COMPACT LORENTZ MANIFOLDS WITH LOCAL SYMMETRY

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Abstract

We prove a structure theorem for compact aspherical Lorentz manifolds with abundant local symmetry. If M is a compact, aspherical, real-analytic, complete Lorentz manifold such that the isometry group of the universal cover has semisimple identity component, then the local isometry orbits in M are roughly fibers of a fiber bundle. A corollary is that if M has an open, dense, locally homogeneous subset, then M is locally homogeneous.

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In loving memory of Laura Sue-Jung Kang

1. Introduction

This work addresses the question, which compact Lorentz manifolds admit a positive-dimensional pseudogroup of local isometries? This question can be loosely rephrased as, which compact Lorentz manifolds have nontrivial local symmetry? For a real-analytic, complete Lorentz manifold, a positive-dimensional pseudogroup of local isometries is equivalent to a positive-dimensional isometry group on the universal cover.

Examples of compact Lorentz manifolds with local symmetry will be discussed below. Given such a Lorentz manifold, one may construct a new compact Lorentz manifold with at least as much local symmetry by forming a warped product.

Definition 1.1. *For two pseudo-Riemannian manifolds* (P, λ) *and* (Q, μ)*, a* warped product $P \times_f Q$ *is given by a positive function* f *on* Q *: the metric at* (p, q) *is* $f(q)\lambda_p + \mu_q$*. The factor* P *is called the* normal factor*.*

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If $Isom(P) = G$, then G also acts isometrically on the warped product $P \times_f Q$ for any f. More generally, let f be any function $Q \to \mathcal{M}$, where $\mathcal{M}$ is the moduli space of G-invariant metrics (of a fixed signature) on P, with $f(q) = \lambda(q)$. Then G acts isometrically on the *twisted product* $P \times_f Q$, where the metric at (p, q) is $\lambda(q)_p + \mu_q$.

Results of Farb and Weinberger stated below give conditions under which a compact Riemannian manifold is a twisted product $P \times_f Q$ with P a locally symmetric space. Our main result (Theorem 1.3 below) gives conditions under which the universal cover of a compact Lorentz manifold has this form with P a Riemannian symmetric space or a complete Lorentz space of constant curvature. In both cases, the conditions are that the manifold have a large pseudogroup of local isometries.

Pseudo-Riemannian metrics are examples of *rigid geometric structures of algebraic type*. For M a compact real-analytic manifold with such a structure, Gromov's Stratification Theorem (stated as Theorem 5.1 below) describes the orbit structure of local symmetries of M. The celebrated Open-Dense Theorem, which is a corollary of this stratification, states that if a point of M has a dense orbit under local isometries, then an open dense subset of M is locally homogeneous. It would be interesting to find conditions on M under which existence of a dense orbit implies that M is locally homogeneous. Dumitrescu has proved in [**D**] that a compact, three-dimensional, real-analytic Lorentz manifold with an open, dense, locally homogeneous subset is locally homogeneous. More generally, one might seek a fibered version: when does existence of a local isometry orbit with positive-dimensional closure imply that M is roughly a fiber bundle with locally homogeneous fibers? Our main theorem (1.3) can be viewed as such a result, under some particular topological and geometric conditions on a compact real-analytic Lorentz manifold.

1.1. Riemannian case. For M a compact Riemannian manifold, $Isom(M)$ is compact. For example, a compact locally symmetric space of noncompact type has finite isometry group (See [**WM2**] 5.43. In fact, any compact M with negative-definite Ricci curvature has finite isometry group—see [**Ko**] II.4.4). While such a group provides some information about M, the isometry group of the universal cover X of M tells much more. For example, if M is a locally symmetric space of noncompact type, then $Isom(X)$ is a semisimple group with no compact factors. A homogeneous, contractible, Riemannian manifold with this isometry group must be a symmetric space.

Recall that an *aspherical* manifold is one with contractible universal cover. Farb and Weinberger studied compact aspherical Riemannian manifolds M with universal cover X having $Isom^0(X) \neq 1$. They proved several results characterizing warped products with locally symmetric factors, and locally symmetric spaces in particular. The following theorem is a weakened statement of their main theorem. Orbibundles will be defined later below in Definition 3.4.

Theorem 1.2 (Farb and Weinberger [**FW**]). *Let M be a compact aspherical Riemannian manifold with universal cover X. Let $G = Isom(X)$. If $G^0 \neq 1$, then M is a Riemannian orbibundle*

$$\Lambda\backslash G^0/K \to M \to Q$$

where $\Lambda \subset G^0$ is a cocompact lattice, K is a maximal compact subgroup of G^0, and Q is aspherical.

Further, if $\pi_1(M)$ contains no normal free abelian subgroup, then $Z(G^0)$ is finite, G^0 is semisimple, and a finite cover of M is isometric to

$$\Lambda\backslash G^0/K \times_f Q$$

for $f : Q \to \mathcal{M}$, the moduli space of locally symmetric metrics on $\Lambda\backslash G^0/K$.

The aspherical assumption is required. A metric on the sphere S^n with a bump at one point, for example, has isometry group containing only rotations fixing that point. However, [**FW**] contains the statement of a similar theorem to the above, under a noncompactness assumption on the connected isometry group of the universal cover, for arbitrary closed Riemannian manifolds.

Their proof relies on the theory of proper transformation groups, Lie theory, and remarkable cohomological dimension arguments.

1.2. Lorentz case. For Lorentz manifolds, a crucial difference from the Riemannian case is that the isometry group need not act properly; in particular, orbits may not be closed, and the group of deck transformations may not intersect G^0 in a lattice. On the other hand, fantastic work has been done on nonproper Lorentz-isometric actions ([**K1**], [**Ze1**], [**Ze2**]), which implies a great deal of structure in that case.

The Lorentz manifolds with the most symmetry are those of constant curvature, modeled on Minkowski space, de Sitter space, or anti-de Sitter space. Any irreducible *Lorentzian locally symmetric space* has constant curvature, as was proved in [**CLPTV**] and independently in [**Ze3**]. Each of the model spaces is a homogeneous space, G/H, where H is the stabilizer of a point. The isometry group, stabilizer, curvature, and diffeomorphism type for each are in the following table.

	Min^n	dS^n	AdS^n
Isom	$O(1,n-1) \ltimes R^n$	$O(1,n)$	$O(2,n-1)$
Stab	$O(1,n-1)$	$O(1,n-1)$	$O(1,n-1)$
Curv	0	1	-1
Diff	$\mathbf{R}^n$	$S^{n-1} \times \mathbf{R}$	$\mathbf{R}^{n-1} \times S^1$

Note that $AdS^2 \cong dS^2 \cong SO(1,2)/A$, where $A \cong \mathbf{R}^*$ is a maximal $\mathbf{R}$-split torus. A result of Calabi and Markus states that no infinite subgroup of $O(1,n)$ acts properly on dS^n, so there are no compact complete de Sitter manifolds ([**CM**]). Kulkarni noted that when n is odd, lattices in $SU(1,(n-1)/2)$ act freely, properly discontinously, and cocompactly on AdS^n. For n even, he proved that there is no cocompact, properly discontinuous, isometric action on AdS^n ([**Ku**]).

Kowalsky, using powerful dynamical techniques, which are treated in detail in Section 4.1 below, proved that a simple group acting nonproperly on an arbitrary Lorentz manifold is locally isomorphic to $O(1,n)$, $n \geq 2$, or $O(2,n)$, $n \geq 3$ ([**K1**]). Adams has characterized groups that admit *orbit nonproper* isometric actions on arbitrary Lorentz manifolds in [**A1**] and [**A2**]; an action $G \times M \to M$ is *orbit nonproper* if for some $x \in M$, the map $g \mapsto g.x$ from G to M is not proper.

There are several recent results on the form of arbitrary Lorentz manifolds admitting isometric actions of certain semisimple groups. Witte Morris showed that a homogeneous Lorentz manifold with isometry group $O(1,n)$ or $O(2,n-1)$ is dS^n or AdS^n, respectively ([**WM1**]). Arouche, Deffaf, and Zeghib, using totally geodesic, lightlike hypersurfaces, showed that if a semisimple group with no local $SL_2(\mathbf{R})$-factors has a Lorentz orbit with noncompact isotropy, then a neighborhood of this orbit is a warped product $N \times_f L$, where N is a complete, constant-curvature Lorentz space, and L is a Riemannian manifold ([**ADZ**]). Deffaf, Zeghib, and the author treat degenerate orbits with noncompact isotropy in [**DMZ**]. We conclude that any nonproper action of a semisimple group with finite center and no local $SL_2(\mathbf{R})$-factors has an open subset isometric to a warped product as in [**ADZ**], and we describe the global structure of such actions.

