ON LEOPOLDT'S AND GROSS'S DEFECTS FOR ARTIN REPRESENTATIONS

ALEXANDRE MAKSOUD

ABSTRACT. We generalize Waldschmidt's bound for Leopoldt's defect and prove a similar bound for Gross's defect for an arbitrary extension of number fields. As an application, we prove new cases of the generalized Gross conjecture (also known as the Gross-Kuz'min conjecture) beyond the classical abelian case, and we show that Gross's p-adic regulator has at least half of the conjectured rank. We also describe and compute non-cyclotomic analogues of Gross's defect.

1. INTRODUCTION

Dirichlet's class number formula classically expresses the residue at $s = 1$ of the Dedekind zeta function of a number field K in terms of its Dirichlet regulator, which is a nonzero determinant involving logarithms of real units in K. Perrin-Riou's influential program [PR95] suggests that this formula has a p-adic analogue which involves a p-adic Dirichlet regulator and an $\mathcal{L}$-invariant, constructed in terms of Leopoldt's and Gross's p-adic regulator maps respectively.

In fact, p-adic class number formulae and their appropriate generalizations to Artin L-functions have already been proposed by Tate (and attributed to Serre) in [Tat84] and Gross in [Gro81] when K is a totally real field or a CM field. In an earlier work [Mak21], we formulated a precise conjecture for more general p-adic Artin L-functions.

Analogously to the complex setting, Leopoldt's and Gross-Kuz'min's conjectures respectively predict that Leopoldt's and Gross's p-adic regulator maps have full rank [Leo62, Gro81, Kuz72]. The purpose of this article is to deduce lower bounds for the ranks of these regulators from classical theorems of Brumer, Waldschmidt and Roy in p-adic transcendence theory [Bru67, Wal81, Roy92]. In order to do this, we make use of Artin formalism.

We recall that the Gross-Kuz'min conjecture is known when $K/\mathbb{Q}$ is abelian thanks to Greenberg [Gre73]. More recently, Kleine [Kle19] showed that this conjecture is true when K has at most two p-adic primes. (We note that Kleine's approach does not use p-adic transcendence theory). In addition to those two results, our bounds imply many new cases of the Gross-Kuz'min conjecture. Besides their potential applications to the Tamagawa number conjecture (as crucially used in [BKS17]) in connection with p-adic Artin L-functions, the Gross-Kuz'min conjecture and its analogue for non-cyclotomic Iwasawa theory also yield information on the fine structure of class groups attached to extensions of K (see e.g. [Kol91, FMD05, Jau17]). These conjectures are now mainstream in Iwasawa theory of number fields.

We now introduce the main notations and results of this paper.

Let K be a number field and fix once and for all a prime number p. We denote by $S_p(K)$ and $S_\infty(K)$ the sets of p-adic places and archimedean places of K respectively. Fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and let $A^\wedge = \overline{\mathbb{Q}}_p \otimes_{\mathbb{Z}_p} \varprojlim_n A/p^n A$ be the p-adic completion of an abelian group

A. Let

$$\iota_K\colon \mathcal{O}_K^{\times,\wedge} \longrightarrow \prod_{\mathfrak{P}|p} \mathcal{O}_{K_{\mathfrak{P}}}^{\times,\wedge} \tag{1}$$

be the $\overline{\mathbb{Q}}_p$-linear map induced by the diagonal embedding of the units of K into all its p-adic completions. For K and p, the classical Leopoldt's conjecture asserts that the map ι_K is injective. The dimension of $\ker \iota_K$ is called Leopoldt's defect and is denoted by $\delta_K^{\mathbf{L}}$ (we will omit its dependence on p). Next, we define the Gross regulator map $\mathscr{L}_K$, which is closely related to ι_K. Consider the $\mathbb{Z}_p$-hyperplane $\mathcal{H}$ of $\bigoplus_{\mathfrak{P}|p} \mathbb{Z}_p$ given by the equation $\sum_{\mathfrak{P}|p} s_{\mathfrak{P}} = 0$, and the map

$$\mathscr{L}_K\colon \left\{ \begin{array}{ccc} \mathcal{O}_K[\frac{1}{p}]^{\times,\wedge} & \longrightarrow & \mathcal{H}^\wedge \\ x & \mapsto & (-\log_p(\mathcal{N}_{\mathfrak{P}}(x)))_{\mathfrak{P}}, \end{array}\right. \tag{2}$$

where $\mathcal{O}_K[\frac{1}{p}]^\times$ is the group of p-units of K, $\mathcal{N}_{\mathfrak{P}}$ is the local norm map for the extension $K_{\mathfrak{P}}/\mathbb{Q}_p$, and $\log_p : \mathbb{Q}_p^\times \to \mathbb{Q}_p$ is the usual Iwasawa p-adic logarithm. By the usual product formula, $\mathscr{L}_K$ is well-defined. The dimension of $\operatorname{coker} \mathscr{L}_K$ is called Gross's defect and is denoted by $\delta_K^{\mathbf{G}}$. Waldschmidt's classical bound for Leopoldt's defect reads $\delta_K^{\mathbf{L}} \le (|S_\infty(K)|-1)/2$ [Wal81]. The following theorem generalizes this to a relative setting and also states a similar bound for $\delta_K^{\mathbf{G}}$.

Theorem 1.1. *Let K/k be an extension of number fields. The following inequalities hold.*

$$\begin{array}{rcl} \delta_K^{\mathbf{L}} & \le & \delta_k^{\mathbf{L}} + (|S_\infty(K)| - |S_\infty(k)|)/2, \\ \delta_K^{\mathbf{G}} & \le & \delta_k^{\mathbf{G}} + \big(|S_p(K)| - |S_p(k)|\big)/2. \end{array}$$

Moreover, if K has at least one real place and $|S_p(K)| \neq |S_p(k)|$, then the second bound is strict.

In particular, the Gross-Kuz'min conjecture holds for all cubic number fields.

A key observation that we use in the computation of $\delta_K^{\mathbf{L}}$ and $\delta_K^{\mathbf{G}}$ is that they are compatible with Artin formalism. For any number field $k \subset \overline{\mathbb{Q}}$ of absolute Galois group $G_k = \mathrm{Gal}(\overline{\mathbb{Q}}/k)$, let $\mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_k)$ be the set of finite dimensional $\overline{\mathbb{Q}}_p$-valued representations of G_k of finite image. We will define defects $\delta_k^{\mathbf{L}}(\rho)$ and $\delta_k^{\mathbf{G}}(\rho)$ associated with $\rho \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_k)$ which satisfy the usual Artin formalism. In particular, when $\rho = \mathrm{Ind}_K^k \mathbb{1}_K$ is the induction from G_K to G_k of the trivial representation, they coincide with $\delta_K^{\mathbf{L}}$ and $\delta_K^{\mathbf{G}}$ respectively. We will also define quantities $d(\rho)$, $d^+(\rho)$ and $f(\rho)$ which compute $[K:\mathbb{Q}]$, $|S_\infty(K)|$ and $|S_p(K)|$, respectively, when $\rho = \mathrm{Ind}_K^k \mathbb{1}_K$ (see (5)). Our main theorem is the following.

Theorem 1.2. *Let $\rho \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_{\mathbb{Q}})$ be an irreducible representation and let $d = d(\rho)$, $d^+ = d^+(\rho)$ and $f = f(\rho)$. If $d^+ = f = 0$, then we have $\delta_{\mathbb{Q}}^{\mathbf{L}}(\rho) = \delta_{\mathbb{Q}}^{\mathbf{G}}(\rho) = 0$. Otherwise, we have the following inequalities.*

$$\delta_{\mathbb{Q}}^{\mathbf{L}}(\rho) \le \frac{(d^+)^2}{d+d^+}, \qquad \delta_{\mathbb{Q}}^{\mathbf{G}}(\rho) \le \frac{f^2}{d^+ + 2f}.$$

By Artin formalism, this yields the upper bound (e.g. for Leopoldt's defect) $\delta_k^{\mathbf{L}}(\rho) \le d^+(\rho)/2$ for an arbitrary representation $\rho \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_k)$. We immediately recover Theorem 1.1 by choosing ρ such that $\mathrm{Ind}_K^k \mathbb{1}_K = \rho \oplus \mathbb{1}_k$.

The first bound in Theorem 1.2 is Laurent's main theorem in [Lau89], but we will provide a much shorter proof of this result via a lemma on local Galois representations (Lemma 3.2.7). The second bound, however, cannot be deduced from the classical methods employed by Laurent [Lau89] and Roy [Roy92] to study the p-adic closure of S-units of K, for a given finite set of places S. Indeed, they need to assume $p \notin S$, because $\log_p$ is not injective on

$\mathcal{O}_K[\frac{1}{p}]^\times$. We circumvent this issue in Proposition 2.1.2 by determining the kernel of the $\overline{\mathbb{Q}}$-linear extension of $\log_p$.

Theorem 1.2 easily implies Ax-Brumer's theorem on the validity of Leopoldt's conjecture for abelian extensions of an imaginary quadratic field [Ax65, Bru67]. In the same vein, we indicate two striking applications of Theorem 1.2.

Corollary 1.3. *Let k be a totally real field and let V be a totally odd Artin representation of G_k. Then Gross's p-adic regulator matrix $R_p(V)$ defined in* [Gro81, (2.10)] *has rank at least half of its size.*

This corollary strengthens Gross's classical result stating that the matrix $R_p(V)$ has positive rank [Gro81, Prop. 2.13].

Corollary 1.4. *The Gross-Kuz'min conjecture holds for abelian extensions of imaginary quadratic fields. It also holds for abelian extensions of real quadratic fields having at least one real place.*

Theorem 4.3.2 provides a more extensive list of number fields for which the Gross-Kuz'min conjecture holds unconditionally.

The last part of this article is devoted to developing some tools to study non-cyclotomic analogues of the Gross-Kuz'min conjecture. Given an arbitrary $\mathbb{Z}_p$-extension K_∞ of K, we will define a map $\mathcal{L}_{K_\infty/K}$ specializing to $\mathscr{L}_K$ if K_∞ is the cyclotomic extension of K. Interestingly, there do exist examples of $\mathbb{Z}_p$-extensions K_∞/K for which $\delta^{\mathbf{G}}_{K_\infty/K} > 0$ [Kis83, JS95], but a conjectural description of all such K_∞/K is still missing.

Theorem 1.5. *Let k be an imaginary quadratic field, and K an abelian extension of k in which p splits completely. Let K^{ab}/k be the maximal subextension of K/k that is abelian over $\mathbb{Q}$. Then there exist at most $[K:\mathbb{Q}]-[K^{ab}:\mathbb{Q}]$ distinct $\mathbb{Z}_p$-extensions k_∞ of k for which $\delta^{\mathbf{G}}_{Kk_\infty/K} > 0$.*

In Proposition 3.1.1 we illustrate Theorem 1.5 with a classical application to the semi-simplicity of Iwasawa modules attached to Kk_∞/K. In addition, we show that the exceptional $\mathbb{Z}_p$-extensions in Theorem 1.5, for which $\delta^{\mathbf{G}}_{Kk_\infty/K} > 0$, necessarily have a transcendental slope (see Example 5.2.3 for the meaning of slope). It turns out that such $\mathbb{Z}_p$-extensions should not exist under the p-adic Schanuel conjecture.

Theorem 1.5 can be generalized to arbitrary base fields k having at most r linearly disjoint $\mathbb{Z}_p$-extensions with $r \le 2$ (Theorem 5.1.1). The main idea is that, under our assumption on p, one can parameterize $\mathbb{Z}_p$-extensions of k by points on a $(r-1)$-dimensional linear subspace L of $\mathbb{P}^{n-1}(\mathbb{Q}_p)$, where $n=[k:\mathbb{Q}]$. The condition $\delta^{\mathbf{G}}_{Kk_\infty/K} > 0$ then cuts out a closed subvariety $\mathscr{C}$ of L given by polynomial equations with coefficients in $\Lambda := \log_p(\mathcal{O}_K[\frac{1}{p}]^\times)$. We then exploit the fact that any linear (resp. algebraic) independence between elements of Λ implies strong conditions on the $\overline{\mathbb{Q}}$-points (resp. the $\overline{\mathbb{Q}}_p$-points) of $\mathscr{C}$.

Theorem 1.5 was inspired by Betina-Dimitrov's work [BD21] where the authors show the non-vanishing of a certain anticyclotomic $\mathcal{L}$-invariant for Katz's p-adic L-function. In fact, their result generalizes to any $\mathbb{Z}_p$-extension with non-transcendental slope. We expect that our techniques can give further results on the non-vanishing of $\mathcal{L}$-invariants in more general contexts.

The paper is structured as follows. In Section 2 we recall all the classical results in p-adic transcendence theory which we make use of. In Section 3 we describe Leopoldt's and Gross's defects via class field theory and we show that they are compatible with Artin formalism. Our main results and corollaries are proven in Section 4, except for Theorem 1.5 whose proof is postponed to Section 5.

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2. p-ADIC TRANSCENDENCE THEORY

Throughout this section we fix an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, allowing us to view algebraic numbers as p-adic numbers. Let Λ be the $\overline{\mathbb{Q}}$-linear subspace of $\overline{\mathbb{Q}}_p$ generated by 1 and by p-adic logarithms of non-zero algebraic numbers. The following very strong conjecture describes the algebraic dependence between elements in Λ.

Conjecture (p-adic Schanuel conjecture)**.** *If $\lambda_1, \dots, \lambda_n \in \Lambda$ are linearly independent over $\mathbb{Q}$, then they are algebraically independent over $\overline{\mathbb{Q}}$.*

We recall some classical results of Brumer, Waldschmidt and Roy and deduce some consequences that turn out to be useful in the study of the Gross-Kuz'min conjecture.

2.1. The Baker-Brumer theorem.

Brumer [Bru67] extended Baker's method to the p-adic setting and proved the following theorem on linear independence of logarithms:

Theorem 2.1.1 (Baker-Brumer theorem)**.** *If $\lambda_1, \dots, \lambda_n \in \Lambda$ are linearly independent over $\mathbb{Q}$, then they are linearly independent over $\overline{\mathbb{Q}}$.*

Recall that $\log_p$ is normalized so that we have $\log_p(p) = 0$.

