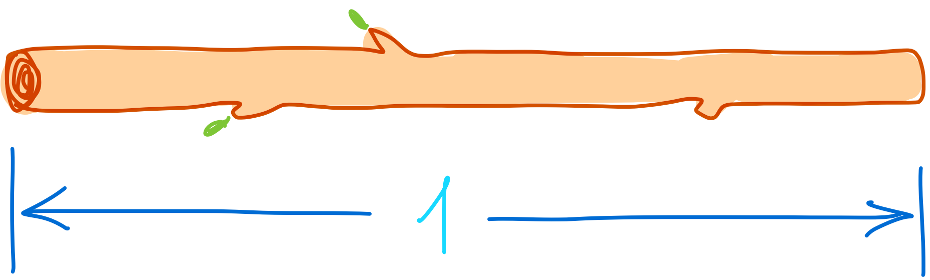
a random component





Poisson - Dirichlet distribution

"stick breaking" process



- you got a random number generator

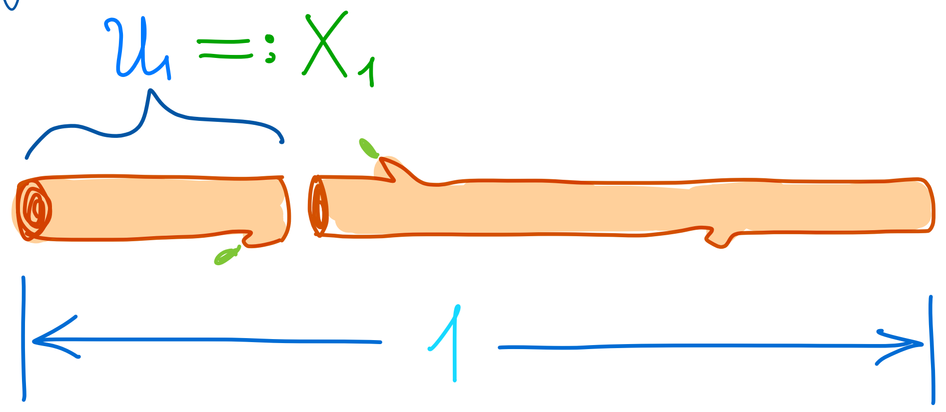
- or formally, you have a sequence of $\underline{\text{iid}}$ random variables

$$U_1, U_2, \ldots \sim \text{Unif}([0,1])$$

independent and identically distributed

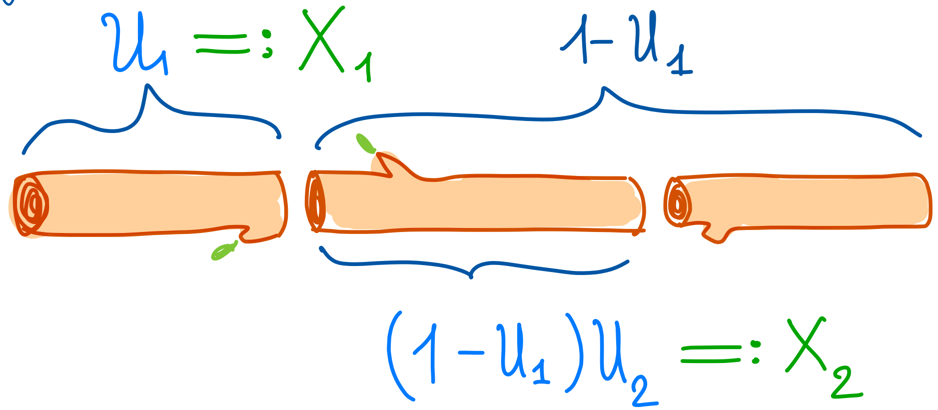
Poisson - Dirichlet distribution

"stick breaking" process $U_1, U_2, \dots$ iid va $\sim \mathrm{Unif}([0,1])$



Poisson – Dirichlet distribution

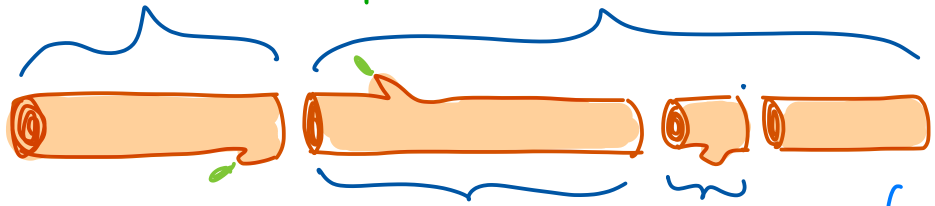
"stick breaking" process $U_1, U_2, \dots$ iid va $\sim \mathcal{U}nif([0,1])$



Poisson – Dirichlet distribution

"stick breaking" process $\quad U_1, U_2, \dots$ iid va $\sim \mathrm{Unif}([0,1])$