The work here combines features and techniques of many of these papers, as well as those of [**FW**]. As in [**FW**], we consider universal covers of compact aspherical Lorentz manifolds and seek to describe those for which the identity component of the isometry group is nontrivial. Here is the main result.

Theorem 1.3. *Let M be a compact, aspherical, real-analytic, complete Lorentz manifold with universal cover X. Let $G = Isom(X)$, and assume G^0 is semisimple.*

(1) Orbibundle. *Then M is an orbibundle*

$$P \to M \to Q$$

where P is aspherical and locally homogeneous, and Q is a good aspherical orbifold.

(2) Splitting. *Further, precisely one of the following holds:*

A. *G^0 acts properly on X:*

Then $P = \Lambda \backslash G^0/K$ where Λ is a lattice in G^0 and K is a maximal compact subgroup of G^0.

Further, if $|Z(G^0)| < \infty$, then a finite cover of M is isometric to

$$P \times_f Q$$

for $f : Q \to \mathcal{M}$, the moduli space of Riemannian locally symmetric metrics on $P = \Lambda \backslash G^0/K$. The Lorentzian manifold Q has $Isom^0(\widetilde{Q}) = 1$.

B. *G^0 acts nonproperly on X:*

Then M is a Lorentzian orbibundle. The metric along G^0-orbits is Lorentzian, with

$$P = \Lambda \backslash (\widetilde{AdS}^k \times G_2/K_2)$$

where $k \geq 3$, $G_2 \lhd G^0$ with maximal compact subgroup K_2, and

$$\Lambda \subset \widetilde{O}^0(2, k-1) \times G_2$$

acts freely, properly discontinuously, and cocompactly on $\widetilde{AdS}^k \times G_2/K_2$. The good Riemannian orbifold Q has $Isom^0(\widetilde{Q}) = 1$. There is a warped product

$$X \cong \widetilde{AdS}^k \times_h L$$

for some real-analytic function $h : L \to \mathbf{R}^+$.

Further, if $|Z(G_2)| < \infty$, then X is isometric to

$$(\widetilde{AdS}^k \times G_2/K_2) \times_f \widetilde{Q}$$

where $f : \widetilde{Q} \to \mathcal{M}$, and $\mathcal{M}$ is the moduli space of G^0-invariant Lorentzian metrics on $\widetilde{AdS}^k \times G_2/K_2$.

Corollary 1.4. *Let M and G^0 be as above. If M has an open, dense, locally homogeneous subset, then M is locally homogeneous.*

The appendix below contains an example illustrating the necessity of the hypothesis of finite center in (2) **A** in order to conclude that M splits locally along G^0-orbits as a metric product. In the example, $Isom^0(X)$ is a noncompact, connected, semisimple group H^0; the center of H^0 is infinite; H^0 acts properly on X; and the metric on H^0-orbits varies among Riemannian, Lorentzian, and degenerate.

Proof Outline for Theorem 1.3:

- The first step involves Gromov's stratification for isometric actions on spaces with rigid geometric structure: there is a closed orbit in $Y \subseteq X$ on which the group of deck transformations acts cocompactly (Propositions 5.4, 5.5).

The stabilizer of a point in this orbit then determines the dynamics of the isometry group on X.

- If the stabilizer is compact, then the group generated by G^0 and the fundamental group acts properly. In this case, techniques of [**FW**] apply (Section 6.1).
- When the stabilizer is noncompact, then G^0 acts nonproperly on X. In this case, we extend work of [**Ze1**] to show that totally geodesic lightlike foliations exist on X (Theorem 4.4). There are two subcases, depending on whether the dynamics of the group on the space of these foliations is strong or weak.
  - In the case of strong dynamics on the space of foliations, results of [**Ze2**] are used to produce the warped product structure on X. From here, the argument resembles the case in which G^0 is proper on X (Sections 6.2.1, 6.2.2).
  - In the case of weak dynamics on the space of foliations, there is an invariant lightlike vector field tangent to the closed orbit Y. We argue by contradiction that this case cannot arise. Techniques of nonproper Lorentz dynamics, including ideas of Kowalsky [**K1**], are applied to give a fairly precise description of Y: it belongs to a family of spaces that do not admit cocompact, properly discontinuous, isometric actions, yielding the contradiction (Section 6.3).

2. Notation

Throughout, M is a compact, aspherical, real-analytic, complete Lorentz manifold. The universal cover of M is X, with $Isom(X) = G$. The group of deck transformations is $\Gamma \cong \pi_1(M)$. The identity component of G is a semisimple group G^0, and $\Gamma_0 = \Gamma \cap G^0$. Note $G^0 \lhd G$ and $\Gamma_0 \lhd \Gamma$.

The Lie algebra of G^0 is $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_l$ be the decomposition of $\mathfrak{g}$ into simple factors. Let G_i be the corresponding subgroups of G^0. The projection $\mathfrak{g} \to \mathfrak{g}_i$ will be denoted π_i, as will the projection $G^0 \to G_i$.

For an arbitrary group H acting on a space Y, the stabilizer of $y \in Y$ will be denoted $H(y)$. In particular, $G_i(y) = G^0(y) \cap G_i$, and $\mathfrak{g}_i(y) = \mathfrak{g}(y) \cap \mathfrak{g}_i$.

3. Background and Terminology

3.1. Proper actions. The following facts follow from the existence of *slices* for smooth proper actions on manifolds. See [**P**] for definitions related to stratified spaces.

Proposition 3.1. *Let H be a Lie group acting smoothly and properly on a connected manifold Y.*

1) *For any compact $\overline{A} \subset H\backslash Y$, there is a compact $A \subset Y$ projecting onto $\overline{A}$.*
2) *In general, $H\backslash Y$ is a Whitney stratified space with*
$$dim(H\backslash Y) = dim\ Y - dim\ H + dim\ H(y)$$
where $dim\ H(y)$ is minimal over $y \in Y$.
3) *If the stabilizers $H(y)$ belong to the same conjugacy class for all $y \in Y$, then $H\backslash Y$ is a smooth manifold.*

Proof:

1) Let π be the projection $Y \to H\backslash Y$. For any $\overline{y} = \pi(y)$ in $H\backslash Y$, the Slice Theorem (see [**P**] 4.2.6) gives a neighborhood $\overline{U}$ of $\overline{y}$ and a diffeomorphism $\varphi_y : H \times_{H(y)} V_y \to \pi^{-1}(\overline{U})$, where V_y is an open ball in some $\mathbf{R}^k$. A disk about $\mathbf{0}$ in V_y corresponds under φ_y to a compact D_y containing y, and projecting to a compact neighborhood of $\overline{y}$ in $H\backslash Y$. For a compact subset $\overline{A}$, there exist $\overline{y}_1, \ldots, \overline{y}_n$ such that $int(\pi(D_{y_1})), \ldots, int(\pi(D_{y_n}))$ cover $\overline{A}$. Then $D_{y_1} \cup \cdots \cup D_{y_n}$ is the desired compact $A \subset Y$.
2) The stratification is by orbit types: for each compact $K \subset H$, let
$$Y_{(K)} = \{y \in Y \ : \ gH(y)g^{-1} = K \text{ for some } g \in H\}$$
and let Y_K be the fixed set of K. Then the pieces of the stratification of $H\backslash Y$ are the components of the quotients
$$H\backslash Y_{(K)} = N_H(K)\backslash Y_K$$
Each piece has the structure of a smooth manifold. See [**P**] 4.3.11 and 4.4.6. When $K = H(y)$ is minimal, then $Y_{(K)}$ is open, and the piece $H\backslash Y_{(K)}$ has maximal dimension $\dim Y - \dim H + \dim H(y)$.
3) If all stabilizers are conjugate to one compact subgroup K, then $H\backslash Y = H\backslash Y_{(K)}$, which consists of a single piece, because Y is connected.

q.e.d.