Proposition 2.1.2. *Let $H \subset \overline{\mathbb{Q}}$ be a number field. The $\overline{\mathbb{Q}}$-linear extension $\log_p : \overline{\mathbb{Q}} \otimes_{\mathbb{Z}} H^\times \to \overline{\mathbb{Q}}_p$, $c \otimes x \mapsto c \log_p(\iota_p(x))$ of the p-adic logarithm has kernel the line $p^{\overline{\mathbb{Q}}}$ spanned by $1 \otimes p$.*

Proof. Let H_p be the completion of $\iota_p(H)$ inside $\overline{\mathbb{Q}}_p$, let $\mathcal{O}_{H_p}^\times$ be its unit group and consider the abelian group $\mathcal{T} = \{x \in H^\times \ : \ \iota_p(x) \in \mathcal{O}_{H_p}^\times\}$. Then we clearly have $\overline{\mathbb{Q}} \otimes H^\times = \left(\overline{\mathbb{Q}} \otimes \mathcal{T}\right) \bigoplus p^{\overline{\mathbb{Q}}}$. Moreover, the p-adic logarithm map is injective on $\mathcal{O}_{H_p}^\times$, so multiplicatively independent numbers $\alpha_1, \dots, \alpha_n \in \mathcal{T}$ have $\overline{\mathbb{Q}}$-linearly independent p-adic logarithms by the Baker-Brumer theorem. This shows that the restriction of $\log_p$ to $\overline{\mathbb{Q}} \otimes \mathcal{T}$ is injective, hence $\ker(\log_p) = p^{\overline{\mathbb{Q}}}$. □

2.2. Waldschmidt's and Roy's theorem.

Extensions of Baker's method due to Waldschmidt and Roy give a lower bound for the rank of matrices with coefficients in Λ. To each matrix M with coefficients in $\overline{\mathbb{Q}}_p$, of size $m \times \ell$, they assign a number $\theta(M)$ defined as the minimum of all ratios $\frac{\ell'}{m'}$ where (m', ℓ') runs among the pairs of integers satisfying $0 < m' \leq m$ and $0 \leq \ell' \leq \ell$, for which there exist matrices $P \in \mathrm{GL}_m(\overline{\mathbb{Q}})$ and $Q \in \mathrm{GL}_\ell(\overline{\mathbb{Q}})$ such that the product PMQ can be written as

$$\begin{pmatrix} M' & 0 \\ N & M'' \end{pmatrix}$$

with M' of size $m' \times \ell'$. Note that $\theta(M) \leq \frac{\ell}{m}$ with equality if all the entries of M are $\overline{\mathbb{Q}}$-linearly independent. The following theorem is due to Waldschmidt [Wal81, Théorème 2.1.p] and Roy [Roy92, Corollary 1].

Theorem 2.2.1. *Let M be a matrix with coefficients in Λ, of size $m \times \ell$ with $m, \ell > 0$, and let n be its rank. We have*

$$n \geq \frac{\theta(M)}{1 + \theta(M)} \cdot m.$$

Roy also deduced a useful corollary for 3×2 matrices from Theorem 2.2.1 in [Roy92, Corollary 2].

Corollary 2.2.2 (Strong six exponentials theorem)**.** *Let M be a (3×2)-matrix with coefficients in Λ. If the rows of M are $\overline{\mathbb{Q}}$-linearly independent, and if the columns of M are also $\overline{\mathbb{Q}}$-linearly independent, then M has rank 2.*

3. Regulator maps and class groups

3.1. **Galois cohomology.** For all fields $L\subset\overline{\mathbb{Q}}$ and all finite sets S of places of L containing $S_p(L)$, we let $X(L)$ (resp. $X'_S(L)$) be the Galois group of the maximal abelian pro-p extension of L which is unramified everywhere (resp. unramified everywhere and totally split at all $v\in S$). If $S=S_p(L)$, we simply put $X'(L)=X'_S(L)$. Given a $\mathbb{Z}_p$-extension $K_\infty=\bigcup_n K_n$ of K with Galois group Γ, we have $X(K_\infty)=\varprojlim_n X(K_n)$ and $X'_S(K_\infty)=\varprojlim_n X'_S(K_n)$, the transition maps being the restriction maps. Therefore, $X(K_\infty)$ and $X'_S(K_\infty)$ are modules over the Iwasawa algebra $\mathbb{Z}_p[[\Gamma]]$. They are finitely generated torsion as shown by Iwasawa [Iwa73]. We let

$$\delta^{\mathbf{G}}_{K_\infty/K} := \mathrm{rk}_{\mathbb{Z}_p} X'(K_\infty)_\Gamma,$$

where $(-)_\Gamma$ means Γ-coinvariants. If K_∞ is the cyclotomic $\mathbb{Z}_p$-extension K_{cyc} of K we simply write $\delta^{\mathbf{G}}_K$ for $\delta^{\mathbf{G}}_{K_\infty/K}$. We will later see that this definition is compatible with that of the introduction. One motivation in classical Iwasawa theory to compute $\delta^{\mathbf{G}}_{K_\infty/K}$ originates in the following simple result by Jaulent and Sands [JS95, Proposition 6].

Proposition 3.1.1. *Let γ be a topological generator of Γ. If no p-adic prime of K splits completely in K_∞ and if $\delta^{\mathbf{G}}_{K_\infty/K}=0$, then $\gamma-1$ acts semi-simply on $X(K_\infty)$. That is, $(\gamma-1)^2$ does not divide the elements $P_i\in\mathbb{Z}_p[[\Gamma]]$ appearing in any elementary module $\bigoplus_i\mathbb{Z}_p[[\Gamma]]/(P_i)$ pseudo-isomorphic to $X(K_\infty)$.*

Let $S_0\supseteq S_p(K)\bigcup S_\infty(K)$ be a finite set of places of K. For any extension L of K and any discrete (resp. compact) G_L-module M which is unramified outside the places of L above S_0, we consider for all $i\geq 0$ the S_0-ramified i-th cohomology group (resp. continuous cohomology group) $\mathrm{H}^i_{S_0}(L,M)=\mathrm{H}^i(\mathrm{Gal}(L_{S_0}/L),M)$, where L_{S_0}/L is the largest extension of L which is unramified outside the places of L above S_0. Given any subset $S\subset S_0$, let

$$Ш^i_S(L,M)=\ker\left[\mathrm{H}^i_{S_0}(L,M)\longrightarrow\prod_{v\in S}\mathrm{H}^i(L_v,M)\times\prod_{v\in S_0-S}\mathrm{H}^i(L_v^{\mathrm{ur}},M)\right],$$

where L_v^{ur}/L_v denotes the maximal unramified extension of L_v (so $\mathbb{R}^{\mathrm{ur}}=\mathbb{R}$ in particular) and the maps above are the usual localization maps. Note that the definition of $Ш^i_S(L,M)$ does not depend on the choice of S_0. We simply write $Ш^i(L,M)$ instead of $Ш^i_S(L,M)$ if $S=S_p(L)\bigcup S_\infty(L)$. We also write M^* for the Pontryagin dual $\mathrm{Hom}_{\mathbb{Z}_p}(M,\mathbb{Q}_p/\mathbb{Z}_p)$ of a $\mathbb{Z}_p$-module M.

Lemma 3.1.2. *Let $L\subset\overline{\mathbb{Q}}$ be a number field and let $S\supset S_p(L)\bigcup S_\infty(L)$ be a finite set of places of L. There are canonical isomorphisms $Ш^2_S(L,\mathbb{Z}_p(1))\simeq Ш^1_S(L,\mathbb{Q}_p/\mathbb{Z}_p)^*\simeq X'_S(L)$.*

Proof. The first isomorphism is given by the Poitou-Tate duality theorem [Mil86, Theorem 4.10 (a)]. Since $\mathrm{H}^1_S(L,\mathbb{Q}_p/\mathbb{Z}_p)=\mathrm{Hom}(\mathrm{Gal}(L_S/L),\mathbb{Q}_p/\mathbb{Z}_p)$ we easily have $Ш^1(L,\mathbb{Q}_p/\mathbb{Z}_p)=\mathrm{Hom}(X'_S(L),\mathbb{Q}_p/\mathbb{Z}_p)$ by class field theory. ☐

The isomorphisms provided by Lemma 3.1.2 are functorial in L in the sense that, given a finite extension L'/L of number fields, the norm map $X(L')\to X(L)$ corresponds to the corestriction map (resp. to the Pontryagin dual of the restriction map) $Ш^2(L',\mathbb{Z}_p(1))\to Ш^2(L,\mathbb{Z}_p(1))$ (resp. $Ш^1(L',\mathbb{Q}_p/\mathbb{Z}_p)^*\to Ш^1(L,\mathbb{Q}_p/\mathbb{Z}_p)^*$).

Given any $\mathbb{Z}_p$-extension $K_\infty = \bigcup_n K_n$ of K and any finite set $S \supset S_p(K) \bigcup S_\infty(K)$, Lemma 3.1.2 provides isomorphisms of $\mathbb{Z}_p[[\Gamma]]$-modules

$$\varprojlim_n Ш_S^2(K_n, \mathbb{Z}_p(1)) \simeq Ш_S^1(K_\infty, \mathbb{Q}_p/\mathbb{Z}_p)^* \simeq X'_S(K_\infty).$$

We now make use of the inflation-restriction exact sequence to study the problem of Galois descent. We have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(\Gamma, \mathbb{Q}_p/\mathbb{Z}_p) \arrow[r] \arrow[d] & \mathrm{H}^1_{\{p,\infty\}}(K, \mathbb{Q}_p/\mathbb{Z}_p) \arrow[r] \arrow[d] & \mathrm{H}^1_{\{p,\infty\}}(K_\infty, \mathbb{Q}_p/\mathbb{Z}_p)^\Gamma \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \bigoplus_v \mathrm{Hom}(\Gamma_v, \mathbb{Q}_p/\mathbb{Z}_p) \arrow[r] & \bigoplus_v \mathrm{H}^1(K_v, \mathbb{Q}_p/\mathbb{Z}_p) \arrow[r] & \bigoplus_w \mathrm{H}^1(K_{\infty,w}, \mathbb{Q}_p/\mathbb{Z}_p), & \end{tikzcd} \tag{3}$$

where the places v (resp. w) of the second row run through all the p-adic and archimedean places of K (resp. of K_∞). Here, we have used the fact that Γ has cohomological dimension one as it is pro-cyclic.

Proposition 3.1.3. *We have* $\delta^{\mathbf{G}}_{K_\infty/K} = \dim\ker(\mathrm{Loc}_{K_\infty/K})$*, where* $\mathrm{Loc}_{K_\infty/K}$ *is the localization map*

$$\mathrm{Loc}_{K_\infty/K} \colon \mathrm{H}^1_{\{p\}}(K, \overline{\mathbb{Q}}_p)/\mathrm{H}^1(\Gamma, \overline{\mathbb{Q}}_p) \longrightarrow \bigoplus_{\mathfrak{P} \in S_p(K)} \mathrm{H}^1(K_{\mathfrak{P}}, \overline{\mathbb{Q}}_p)/\mathrm{H}^1(\Gamma_{\mathfrak{P}}, \overline{\mathbb{Q}}_p).$$

Proof. By Lemma 3.1.2, the kernel of the right vertical map of (3) is equal to the Pontryagin dual of $X'(K_\infty)_\Gamma$. Since $\mathrm{H}^1(K_v, \mathbb{Q}_p/\mathbb{Z}_p)$ for $v|\infty$ is finite, this implies that $\delta^{\mathbf{G}}_{K_\infty/K}$ is equal to the rank of the Pontryagin dual of the kernel of the natural map

$$\mathrm{H}^1_{\{p\}}(K, \mathbb{Q}_p/\mathbb{Z}_p)/\mathrm{H}^1(\Gamma, \mathbb{Q}_p/\mathbb{Z}_p) \to \bigoplus_{\mathfrak{P} \in S_p(K)} \mathrm{H}^1(K_{\mathfrak{P}}, \mathbb{Q}_p/\mathbb{Z}_p)/\mathrm{H}^1(\Gamma_{\mathfrak{P}}, \mathbb{Q}_p/\mathbb{Z}_p).$$

To end the proof, it suffices to notice that for $\mathcal{G} = \mathrm{Gal}(K_S/K)$, $\mathrm{Gal}(\overline{K}_{\mathfrak{P}}/K_{\mathfrak{P}})$, Γ or $\Gamma_{\mathfrak{P}}$, the natural map $\mathrm{H}^1(\mathcal{G}, \mathbb{Z}_p) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{H}^1(\mathcal{G}, \mathbb{Q}_p/\mathbb{Z}_p)$ (resp. $\mathrm{H}^1(\mathcal{G}, \mathbb{Z}_p) \otimes \overline{\mathbb{Q}}_p \to \mathrm{H}^1(\mathcal{G}, \overline{\mathbb{Q}}_p)$) has finite kernel and cokernel (resp. is an isomorphism), see [Rub00, Appendix B § 2]. □

Remark 3.1.4. Since $\mathrm{H}^1_{\{p\}}(K, \overline{\mathbb{Q}}_p) = \mathrm{Hom}(G_K, \overline{\mathbb{Q}}_p)$ parameterizes the $\mathbb{Z}_p$-extensions of K, the domain of $\mathrm{Loc}_{K_\infty/K}$ has dimension $r_2 + \delta^{\mathbf{L}}_K$, where r_2 is the number of complex places of K ([Was97, Theorem 13.4]). In particular, Proposition 3.1.3 yields an upper bound $\delta^{\mathbf{G}}_{K_\infty/K} \leq r_2 + \delta^{\mathbf{L}}_K$. Therefore, Leopoldt's conjecture for a totally real field K implies the Gross-Kuz'min conjecture for K, as already noticed by Kolster in [Kol91, Corollary 1.3].

Given a prime $\mathfrak{P} \in S_p(K)$, let $\Gamma_{\mathfrak{P}}$ be the decomposition subgroup of Γ at $\mathfrak{P}$ and denote by $\mathrm{rec}_{\Gamma_{\mathfrak{P}}} \colon K_{\mathfrak{P}}^\times \to \Gamma_{\mathfrak{P}}$ the corresponding local reciprocity map. Define also the $\mathbb{Z}_p$-module

$$\mathcal{H}_{K_\infty/K} := \ker\left(\bigoplus_{\mathfrak{P} \in S_p(K)} \Gamma_{\mathfrak{P}} \longrightarrow \Gamma\right).$$

By the usual product formula in class field theory the regulator map

$$\mathscr{L}_{K_\infty/K} \colon \left\{ \begin{array}{ccc} \mathcal{O}_K[\frac{1}{p}]^\times & \longrightarrow & \mathcal{H}_{K_\infty/K} \\ x & \mapsto & (\mathrm{rec}_{\Gamma_{\mathfrak{P}}}(x))_{\mathfrak{P}}, \end{array} \right. \tag{4}$$

is well-defined, and it extends to a $\overline{\mathbb{Q}}_p$-linear map $\mathcal{O}_K[\frac{1}{p}]^{\times,\wedge} \to \mathcal{H}^\wedge_{K_\infty/K}$ which we still denote by $\mathscr{L}_{K_\infty/K}$. If $K_\infty = K_{\mathrm{cyc}}$, then the character $\log_p \circ \chi_{\mathrm{cyc}} \circ \mathrm{rec}_{\Gamma_{\mathfrak{P}}} \colon K_{\mathfrak{P}}^\times \longrightarrow \mathbb{Q}_p$ coincides with

$-\log_p \circ \mathcal{N}_{\mathfrak{P}}$, where χ_{cyc} is the cyclotomic character. Therefore, $\mathscr{L}_{K_{\text{cyc}}/K}$ is essentially the same as the map $\mathscr{L}_K$ of the introduction, and we easily see that $\delta^{\mathbf{G}}_{K_{\text{cyc}}/K} = \delta^{\mathbf{G}}_K$.

