Hopf fibration and quaternions

Bachelor thesis proposal

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> In these days the angel of topology and the devil of abstract algebra fight for the soul of each individual mathematical domain.
>
> *Invariants*, Hermann Weyl

Roughly speaking, the *Hopf fibration* describes how a 3-dimensional sphere can be decomposed into a family of disjoint 1-dimensional spheres (circles) indexed by the points of a 2-dimensional sphere (an ordinary sphere). Pointed out by Heinz Hopf in 1931, the Hopf fibration is one of the first non-trivial examples of a *fiber bundle*, and has board-ranging applications in mathematics (algebraic topology, K-theory) and physics (quantum information, magnetic monopoles, twistor theory).

In this project, we shall construct the Hopf fibration by means of *quaternions*, which extends the complex numbers and were discovered in 1943 by William Rowan Hamilton after having been puzzled long and deeply. Among other unexpected applications in mathematics and physics, quaternions turn out to be particularly useful to spatial rotations, and we shall be seeing later how this applies to the construction of the Hopf fibration.

Mostly following [Lyons], we shall start by defining the Hopf map, and the quaternions. For some beautiful visualizations of the Hopf fibration, see [Johnson] and [Belmonte]. Other references will be given later.

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Prerequisites

Linear algebra.

References

[Belmonte] Nico Belmonte, *Hopf fibration*, https://philogb.github.io/page/hopf/.

[Lyons] David W. Lyons, *An Elementary Introduction to the Hopf Fibration*, https://nilesjohnson.net/hopf-articles/Lyons_Elem-intro-Hopf-fibration.pdf.

[Johnson] Niles Johnson, *Hopf fibration – fibers and base*, https://www.youtube.com/watch?v=AKotMPGFJYk.