3.2. Orbifolds and orbibundles.

Definition 3.2. *An n-dimensional* orbifold *is a Hausdorff, paracompact space with an open cover $\{U_i\}$, closed under finite intersections, with homeomorphisms*
$$\varphi_i : \widetilde{U}_i/\Lambda_i \to U_i$$

where $\widetilde{U}_i$ *is an open subset of* $\mathbf{R}^n$ *and* Λ_i *is a finite group. The atlas* (U_i, φ_i) *must additionally satisfy the compatibility condition: whenever* $U_j \subset U_i$*, then there is a monomorphism* $\Lambda_j \to \Lambda_i$ *and an equivariant embedding* $\widetilde{U}_j \to \widetilde{U}_i$ *inducing a commutative diagram*

$$\begin{array}{ccc} \widetilde{U}_j & \to & \widetilde{U}_i \\ \downarrow & & \downarrow \\ \widetilde{U}_j/\Lambda_j & \to & \widetilde{U}_i/\Lambda_i \\ \downarrow & & \downarrow \\ U_j & \to & U_i \end{array}$$

A smooth orbifold *is an orbifold for which the action of each* Λ_i *is smooth, and the embeddings* $\widetilde{U}_j \to \widetilde{U}_i$ *are smooth.*

Definition 3.3. *A* good (pseudo-Riemannian) orbifold *is the quotient of a (pseudo-Riemannian) manifold by a smooth, properly discontinuous, (isometric) action.*

It is not hard to see using proper discontinuity that a good orbifold is a smooth orbifold.

Definition 3.4. *A smooth* orbibundle *is a manifold* M *with a projection* π *to a good orbifold* B*, written*

$$N \to M \xrightarrow{\pi} B$$

where N *is a manifold, and the orbifold charts* (U_i, φ_i) *on* B *lift to*

$$\psi_i : \pi^{-1}(U_i) \to N \times_{\Lambda_i} \widetilde{U}_i$$

where Λ_i *acts freely and smoothly on* $N \times \widetilde{U}_i$.

A pseudo-Riemannian orbibundle *is a pseudo-Riemannian manifold* M *with a projection* π *to a good pseudo-Riemannian orbifold* B *as above, such that the maps arising from* ψ_i

$$\widetilde{U}_i \to N \times_{\Lambda_i} \widetilde{U}_i \to \pi^{-1}(U_i)$$

are isometric immersions.

Note that, for a pseudo-Riemannian orbibundle M the type of the metric on M may be different from the type of the metric on the quotient orbifold B.

3.3. Rational cohomological dimension.

Definition 3.5. *The* rational cohomological dimension *of* Λ *is*

$$cd_{\mathbf{Q}}\Lambda = sup\{n \ : \ H^n(\Lambda, A) \neq 0, \ A \text{ a } \mathbf{Q}\Lambda\text{-module}\}$$

References for the following facts about rational cohomological dimension are [**FW**] and [**Me**].

Proposition 3.6. *Let* Λ *be a discrete group.*

1) *Let Y be a contractible space on which Λ acts freely and properly with $\Lambda \backslash Y$ a finite CW-complex. Then*
$$cd_{\mathbf{Q}}\Lambda \leq dim\ Y$$
If $\Lambda \backslash Y$ is a manifold, then there is equality.
2) *If Λ is finite, then $cd_{\mathbf{Q}}\Lambda = 0$.*
3) *Let $\Lambda_0 \lhd \Lambda$. Then*
$$cd_{\mathbf{Q}}\Lambda \leq cd_{\mathbf{Q}}\Lambda_0 + cd_{\mathbf{Q}}(\Lambda/\Lambda_0)$$
4) *Let Λ act on a contractible CW complex Y properly and cellularly. Then*
$$cd_{\mathbf{Q}}\Lambda \leq dim\ Y$$

3.4. Symmetric spaces. Recall that any connected Lie group G has a maximal compact subgroup K, unique up to conjugacy. This subgroup is always connected; further, the quotient G/K is contractible ([**I**] 6).

We collect here some facts about symmetric spaces of noncompact type, which are homogeneous Riemannian manifolds of the form G/K, where G is semisimple with no compact local factors, connected, and has finite center. References for this proposition are [**E**], [**Me**], [**WM2**], or [**Wo**]. Recall that a *lattice* Γ in a Lie group G is a discrete subgroup such that G/Γ has finite volume with respect to Haar measure.

Proposition 3.7. *Let G be a connected semisimple Lie group with finite center. Let K be a maximal compact subgroup of G.*
1) *There is an $Ad(K)$-invariant decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. The $Ad(K)$-irreducible subspaces of $\mathfrak{p}$ correspond to the simple factors of $\mathfrak{g}$, and there is no one-dimensional $Ad(K)$-invariant subspace of $\mathfrak{p}$.*
2) $N_G(K) = K$
3) *For any torsion-free lattice $\Gamma \subset G$, the center $Z(\Gamma) = 1$, and $|N_G\Gamma/\Gamma| < \infty$, where $N_G\Gamma$ is the normalizer in G of Γ.*

4. Lorentz dynamics

4.1. Kowalsky's argument. In [**K1**], Kowalsky relates the dynamics of Lorentz-isometric actions of a semisimple Lie group G with the adjoint representation on $Sym^2(\mathfrak{g}^*)$.

For each $x \in X$, there are linear maps
$$\begin{aligned} f_x &: \mathfrak{g} \to T_xX \\ f_x &: Y \mapsto \left.\frac{\partial}{\partial t}\right|_0 e^{tY}x \end{aligned}$$

Differentiating $ge^{tY}x = ge^{tY}g^{-1}(gx)$ gives the relation
$$g_{*x}f_x(Y) = f_{gx} \circ \mathrm{Ad}(g)(Y)$$

Let $<,>_x$ denote the inner product on $\mathfrak{g}$ obtained by pulling back the Lorentz inner product on T_xX by f_x. Since the G^0-action is isometric,

$$< Y, Z >_{gx} = < \mathrm{Ad}(g^{-1})Y, \mathrm{Ad}(g^{-1})Z >_x$$

In Kowalsky's argument, the dynamics of the nonproper group action imply that many root spaces of $\mathfrak{g}$ belong to the same maximal isotropic subspace for some $<,>_x$. We adapt this argument to obtain the following result. Recall that π_i is the projection of G or $\mathfrak{g}$ on the i^{th} (local) factor.

Proposition 4.1. *Let G be a connected semisimple group acting isometrically on a Lorentz manifold. Suppose that for $y \in X$, there is a sequence $g_n \in G(y)$ with $Ad(g_n) \to \infty$. Then $\mathfrak{g}$ has a root system Δ and an $\mathbf{R}$-split element A such that*

$$\bigoplus_{\alpha \in \Delta, \alpha(A) > 0} \mathfrak{g}_\alpha$$

is an isotropic subspace for $<,>_y$.

Suppose further that G^0 preserves an isotropic vector field S^* along the orbit G^0y, and let $S \in \mathfrak{g}$ be such that $f_y(S) = S^*(y)$. Then, with respect to $<,>_y$,

$$\left(\bigoplus_{\alpha(A) > 0} \mathfrak{g}_\alpha \right) \perp S$$

Proof: Let $g_n = \widehat{k}_n \widehat{a}_n \widehat{l}_n$ be the KTK decomposition of g_n, where T is a maximal $\mathbf{R}$-split torus in G, and $K = \mathrm{Ad}^{-1}(\mathrm{Ad}(K))$, for $\mathrm{Ad}(K)$ a maximal compact subgroup of $\mathrm{Ad}(G)$. Let $\mathrm{Ad}(g_n) = k_n a_n l_n$ be the corresponding decomposition in $\mathrm{Ad}(G)$. The condition $\mathrm{Ad}(g_n) \to \infty$ implies $a_n \to \infty$. Let $A_n = \ln a_n$. By passing to a subsequence, we may assume

- $A_n/|A_n| \to A$ for some $\mathbf{R}$-split $A \in \mathfrak{g}$
- $k_n \to k$
- $l_n \to l$

Let Δ be a root system with respect to $\mathfrak{a} = \ln T$. Let $\alpha, \beta \in \Delta$ be such that $\alpha(A), \beta(A) > 0$. Let $U \in \mathfrak{g}_\alpha$ and $V \in \mathfrak{g}_\beta$. We have, for all n,

$$< U, V >_{\hat{a}_n \hat{l}_n y} = < U, V >_{\hat{k}_n^{-1} y} \tag{1}$$

The left hand side is

$$\begin{aligned} < a_n^{-1}(U), a_n^{-1}(V) >_{\hat{l}_n y} &= e^{-\alpha(A_n) - \beta(A_n)} < U, V >_{\hat{l}_n y} \\ &= e^{-\alpha(A_n) - \beta(A_n)} < l_n^{-1}(U), l_n^{-1}(V) >_y \end{aligned}$$

The inner products $< l_n^{-1}(U), l_n^{-1}(V) >_y$ converge to $< l^{-1}(U), l^{-1}(V) >_y$; in particular, they are bounded. The factors $e^{-\alpha(A_n)-\beta(A_n)}$ converge to 0. Then the left side of (1) converges to 0.