Proposition 3.1.5. *We have* $\delta^{\mathbf{G}}_{K_\infty/K} = \dim\operatorname{coker}(\mathscr{L}_{K_\infty/K})$.

Proof. By Kummer theory and local class field theory, Tate's local pairing $\mathrm{H}^1(K_{\mathfrak{P}},\overline{\mathbb{Q}}_p) \times \mathrm{H}^1(K_{\mathfrak{P}},\overline{\mathbb{Q}}_p(1)) \to \overline{\mathbb{Q}}_p$ can be identified with the evaluation map $\operatorname{Hom}(K^\times_{\mathfrak{P}},\overline{\mathbb{Q}}_p) \times K^{\times,\wedge}_{\mathfrak{P}} \to \overline{\mathbb{Q}}_p$. Therefore, the orthogonal complement of $\mathrm{H}^1(\Gamma_{\mathfrak{P}},\overline{\mathbb{Q}}_p) \subseteq \mathrm{H}^1(K_{\mathfrak{P}},\overline{\mathbb{Q}}_p)$ is simply the kernel of $\operatorname{rec}_{\Gamma_{\mathfrak{P}}} : K^{\times,\wedge}_{\mathfrak{P}} \twoheadrightarrow \Gamma^\wedge_{\mathfrak{P}}$. Using the fact that $\text{Ш}^1(K,\mathbb{Q}_p) = 0$, Poitou-Tate's duality then yields an isomorphism

$$\ker\left(\mathrm{H}^1_{\{p\}}(K,\overline{\mathbb{Q}}_p) \to \bigoplus_{\mathfrak{P}\in S_p(K)} \frac{\mathrm{H}^1(K_{\mathfrak{P}},\overline{\mathbb{Q}}_p)}{\mathrm{H}^1(\Gamma_{\mathfrak{P}},\overline{\mathbb{Q}}_p)}\right) \simeq \operatorname{coker}\left(\oplus_{\mathfrak{P}} \operatorname{rec}_{\Gamma_{\mathfrak{P}}} : \mathcal{O}_K[\tfrac{1}{p}]^{\times,\wedge} \to \bigoplus_{\mathfrak{P}\in S_p(K)} \Gamma^\wedge_{\mathfrak{P}}\right).$$

In particular, $\dim\ker(\operatorname{Loc}_{K_\infty/K}) + 1 = \dim\operatorname{coker}(\mathscr{L}_{K_\infty/K}) + 1$, so Proposition 3.1.3 yields the desired equality. □

3.2. **Isotypic components.** We consider in this paragraph the situation where the $\mathbb{Z}_p$-extension K_∞/K comes from the $\mathbb{Z}_p$-extension k_∞/k of a subfield k of K, which means that $K_\infty = Kk_\infty$. Assume that K/k is Galois with Galois group G. Given an algebraically closed field $\mathbf{Q}$ of characteristic zero (typically, $\overline{\mathbb{Q}}$ or $\overline{\mathbb{Q}}_p$), the $\mathbf{Q}$-valued representations of G are semi-simple and the regular representation of G splits as

$$\mathbf{Q}[G] = \bigoplus_\rho e(\rho)\cdot\mathbf{Q}[G] = \bigoplus_\rho W^{\oplus\dim\rho},$$

where (W,ρ) runs through the set of all the $\mathbf{Q}$-valued irreducible representations of G and $e(\rho) = \frac{\dim\rho}{|G|}\cdot\sum_{g\in G}\operatorname{Tr}(\rho(g^{-1}))g \in \mathbf{Q}[G]$ is the usual idempotent attached to ρ.

For any finite set of places S of k containing $S_\infty(k)$, let $\mathcal{O}_K[1/S]^\times$ be the group of S-units of K. Dirichlet's unit theorem implies that we have a decomposition of $\mathbf{Q}[G]$-modules

$$\mathbf{Q}\otimes_{\mathbb{Z}} \mathcal{O}_K[1/S]^\times = \left(\mathbf{Q}\otimes_{\mathbb{Z}}\mathcal{O}_k[1/S]^\times\right)\bigoplus\left(\bigoplus_{\mathbb{1}\neq\rho} W^{\oplus d^+_S(\rho)}\right),$$

where (W,ρ) runs through the set of all non-trivial irreducible representations of G and $d^+_S(\rho) = \sum_{v\in S}\dim\mathrm{H}^0(k_v,W)$. It will be convenient to introduce the following invariants:

$$(5)\qquad d(\rho) := [k:\mathbb{Q}]\cdot\dim\rho,\qquad d^+(\rho) := \sum_{v|\infty}\dim\mathrm{H}^0(k_v,W),\qquad f(\rho) := \sum_{\mathfrak{p}|p}\dim\mathrm{H}^0(k_{\mathfrak{p}},W),$$

so that $d^+(\rho) = d^+_{S_\infty(k)}(\rho)$ and $f(\rho) = d^+_{S_\infty(k)\cup S_p(k)}(\rho) - d^+_{S_\infty(k)}(\rho)$.

We record in the next lemma a list of useful properties satisfied by the invariants introduced in (5) and which we make use of in Sections 4 and 5. Recall that a rule $\rho\mapsto a(\rho)\in\mathbb{Z}$, where ρ runs among all the representations of Galois groups of finite extensions of number fields, is said to be compatible with Artin formalism if, for all finite Galois extensions $M/L/E$:

(a) $a(\tilde\rho) = a(\rho)$ if $\tilde\rho \in \operatorname{Art}_{\overline{\mathbb{Q}}_p}(\operatorname{Gal}(M/E))$ is the inflation of $\rho \in \operatorname{Art}_{\overline{\mathbb{Q}}_p}(\operatorname{Gal}(L/E))$,
(b) $a(\rho_1\oplus\rho_2) = a(\rho_1) + a(\rho_2)$ for all $\rho_1,\rho_2 \in \operatorname{Art}_{\overline{\mathbb{Q}}_p}(\operatorname{Gal}(M/E))$, and
(c) $a(\rho') = a(\rho)$ if ρ' is the induction of ρ from $\operatorname{Gal}(M/L)$ to $\operatorname{Gal}(M/E)$.

Lemma 3.2.1. *Let L/E be a finite extension of number fields and let $a\in\{d,d^+,f\}$.*

(1) The rule $\rho\mapsto a(\rho)$ is compatible with Artin formalism.

(2) We have $d(\mathbb{1}_L)=[L:\mathbb{Q}]$, $d^+(\mathbb{1}_L)=|S_\infty(L)|$ *and* $f(\mathbb{1}_L)=|S_p(L)|$. *For any representation* $(W,\rho)\in Art_{\overline{\mathbb{Q}}_p}(\mathrm{Gal}(M/L))$ *with* $\mathbb{1}_L\not\subset\rho$, *we have* $\dim\mathrm{Hom}_{\mathrm{Gal}(M/L)}(W,\mathcal{O}_M^{\times,\wedge})=d^+(\rho)$ *and* $\dim\mathrm{Hom}_{\mathrm{Gal}(M/L)}(W,\mathcal{O}_M[\frac{1}{p}]^{\times,\wedge})=d^+(\rho)+f(\rho)$.

(3) Let $\rho\in Art_{\overline{\mathbb{Q}}_p}(G_L)$ *and let* $M=\overline{\mathbb{Q}}^{\ker\rho}$ *be the field extension cut out by* ρ. *Then* M *is totally real (resp.* L *is totally real and* M *is a CM field) if and only if* $d^+(\rho)=d(\rho)$ *(resp.* $d^+(\rho)=0$*). If* M *has at least one real place, then* $d^+(\rho)\geq\dim\rho$ *and any subrepresentation* θ *of* $\mathrm{Ind}_L^E\rho$ *satisfies* $d^+(\theta)\geq 1$. *If* L *has* r *complex places, then* $d^+(\rho)\geq r\cdot\dim\rho$.

(4) Let $\rho\in Art_{\overline{\mathbb{Q}}_p}(G_L)$. *Then we have* $a(\rho)\leq(\dim\rho)\cdot a(\mathbb{1}_L)$.

(5) Let M/L *be a finite Galois extension, let* $\theta\in Art_{\overline{\mathbb{Q}}_p}(\mathrm{Gal}(M/L))$ *be irreducible and let* χ *be a multiplicative character of* G_L. *Then we have* $(\dim\theta)\cdot a(\theta\otimes\chi)\leq a(\chi_{|G_M})\leq a(\mathbb{1}_M)$. *Moreover, if* $\theta\neq\mathbb{1}_L$, *then we have* $(\dim\theta)\cdot a(\theta\otimes\chi)\leq a(\chi_{|G_M})-a(\chi)\leq a(\mathbb{1}_M)-a(\chi)$.

Proof. The claims (1), (2) and (4) are easy to deduce from the definitions and Dirichlet's unit theorem. Let us justify claim (3). It is clear that $d^+(\rho)\leq d(\rho)$, and we have $d^+(\rho)=d(\rho)$ (resp. $d^+(\rho)=0$) if and only if all archimedean places of L are real and $\rho(\sigma)=\mathrm{id}$ (resp. $\rho(\sigma)=-\mathrm{id}$) for all complex conjugations $\sigma\in\mathrm{Gal}(M/L)$. Since ρ is faithful on $\mathrm{Gal}(M/L)$, this is equivalent to M being totally real (resp. M being CM and L totally real). Assume now that M has at least one real place w and let v (resp. v_0) be the place of L (resp. of E) lying below w. Then we clearly have $d^+(\rho)\geq\dim\mathrm{H}^0(L_v,\rho)=\dim\rho$. Moreover, if $\theta\subset\mathrm{Ind}_L^E\rho$, then the Frobenius reciprocity implies that there exists a subrepresentation $\rho'\subset\rho$ such that $\rho'\subset\theta_{|G_L}$, yielding $d^+(\theta)\geq\dim\mathrm{H}^0(E_{v_0},\theta)=\dim\mathrm{H}^0(L_v,\theta)\geq\mathrm{H}^0(L_v,\rho')=\dim\rho'\geq 1$ as claimed. Suppose finally that L has r complex places $v_1,\dots,v_r$. Then $d^+(\rho)\geq\sum_{i=1}^r\dim\mathrm{H}^0(L_{v_i},\rho)=r\cdot\dim\rho$, so this ends the proof of claim (3).

We now prove claim (5). The upper bounds on $a(\chi_{|G_M})$ directly follow from claim (4), so we only prove the lower bounds. Since θ is irreducible, the representation $(\theta\otimes\chi)^{\oplus\dim\theta}$ (and even $(\theta\otimes\chi)^{\oplus\dim\theta}\oplus\chi$ if $\theta\neq\mathbb{1}_L$) occurs as a subrepresentation of $(\mathrm{Ind}_M^L\mathbb{1}_M)\otimes\chi=\mathrm{Ind}_M^L\chi_{|G_M}$. Artin formalism then yields the lower bounds of claim (5), as $a(\mathrm{Ind}_M^L\chi_{|G_M})=a(\chi_{|G_M})$ and a takes non-negative values. ☐

We now describe the isotypic components of the map $\mathscr{L}_{K_\infty/K}$. For $g\in G$, $\mathfrak{P}\in S_p(K)$ and η a place of K_∞ above $\mathfrak{P}$, the map $K_{\mathfrak{P}}\to K_{g(\mathfrak{P})}$ (resp. $K_{\infty,\eta}\to K_{\infty,\tilde{g}(\eta)}$) induced by g (resp. by a lift $\tilde{g}\in\mathrm{Gal}(K_\infty/k)$ of g) is a field isomorphism which yields a left G-action $x\mapsto g(x)$ (resp. $\gamma\mapsto\tilde{g}\gamma\tilde{g}^{-1}$) on $\bigoplus_{\mathfrak{P}|p}K_{\mathfrak{P}}^\times$ and on $\bigoplus_{\mathfrak{P}|p}\Gamma_{\mathfrak{P}}$ respectively. This action also restricts to $\mathcal{H}_{K_\infty/K}$, and G acts trivially on the quotient $(\bigoplus_{\mathfrak{P}|p}\Gamma_{\mathfrak{P}})/\mathcal{H}_{K_\infty/K}$. Moreover, the map $\mathscr{L}_{K_\infty/K}$ is G-equivariant for the natural G-action on $\mathcal{O}_K[\frac{1}{p}]^\times$ and the action on $\mathcal{H}_{K_\infty/K}$ described above.

Fix any $(W,\rho)\in\mathrm{Art}_{\overline{\mathbb{Q}}_p}(G)$ and let $\mathrm{Hom}_G(X,Y)$ be the $\overline{\mathbb{Q}}_p$-vector space of all G-equivariant linear maps between two $\overline{\mathbb{Q}}_p[G]$-modules X and Y. By definition, the ρ-isotypic component of a G-equivariant $\overline{\mathbb{Q}}_p$-linear map $f:X\to Y$ is the linear map $\mathrm{Hom}_G(W,X)\to\mathrm{Hom}_G(W,Y)$ obtained by post-composing by f. Write $\mathscr{L}_{k_\infty/k}(\rho)$ for the ρ-isotypic component of $\mathscr{L}_{K_{\mathrm{cyc}}/K}$ and define

$$\delta_{k_\infty/k}^{\mathbf{G}}(\rho):=\dim\mathrm{coker}\left(\mathscr{L}_{k_\infty/k}(\rho)\right).$$

If $k_\infty=k_{\mathrm{cyc}}$, we abbreviate $\mathscr{L}_{k_\infty/k}(\rho)$ and $\delta_{k_\infty/k}^{\mathbf{G}}(\rho)$ as $\mathscr{L}_k(\rho)$ and $\delta_k^{\mathbf{G}}(\rho)$ respectively.

For all $\mathfrak{p}\in S_p(k)$, fix a place $\mathfrak{P}_0$ of K above $\mathfrak{p}$, let $G_{\mathfrak{p}}$ be the decomposition subgroup of G at $\mathfrak{P}_0$, let $W_{\mathfrak{p}}^0=W^{G_{\mathfrak{p}}}$ and denote by $\mathrm{res}_{\mathfrak{p}}$ the restriction-to-$W_{\mathfrak{p}}^0$ map.