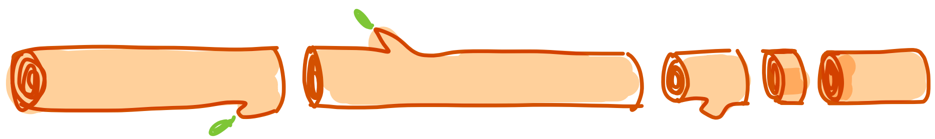
$U_1 =: X_1 \qquad\qquad 1-U_1$



$X_2 := (1-U_1)U_2 \qquad\qquad (1-U_1)(1-U_2)U_3 =: X_3$

Poisson - Dirichlet distribution

"stick breaking" process $\quad U_1, U_2, \ldots$ iid va $\sim \mathrm{Unif}([0,1])$



$X_1 = U_1$

$X_2 = (1-U_1)U_2$

$X_3 = (1-U_1)(1-U_2)U_3$

$X_4 = (1-U_1)(1-U_2)(1-U_3)U_4$

$\vdots$

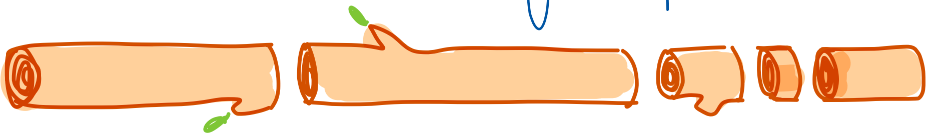
almost surely, $X_1 + X_2 + \ldots = 1$

$\Rightarrow X = (X_1, X_2, \ldots) \in \Delta^\infty$

$X^\downarrow \overset{\text{def}}{\sim} PD(1)$

Poisson – Dirichlet distribution

"stick breaking" process



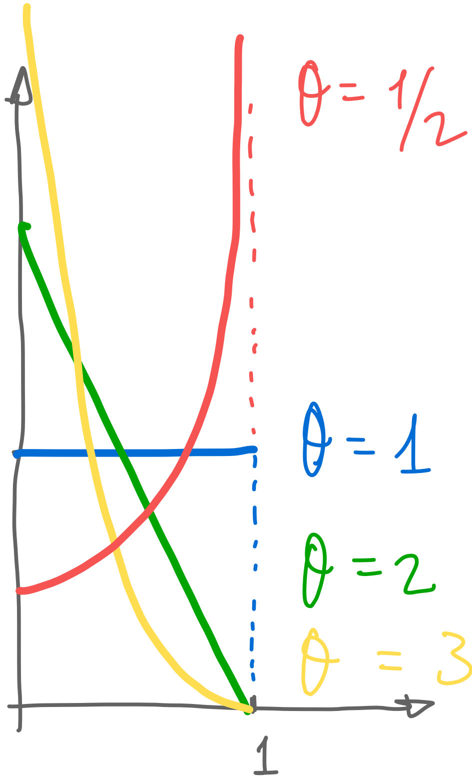
$U_1, U_2, \dots$ iid va $\sim$ ~~Unif([0,1])~~ Beta(0, θ)

$\mathbb{1}_{[0,1]}(x) \cdot \theta (1-x)^{1-\theta}$

$X_1 = U_1$

$X_2 = (1-U_1) U_2$

$\vdots$



almost surely, $X_1 + X_2 + \dots = 1$

$\Rightarrow X = (X_1, X_2, \dots) \in \Delta^{\infty}$

$\underline{\text{def}}$ $X^{\downarrow} \sim PD(\cancel{1})$

$\theta > 0$

Example 1 — Integer factorization

$k \geq 2$ integer

$$k = 7182 = 19 \times 7 \times 3 \times 3 \times 3 \times 2$$

$$19 = P_1(k), \quad 7 = P_2(k), \quad 3 = P_3(k), \quad 3 = P_4(k)$$

uniform proba measure

$$\hat{\ell}_n^{\downarrow} : \{2, 3, 4, \dots, n\} \longrightarrow \Delta^{\infty}$$

$$k \longmapsto \frac{1}{\log k}\left(\log P_1(k), \log P_2(k), \log P_3(k), \dots\right)$$