The right hand side of (1) converges to

$$< k(U), k(V) >_y$$

Therefore, the sum of root spaces

$$\bigoplus_{\alpha(A)>0} k(\mathfrak{g}_\alpha)$$

is an isotropic subspace for $<,>_y$. Now replace Δ with $\Delta \circ k^{-1}$ and A with $k(A)$ to obtain the first assertion of the proposition.

Now let S^* be a G^0-invariant vector field along the orbit $G^0 y$. If S is such that $f_y(S) = S^*(y)$, then $f_{gy}(\mathrm{Ad}(g)(S)) = S^*(gy)$ for any $g \in G^0$. Let $\mathrm{Ad}(g_n) = k_n a_n l_n$ be the KTK decomposition as above, and let $A = \lim(A_n/|A_n|)$. Now suppose α is a root with $\alpha(A) > 0$. For $U \in \mathfrak{g}_\alpha$

$$< k_n U, k_n a_n l_n(S) >_{\widehat{k}_n \widehat{a}_n \widehat{l}_n y} = < k_n U, S >_y$$

The left hand side is

$$< a_n^{-1}(U), l_n(S) >_{\widehat{l}_n y} = e^{-\alpha(A_n)} < U, l_n(S) >_{\widehat{l}_n y}$$

This sequence converges to 0. The right hand side converges to

$$< k(U), S >_y$$

Then $k(\mathfrak{g}_\alpha) \perp S$ with respect to $<,>_y$, yielding the desired result when A is replaced with $k(A)$ and Δ with $\Delta \circ k^{-1}$. q.e.d.

Remark 4.2. *Note that if, for the* $\mathbf{R}$*-split element A given by Proposition* 4.1*, $\pi_i(A) \neq \mathbf{0}$, then $\pi_i(g_n) \to \infty$.*

Remark 4.3. *In the proof above, if we start with a KTK decomposition with $\mathfrak{a} = \ln T$, then the element A given by Proposition* 4.1 *belongs to $Ad(K)(\mathfrak{a})$.*

4.2. Totally geodesic codimension-one lightlike foliations. A *lightlike* submanifold of a Lorentz manifold is a submanifold on which the restriction of the metric is degenerate. A foliation is lightlike if each leaf is lightlike. In [**Ze1**], Zeghib shows that a compact Lorentz manifold M with a noncompact group $G \subset Isom(M)$ has totally geodesic codimension-one lightlike (tgl) foliations. Fix a smooth Riemannian metric σ on M giving rise to a norm $|\cdot|$ and a distance d on M. Let $x \in M$ and g_n be a sequence in G. The *approximately stable set of g_n at x* is

$$AS(x, g_n) = \{v \in T_x M : v = \lim v_n \text{ where } v_n \in TM \text{ and } |g_{n*} v_n| \text{ is bounded}\}$$

Zeghib proves that any unbounded g_n has a subsequence for which the approximately stable set in TM forms an integrable codimension-one lightlike distribution with totally geodesic leaves. The resulting foliation $\mathcal{F}$ is Lipschitz, in the sense that there exists $C > 0$ such that

$$\angle(T\mathcal{F}_x, T\mathcal{F}_y) \leq C \cdot d(x, y)$$

for all sufficiently close $x, y \in M$. Provided x and y are in a common normal neighborhood, we can define the angle above as

$$\angle(T\mathcal{F}_x, T\mathcal{F}_y) = \angle_\sigma(P_\gamma T_x\mathcal{F}_x, T_y\mathcal{F}_y)$$

where P_γ is parallel transport with respect to the Lorentzian connection along the geodesic γ from x to y. In fact, there exists C that serves as a uniform Lipschitz constant for all totally geodesic codimension-one foliations.

We extend this work to obtain tgl foliations on X associated to a sequence $g_n \in G$ unbounded modulo Γ. Let $|\cdot|$ be a smooth norm on X that is Γ-invariant; such a norm can be obtained by lifting an arbitrary smooth norm from M. For $x \in X$ and a sequence $g_n \in G$, define

$$AS(x, g_n) = \{v \in T_xX : v = \lim v_n \text{ where } v_n \in TX \text{ and } |g_{n*}v_n| \text{ is bounded}\}$$

Note that $AS(gx, g_n) = AS(x, \gamma_n g_n)$ for any sequence γ_n in Γ, so this set can be considered associated to a sequence in $\Gamma\backslash G$. On the other hand, for $g \in G$,

$$AS(x, g_n g^{-1}) = g_*(AS(x, g_n))$$

Theorem 4.4. *Let $g_n \in G$ be unbounded modulo Γ. Then there is a subsequence such that the set of $AS(x, g_n)$, for $x \in X$, form an integrable distribution with totally geodesic codimension-one lightlike leaves. Moreover, the set $\mathcal{TGL}(X)$ of tgl foliations is uniformly Lipschitz: there exist $C, \delta > 0$, such that, for any foliation $\mathcal{F} \in \mathcal{TGL}(X)$, for any $x, y \in X$ with $d(x, y) < \delta$,*

$$\angle(\mathcal{F}_x, \mathcal{F}_y) \leq C \cdot d(x, y)$$

The proof is essentially the same as that in [**Ze1**]. We outline that proof and provide the observations relevant to our generalization in [**Me**]. For completeness, the uniformly Lipschitz property is proved in detail in the Appendix of [**Me**].

We begin with a definition.

Definition 4.5. *Let X be a k-dimensional manifold endowed with a smooth, torsion-free connection ∇ and a smooth Riemannian metric σ. A* radius-r codimension-one geodesic lamination *on X consists of a subset $X' \subset X$ and a section $f : X' \to Gr^{k-1}(TX)|_{X'}$, satisfying*

1) *$\mathcal{L}_x = exp^\nabla(f(x) \cap B_\sigma(\mathbf{0}, r))$ is ∇-geodesic for each $x \in X'$*
2) *$\mathcal{L}_x \cap \mathcal{L}_y$ is open in both $\mathcal{L}_x$ and $\mathcal{L}_y$ for all $x, y \in X'$*

Proposition 4.6. *Let X be the universal cover of a compact manifold M. Let ∇ be a smooth connection and σ a smooth Riemannian metric, both lifted from M. For any $r > 0$, there exist $C, \delta > 0$ such that any radius-r, codimension-one geodesic lamination (X', f) on X is (C, δ)-Lipschitz: any $x, y \in X'$ with $d_\sigma(x, y) < \delta$ are connected by a unique ∇-geodesic γ, and*

$$\angle_\sigma(P_\gamma f(x), f(y)) \leq C \cdot d_\sigma(x, y)$$

We record two consequences.

Corollary 4.7. *For any radius-r codimension-one geodesic lamination (X', f), the function f is uniformly continuous on X'.*

Assuming $C \geq 1$, two values $f(x), f(y)$ will be ϵ-close in $Gr^{k-1}(TX)$ provided $d_\sigma(x, y) \leq \min\{\epsilon/2C, \delta\}$.

Corollary 4.8. *The space $\mathcal{TGL}(X)$ is compact.*

The space $\mathcal{TGL}(X)$ can be identified with a closed subset of the space of sections $f : X \to Gr^{k-1}TX$. Given any sequence $f_n \in \mathcal{TGL}(X)$, a diagonalization procedure gives a pointwise limit f_∞ defined on a countable dense subset $X' \subset X$. Since f_∞ is uniformly continuous on X', it extends uniquely to X. Because the f_n are equicontinuous, they converge uniformly on compact sets to f_∞.

5. A closed orbit

In this section, we consider a slightly more general setting. Let M be a compact, connected, real-analytic manifold with a real-analytic rigid geometric structure of algebraic type defining a connection (See [**B**], [**Gr**] or [**DAG**] for an introduction to Gromov's theory of rigid geometric structures). Assume that this connection is *complete*—that is, that the exponential map is defined on all of TM. An example of a rigid geometric structure of algebraic type defining a connection is a pseudo-Riemannian metric. We will make use of Gromov's stratification theorem and its consequences for real-analytic rigid geometric structures of algebraic type. Let G be the group of automorphisms of the lifted structure on the universal cover X; it is a finite-dimensional Lie group ([**Gr**] 1.6.H). As usual, let G^0 be the identity component of G; let $\Gamma \subset G$ be the group of deck transformations of X; and let $\Gamma_0 = \Gamma \cap G^0$.