Proposition 3.2.2. *The map $\mathscr{L}_{k_\infty/k}(\mathbb{1})$ can be naturally identified with $\mathscr{L}_{k_\infty/k}$. If $\mathbb{1} \not\subset \rho$, then the map $\mathscr{L}_{k_\infty/k}(\rho)$ can be naturally identified with the composite map*
(6)
$$\mathrm{Hom}_G(W, \mathcal{O}_K[\tfrac{1}{p}]^{\times,\wedge}) \xrightarrow{\oplus \mathrm{loc}_{\mathfrak{p}}} \bigoplus_{\mathfrak{p}} \mathrm{Hom}_{G_{\mathfrak{p}}}(W, K_{\mathfrak{P}_0}^{\times,\wedge}) \xrightarrow{\oplus \mathrm{res}_{\mathfrak{p}}} \bigoplus_{\mathfrak{p}} \mathrm{Hom}(W_{\mathfrak{p}}^0, k_{\mathfrak{p}}^{\times,\wedge}) \xrightarrow{\oplus \mathrm{rec}_{\Gamma_{\mathfrak{p}}}} \bigoplus_{\mathfrak{p}} \mathrm{Hom}(W_{\mathfrak{p}}^0, \Gamma_{\mathfrak{p}}^{\wedge}).$$

Here, $\mathfrak{p}$ runs over $S_p(k)$ in each sum, and we implicitly used the fact that $\left(K_{\mathfrak{P}_0}^{\times,\wedge}\right)^{G_{\mathfrak{p}}} = k_{\mathfrak{p}}^{\times,\wedge}$.

Proof. Let $j\colon \mathcal{H}_{K_\infty/K} \hookrightarrow \bigoplus_{\mathfrak{P}|p} \Gamma_{\mathfrak{P}}$ be the inclusion map and let $j(\rho)$ be its ρ-isotypic component.

Given a prime $\mathfrak{p} \in S_p(k)$ and a fixed prime $\mathfrak{P}_0|\mathfrak{p}$ of K as before, we have $\bigoplus_{\mathfrak{P}|\mathfrak{p}} K_{\mathfrak{P}}^{\times} = \mathrm{Ind}_{G_{\mathfrak{p}}}^{G} K_{\mathfrak{P}_0}^{\times}$ and $\bigoplus_{\mathfrak{P}|\mathfrak{p}} \Gamma_{\mathfrak{P}} = \mathrm{Ind}_{G_{\mathfrak{p}}}^{G} \Gamma_{\mathfrak{P}_0}$ as G-modules and Frobenius reciprocity shows that $\mathrm{Hom}_G(W, \bigoplus_{\mathfrak{P}|\mathfrak{p}} K_{\mathfrak{P}}^{\times,\wedge}) \simeq \mathrm{Hom}_{G_{\mathfrak{p}}}(W, K_{\mathfrak{P}_0}^{\times,\wedge})$ and $\mathrm{Hom}_G(W, \bigoplus_{\mathfrak{P}|\mathfrak{p}} \Gamma_{\mathfrak{P}}^{\times,\wedge}) \simeq \mathrm{Hom}_{G_{\mathfrak{p}}}(W, \Gamma_{\mathfrak{P}_0}^{\times,\wedge})$, the isomorphisms being the natural projection maps. Therefore, $j(\rho) \circ \mathscr{L}_{k_\infty/k}(\rho)$ can be identified with the composite map

$$\mathrm{Hom}_G(W, \mathcal{O}_K[\tfrac{1}{p}]^{\times,\wedge}) \xrightarrow{\oplus \mathrm{loc}_{\mathfrak{p}}} \bigoplus_{\mathfrak{p}|p} \mathrm{Hom}_{G_{\mathfrak{p}}}(W, K_{\mathfrak{P}_0}^{\times,\wedge}) \xrightarrow{\oplus \mathrm{rec}_{\Gamma_{\mathfrak{P}_0}}} \bigoplus_{\mathfrak{p}|p} \mathrm{Hom}_{G_{\mathfrak{p}}}(W, \Gamma_{\mathfrak{P}_0}^{\wedge}) = \bigoplus_{\mathfrak{p}|p} \mathrm{Hom}(W_{\mathfrak{p}}^0, \Gamma_{\mathfrak{P}_0}^{\wedge}),$$

where the last identification is induced by $\oplus \mathrm{res}_{\mathfrak{p}}$. Note that $\mathrm{res}_{\mathfrak{p}}$ and $\mathrm{rec}_{\Gamma_{\mathfrak{P}_0}}$ commute. Furthermore, letting $[n_{\mathfrak{p}}]\colon \Gamma_{\mathfrak{P}_0}^{\wedge} \to \Gamma_{\mathfrak{p}}^{\wedge}$ be the multiplication by $n_{\mathfrak{p}} = [K_{\mathfrak{P}_0} : k_{\mathfrak{p}}]$ map, the functoriality of Artin's reciprocity law shows that $\mathrm{rec}_{\mathfrak{p}}$ coincides with $[n_{\mathfrak{p}}] \circ \mathrm{rec}_{\mathfrak{P}_0}$ on $k_{\mathfrak{p}}^{\times,\wedge}$. Hence, if $\mathbb{1} \not\subset \rho$, then the map $j(\rho) \circ \mathscr{L}_{k_\infty/k}(\rho)$ coincides with the map of (6) under the identification $[n_{\mathfrak{p}}]\colon \Gamma_{\mathfrak{P}_0}^{\wedge} \simeq \Gamma_{\mathfrak{p}}^{\wedge}$. Since $j(\rho)$ is an isomorphism in this case, we obtain the desired description of $\mathscr{L}_{k_\infty/k}(\rho)$. In the case where $\rho = \mathbb{1}$, the map $\mathscr{L}_{K_\infty/K}(\mathbb{1})$ is nothing but the restriction $(\mathscr{L}_{K_\infty/K})^G\colon \mathcal{O}_k[\tfrac{1}{p}]^{\times,\wedge} \to (\mathcal{H}_{K_\infty/K}^{\wedge})^G$ of $\mathscr{L}_{K_\infty/K}$ to the G-invariants. Under the identifications

$$(\bigoplus_{\mathfrak{P}|p} \Gamma_{\mathfrak{P}}^{\wedge})^G = (\bigoplus_{\mathfrak{p}|p} \mathrm{Ind}_{G_{\mathfrak{p}}}^{G} \Gamma_{\mathfrak{P}_0}^{\wedge})^G = \bigoplus_{\mathfrak{p}|p} \Gamma_{\mathfrak{P}_0}^{\wedge} \simeq \bigoplus_{\mathfrak{p}|p} \Gamma_{\mathfrak{p}}^{\wedge}$$

induced by by Frobenius reciprocity and by $\oplus_{\mathfrak{p}}[n_{\mathfrak{p}}]$, it is clear that $(\mathcal{H}_{K_\infty/K}^{\wedge})^G$ is mapped onto $\mathcal{H}_{k_\infty/k}^{\wedge}$, and that $\mathscr{L}_{K_{\mathrm{cyc}}/K}(\mathbb{1})$ can be naturally identified with $\mathscr{L}_{k_\infty/k}$.

∎

Remark 3.2.3. The map $\mathscr{L}_{k_\infty/k}(\rho)$ admits a more intrinsic description in terms of Bloch-Kato Selmer groups. Namely, let $\mathrm{H}^1_{\mathrm{f},p}(k, \check{W}) \subset \mathrm{H}^1(k, \check{W})$ be the Selmer group of $\check{W}$ defined by the Bloch-Kato condition at all places not dividing p (see [BK90, § 3]), and let $\mathrm{H}^1_{\Gamma}(k_{\mathfrak{p}}, \check{W}_{\mathfrak{p}}^0)$ be the orthogonal complement of $\mathrm{H}^1(\Gamma_{\mathfrak{p}}, W_{\mathfrak{p}}^0) \subset \mathrm{H}^1(k_{\mathfrak{p}}, W)$ under Tate's local pairing. Then Kummer theory and the Inflation-Restriction exact sequence provide natural isomorphisms $\mathrm{H}^1_{\mathrm{f},p}(k, \check{W}) \simeq \mathrm{Hom}_{G_k}(W, \mathcal{O}_K[\tfrac{1}{p}]^{\times,\wedge})$ and $\mathrm{H}^1(k_{\mathfrak{p}}, \check{W})/\mathrm{H}^1_{\Gamma}(k_{\mathfrak{p}}, \check{W}_{\mathfrak{p}}^0) \simeq \mathrm{Hom}(W_{\mathfrak{p}}^0, \Gamma_{\mathfrak{p}})$. An easy adaptation of the proof of Proposition 3.1.5 then shows that $\mathscr{L}_{k_\infty/k}(\rho)$ coincides with the localization map $\mathrm{H}^1_{f,p}(k, \check{W}) \to \bigoplus_{\mathfrak{p}|p} \mathrm{H}^1(k_{\mathfrak{p}}, \check{W})/\mathrm{H}^1_{\Gamma}(k_{\mathfrak{p}}, \check{W}_{\mathfrak{p}}^0)$ under these identifications.

Corollary 3.2.4. *(1) If $\rho = \mathbb{1}$ then we have $\delta_{k_\infty/k}^{\boldsymbol{G}}(\mathbb{1}) = \delta_{k_\infty/k}^{\boldsymbol{G}}$.*

(2) Assume $k = \mathbb{Q}$, $k_\infty = \mathbb{Q}_{cyc}$ and $\mathbb{1} \not\subset \rho$. Fix an embedding $\iota_p\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and let $W_p^0 = W^{G_{\mathbb{Q}_p}}$. Then,

$$\delta_{\mathbb{Q}}^{\boldsymbol{G}}(\rho) = \dim \operatorname{coker}\left[\mathrm{Hom}_G(W, \mathcal{O}_K[\tfrac{1}{p}]^{\times,\wedge}) \longrightarrow \mathrm{Hom}(W_p^0, \overline{\mathbb{Q}}_p)\right],$$

the map being the restriction-to-W_p^0 map followed by the post-composition by $\log_p \circ \iota_p$.

Proof. The first claim directly follows from Propositions 3.1.5 and 3.2.2. The second claim follows from from Proposition 3.2.2 and from the fact that, if $\Gamma_p = \mathrm{Gal}(\mathbb{Q}_{p,\mathrm{cyc}}/\mathbb{Q}_p)$, then the composite map $\log_p \circ \chi_{\mathrm{cyc}} \circ \mathrm{rec}_{\Gamma_p} : \mathbb{Q}_p^{\times,\wedge} \to \Gamma_p^{\wedge} \simeq \overline{\mathbb{Q}}_p \otimes_{\mathbb{Z}_p} (1+p\mathbb{Z}_p) \simeq \overline{\mathbb{Q}}_p$ coincides with $-\log_p$. $\square$

Corollary 3.2.5. *The assignment $\rho \mapsto \delta^{\boldsymbol{G}}_{k_\infty/k}(\rho)$ is compatible with Artin formalism. More precisely,*

(a) $\delta^{\boldsymbol{G}}_{k_\infty/k}(\rho)$ does not depend on the choice of the splitting field K of $\rho \in Art_{\overline{\mathbb{Q}}_p}(G_k)$.

(b) $\delta^{\boldsymbol{G}}_{k_\infty/k}(\rho_1 \oplus \rho_2) = \delta^{\boldsymbol{G}}_{k_\infty/k}(\rho_1) + \delta^{\boldsymbol{G}}_{k_\infty/k}(\rho_2)$ for any $\rho_1, \rho_2 \in Art_{\overline{\mathbb{Q}}_p}(G_k)$.

(c) If $k_\infty = kk'_\infty$ for some subfield $k' \subset k$ and $\mathbb{Z}_p$-extension k'_∞/k' and if $\rho' = \mathrm{Ind}_k^{k'}\rho$ is induced from $\rho \in Art_{\overline{\mathbb{Q}}_p}(G_k)$, then $\delta^{\boldsymbol{G}}_{k'_\infty/k'}(\rho') = \delta^{\boldsymbol{G}}_{k_\infty/k}(\rho)$.

Proof. Part (b) is obvious from the definition of $\mathscr{L}_{k_\infty/k}(\rho)$. Part (a) is true if ρ is trivial by Corollary 3.2.4 (1). Let $K'/K/k$ be Galois extensions, take $\mathbb{1} \not\subseteq \rho \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(\mathrm{Gal}(K/k))$ and denote by $\tilde{\rho}$ its inflation to $\mathrm{Gal}(K'/k)$. Then the maps of (6) for ρ and $\tilde{\rho}$ coincide on $\mathrm{Hom}_{\mathrm{Gal}(K/k)}(W, \mathcal{O}_K[\frac{1}{p}]^{\times,\wedge}) = \mathrm{Hom}_{\mathrm{Gal}(K'/k)}(W, \mathcal{O}_{K'}[\frac{1}{p}]^{\times,\wedge})$, so $\delta^{\mathbf{G}}_{k_\infty/k}(\rho) = \delta^{\mathbf{G}}_{k_\infty/k}(\tilde{\rho})$. Hence, (a) holds for every $\rho \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_k)$. For (c), take any splitting field K of ρ which is Galois over k' and put $G = \mathrm{Gal}(K/k)$, $G' = \mathrm{Gal}(K/k')$. Then Frobenius reciprocity identifies the ρ'-isotypic component of $\mathscr{L}_{K_\infty/K}$ (seen as a G'-equivariant map) with the ρ-isotypic component of $\mathscr{L}_{K_\infty/K}$ (seen as a G-equivariant map). Hence, the last property follows from Proposition 3.2.2. $\square$

We next define an analogous invariant for Leopoldt's conjecture. For any Galois extension K/k with Galois group G, the localization map

$$\iota_K : \mathcal{O}_K^{\times,\wedge} \longrightarrow \bigoplus_{\mathfrak{P} \in S_p(K)} \mathcal{O}_{K_{\mathfrak{P}}}^{\times,\wedge}$$

is clearly G-equivariant. For any $(W,\rho) \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G)$ we let $\delta^{\mathbf{L}}_k(\rho)$ be the dimension of the kernel of the ρ-isotypic component

$$(7) \qquad \iota_k(\rho) : \ \mathrm{Hom}_G(W, \mathcal{O}_K^{\times,\wedge}) \longrightarrow \bigoplus_{\mathfrak{p} \in S_p(k)} \mathrm{Hom}_G(W, \bigoplus_{\mathfrak{P}|\mathfrak{p}} \mathcal{O}_{K_{\mathfrak{P}}}^{\times,\wedge}) \simeq \bigoplus_{\mathfrak{p} \in S_p(k)} \mathrm{Hom}_{G_{\mathfrak{p}}}(W, \mathcal{O}_{K_{\mathfrak{P}_0}}^{\times,\wedge})$$

of ι_K. Here (and as in the definition of $\mathscr{L}_{k_\infty/k}(\rho)$), $\mathfrak{P}_0$ is a fixed place of K above $\mathfrak{p}$ for every place $\mathfrak{p}$ of k. The last isomorphism is given by Frobenius reciprocity and is induced by the natural projection map. As in Corollary 3.2.5, it is easy to see that the rule $\rho \mapsto \delta^{\mathbf{L}}_k(\rho)$ is compatible with Artin formalism.