Theorem

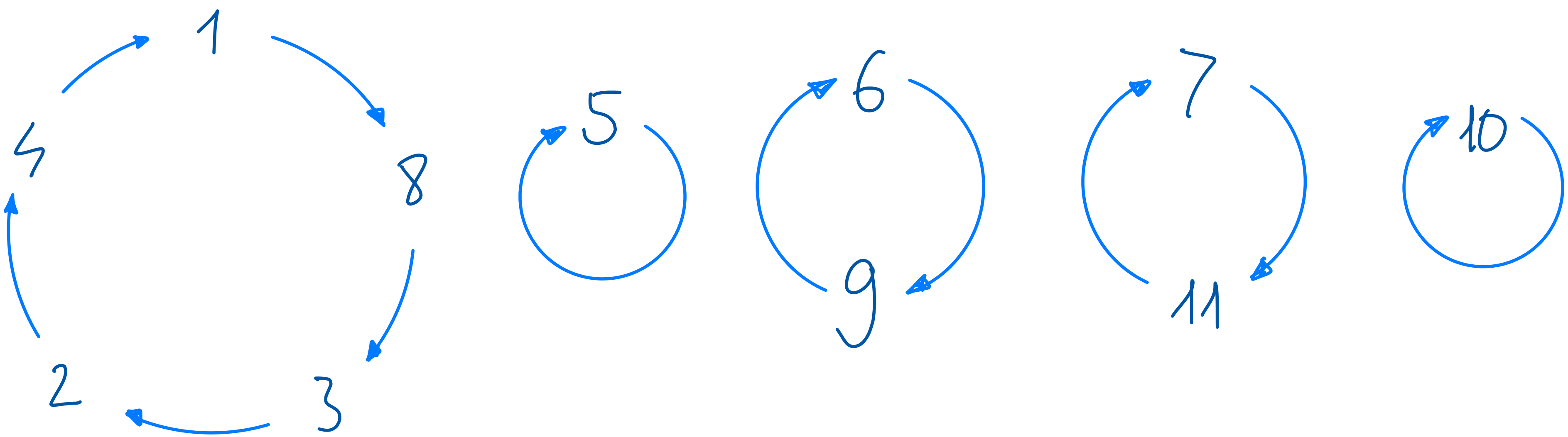
$$\boxed{\hat{\ell}_n^{\downarrow} \xrightarrow[n \to \infty]{(d)} PD(1).}$$

Example 2 Cycle decomposition for permutations

$$\sigma \in S_n := \{ f : \{1, 2, \dots, n\} \circlearrowleft \mid f \text{ is bijective} \}$$

ex.

k	1	2	3	4	5	6	7	8	9	10	11
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\sigma(k)$	8	4	2	1	5	9	11	3	6	10	7



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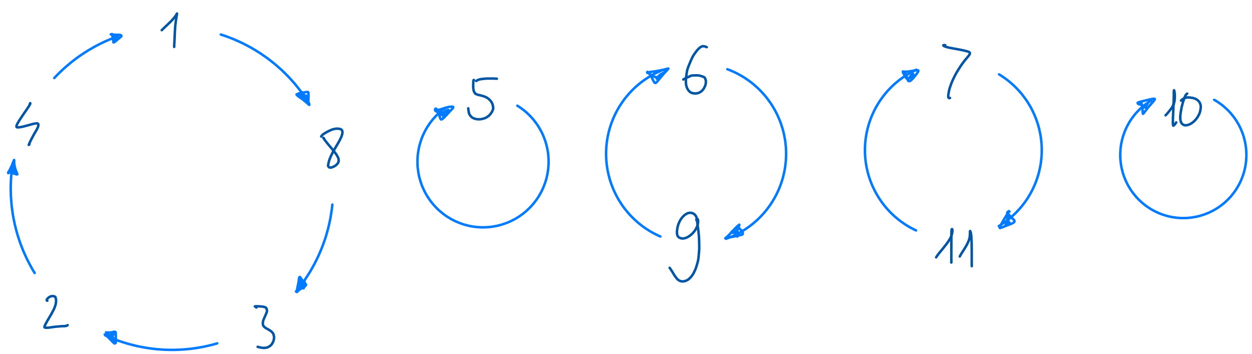

$$C_1(\sigma) = 5,\ C_2(\sigma) = C_3(\sigma) = 2,\ C_4(\sigma) = C_5(\sigma) = 1$$

uniform proba measure

$$\hat{\ell}_n^{\downarrow} : S_n \longrightarrow \Delta^\infty$$

$$\sigma \longmapsto \frac{1}{n}\left(C_1(\sigma), C_2(\sigma), C_3(\sigma), \dots\right)$$

Theorem $\boxed{\hat{\ell}_n^{\downarrow} \xrightarrow[n\to\infty]{(d)} PD(1)}$

PD(θ)?

S_n

Weight : $w_\theta(\sigma) := \theta^{K_n(\sigma)}$ ← number of cycles in σ

ex $K_{11}\big((1\,8\,3\,2\,4)(5)(6\,9)(7\,11)(10)\big) = 5$

$$\boxed{\mathbb{P}_\theta(\sigma) := \frac{w_\theta(\sigma)}{\sum_{\tau \in S_n} w_\theta(\tau)}}$$

$$\sum_{\tau \in S_n} w_\theta(\tau) = \theta(\theta+1)\cdots(\theta+n-1)$$

$$= n! \text{ if } \theta = 1 \Rightarrow \text{uniform}$$

Ewens measure

Theorem $\boxed{\hat{\ell}^{\downarrow}_{n,\theta} \xrightarrow[n\to\infty]{(d)} \mathrm{PD}(\theta)}$

[DGZZ]: number of components of a random multi-geod $\underset{g\to\infty}{\sim} \frac{1}{2}\log g$

By the way, $\boxed{K_{n,\theta} \underset{n\to\infty}{\sim} \theta \cdot \log n}$