Let J be the pseudogroup of germs of local automorphisms of M. For $x \in M$, let J_x be the pseudogroup of germs at x of local isometries. Call the *J-orbit* of $x \in M$ the equivalence class of x under the relation $x \sim y$ when $jx = y$ for some $j \in J_x$. Gromov's stratification theorem says the following:

Theorem 5.1 ([**Gr**] 3.4)**.** *There is a J-invariant stratification*

$$\emptyset = M_{-1} \subset M_0 \subset \cdots \subset M_k = M$$

such that, for each i, $0 \leq i \leq k$, the complement $M_i \backslash M_{i-1}$ is an analytic subset of M_i. Further, each $M_i \backslash M_{i-1}$ is foliated by J-orbits, and the J-orbits are properly embedded in $M_i \backslash M_{i-1}$.

Corollary 5.2 ([**Gr**] 3.4.B, c.f. [**DAG**] 3.2.A (iii))**.** *There exists a closed J-orbit in M.*

The stratification above is obtained from similar stratifications invariant by infinitesimal isometries of order k, for arbitrary sufficiently large k. It is shown in [**Gr**] 1.7.B that orbits of infinitesimal isometries of increasing order eventually stabilize to J-orbits. For any $x \in M$, the infinitesimal isometries of order k fixing x form an algebraic subgroup of $GL(T_xM)$, because the given H-structure is of algebraic type. Then stabilization of infinitesimal isometries to local isometries implies that the group $J(x)$ of germs in J_x fixing x has algebraic isotropy representation on T_xM (see [**DAG**] 3.5, [**Gr**] 3.4.A); in particular, $J(x)$ has finitely-many components.

The aim of this section is to establish that the properties of J-orbits discussed above apply also to images in M of G^0-orbits on X. The main reason for this correspondence is the fact, proved by Nomizu [**N**], Amores [**Am**], and, in full generality, Gromov [**Gr**], that local Killing fields on X can be uniquely extended to global Killing fields. Because the connection on X is complete, any global Killing field integrates to a one-parameter subgroup of G (see [**KN**] VI.2.4). Thus there is a correspondence between local Killing fields near any point of M and elements of $\mathfrak{g}$. A group $H \subseteq GL(V)$ will be called *locally algebraic* if $\mathfrak{h}$ is the Lie algebra of an algebraic subgroup of $GL(V)$.

Proposition 5.3. *For any $y \in X$, the image of the isotropy representation of $G(y)$ is a finite-index subgroup of an algebraic subgroup of $GL(T_yX)$; the same is true for $G^0(y)$. In particular, $G^0(y)$ is locally algebraic.*

Proof: Denote by π the covering map from X to M. There is an obvious homomorphism $\varphi : G(y) \to J(z)$, where $z = \pi(y)$. A tangent vector at the identity to $J(z)$ corresponds to the germ of a local Killing field at z. Local Killing fields near z can be lifted to X, extended, and integrated, giving a linear homomorphism $T_e(J(z)) \to \mathfrak{g}$ inverse to $D_e\varphi$. Then φ is a local diffeomorphism near the identity, and so it is a local isomorphism $G(y) \to J(z)$. By rigidity, any $g \in G(y)$ with trivial germ at y is trivial, so φ is an isomorphism onto its image. The image is a union of components of $J(z)$; because the latter group is algebraic, the proposition follows for $G(y)$. The restriction of φ to $G^0(y)$ is also an isomorphism onto its image. q.e.d.

Proposition 5.4. *There is an orbit G^0y in X with closed image in M.*

Proof: Let $z \in M$ have closed J-orbit, and choose any $y \in X$ with $\pi(y) = z$. The image $\pi(G^0 y)$ is a connected submanifold of Jz, though it is not *a priori* closed. Denote by $J^0 z$ the component of z in Jz. This is the orbit of z under local Killing fields on M—that is, all points of M that can be reached from z by flowing along a finite sequence of local Killing fields. Because each local Killing field on M corresponds to a 1-parameter subgroup of G^0, this component $J^0 z$ is contained in $\pi(G^0 y)$. They are therefore equal, and closed in M, because Jz has finitely-many components and is closed in M. q.e.d.

Proposition 5.5. *Let y be as in the previous proposition, so $G^0 y$ has closed image in M. The subgroup $\Gamma_0 = G^0 \cap \Gamma \subset G^0$ acts freely, properly discontinuously, and cocompactly on $G^0/G^0(y)$.*

Proof: Let G_y be the subgroup of G leaving invariant the orbit $G^0 y$, and $\Gamma_y = G_y \cap \Gamma$; note that Γ_y acts cocompactly on $G^0 y$. Because $G^0 y$ is a closed submanifold of X, the orbit map $G^0/G^0(y) \to G^0 y$ is a homeomorphism onto its image (See [**Gl**]). It therefore suffices to show that Γ_0 has finite index in Γ_y.

Now $G^0 y = G_y y$ is also the homeomorphic image of $G_y/G(y)$, which is then connected. As in Proposition 5.3, $G(y)$ has finitely-many components; then so does G_y. Thus G^0 is a finite-index subgroup of G_y, so Γ_0 is a finite-index subgroup of Γ_y, as desired. q.e.d.

Corollary 5.6. *If G^0 has no compact orbits on X, then Γ_0 is an infinite normal subgroup of Γ.*

6. Proof of main theorem

Let $Y = G^0 y$ be the orbit given by Proposition 5.4 with closed projection to M.

6.1. Proper case. If $G^0(y)$ is compact, then G^0 acts properly; in fact, so does the group G' generated by G^0 and Γ.

Proposition 6.1. *Let G' be the closed subgroup of G generated by G^0 and Γ. If $G^0(y)$ is compact, then G' acts properly on X.*

Proof: If $G^0(y)$ is compact, then by Proposition 5.5, Γ_0 is a cocompact lattice in G^0. Let F be a compact fundamental domain for Γ_0 containing the identity in G^0; note F is also a compact fundamental domain for Γ in G'. Let A be a compact subset of X, and G'_A the set of all g in G' with $gA \cap A \neq \emptyset$. Any $g \in G'_A$ is a product γf where $f \in F$ and $\gamma \in \Gamma_{FA}$. Since FA is compact, Γ_{FA} is a finite set $\{\gamma_1, \dots, \gamma_l\}$. Then G'_A is a closed subset of the compact set $\gamma_1 F \cup \cdots \cup \gamma_l F$, so it is compact. q.e.d.

The first statement in the proper case of Theorem 1.3 is that M is an orbibundle

$$\Lambda\backslash G^0/K_0 \to M \to Q$$

We prove this statement, with $\Lambda = \Gamma_0$, in three steps.

Step 1: Γ/Γ_0 proper on $G^0\backslash X$.

Let $\overline{A}$ be a compact subset of $G^0\backslash X$, and let

$$(\Gamma/\Gamma_0)_{\overline{A}} = \{[\gamma] \in \Gamma/\Gamma_0 : [\gamma]\overline{A} \cap \overline{A} \neq \emptyset\}$$

The aim is to show this set is finite. There is a compact subset A of X projecting onto $\overline{A}$ by Proposition 3.1 (1) because G^0 acts properly on X. Let

$$\Gamma_{A,G^0A} = \{\gamma \in \Gamma : \gamma A \cap G^0 A \neq \emptyset\}$$

Note that Γ_{A,G^0A} is invariant under right multiplication by Γ_0, and

$$(\Gamma/\Gamma_0)_{\overline{A}} = \Gamma_{A,G^0A}/\Gamma_0$$

Let F be a compact fundamental domain for Γ_0 in G^0. Since FA is compact and Γ acts properly, the set $\Gamma_{A,FA}$ is finite. Then $(\Gamma_{A,FA} \cdot \Gamma_0)/\Gamma_0 = \Gamma_{A,G^0A}/\Gamma_0$ is finite, as well.

Let K_0 be a maximal compact subgroup of G^0.

Step 2: $G^0(x) \cong K_0$ for all $x \in X$.

Any stabilizer $G^0(x)$ is compact, so conjugate to a subgroup of K_0. Since K_0 is connected, it suffices to show that $\dim K_0 \leq \dim G^0(x)$. We follow the cohomological dimension arguments of Farb and Weinberger [**FW**]. By Proposition 3.6 (2) and 5.5,

$$\mathrm{cd}_{\mathbf{Q}}\Gamma = \dim X \qquad \text{and} \qquad \mathrm{cd}_{\mathbf{Q}}\Gamma_0 = \dim(G^0/K_0)$$

By the extension of [**FW**] (2.2) of the Conner conjecture ([**O**]), the quotient space $G^0\backslash X$ is contractible because G^0 acts properly and X is contractible. For any $x \in X$,

$$\dim X - \dim(G^0/G^0(x)) \geq \dim(G^0\backslash X)$$

by 3.1 (2).