Remark 3.2.6. In terms of Bloch-Kato Selmer groups for $\breve{W}$, the injectivity of $\iota_k(\rho)$ is equivalent to that of the localization map $\mathrm{H}^1_{\mathrm{f}}(k, \breve{W}) \to \prod_{\mathfrak{p}|p} \mathrm{H}^1_{\mathrm{f}}(k_{\mathfrak{p}}, \breve{W})$. This last statement is Jannsen's conjecture for $\breve{W}$ ([Jan89]).

The following lemma on local Galois representations with finite image will help us describe $\delta^{\mathbf{L}}_k(\rho)$ in another way if $k = \mathbb{Q}$.

Lemma 3.2.7. *Let $(W,\rho) \in Art_{\overline{\mathbb{Q}}_p}(G_{\mathbb{Q}_p})$ be a local representation factoring through the Galois group of a finite extension $E \subset \overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Then the internal multiplication map $\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} E \to \overline{\mathbb{Q}}_p$ induces an isomorphism*

$$m \ : \ \mathrm{Hom}_{G_{\mathbb{Q}_p}}(W, \overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} E) \simeq \mathrm{Hom}_{\overline{\mathbb{Q}}_p}(W, \overline{\mathbb{Q}}_p).$$

Here, we let $G_{\mathbb{Q}_p}$ act on $\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} E$ via $g(a \otimes x) = a \otimes g(x)$.

Proof. Since $E \simeq \mathbb{Q}_p[\mathrm{Gal}(E/\mathbb{Q}_p)]$ as a Galois module, it is enough to show that m is injective. Choose any finite Galois extension $L/\mathbb{Q}_p$ which contains E and over which ρ is realizable, *i.e.*, there exist a $L[G_{\mathbb{Q}_p}]$-module W_L and an isomorphism $W_L \otimes_L \overline{\mathbb{Q}}_p \simeq W$. Then we have to show that the map $\mathrm{Hom}_{L[G_{\mathbb{Q}_p}]}(W_L, L\otimes_{\mathbb{Q}_p} E) \to \mathrm{Hom}_L(W_L, L)$ is injective. By considering $\mathbb{Q}_p$-linear homomorphisms instead of L-linear ones in the previous map, it suffices to prove that the map $(V\otimes_{\mathbb{Q}_p}\overline{\mathbb{Q}}_p)^{G_{\mathbb{Q}_p}}\otimes_{\mathbb{Q}_p}\overline{\mathbb{Q}}_p \to V\otimes_{\mathbb{Q}_p}\overline{\mathbb{Q}}_p$ given by $(v\otimes a)\otimes b \mapsto v\otimes(ab)$ is injective, where $V = \mathrm{Hom}_{\mathbb{Q}_p}(W_L, \mathbb{Q}_p)$ and $G_{\mathbb{Q}_p}$ acts on both factors of $V\otimes_{\mathbb{Q}_p}\overline{\mathbb{Q}}_p$. But V is finite dimensional over $\mathbb{Q}_p$, so this follows from the $\overline{\mathbb{Q}}_p$-admissibility in the sense of Fontaine of Galois representations of finite image (see e.g. [FO08, Proposition 2.7]). □

Proposition 3.2.8. *Let $(W,\rho)\in Art_{\overline{\mathbb{Q}}_p}(G_{\mathbb{Q}})$, let $K\subset\overline{\mathbb{Q}}$ be a splitting field of ρ and let $\mathfrak{P}_0$ be the p-adic place of K defined by a fixed embedding $\iota_p:\overline{\mathbb{Q}}\hookrightarrow\overline{\mathbb{Q}}_p$. We have*

$$\delta^{\boldsymbol{L}}_{\mathbb{Q}}(\rho) = \dim\ker\left[\mathcal{L}\ :\ \mathrm{Hom}_{G_{\mathbb{Q}}}(W, \mathcal{O}_K^{\times,\wedge}) \to \mathrm{Hom}(W,\overline{\mathbb{Q}}_p)\right],$$

the map $\mathcal{L}$ being the post-composition by $\log_p:\mathcal{O}_K^{\times,\wedge}\to\overline{\mathbb{Q}}_p\otimes_{\mathbb{Q}_p}K_{\mathfrak{P}_0}$ followed by the internal multiplication $\overline{\mathbb{Q}}_p\otimes_{\mathbb{Q}_p}K_{\mathfrak{P}_0}\to\overline{\mathbb{Q}}_p$.

Proof. First note that the p-adic logarithm $\log_p:\mathcal{O}_E^\times\to E$ over a finite extension E of $\mathbb{Q}_p$ induces an isomorphism $\mathcal{O}_E^{\times,\wedge}\simeq\overline{\mathbb{Q}}_p\otimes_{\mathbb{Q}_p}E$. Applying Lemma 3.2.7 to $E=K_{\mathfrak{P}_0}$ we obtain isomorphisms $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(W,\mathcal{O}_{K_{\mathfrak{P}_0}}^{\times,\wedge})\simeq\mathrm{Hom}_{G_{\mathbb{Q}_p}}(W,\overline{\mathbb{Q}}_p\otimes_{\mathbb{Q}_p}K_{\mathfrak{P}_0})\simeq\mathrm{Hom}(W,\overline{\mathbb{Q}}_p)$. The map $\mathcal{L}$ is simply the map $\iota_{\mathbb{Q}}(\rho)$ composed with these isomorphisms, so the claim follows. □

4. Bounds on Leopoldt's and Gross's defects

Throughout this section we fix an embedding $\iota_p:\overline{\mathbb{Q}}\hookrightarrow\overline{\mathbb{Q}}_p$.

4.1. Bounds on Leopoldt's defect.

Theorem 4.1.1. *Let $\rho\in Art_{\overline{\mathbb{Q}}_p}(G_{\mathbb{Q}})$ be irreducible and let $d=d(\rho)$, $d^+=d^+(\rho)$. We have $\delta^{\boldsymbol{L}}_{\mathbb{Q}}(\rho)\le\frac{(d^+)^2}{d+d^+}$.*

Proof. Since $\delta^{\mathbf{L}}_{\mathbb{Q}}(\mathbb{1})=\delta^{\mathbf{L}}_{\mathbb{Q}}=0$, we may assume that $\rho\neq\mathbb{1}$. Let K be a splitting field of ρ and let $G=\mathrm{Gal}(K/\mathbb{Q})$. Recall from Lemma 3.2.1 that $\dim\mathrm{Hom}_G(W,\mathcal{O}_K^{\times,\wedge})=d^+$. By Proposition 3.2.8 it is enough to show that the rank of the map $\mathrm{Hom}_G(W,\mathcal{O}_K^{\times,\wedge})\to\mathrm{Hom}(W,\overline{\mathbb{Q}}_p)$ induced by $a\otimes x\mapsto a\log_p(\iota_p(x))$ on $\mathcal{O}_K^{\times,\wedge}$ is at least $\frac{d\cdot d^+}{d+d^+}$.

Consider a $\overline{\mathbb{Q}}$-structure on W, that is, a $\overline{\mathbb{Q}}$-linear representation $W_{\overline{\mathbb{Q}}}$ of G such that $W_{\overline{\mathbb{Q}}}\otimes_{\overline{\mathbb{Q}}}\overline{\mathbb{Q}}_p\simeq W$. Let also $w_1,\dots,w_d$ be a $\overline{\mathbb{Q}}$-basis of $W_{\overline{\mathbb{Q}}}$. Using an isomorphism $e(\rho)\cdot\left(\overline{\mathbb{Q}}\otimes\mathcal{O}_K^\times\right)\simeq W_{\overline{\mathbb{Q}}}^{\oplus d^+}$ we may define G-equivariant morphisms $\Psi_1,\dots,\Psi_{d^+}:W_{\overline{\mathbb{Q}}}\longrightarrow\overline{\mathbb{Q}}\otimes\mathcal{O}_K^\times$ which form a basis of $\mathrm{Hom}_G(W,\mathcal{O}_K^{\times,\wedge})$. Moreover, the elements $\Psi_j(w_i)\in\overline{\mathbb{Q}}\otimes\mathcal{O}_K^\times$ for $1\le i\le d$, $1\le j\le d^+$ are $\overline{\mathbb{Q}}$-linearly independent by construction, as well as their p-adic logarithms by Proposition 2.1.2. Hence, the matrix $M=(\log_p(\iota_p(\Psi_j(w_i))))_{i,j}$ of size $d\times d^+$ satisfies $\theta(M)=\frac{d^+}{d}$ and Theorem 2.2.1 implies $\mathrm{rk}\,M\ge\frac{d\cdot d^+}{d+d^+}$ as claimed. □

Corollary 4.1.2. *(1) Let $\rho\in Art_{\overline{\mathbb{Q}}_p}(G_k)$ and let $d^+=d^+(\rho)$. We have $\delta^{\boldsymbol{L}}_k(\rho)\le d^+/2$.*

(2) For every finite extensions K/k of number fields, we have $\delta^{\boldsymbol{L}}_K\le\delta^{\boldsymbol{L}}_k+\left(\mathrm{rk}\,\mathcal{O}_K^\times-\mathrm{rk}\,\mathcal{O}_k^\times\right)/2$.

Proof. Since $d^+ \le d$, the first inequality obviously follows from Theorem 4.1.1 for irreducible $\rho \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_\mathbb{Q})$, hence it follows for general ρ from Artin formalism. By Lemma 3.2.1, the unique representation ρ_0 of G_k such that $\mathrm{Ind}_K^k \mathbb{1} = \mathbb{1} \oplus \rho_0$ satisfies $d^+(\rho_0) = \mathrm{rk}\,\mathcal{O}_K^\times - \mathrm{rk}\,\mathcal{O}_k^\times$, so the second inequality follows from the first one applied to ρ_0. $\square$

4.2. **Bounds on Gross's defect.**

Theorem 4.2.1. *Let $\rho \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_\mathbb{Q})$ be irreducible and let $f = f(\rho)$, $d^+ = d^+(\rho)$. If $d^+ = f = 0$, then $\delta_\mathbb{Q}^{\mathbf{G}}(\rho) = 0$. Otherwise, we have $\delta_\mathbb{Q}^{\mathbf{G}}(\rho) \le \frac{f^2}{d^+ + 2f}$.*

Proof. Recall from Section 3.2 that $\dim \mathrm{Hom}_G(W, \mathcal{O}_K[\frac{1}{p}]^{\times,\wedge}) = d^+ + f$ and that $\dim W_p^0 = f$. If $\rho = \mathbb{1}$ or $f = 0$, then the codomain of $\mathscr{L}_\mathbb{Q}(\rho)$ is $\{0\}$, yielding $\delta_\mathbb{Q}^{\mathbf{G}}(\rho) = 0$. We assume henceforth that $\rho \ne \mathbb{1}$ and $f > 0$. Let K be a splitting field of ρ and let $G = \mathrm{Gal}(K/\mathbb{Q})$. By Corollary 3.2.4 (2) and Proposition 3.2.2 it is enough to show that the rank of the map $\mathrm{Hom}_G(W, \mathcal{O}_K[\frac{1}{p}]^{\times,\wedge}) \to \mathrm{Hom}(W_p^0, \overline{\mathbb{Q}}_p)$ induced by $a \otimes x \mapsto a \log_p(\iota_p(x))$ on $\mathcal{O}_K[\frac{1}{p}]^{\times,\wedge}$ is at least $\frac{(d^+ + f)\cdot f}{d^+ + 2f}$.

As in the proof of Theorem 4.1.1, fix a $\overline{\mathbb{Q}}$-structure $W_{\overline{\mathbb{Q}}}$ of W. Fix also a basis $w_1, \dots, w_f$ of the subspace $W_{\overline{\mathbb{Q}},p}^0$ of $G_{\mathbb{Q}_p}$-invariants of $W_{\overline{\mathbb{Q}}}$ and an isomorphism $e(\rho)\cdot\left(\overline{\mathbb{Q}} \otimes \mathcal{O}_K[\frac{1}{p}]^\times\right) \simeq W_{\overline{\mathbb{Q}}}^{\oplus(d^+ + f)}$. These choices yield a basis $\Psi_1, \dots, \Psi_{d^+ + f}$ of $\mathrm{Hom}_G(W, \mathcal{O}_K[\frac{1}{p}]^{\times,\wedge})$ such that the elements $\Psi_j(w_i) \in \overline{\mathbb{Q}} \otimes \mathcal{O}_K^\times$ for $1 \le i \le f$, $1 \le j \le d^+ + f$ are $\overline{\mathbb{Q}}$-linearly independent. Since $e(\rho)$ kills $p^{\overline{\mathbb{Q}}}$, we deduce from Proposition 2.1.2 that the entries of the matrix $M' = (\log_p(\iota_p(\Psi_j(w_i))))_{i,j}$ of size $f \times (d^+ + f)$ are $\overline{\mathbb{Q}}$-linearly independent as well. Therefore, $\theta(M') = \frac{d^+ + f}{f}$ and Theorem 2.2.1 implies $\mathrm{rk}\, M' \ge \frac{(d^+ + f)\cdot f}{d^+ + 2f}$. $\square$

Corollary 4.2.2. *(1) Let $\rho \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_k)$, let K be the field cut out by ρ and let $f = f(\rho)$. Then the following inequalities hold true:*

$$\delta_k^{\boldsymbol{G}}(\rho) \begin{cases} \le & f/2 \\ < & f/2 \quad \text{if } f \ne 0 \text{ and } K \text{ contains at least one real place} \\ \le & f/3 \quad \text{if } K \text{ is totally real} \end{cases}$$

(2) Let K/k be a finite extension of number fields. Then the following inequalities hold true:

$$\delta_K^{\boldsymbol{G}} \begin{cases} \le & \delta_k^{\boldsymbol{G}} + \big(|S_p(K)| - |S_p(k)|\big)/2 \\ < & \delta_k^{\boldsymbol{G}} + \big(|S_p(K)| - |S_p(k)|\big)/2 \quad \text{if } |S_p(K)| \ne |S_p(k)| \text{ and } K \text{ contains at least one real place} \\ \le & \delta_k^{\boldsymbol{G}} + \big(|S_p(K)| - |S_p(k)|\big)/3 \quad \text{if } K \text{ is totally real} \end{cases}$$

Proof. We know that $\rho \mapsto \delta_k^{\mathbf{G}}(\rho)$ is compatible with Artin formalism by Corollary 3.2.5. We now explain how to prove (1). Again by Artin formalism, it suffices to prove (1) with ρ replaced by any irreducible subrepresentation $\theta \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_\mathbb{Q})$ of $\mathrm{Ind}_k^\mathbb{Q} \rho$. For such a θ, Lemma 3.2.1 implies that $d^+(\theta) \ge 1$ if K contains at least one real place, and $d^+(\theta) = d(\theta)$ if K is totally real. Therefore, the inequalities in (1) for θ directly follow from Theorem 4.2.1. Finally, (2) follows from (1) as in the proof of Corollary 4.1.2. $\square$

Remark 4.2.3. The matrices M and M' appearing in the course of the proof of Theorems 4.1.1 and 4.2.1 have full rank under the p-adic Schanuel conjecture. Therefore, Artin formalism shows that Leopoldt's and Gross-Kuz'min's conjectures hold in great generality under the p-adic Schanuel conjecture.

4.3. **Applications.**

Theorem 4.3.1. *Let k be a totally real number field and let $(V,\rho) \in Art_{\overline{\mathbb{Q}}_p}(G_k)$ be such that $d^+(\rho) = 0$. Then Gross's p-adic regulator matrix $R_p(V)$* [Gro81, (2.10)] *is of size $f(\rho)$ and of rank at least $f(\rho)/2$.*

Proof. Let K be the CM field cut out by ρ and let $(\mathrm{Hom}_{\overline{\mathbb{Q}}_p}(V,\overline{\mathbb{Q}}_p),\rho^*)$ be the contragredient representation of ρ. Gross's regulator map λ_p defined in [Gro81, (1.18)] can be identified with the "minus part" of $\mathscr{L}_K$, which is, by definition, the restriction of $\mathscr{L}_K$ to the subspace where the complex conjugation acts by -1. This means that λ_p and $\mathscr{L}_K$ share the same θ-isotypic component for every representation $\theta \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(\mathrm{Gal}(K/k))$ such that $d^+(\theta) = 0$. Since taking $(V \otimes -)^{G_k}$ amounts to taking ρ^*-isotypic components, we conclude that $\mathrm{rk} R_p(V) = \mathrm{rk}\,\mathscr{L}_k(\rho^*) = f(\rho^*) - \delta_k^{\mathbf{G}}(\rho^*)$, so $\mathrm{rk} R_p(V) \geq f(\rho^*)/2 = f(\rho)/2$ by Theorem 4.2.1. □

In the next theorem, we write k^+ for the maximal totally real subfield of a number field k, and $\mathbb{Q}^{\mathrm{ab}}$ for the maximal abelian extension of $\mathbb{Q}$.

Theorem 4.3.2. *Let K/k be an abelian extension of number fields. The Gross-Kuz'min conjecture holds for K in each of the following cases.*

- *(a) Either $|S_p(K)| \leq 2$, or $|S_p(K)| \leq 3$ and K has at least one real place, or $|S_p(K)| \leq 4$ and $K/\mathbb{Q}$ is Galois, or $|S_p(K)| \leq 6$ and $K/\mathbb{Q}$ is a real Galois extension.*
- *(b) $|S_p(k)| = 1$, or $|S_p(k)| \leq 2$ and K has at least one real place.*
- *(c) $K \subset k \cdot \mathbb{Q}^{ab}$, $k/\mathbb{Q}$ is Galois, and either $|S_p(k)| \leq 3$, or $|S_p(k)| \leq 5$ and K is real.*
- *(d) k is an imaginary quadratic field, or k is a real quadratic field and K has at least one real place.*
- *(e) $k/\mathbb{Q}$ is Galois, $|S_p(k)| \leq 2$, $|S_p(k^+)| = 1$ and $[K : k]$ and $[k : \mathbb{Q}]$ are coprime.*

Proof. Recall that $\delta^{\mathbf{G}}(-)$ is compatible with Artin formalism by Corollary 3.2.5. We shall often appeal to Lemma 3.2.1 and to the following consequence of Theorem 4.2.1 without further notice. For any irreducible representation $\theta \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_{\mathbb{Q}})$, we have $\delta_{\mathbb{Q}}^{\mathbf{G}}(\theta) = 0$ if $f(\theta) \leq 1$, or if $f(\theta) = 2$ and $d^+(\theta) \geq 1$. In particular, $\delta_{\mathbb{Q}}^{\mathbf{G}}(\theta) = 0$ if θ is a multiplicative character of $G_{\mathbb{Q}}$, so $\delta_M^{\mathbf{G}} = 0$ for any abelian extension $M/\mathbb{Q}$ by Artin formalism.

Since $\delta_{\mathbb{Q}}^{\mathbf{G}} = 0$, it follows from Corollary 4.2.2 that $\delta_K^{\mathbf{G}} = 0$ for K satisfying one of the two first assumptions in case (a). Consider the two last assumptions in (a) and assume that $K/\mathbb{Q}$ is Galois. We claim that $\delta_{\mathbb{Q}}^{\mathbf{G}}(\theta) = 0$ for all irreducible $\theta \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(\mathrm{Gal}(K/\mathbb{Q}))$. We may assume that $\dim\theta \geq 2$, so $f(\theta) \leq (f(\mathbb{1}_K) - f(\mathbb{1}_{\mathbb{Q}}))/(\dim\theta) \leq (|S_p(K)| - 1)/2$. The two last assumptions in (a) ensure that we either have $f(\theta) \leq 1$, or $f(\theta) \leq 2$ and $d^+(\theta) = d(\theta) \geq 1$, so we indeed have $\delta_{\mathbb{Q}}^{\mathbf{G}}(\theta) = 0$. Therefore, $\delta_K^{\mathbf{G}} = 0$ in case (a).

Let $G = \mathrm{Gal}(K/k)$ and let $\hat{G} = \mathrm{Hom}(G, \overline{\mathbb{Q}}_p^\times)$ be the group of characters of G. We place ourselves in cases (b), (c) and (d), we fix $\chi \in \hat{G}$ and we show that $\delta_k^{\mathbf{G}}(\chi) = 0$. Since $f(\chi) \leq |S_p(k)|$, Corollary 4.2.2 (1) implies $\delta_k^{\mathbf{G}}(\chi) = 0$ in case (b). Suppose now we are in case (c). Then χ descends to a character $\chi_{\mathbb{Q}}$ of $G_{\mathbb{Q}}$. Moreover, as $k/\mathbb{Q}$ is Galois, any irreducible subrepresentation ρ of $\mathrm{Ind}_k^{\mathbb{Q}}\chi \simeq (\mathrm{Ind}_k^{\mathbb{Q}}\mathbb{1}_k) \otimes \chi_{\mathbb{Q}}$ occurs $(\dim\rho)$ times, so it satisfies $f(\rho) \leq |S_p(k)|/(\dim\rho)$. Moreover, if K is totally real, then any such ρ satisfies $d^+(\rho) = d(\rho) \geq 1$, so we can conclude $\delta_{\mathbb{Q}}^{\mathbf{G}}(\rho) = 0$. Therefore, $\delta_k^{\mathbf{G}}(\chi) = 0$ in case (c). We now assume to be in case (d). Then $\mathrm{Ind}_k^{\mathbb{Q}}\chi$ has dimension 2, so it is either irreducible or it is the sum of two characters of $G_{\mathbb{Q}}$, say η_1 and η_2. In the latter case, we already know that $\delta_{\mathbb{Q}}^{\mathbf{G}}(\eta_i) = 0$ for $i = 1,2$, so

$\delta_k^{\mathbf{G}}(\chi) = \delta_{\mathbb{Q}}^{\mathbf{G}}(\mathrm{Ind}_k^{\mathbb{Q}}\chi) = 0$. If $\mathrm{Ind}_k^{\mathbb{Q}}\chi$ is irreducible, then the assumptions on K and k imply that $f(\chi) \leq 2$ and $d^+(\chi) \geq 1$, yielding $\delta_k^{\mathbf{G}}(\chi) = 0$. Therefore, $\delta_K^{\mathbf{G}} = 0$ in the cases (b), (c) and (d).

We now make the assumptions in (e) and we assume without loss of generality that K is Galois over $\mathbb{Q}$ with Galois group $\mathcal{G}$. By the Schur-Zassenhaus theorem, $\mathcal{G}$ is the semi-direct product of $H := \mathrm{Gal}(k/\mathbb{Q})$ acting on G. Let $\rho \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(\mathcal{G})$ be irreducible and let us prove that $\delta_{\mathbb{Q}}^{\mathbf{G}}(\rho) = 0$. By [Ser78, Chap II § 8.2], ρ can be written as $\mathrm{Ind}_{k'}^{\mathbb{Q}}(\theta \otimes \chi)$, where $k'/\mathbb{Q}$ is a subextension of $k/\mathbb{Q}$, θ an irreducible representation of $\mathrm{Gal}(k/k')$ and χ a character of $\mathrm{Gal}(K/k')$. Note that $f(\rho) \leq |S_p(k)|/(\dim\theta) \leq 2/(\dim\theta)$, so we may assume without loss of generality that $\dim\theta = 1$. If k' is totally real, then $f(\rho) \leq |S_p(k')| \leq |S_p(k^+)| = 1$, and otherwise we have $d^+(\rho) \geq 1$. In any case, $\delta_{\mathbb{Q}}^{\mathbf{G}}(\rho) = 0$ so we may infer $\delta_K^{\mathbf{G}} = 0$ in case (e) as well. □

5. Vanishing locus of Gross's defect

5.1. **Preliminaries.** This section is devoted to the proof of the following theorem which, in turn, implies Theorem 1.5 stated in the introduction.

Theorem 5.1.1. *Let k be a number field and let $\varphi\colon G_k \to \overline{\mathbb{Q}}_p^\times$ be a finite-order character. Assume also that p completely splits in the number field cut out by φ.*

(1) Assume that $r_2 + \delta_k^{\mathbf{L}} \leq 1$, where r_2 is the number of complex places of k. If there exists at least one $\mathbb{Z}_p$-extension k_∞ of k such that $\delta_{k_\infty/k}^{\mathbf{G}}(\varphi) = 0$, then there are at most $[k:\mathbb{Q}]$ (and at most $[k:\mathbb{Q}]-1$ if $\varphi = \mathbb{1}$) $\mathbb{Z}_p$-extensions k_∞/k such that $\delta_{k_\infty/k}^{\mathbf{G}}(\varphi) \neq 0$.

(2) Assume that k is an imaginary quadratic field. There is at most one $\mathbb{Z}_p$-extension k_∞/k for which $\delta_{k_\infty/k}^{\mathbf{G}}(\varphi) \neq 0$, and it has a transcendental slope (see Example 5.2.3 for a definition of the slope of k_∞/k). Moreover, if φ cuts out an abelian extension of $\mathbb{Q}$ or if the p-adic Schanuel conjecture holds, then $\delta_{k_\infty/k}^{\mathbf{G}}(\varphi) = 0$ for any $\mathbb{Z}_p$-extension k_∞ of k.

Proof of Theorem 1.5, assuming Theorem 5.1.1. Let K be an abelian extension of an imaginary quadratic field k and let $K^{\mathrm{ab}} \subset K$ be its maximal absolutely abelian subfield. By Artin formalism (Corollary 3.2.5) and by Theorem 5.1.1, we have $\delta_{Kk_\infty/K}^{\mathbf{G}} > 0$ for a given $\mathbb{Z}_p$-extension k_∞/k if and only if there exists a character φ of $\mathrm{Gal}(K/k)$ such that $\delta_{k_\infty/k}^{\mathbf{G}}(\varphi) > 0$. Such a character cannot be a character of $\mathrm{Gal}(K^{\mathrm{ab}}/k)$, and moreover, k_∞ is uniquely determined by φ. Therefore, we have $\delta_{Kk_\infty/K}^{\mathbf{G}} > 0$ for at most $[K:k]-[K^{\mathrm{ab}}:k]$ distinct $\mathbb{Z}_p$-extensions of k. □

In the rest of Section 5, we fix once and for all an abelian extension K/k with Galois group G such that p totally splits in K. We let n be the degree of k and (r_1, r_2) its signature, and we put $r = r_1 + r_2 - 1 - \delta_k^{\mathbf{L}}$ so that the maximal multiple $\mathbb{Z}_p$-extension of k has rank $n - r$. Write $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ for the p-adic primes of k. Finally, denote by $\mathcal{Z}(k)$ the set of all $\mathbb{Z}_p$-extensions of k.

Instead of working with the map $\mathscr{L}_{k_\infty/k}(\varphi)$, it will be more convenient to consider the following alternative description of $\delta_{k_\infty/k}^{\mathbf{G}}(\varphi)$.

Lemma 5.1.2. *Let k_∞/k be a $\mathbb{Z}_p$-extension with Galois group Γ. The quantity $\delta_{k_\infty/k}^{\mathbf{G}}(\varphi)$ is the dimension of the kernel of the φ-isotypic component of the map $\mathrm{Loc}_{K_\infty/K}$ of Proposition 3.1.3.*

Proof. This follows from Poitou-Tate duality as in the proof of Proposition 3.1.5 where one replaces the G_K-module $\mathbb{Q}_p$ by the module $\overline{\mathbb{Q}}_p(\varphi)$ on which G_k acts by φ. □

Since φ is a multiplicative character, the φ-isotypic component of a $\overline{\mathbb{Q}}_p[G]$-module X is canonically isomorphic to the linear subspace $X[\varphi]$ of X consisting of elements $x \in X$ such that $g\cdot x = \varphi(g)x$ for all $g \in G$. Hence Lemma 5.1.2 asserts that, if $\varphi \neq \mathbb{1}_k$, then $\delta^{\mathbf{G}}_{k_\infty/k}(\varphi)$ is equal to the dimension of the kernel of the localization map

$$\mathrm{Hom}(G_K, \overline{\mathbb{Q}}_p)[\varphi] \longrightarrow \bigoplus_{i=1}^{n} \left(\bigoplus_{\mathfrak{P}|\mathfrak{p}_i} \mathrm{H}^1(K_{\mathfrak{P}}, \overline{\mathbb{Q}}_p)/\mathrm{Hom}(\Gamma_{\mathfrak{P}}, \overline{\mathbb{Q}}_p) \right)[\varphi]. \tag{8}$$

5.2. **Matrices in logarithms of algebraic numbers.** For any $\mathfrak{P} \in S_p(K)$, we identify $\mathrm{H}^1(K_{\mathfrak{P}}, \overline{\mathbb{Q}}_p)$ with $\mathrm{Hom}(K_{\mathfrak{P}}^\times, \overline{\mathbb{Q}}_p) \simeq \mathrm{Hom}(\mathbb{Q}_p^\times, \overline{\mathbb{Q}}_p)$ via local class field theory. We also see $\log_p$ and the p-adic valuation map ord_p as additive characters $K_{\mathfrak{P}}^\times \simeq \mathbb{Q}_p^\times \to \overline{\mathbb{Q}}_p$. In order to describe elements in the domain of the map (8) we make use of the short exact sequence of $\overline{\mathbb{Q}}_p[G]$-modules

$$0 \longrightarrow \mathrm{Hom}(G_K, \overline{\mathbb{Q}}_p) \xrightarrow{\ A\ } \oplus_{i=1}^n \mathrm{Hom}(\textstyle\prod_{\mathfrak{P}|\mathfrak{p}_i} K_{\mathfrak{P}}^\times, \overline{\mathbb{Q}}_p) \longrightarrow \mathrm{Hom}(\mathcal{O}_K[\tfrac{1}{p}]^\times, \overline{\mathbb{Q}}_p), \tag{9}$$

where A is induced by the Artin map.