Because G' acts properly and smoothly on X, the quotient $G'\backslash X$ is Whitney stratified, and so triangulable by [**Go**]. Then the action of Γ/Γ_0 on $G^0\backslash X$ is proper and cellular (see [**Me**] 4.13). Then Proposition 3.6 (4) gives

$$\mathrm{cd}_{\mathbf{Q}}(\Gamma/\Gamma_0) \leq \dim(G^0\backslash X)$$

Now the inequality 3.6 (3) gives for any $x \in X$,

$$\begin{aligned} \dim X &\leq \dim(G^0/K_0) + \dim(G^0\backslash X) \\ &\leq -\dim K_0 + \dim X + \dim G^0(x) \end{aligned}$$

so $\dim K_0 \leq \dim G^0(x)$ for any $x \in X$, as desired.

Step 3: Orbibundle.

Now from Step 2 and proposition 3.1 (3), $G^0\backslash X$ is a manifold on which Γ/Γ_0 acts properly discontinuously. The foliation of X by G^0-orbits descends to M, and all leaves in M are closed. The leaf space is $Q = (\Gamma/\Gamma_0)\backslash(G^0\backslash X)$, a smooth orbifold. Given U open in Q, lift it to a connected $\widetilde{U}$ in $G^0\backslash X$. For U sufficiently small, the fibers of M over U are

$$\widetilde{U} \times_{\Lambda_{\widetilde{U}}} (\Gamma_0\backslash G^0/K_0)$$

where $\Lambda_{\widetilde{U}} = \{[\gamma] \in \Gamma/\Gamma_0 \;:\; [\gamma]\widetilde{U} \cap \widetilde{U} \neq \emptyset\}$ is a finite group. We have an orbibundle

$$\Gamma_0\backslash G^0/K_0 \to M \to Q$$

Now it remains to prove the second part of the theorem in the proper case, giving the metric on M, assuming $Z(G^0)$ is finite.

Step 4: Splitting of X.

Let

$$\rho : X \to G^0/K_0$$
$$\rho(x) = [g] \qquad \text{where} \qquad gK_0g^{-1} = G^0(x)$$

This map is well-defined and injective along each orbit because $N(K_0) = K_0$ (3.7 (2)). Each fiber $\rho^{-1}([g])$ equals the fixed set $Fix(gK_0g^{-1})$. Each orbit is mapped surjectively onto G^0/K_0. Let $L = \rho^{-1}([e]) = Fix(K_0)$, a totally geodesic submanifold of X. Under the quotient, L maps diffeomorphically to $G^0\backslash X$, so L is connected. The map

$$G^0/K_0 \times L \to X \qquad ([g], l) \to gl$$

is a well-defined diffeomorphism.

The restriction of the metric to each G^0-orbit must be Riemannian. Indeed, let $x \in L$ and consider the isotropy representation of K_0. The map $f_x : \mathfrak{g} \to T_xX$ gives a K_0-equivariant isomorphism $\mathfrak{g}/\mathfrak{k} \to T_x(G^0x)$, where K_0 acts on $\mathfrak{g}/\mathfrak{k}$ via the adjoint representation. If the inner-product on $T_x(G^0x)$ is degenerate, then the kernel is 1-dimensional, and K_0 is trivial on it. If $T(G^0x)$ is Lorentzian, then K_0 preserves a norm, so it fixes a minimal length timelike vector. Either way, the isotropy representation of K_0 has a fixed vector. But $\mathrm{Ad}(K_0)$ has no one-dimensional invariant subspace in $\mathfrak{p} \cong \mathfrak{g}/\mathfrak{k}$ (3.7 (1)), a contradiction.

For the same reason, L is orthogonal to each G^0-orbit. Indeed, let $x \in L$. The subspaces $T_x(G^0x)^\perp$ and T_xL are both K_0-invariant complements to $T_x(G^0x)$ in T_xX. If they are unequal, then there are nonzero vectors $v \in T_x(G^0x)$ and $w \in T_xL$ such that $v - w \in T_x(G^0x)^\perp$. Then

$$\begin{aligned} & k(v-w) = kv - w \in T_x(G^0x)^\perp \\ \Rightarrow \quad & kv - v \in T_x(G^0x)^\perp \\ \Rightarrow \quad & kv = v \end{aligned}$$

again contradicting that K_0 has no one-dimensional invariant subspace in $T_x(G^0x)$.

Step 5: Splitting of Γ.

The argument here is the same as in [**FW**]. The extension

$$\Gamma_0 \to \Gamma \to \Gamma/\Gamma_0$$

is a subextension of

$$G^0 \to G' \to \Gamma/\Gamma_0$$

so the action $\Gamma/\Gamma_0 \to Out(\Gamma_0)$ is the restriction of $\Gamma/\Gamma_0 \to Out(G^0)$. Since G^0 is semisimple, $Out(G^0)$ is finite. Thus there is a finite-index subgroup Γ' of Γ containing Γ_0 such that conjugation by any $\gamma \in \Gamma'$ is an inner automorphism of Γ_0. The extension

$$\Gamma_0 \to \Gamma' \to \Gamma'/\Gamma_0$$

also determines a cocycle in $H^2(\Gamma'/\Gamma_0, Z(\Gamma_0))$. But $Z(\Gamma_0)$ is trivial (3.7 (3)). This extension is therefore a product

$$\Gamma' \cong \Gamma_0 \times \Gamma'/\Gamma_0$$

Since Γ' has finite integral cohomological dimension, it is torsion-free, and thus so is Γ'/Γ_0. Then Γ'/Γ_0 acts freely on $G^0\backslash X$, and the quotient, which is a finite cover of Q, is a manifold Q'. The finite cover $M' = \Gamma'\backslash X$ is diffeomorphic to $\Gamma_0\backslash G^0/K_0 \times Q'$. The metric descends from X to M' and has the form claimed in the theorem.

6.2. Nonproper case: if G^0 has infinite orbit in $\mathcal{TGL}(X)$. Now suppose that $G^0(y)$ is noncompact, so G^0 acts nonproperly; further, $\Gamma\backslash G$ is noncompact. By Theorem 4.4, there are tgl foliations on X. The set $\mathcal{TGL}(X)$ of all these foliations forms a G-space. Pick any $\mathcal{F} \in \mathcal{TGL}(X)$ and let $\mathcal{O}$ be the G^0-orbit of $\mathcal{F}$. Because G^0 is connected, this orbit is connected, so it either equals $\{\mathcal{F}\}$ or is infinite. We first deduce the conclusion of the main theorem in case $\mathcal{O}$ is infinite.

6.2.1. Warped product. Consider the continuous map

$$\begin{aligned} \varphi &: \mathcal{TGL}(X) \times X \to \mathbf{P}(TX) \\ \varphi &: (\mathcal{F}, x) \mapsto (x, (T\mathcal{F}_x)^{\perp}) \end{aligned}$$

For each $x \in X$, the image $\varphi(\mathcal{O} \times \{x\})$ is connected, so it is either infinite or just one point. The set D of all x for which $|\varphi(\mathcal{O} \times \{x\})| = 1$ is closed. The complement $D^c \neq \emptyset$ because $\mathcal{O}$ is infinite. For $x \in X$, let C_x be the set of lightlike lines in T_xX normal to leaves through x of codimension-one, totally goedesic, lightlike hypersurfaces. For all $x \in D^c$, the set C_x is infinite.

Now Theorem 1.1 of [**Ze2**] applies to give an open set $U \subseteq D^c$ locally isometric to a warped product $N \times_h L$, where N is Lorentzian of constant curvature, and L is Riemannian. For each $x \in U$, the subspace generated by C_x equals T_xN_x, where N_x is the N-fiber through x (see the intermediate result [**Ze2**] 3.3). Since X is the universal cover of a compact, real-analytic manifold, Theorem 1.2 of [**Ze2**] implies that X is a global warped product $N \times_h L$, and both N and L are complete.

Because G^0 preserves the cone field $x \mapsto C_x$, it also preserves the N-foliation. Then $G_1 = Isom^0(N) \lhd G^0$, so it is semisimple. Since X is contractible, N and L are, as well. Then N must be isometric to $\widetilde{AdS}^k$ for some k, and $G_1 \cong \widetilde{O}^0(2, k-1)$. The assumption that $C_x \subset T_xN$ is infinite implies $k \geq 3$.