Let $1 \leq i \leq n$ and fix a prime $\mathfrak{P}_i$ of K above $\mathfrak{p}_i$. We define a basis $\{\eta_{i,\varphi}, \tilde{\eta}_{i,\varphi}\}$ of the φ-component of $\mathrm{Hom}(\prod_{\mathfrak{P}|\mathfrak{p}_i} K_{\mathfrak{P}}^\times, \mathbb{Q}_p)$ as follows. First define characters η_i and $\tilde{\eta}_i$ of $\prod_{\mathfrak{P}|\mathfrak{p}_i} K_{\mathfrak{P}}^\times$ by imposing that they are supported on $K_{\mathfrak{P}_i}^\times$ and that $\eta_{i|K_{\mathfrak{P}_i}^\times} = -\log_p$ and $\tilde{\eta}_{i|K_{\mathfrak{P}_i}^\times} = \mathrm{ord}_p$. We then define

$$\eta_{i,\varphi} = \sum_{\sigma \in G} \varphi(\sigma)\cdot \eta_i \circ \sigma, \qquad \tilde{\eta}_{i,\varphi} = \sum_{\sigma \in G} \varphi(\sigma)\cdot \tilde{\eta}_i \circ \sigma.$$

Let u_i be any $\mathfrak{P}_i$-unit of K which is not a unit (take for example a generator of $\mathfrak{P}_i^h$, where h is the class number of K). The choice of u_i with a given $\mathfrak{P}_i$-valuation is unique, up to multiplication by a unit of K. Consider

$$u_{i,\varphi} = \prod_{\sigma \in G} \varphi(\sigma) \otimes \sigma^{-1}(u_i) \in \overline{\mathbb{Q}} \otimes K^\times.$$

It is clear that $u_{i,\varphi}$ is a unit away from the primes above $\mathfrak{p}_i$, and $u_{1,\varphi}, \ldots, u_{n,\varphi}$ form a basis of $(\overline{\mathbb{Q}} \otimes \mathcal{O}_K[1/\mathfrak{p}_i]^\times)[\varphi]$ modulo $(\overline{\mathbb{Q}} \otimes \mathcal{O}_K^\times)[\varphi]$. We also fix a basis $\{\varepsilon_{1,\varphi}, \ldots, \varepsilon_{r(\varphi),\varphi}\}$ of $(\overline{\mathbb{Q}} \otimes \mathcal{O}_K^\times)[\varphi]$ modulo the kernel of Leopoldt's map $\iota_k(\varphi)$ of (7), where $r(\varphi) = d^+(\varphi) - \delta_k^{\mathbf{L}}(\varphi)$. For all $j = 1, \ldots, n$, one can see via $\iota_{\mathfrak{P}_j} : K \hookrightarrow K_{\mathfrak{P}_j} = \mathbb{Q}_p$ the elements $u_{i,\varphi}$ and $\varepsilon_{i,\varphi}$ inside $\overline{\mathbb{Q}} \otimes \mathbb{Q}_p^\times$. We then define two matrices $L_\varphi = (L_{i,j,\varphi})$ and $M_\varphi = (M_{i,j,\varphi})$ of respective sizes $n \times n$ and $r(\varphi) \times n$ by letting

$$L_{i,j,\varphi} = \frac{\log_p(\iota_{\mathfrak{P}_j}(u_{i,\varphi}))}{\mathrm{ord}_p(\iota_{\mathfrak{P}_i}(u_i))}, \qquad M_{i,j,\varphi} = \log_p(\iota_{\mathfrak{P}_j}(\varepsilon_{i,\varphi})),$$

where we extended $\log_p$ to $\overline{\mathbb{Q}} \otimes \mathbb{Q}_p^\times$ by linearity. Notice that M_φ has full rank by construction.

Let η' be an element in the φ-component of $\oplus_{i=1}^n \mathrm{Hom}(\prod_{\mathfrak{P}|\mathfrak{p}_i} K_{\mathfrak{P}}^\times, \overline{\mathbb{Q}}_p)$, which we write as $\sum t_i \eta_{i,\varphi} + \tilde{t}_i \tilde{\eta}_{i,\varphi}$ in the basis $\{\eta_{i,\varphi}, \tilde{\eta}_{i,\varphi} \,:\, 1 \leq i \leq n\}$. Denote by T and $\tilde{T}$ the column matrices of respective coordinates $(t_1, \ldots, t_n)$ and $(\tilde{t}_1, \ldots, \tilde{t}_n)$.

Lemma 5.2.1. *η' belongs to the image of the map A of (9) if and only if $\tilde{T} = L_\varphi T$ and $M_\varphi T = 0$.*

Proof. By the exactness of (9) such an η' is characterized by its vanishing at all the u_i's and the $\varepsilon_{i,\varphi}$'s. The lemma then follows from a straightforward computation, using that $\eta_{j,\varphi}(\iota_{\mathfrak{P}_j}(u_i)) = -\log_p(\iota_{\mathfrak{P}_j}(u_{i,\varphi}))$ and $\tilde{\eta}_{j,\varphi}(\iota_{\mathfrak{P}_j}(u_i)) = \mathrm{ord}_p(\iota_{\mathfrak{P}_j}(u_i))$ for all $1 \leq i, j \leq n$. ☐

In what follows we repeatedly use the following elementary fact. For all compact topological groups $\mathcal{G}$, any non-trivial continuous group homomorphism $\eta \colon \mathcal{G} \to \mathbb{Q}_p$ factors through a quotient Z_η isomorphic to $\mathbb{Z}_p$, and two such homomorphisms η and η' are proportional if and only if $Z_\eta = Z_{\eta'}$. Conversely, any topological group Z isomorphic to $\mathbb{Z}_p$ which arises as a quotient of $\mathcal{G}$ defines a continuous homomorphism $\eta\colon \mathcal{G} \to \mathbb{Q}_p$, which is unique up to scaling.

Fix $k_\infty \in \mathscr{Z}(k)$ and put $\Gamma = \mathrm{Gal}(k_\infty/k)$. The above argument attaches to Γ a non-zero element $\eta \in \mathrm{Hom}(G_k, \mathbb{Q}_p)$, unique up to scaling. Since the restriction map induces an isomorphism $\mathrm{Hom}(G_k, \mathbb{Q}_p) \simeq \mathrm{Hom}(G_K, \mathbb{Q}_p)[\mathbb{1}]$, one can write $A(\eta)$ as $\sum_i (s_i \eta_{i,\mathbb{1}} + \tilde{s}_i \tilde{\eta}_{i,\mathbb{1}})$. We shall refer to the column matrices $S = (s_1, \ldots, s_n)^{\mathbf{t}} \in \mathbb{P}^{n-1}(\mathbb{Q}_p)$ and $\tilde{S} = (\tilde{s}_1, \ldots, \tilde{s}_n)^{\mathbf{t}} \in \mathbb{P}^{n-1}(\mathbb{Q}_p)$ as the coordinates of k_∞.

Proposition 5.2.2. *(1) The map sending a $\mathbb{Z}_p$-extension k_∞/k to its coordinates S defines a bijection between $\mathscr{Z}(k)$ and $\{S \in \mathbb{P}^{n-1}(\mathbb{Q}_p) : M_{\mathbb{1}} S = 0\}$.*

(2) Let $k_\infty \in \mathscr{Z}(k)$ with coordinates $S \in \ker(M_{\mathbb{1}})$ and let $N_\varphi(S)$ be the matrix of size $(n + r(\varphi)) \times n$ given in block notation by:

$$\begin{bmatrix} \mathrm{Diag}(S) L_\varphi - \mathrm{Diag}(L_{\mathbb{1}} S) \\ M_\varphi \end{bmatrix},$$

where $\mathrm{Diag}(U)$ denotes the diagonal matrix associated with a column matrix U. Then

$$\delta^{\mathbf{G}}_{k_\infty/k}(\varphi) = \dim \ker N_\varphi(S) - \gamma,$$

where $\gamma = 1$ if $\varphi = \mathbb{1}$ and $\gamma = 0$ otherwise.

Proof. The first point follows from Lemma 5.2.1 applied to $\varphi = \mathbb{1}$. More precisely, S and $\tilde{S}$ are uniquely determined by the relations $\tilde{S} = L_{\mathbb{1}} S$ and $M_{\mathbb{1}} S = 0$. Conversely, any $S, \tilde{S} \in \mathbb{P}^{n-1}(\mathbb{Q}_p)$ satisfying these relations define a non-zero element $\eta \in \mathrm{Hom}(G_k, \mathbb{Q}_p)$ which cuts out the Galois group of a $\mathbb{Z}_p$-extension k_∞/k.

Let us prove the second claim. Fix $k_\infty \in \mathscr{Z}(k)$ and denote by $\eta \in \mathrm{Hom}(G_K, \mathbb{Q}_p)[\mathbb{1}]$ the corresponding continuous homomorphism. By Lemma 5.1.2 and by (8), $\delta^{\mathbf{G}}_{k_\infty/k}(\varphi) + \gamma$ is the dimension of the space consisting of all elements $\eta' \in \bigoplus_{i=1}^n \left(\mathrm{Hom}(\prod_{\mathfrak{P}|\mathfrak{p}_i} K^\times_{\mathfrak{P}}, \overline{\mathbb{Q}}_p)\right)[\varphi]$ satisfying the conditions of Lemma 5.2.1 and which are proportional to $A(\eta)$. This last condition means that for all $1 \le i \le n$, the restriction to $K^\times_{\mathfrak{P}_i}$ of η' and $A(\eta)$ are proportional, *i.e.*,

$$\eta'(\varpi_i) \cdot A(\eta)_{|\mathcal{O}^\times_{K_{\mathfrak{P}_i}}} = A(\eta)(\varpi_i) \cdot \eta'_{|\mathcal{O}^\times_{K_{\mathfrak{P}_i}}},$$

where ϖ_i is a uniformizer of $K_{\mathfrak{P}_i}$. In terms of coordinates $S, \tilde{S}$ and $T, \tilde{T}$, this last equality is equivalent to $\tilde{t}_i s_i = \tilde{s}_i t_i$, an equality for all i which can be rephrased as $\mathrm{Diag}(S) L_\varphi T = \mathrm{Diag}(L_{\mathbb{1}} S) T$. Therefore, $\delta^{\mathbf{G}}_{k_\infty/k}(\varphi) + \gamma$ is the dimension of the space of all $T \in \overline{\mathbb{Q}}_p^n$ in the kernel of both $\mathrm{Diag}(S) L_\varphi - \mathrm{Diag}(L_{\mathbb{1}} S)$ and M_φ, as claimed. □

Example 5.2.3. If k is an imaginary quadratic field, then $M_{\mathbb{1}}$ is of size 0 and Proposition 5.2.2 (1) provides a bijection between $\mathscr{Z}(k)$ and $\mathbb{P}^1(\mathbb{Q}_p)$. The ratio $s_1/s_2 \in \mathbb{Q}_p \cup \{\infty\}$ attached to any $k_\infty \in \mathscr{Z}(k)$ of coordinates $S = (s_1, s_2) \in \mathbb{P}^1(\mathbb{Q}_p)$ is referred to as the *slope* of k_∞. For instance, the cyclotomic extension of k has slope 1, whereas its anticyclotomic extension has slope -1.

5.3. **Proof of Theorem 5.1.1.** We keep the notations of the preceding sections and we abbreviate $L_{\mathbb{1}}, M_{\mathbb{1}}$ and $N_{\mathbb{1}}(S)$ as L, M and $N(S)$ respectively. Note that, by Proposition 5.2.2 (1), the set $\mathscr{Z}(k)$ can be identified with a closed linear subvariety of $\mathbb{P}^{n-1}(\mathbb{Q}_p)$ of dimension $n - r - 1 = r_2 + \delta^{\mathbf{L}}_k$.

Proposition 5.3.1. *Assume that $n - r \le 2$.*

(1) Let $\mathscr{C}(k) = \{k_\infty \in \mathscr{Z}(k) \ : \ \delta^{\boldsymbol{G}}_{k_\infty/k} \neq 0\}$. If $\mathscr{C}(k) \neq \mathscr{Z}(k)$, then $\mathscr{C}(k)$ is finite with $|\mathscr{C}(k)| \leq n-1$. If, moreover, k is imaginary quadratic, then $\mathscr{C}(k) = \emptyset$.

(2) Let $\mathscr{C}_\varphi(k) = \{k_\infty \in \mathscr{Z}(k) \ : \ \delta^{\boldsymbol{G}}_{k_\infty/k}(\varphi) \neq 0\}$. If $\mathscr{C}_\varphi(k) \neq \mathscr{Z}(k)$, then $\mathscr{C}_\varphi(k)$ is finite with $|\mathscr{C}(k)| \leq n$.

Proof. Let us prove (1). Notice first that all $S \in \ker M$ lie in the kernel of $N(S)$. By Proposition 5.2.2 with $\varphi = \mathbb{1}$ and $K = k$, $\mathscr{C}(k)$ is in bijection with the set of all $S \in \mathbb{P}^{n-1}(\mathbb{Q}_p)$ such that $MS = 0$ and $\operatorname{rk} N(S) < n-1$.

Assume first that k is quadratic, so $n = 2$ and $r = 0$. Given any $S = (s_1, s_2) \in \mathbb{P}^1(\mathbb{Q}_p)$, the matrix $N(S) = \operatorname{Diag}(S)L - \operatorname{Diag}(LS)$ has the form

$$\begin{pmatrix} -s_2 L_{1,2} & s_1 L_{1,2} \\ s_2 L_{2,1} & -s_1 L_{2,1} \end{pmatrix}.$$

As $\log_p$ is injective on $\mathbb{Z}_p^\times$, $\log_p(\iota_{\mathfrak{p}_1}(u_2))$ and $\log_p(\iota_{\mathfrak{p}_2}(u_1))$ both are non-zero, so at least one of the two non-diagonal entries of $N(S)$ is non-zero. Therefore, this matrix has rank one for any $S \in \mathbb{P}^1(\mathbb{Q}_p)$, and $\mathscr{C}(k) = \emptyset$.

We no longer assume that k is imaginary quadratic, but we still assume that $n - r \leq 2$. The case where $n - r = 1$ is trivial, because it forces $\mathscr{Z}(k) = \{k_{\mathrm{cyc}}\}$. We may then assume that $n - r = 2$. Since M has full rank r, there exist invertible matrices P, Q such that $PMQ = (I_r \mid 0)$, where I_r is the identity matrix of size r. The change of variables $S' = Q^{-1}S$ induces a linear bijection between $\ker M \subset \mathbb{P}^{n-1}(\overline{\mathbb{Q}}_p)$ and the projective line $\{0\} \times \mathbb{P}^1(\overline{\mathbb{Q}}_p) \subset \mathbb{P}^{n-1}(\overline{\mathbb{Q}}_p)$. Now consider the list $P_1(S'), \ldots, P_t(S')$ of all $(n-1) \times (n-1)$-minors of the matrix

$$N(S) = N(QS') = \begin{bmatrix} \operatorname{Diag}(QS')L - \operatorname{Diag}(LQS') \\ M \end{bmatrix}.$$

All the P_k's are two-variable homogeneous polynomials of degree $\leq n-1$. In particular, if $\mathscr{C}(k) \neq \mathscr{Z}(k)$, then at least one of the P_k's is not the zero polynomial and hence, it has at most $n-1$ zeros in $\{0\} \times \mathbb{P}^1(\overline{\mathbb{Q}}_p)$, so we can conclude that $|\mathscr{C}(k)| \leq n-1$.

The proof of point (2) very similar to the previous one. Indeed, by Proposition 5.2.2, $\mathscr{C}_\varphi(k)$ is in bijection with the set of all $S \in \mathbb{P}^{n-1}(\mathbb{Q}_p)$ such that $MS = 0$ and $\operatorname{rk} N_\varphi(S) < n$. The same argument with the $(n-1) \times (n-1)$ minors of $N(S)$ replaced by the $n \times n$ minors of $N_\varphi(S)$ shows that, if $\mathscr{C}_\varphi(k) \neq \mathscr{Z}(k)$, then $|\mathscr{C}(k)| \leq n$. □

We end the proof of Theorem 5.1.1 with the case where k is imaginary quadratic. We let τ be the complex conjugation of k. Recall that τ acts on φ via $\varphi^\tau(g) = \varphi(\tau g \tau)$ and that $\varphi^\tau = \varphi$ if and only if φ cuts out an extension of k which is abelian over $\mathbb{Q}$.

Proposition 5.3.2. *Assume that k is imaginary quadratic and that $\varphi \neq \mathbb{1}$.*

(1) If $\varphi^\tau = \varphi$, then $\mathscr{C}_\varphi(k) = \emptyset$.

(2) If $\varphi^\tau \neq \varphi$, then any $k_\infty \in \mathscr{C}_\varphi(k)$ has transcendental slope.

(3) If $\varphi^\tau \neq \varphi$, then $|\mathscr{C}_\varphi(k)| \leq 1$. Moreover, if the p-adic Schanuel conjecture holds, then $\mathscr{C}_\varphi(k) = \emptyset$.

Proof. Let $k_\infty \in \mathscr{Z}(k)$ of coordinates $S = (s_1, s_2)$. Take K to be the Galois closure over $\mathbb{Q}$ of the field cut out by φ. Note that $K \neq k$ and that p totally splits in K. By Proposition 5.2.2, k_∞ belongs to $\mathscr{C}_\varphi(k)$ if and only if the matrix

$$N_\varphi(S) = \begin{pmatrix} s_1(L_{1,1,\varphi} - L_{1,1}) - s_2 L_{1,2} & s_1 L_{1,2,\varphi} \\ s_2 L_{2,1,\varphi} & s_2(L_{2,2,\varphi} - L_{2,2}) - s_1 L_{2,1} \\ M_{1,1,\varphi} & M_{1,2,\varphi} \end{pmatrix}$$

has rank 1. The definition of L_φ (and also L_{φ^τ}) involves the choice, for $i = 1,2$, of a prime $\mathfrak{P}_i$ of K above $\mathfrak{p}_i$ and a $\mathfrak{P}_i$-unit u_i of non-zero valuation. Since $\tau(\mathfrak{p}_1) = \mathfrak{p}_2$, one may take $\tau(\mathfrak{P}_1) = \mathfrak{P}_2$ and $\tau(u_1) = u_2$, so that $L_{1,1} = L_{2,2}$, $L_{1,2} = L_{2,1}$, $L_{1,1,\phi} = L_{2,2,\phi^\tau}$ and $L_{1,2,\phi} = L_{2,1,\phi^\tau}$ for $\phi \in \{\varphi,\varphi^\tau\}$. We may also take $\tau(\varepsilon_{1,\varphi})$ to be $\varepsilon_{1,\varphi^\tau}$, so that $M_{1,2,\varphi} = M_{1,1,\varphi^\tau}$. Moreover, the sets $\{u_{1,\varphi}, u_{1,\mathbb{1}}, u_{2,\varphi}, u_{2,\mathbb{1}}, \varepsilon_{1,\varphi}\}$ and, under the additional condition $\varphi \neq \varphi^\tau$, $\{u_{1,\varphi}, u_{1,\varphi^\tau}, u_{1,\mathbb{1}}, u_{2,\varphi}, u_{2,\varphi^\tau}, u_{2,\mathbb{1}}, \varepsilon_{1,\varphi}, \varepsilon_{1,\varphi^\tau}\}$ are sets of linearly independent p-units. To see this, it suffices to consider their valuations at $\mathfrak{P}_1$ and $\mathfrak{P}_2$, and use the simple fact that units belonging to distinct isotypic components are linearly independent.

Now, consider first the case where $\varphi = \varphi^\tau$. Then $N_\varphi(S)$ has rank 1 if and only if its two columns are equal. This last condition easily implies that $s_1 = \pm s_2$ and that $u_{1,\varphi}/u_{1,\mathbb{1}}$, $u_{2,\mathbb{1}}$ and $u_{2,\varphi}$ have $\overline{\mathbb{Q}}$-linearly dependent $\mathfrak{P}_1$-adic logarithms. But all of these units have trivial $\mathfrak{P}_1$-valuation, so they must be linearly dependent by Proposition 2.1.2. We have already justified that this is not the case, so $N_\varphi(S)$ has rank 2 and $\mathscr{C}_\varphi(k) = \emptyset$.

Assume now that $\varphi \neq \varphi^\tau$ and that k_∞ has an algebraic slope, *i.e.*, $s_1/s_2 \in \mathbb{P}^1(\overline{\mathbb{Q}})$. We may assume that both s_1 and s_2 are algebraic numbers, hence $N_\varphi(S)$ has coefficients in the $\overline{\mathbb{Q}}$-linear subspace Λ of $\overline{\mathbb{Q}}_p$ introduced in Section 2. Moreover, $N_\varphi(S)$ has $\overline{\mathbb{Q}}$-linearly independent rows and columns. Indeed, the sets $\{\varepsilon_{1,\varphi}, \varepsilon_{1,\varphi^\tau}\}$ and $\{u_{1,\varphi}/u_{1,\mathbb{1}}, u_{2,\mathbb{1}}, u_{2,\varphi}, \varepsilon_{1,\varphi}\}$ are two sets of independent units with trivial $\mathfrak{P}_1$-valuation. Therefore, their images under $\log_p \circ \iota_{\mathfrak{P}_1}$ are again linearly independent by Proposition 2.1.2. We thus may apply Corollary 2.2.2 and conclude that $\operatorname{rk} N_\varphi(S) = 2$.

Finally, assume that $\varphi \neq \varphi^\tau$ and that $k_\infty \in \mathscr{C}_\varphi(k)$, *i.e.*, $N_\varphi(S)$ has rank 1. We already know that $s_1/s_2 \in \mathbb{P}^1(\mathbb{Q}_p) - \mathbb{P}^1(\overline{\mathbb{Q}})$ and in particular, both s_1 and s_2 are non-zero. Since $L_{1,2} \cdot M_{1,2,\varphi} \neq 0$, the vanishing of the minor obtained by removing the second row uniquely determines the slope s_1/s_2, so $|\mathscr{C}_\varphi(k)| \leq 1$. Using the vanishing of the other minors, an elementary computation yields the polynomial relation

$$M_{1,1,\varphi} \cdot M_{1,1,\varphi^\tau} \cdot L_{2,1}^2 = \left(M_{1,1,\varphi} \cdot (L_{1,1,\varphi^\tau} - L_{1,1}) - M_{1,1,\varphi^\tau} \cdot L_{2,1,\varphi}\right)\left(M_{1,1,\varphi^\tau} \cdot (L_{1,1,\varphi} - L_{1,1}) - M_{1,1,\varphi} \cdot L_{2,1,\varphi^\tau}\right).$$

The elements of the set $\{u_{1,\varphi}/u_{1,\mathbb{1}}, u_{1,\varphi^\tau}/u_{1,\mathbb{1}}, u_{2,\varphi}/u_{2,\mathbb{1}}, u_{2,\varphi^\tau}/u_{2,\mathbb{1}}, \varepsilon_{1,\varphi}, \varepsilon_{1,\varphi^\tau}\}$ are linearly independent, and they all have a trivial $\mathfrak{P}_1$-adic valuation, so their images under $\log_p \circ \iota_{\mathfrak{P}_1}$ are also $\overline{\mathbb{Q}}$-linearly independent. Therefore, the above polynomial identity contradicts the p-adic Schanuel conjecture. This shows that $\mathscr{C}_\varphi(k) = \emptyset$ under the p-adic Schanuel conjecture. □

References

[Ax65] James Ax. On the units of an algebraic number field. *Illinois J. Math.*, 9:584–589, 1965.

[BD21] Adel Betina and Mladen Dimitrov. Geometry of the eigencurve at CM points and trivial zeros of Katz p-adic L-functions. *Advances in Mathematics*, 384:107724, 2021.

[BK90] Spencer Bloch and Kazuya Kato. L-functions and Tamagawa numbers of motives. In *The Grothendieck Festschrift, Vol. I*, volume 86 of *Progr. Math.*, pages 333–400. Birkhäuser Boston, Boston, MA, 1990.

[BKS17] David Burns, Masato Kurihara, and Takamichi Sano. On Iwasawa theory, zeta elements for $\mathbb{G}_m$, and the equivariant Tamagawa number conjecture. *Algebra Number Theory*, 11(7):1527–1571, 2017.

[Bru67] Armand Brumer. On the units of algebraic number fields. *Mathematika*, 14:121–124, 1967.

[FMD05] Matthieu Le Floc'h, Abbas Movahhedi, and Thong Nguyen Quang Do. On capitulation cokernels in Iwasawa theory. *Amer. J. Math.*, 127(4):851–877, 2005.

[FO08] Jean-Marc Fontaine and Yi Ouyang. Theory of p-adic Galois representations. https://www.imo.universite-paris-saclay.fr/~fontaine/galoisrep.pdf, 2008.

[Gre73] Ralph Greenberg. On a certain l-adic representation. *Invent. Math.*, 21:117–124, 1973.

[Gro81] Benedict H. Gross. p-adic L-series at $s = 0$. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 28(3):979–994 (1982), 1981.

[Iwa73] Kenkichi Iwasawa. On $\mathbb{Z}_l$-extensions of algebraic number fields. *Annals of Mathematics*, pages 246–326, 1973.

[Jan89] Uwe Jannsen. On the l-adic cohomology of varieties over number fields and its Galois cohomology. In *Galois groups over* $\mathbf{Q}$ *(Berkeley, CA, 1987)*, volume 16 of *Math. Sci. Res. Inst. Publ.*, pages 315–360. Springer, New York, 1989.

[Jau17] Jean-François Jaulent. Sur les normes cyclotomiques et les conjectures de Leopoldt et de Gross-Kuz'min. *Ann. Math. Qué.*, 41(1):119–140, 2017.

[JS95] Jean-François Jaulent and Jonathan W. Sands. Sur quelques modules d'Iwasawa semi-simples. *Compositio Math.*, 99(3):325–341, 1995.

[Kis83] H. Kisilevsky. Some nonsemisimple Iwasawa modules. *Compositio Math.*, 49(3):399–404, 1983.

[Kle19] Sören Kleine. T-ranks of Iwasawa modules. *J. Number Theory*, 196:61–86, 2019.

[Kol91] Manfred Kolster. An idelic approach to the wild kernel. *Invent. Math.*, 103(1):9–24, 1991.

[Kuz72] L. V. Kuz'min. The Tate module of algebraic number fields. *Izv. Akad. Nauk SSSR Ser. Mat.*, 36:267–327, 1972.

[Lau89] Michel Laurent. Rang p-adique d'unités et action de groupes. *J. Reine Angew. Math.*, 399:81–108, 1989.

[Leo62] Heinrich-Wolfgang Leopoldt. Zur Arithmetik in abelschen Zahlkörpern. *J. Reine Angew. Math.*, 209:54–71, 1962.

[Mak21] Alexandre Maksoud. On generalized Iwasawa main conjectures and p-adic Stark conjectures for Artin motives. *ArXiv:2103.06864*, 2021.

[Mil86] J. S. Milne. *Arithmetic duality theorems*, volume 1 of *Perspectives in Mathematics*. Academic Press, Inc., Boston, MA, 1986.

[PR95] Bernadette Perrin-Riou. Fonctions L p-adiques des représentations p-adiques. *Astérisque*, 1995.

[Roy92] Damien Roy. Matrices whose coefficients are linear forms in logarithms. *J. Number Theory*, 41(1):22–47, 1992.

[Rub00] Karl Rubin. *Euler systems*, volume 147 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2000. Hermann Weyl Lectures. The Institute for Advanced Study.

[Ser78] Jean-Pierre Serre. *Représentations linéaires des groupes finis*. Hermann, Paris, revised edition, 1978.

[Tat84] John Tate. *Les conjectures de Stark sur les fonctions L d'Artin en* $s = 0$, volume 47 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1984. Lecture notes edited by Dominique Bernardi and Norbert Schappacher.

[Wal81] Michel Waldschmidt. Transcendance et exponentielles en plusieurs variables. *Invent. Math.*, 63(1):97–127, 1981.

[Was97] Lawrence C Washington. *Introduction to cyclotomic fields*, volume 83. Springer Science & Business Media, 1997.

UNIVERSITÉ DU LUXEMBOURG, UNITÉ DE RECHERCHE EN MATHÉMATIQUES, MAISON DU NOMBRE, 6 AVENUE DE LA FONTE, 4364 ESCH-SUR-ALZETTE, LUXEMBOURG

Email address: alexandre.maksoud@uni.lu