6.2.2. Orbibundle. Now it remains to show that $X \to G^0\backslash X$ is a fiber bundle, and that M is an orbibundle. Let G_2 be the kernel of the homomorphism $G^0 \to Isom^0(N)$; it is semisimple, and $G^0 \cong G_1 \times G_2 \subseteq Isom(N) \times Isom(L)$. The G^0-orbit Y is isometric to $N \times L_2$ for a Riemannian submanifold L_2 of L, and G_2 is isomorphic to a connected subgroup of $Isom(L_2)$. Clearly, $G_2(x)$ is compact for all $x \in X$, so it is conjugate into K_2. We will show, using cohomological dimension, that $G_2(x) \cong K_2$ for all $x \in X$, where K_2 is a maximal compact subgroup of G_2.

Since Γ_0 acts properly discontinuously and cocompactly on $Y \cong N \times G_2/G_2(y)$, it is also properly discontinuous and cocompact on $N \times G_2/K_2$, where we assume $G_2(y) \subseteq K_2$. This latter space is contractible, so by Proposition 3.6 (2),

$$\mathrm{cd}_{\mathbf{Q}}\Gamma_0 = k + \dim(G_2/K_2)$$

Next, the quotient $G_2\backslash L$ can be identified with $G^0\backslash X$. Since L is contractible and G_2 acts properly on it, either quotient is contractible by [**FW**] 2.2. We want to show that Γ/Γ_0 acts properly discontinuously on this quotient. Suppose that a compact $\overline{C} \subset G_2\backslash L$ is given. The goal is to show that

$$(\Gamma/\Gamma_0)_{\overline{C}} = \{[\gamma] \in \Gamma/\Gamma_0 : [\gamma]\overline{C} \cap \overline{C} \neq \emptyset\}$$

is finite.

Denote by L_y the L-leaf containing y. There is a compact $C \subset L_y \subset X$ projecting onto $\overline{C}$ by 3.1 (1). Let $\mathcal{L}X$ be the bundle of Lorentz frames on X. We may assume C is small enough that $\mathcal{L}X|_C \cong C \times O(1, n-1)$. Let A be the image of a continuous section of $\mathcal{L}X|_C$ split along the product $X = N \times L$—that is, each frame in A has the first k vectors tangent to the N-foliation, and the succeeding vectors tangent to the L-foliation. Let B be the saturation $A \cdot (\mathbf{Z}_2 \times \mathbf{Z}_2 \times O(n-k))$, where $\mathbf{Z}_2 \times \mathbf{Z}_2 \subseteq O(1, k-1)$ acts transitively on orientation and time orientation of Lorentz bases, and $O(n-k) \subset O(1, n-1)$ is trivial on the first k basis vectors; now B is still compact. Since G acts properly on $\mathcal{L}X$ (see [**Gr**] 1.5.B or [**Ko**] 3.2), the set $G_{A,B}$ is compact in G. Because N has constant curvature, $G_1 \cong Isom^0(N)$ is transitive on Lorentz frames along N, up to orientation and time orientation. Then it is not hard to see

$$G_{C,G^0C} = G^0 \cdot G_{A,B} = G_{A,B} \cdot G^0$$

Then G_{C,G^0C} consists of finitely many components of G. Now

$$(\Gamma/\Gamma_0)_{\overline{C}} = (\Gamma_{C,G^0C} \cdot \Gamma_0)/\Gamma_0$$

Distinct Γ_0-cosets in Γ occupy distinct components of G. Then Γ_{C,G^0C} consists of finitely many cosets of Γ_0, and $(\Gamma/\Gamma_0)_{\overline{C}}$ is finite, as desired.

Now, as in Step 2 of Section 6.1,

$$\mathrm{cd}_{\mathbf{Q}}(\Gamma/\Gamma_0) \leq \dim(G^0\backslash X) = \dim(G_2\backslash L)$$

The inequality 3.6 (3) gives

$$k + \dim L \leq k + \dim(G_2/K_2) + \dim(G_2\backslash L)$$

so $\dim G_2(x) = \dim K_2$, and $G_2(x)$ is conjugate in G_2 to K_2 for all x. Then the quotient $\widetilde{Q} = G^0\backslash X$ is a contractible manifold by 3.1 (2). Since Γ/Γ_0 acts properly discontinuously here, M is an orbibundle

$$\Gamma_0\backslash G^0/H_0 \to M \to Q$$

The homogeneous space $G^0/H_0 \cong \widetilde{AdS}^k \times G_2/K_2$.

6.2.3. Splitting. From section 6.2.1, we have

$$X \cong \widetilde{AdS}^k \times_h L$$

where the warping function h on L is G_2-invariant. The function h descends to a function h_1 on $\widetilde{Q}$. From the previous section, all G_2-orbits in L are equivariantly diffeomorphic to G_2/K_2. As in the proper case, if $Z(G_2)$ is finite, we can define

$$\rho : X \to G_2/K_2$$
$$\rho(x) = [g] \qquad \text{where} \qquad gK_2g^{-1} = G_2(x)$$

This map factors through the projection to L. As in the proper case, we can show that $L \cong G_2/K_2 \times_{h_2} \widetilde{Q}$ for some $h_2 : \widetilde{Q} \to \mathcal{M}$, the moduli space of G_2-invariant Riemannian metrics on G_2/K_2. Now $h = (h_1, h_2)$ can be viewed as a function from $\widetilde{Q}$ to the moduli space of G^0-invariant Lorentz metrics on $\widetilde{AdS}^k \times G_2/K_2$.

6.3. Nonproper case: fixed point in $\mathcal{TGL}(X)$. Now suppose, as above, that $G^0(y)$ is noncompact, so $\mathcal{TGL}(X) \neq \emptyset$, but every G^0-orbit in $\mathcal{TGL}(X)$ is a fixed point. Then G^0 preserves a tgl foliation on X, so it preserves a lightlike line field on X. We will show that this is impossible.

First, we may assume that this lightlike line field along Y is tangent to Y. Suppose that Y is either a fixed point or Riemannian. Then the kernel of the restriction of G^0 to Y contains a noncompact semisimple local factor G_1. Recall that C_y is the set of lightlike lines in T_yX normal to codimension-one, totally geodesic, lightlike hypersurfaces through y. Now G_1 acts on C_y via the isotropy representation, and by assumption, it preserves an isotropic line in C_y, but this is impossible if G_1 is semisimple and noncompact. Therefore, the orbit Y is either Lorentzian or *degenerate*—$T_yY^\perp \cap T_yY \neq \mathbf{0}$ for all $y \in Y$. If Y is degenerate, then G^0 preserves the lightlike line field $T_yY^\perp$ along Y. Suppose Y is Lorentzian. Now G^0 preserves the projections of the isotropic line field $y \mapsto C_y$ onto TY and $(TY)^\perp$. If the second projection is nonzero, then the first is necessarily timelike. But if $G^0(y)$ preserves a timelike vector in T_yY, then $G^0(y)$ must be compact, a contradiction. Therefore, we may assume G^0 preserves a lightlike line field tangent to Y, and that Y is either a degenerate or Lorentz submanifold.

We first collect some facts about the isotropy representation. Lemma 6.3 below says that the isotropy is either reductive or unimodular, in each case with a rather specific form. In the reductive case, Proposition 6.5 says that $\mathfrak{g}(y)$ contains no nilpotents. On the other hand, Kowalsky's argument almost always yields nilpotents in $\mathfrak{g}(y)$. The only possibility then is that G has a direct factor locally isomorphic to $SL(2, \mathbf{R})$. In this case, we show that the orbit Y is roughly dS^2. A generalization of the Calabi-Markus phenomenon implies Y admits no cocompact isometric actions, contradicting that $\Gamma_0 \backslash Y$ is compact. In the unimodular case, Kowalsky's argument yields root spaces in $\mathfrak{g}(y)$, and some factor G_1 of G acts nonproperly on Y. Kowalsky's Theorem [**K1**] says that G_1 is locally isomorphic to $O(1, k)$ or $O(2, k)$ for some k. We argue that the root lattice of $O(2, k)$ is incompatible with properties of the adjoint representation of G_1 established in Proposition 6.6. We conclude that Y is roughly equivalent to the light cone in Minkowski space, which also admits no cocompact isometric actions, a contradiction.

6.3.1. Properties of the isotropy respresentation. Fix an isometric isomorphism of T_yX with $\mathbf{R}^{1,n-1}$, determining an isomorphism $O(T_yX) \cong O(1, n-1)$. Let V be the image of T_yY under this isomorphism, and let $k = \dim V$. Let $\Phi : G^0(y) \to O(1, n-1)$ be the resulting isotropy representation. There is a filtration on V preserved by Φ. The notation $U \subset_i V$ means U is a subspace of V with $\dim(V/U) = i$. The invariant filtration is

$$\mathbf{0} \subset_1 V_0 \subset_{k-1-i} V_1 \subset_i V$$

where $i = 0$ or 1 depending on whether V is degenerate or Lorentz. The subspaces V_0 and V_1 are degenerate. Because Φ preserves the isotropic line V_0 it descends to a quotient representation on V_1/V_0, which is orthogonal. The image of Φ is conjugate in $O(1, n-1)$ to the minimal parabolic

$$P = (M \times A) \ltimes U$$

where $U \cong \mathbf{R}^{n-2}$ is unipotent, $A \cong \mathbf{R}^*$, and $M \cong O(n-2)$, with the conjugation action of $M \times A$ on U equivalent to the standard conformal representation of $O(n-2) \times \mathbf{R}^*$ on $\mathbf{R}^{n-2}$. Denote by $\mathfrak{p}$ the Lie algebra of P, and by $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{u}$, the subalgebras corresponding to M, A, and U.

Because G^0 acts properly and freely on the bundle of Lorentz frames of X, the isotropy representation is an injective, proper map. By Corollary 5.3, the image $\Phi(G^0(y))$ is locally algebraic. Let $\varphi : \mathfrak{g}(y) \to \mathfrak{o}(1, n-1)$ be the Lie algebra representation tangent to Φ. Because $im(\varphi)$ is algebraic, there is a Lie algebra decomposition

$$im(\varphi) \cong \mathfrak{r}' \ltimes \mathfrak{u}'$$

where $\mathfrak{r}'$ is reductive and $\mathfrak{u}'$ is unipotent ([**WM3**] 4.4.7). Any unipotent subalgebra of $\mathfrak{p}$ lies in $\mathfrak{u}$, so $\mathfrak{u}' \subset \mathfrak{u}$. The reductive complement $\mathfrak{r}'$ is contained in a maximal reductive subalgebra, which is then conjugate into $\mathfrak{a} \times \mathfrak{m}$.

Note that T_yY can be identified with $\mathfrak{g}/\mathfrak{g}(y)$ by the map f_y as in Section 4.1, and there is the relation

$$g_{*y} \circ f_y(B) = f_y \circ \mathrm{Ad}(g)(B)$$

for $B \in \mathfrak{g}$ and $g \in G^0(y)$. In other words, Φ restricted to V is equivalent to the representation $\overline{\mathrm{Ad}}$ of $G^0(y)$ on $\mathfrak{g}/\mathfrak{g}(y)$ arising from the adjoint representation. Let $\overline{\mathrm{ad}}$ be the representation tangent to $\overline{\mathrm{Ad}}$.

Proposition 6.2. *There is a filtration of $\mathfrak{g}$ invariant by the adjoint of $\mathfrak{g}(y)$:*

$$\mathbf{0} \subset \mathfrak{g}(y) \subset_1 \mathfrak{s}(y) \subset_{k-1-i} \mathfrak{t}(y) \subset_i \mathfrak{g}$$

where $i = 0$ or 1 depending on whether Y is degenerate or Lorentz. The subspace $\mathfrak{s}(y)$ is a subalgebra. The quotient representation for $\overline{\mathrm{ad}}$ on $\mathfrak{t}(y)/\mathfrak{s}(y)$ is skew-symmetric.

Proof: The φ-invariant filtration $\mathbf{0} \subset V_0 \subset V_1 \subset V$ of V corresponds to an $\overline{\mathrm{ad}}$-invariant filtration of $\mathfrak{g}/\mathfrak{g}(y)$. Lifting to $\mathfrak{g}$ gives the desired $\mathrm{ad}(\mathfrak{g}(y))$-invariant filtration. That $\mathfrak{s}(y)$ is a subalgebra follows from the facts that $[\mathfrak{g}(y), \mathfrak{s}(y)] \subset \mathfrak{s}(y)$ and $\dim(\mathfrak{s}(y)/\mathfrak{g}(y)) = 1$. Orthogonality of Φ on V_1/V_0 implies φ is skew-symmetric on V_1/V_0; skew-symmetry of $\overline{\mathrm{ad}}$ on $\mathfrak{t}(y)/\mathfrak{s}(y)$ follows. q.e.d.

Now we show that the image of Φ is either contained in $A \times M$ or $M \ltimes U$.

Lemma 6.3. *The image of φ is either reductive or consists of endomorphisms with no nonzero real eigenvalues.*

Proof: Suppose there is $B \in \mathfrak{g}(y)$ such that $\varphi(B)$ has nonzero eigenvalue λ for some eigenvector $\mathbf{v} \in T_yX$. The vector $\mathbf{v}$ is necessarily isotropic, and we may assume that $\mathbf{v} \in V_0$. Otherwise, for any nonzero

$\mathbf{w} \in V_0$, the inner product $< \mathbf{v}, \mathbf{w} > \neq 0$, which implies that $\varphi(B)$ has nonzero real eigenvalue on $\mathbf{w}$, as well.

Assume $\lambda > 0$; the case $\lambda < 0$ is similar. We may assume $B \in \mathfrak{r}'$. By considering $\varphi(B)$ on the subquotients of the invariant filtration $V_0 \subset V_1 \subset V$, one sees that the trace of $\varphi(B)|_V$ is nonnegative, and equals 0 if and only if V is Lorentz. Correspondingly, the trace of $\overline{\mathrm{ad}}(B)$ on $\mathfrak{g}/\mathfrak{g}(y)$ is nonnegative.

If $\varphi(B) \in \mathfrak{p}$ has eigenvalue $\lambda > 0$, then the adjoint $\mathrm{ad}(\varphi(B))$ has no negative eigenvalues on $\mathfrak{p}$. To simplify the argument, we will use that $\varphi(B) = B_1 + B_2$, where $\mathbf{0} \neq B_1 \in \mathfrak{a}$ and $B_2 \in \mathfrak{m}$. It is easy to see that $\mathrm{ad}(B_1)$ has only real nonnegative eigenvalues on $\mathfrak{p}$. All eigenvalues of $\mathrm{ad}(B_2)$ are purely imaginary. Since $\mathrm{ad}(B_1)$ and $\mathrm{ad}(B_2)$ are simultaneously diagonalizable, their sum $\mathrm{ad}(\varphi(B))$ cannot have a negative eigenvalue.

Now suppose that $im(\varphi)$ is not reductive, so $\mathfrak{u}' \neq \mathbf{0}$. Let $m = \dim(\mathfrak{u}')$. It is easy to compute that the trace of $\mathrm{ad}(\varphi(B))$ on $\mathfrak{u}'$ is $m\lambda$. Since $\mathrm{ad}(\varphi(B))$ has no negative eigenvalues, the trace of $\mathrm{ad}(\varphi(B))$ on $im(\varphi) \subseteq \mathfrak{p}$ is positive. Then the trace of $\mathrm{ad}(B)$ on $\mathfrak{g}(y)$ is positive.

Finally, the trace of $\mathrm{ad}(B)$ on $\mathfrak{g}$ is positive, which is impossible because $\mathfrak{g}$ is semisimple, hence unimodular. q.e.d.

Now we have that $im(\Phi)$ is either a reductive subgroup of $A \times M$ or has the form $M' \ltimes U'$, where $M' \subset M$ and $U' \subset U$.

6.3.2. Two examples with no compact quotient. Two-dimensional de Sitter space.

The 2-dimensional de Sitter space dS^2 has isometry group $O(1,2)$ and isotropy $O(1,1)$, which has an index-two subgroup isomorphic to $\mathbf{R}^*$. It is a well-known result of Calabi and Markus that no infinite subgroup of $O(1,2)$ acts properly on dS^2, so it has no compact quotient [**CM**]. More generally, if $Y = dS^2 \times L$ for some Riemannian manifold L, then no subgroup of the product $O(1,2) \times Isom(L)$ acts properly discontinuously and cocompactly on Y; this is proved in [**Ze1**] §15.1.

We will need an analogous result that also applies to the universal cover $\widetilde{dS}^2$.

Proposition 6.4. *Let S be a Lorentz manifold with universal cover $\widetilde{dS}^2$. Let $G \cong Isom^0(S)$, and H be a connected Lie group. There is no subgroup $\Gamma \subset G \times H$ acting properly discontinuously and cocompactly on $S \times H$.*

Proof: It suffices to prove the proposition assuming $S = \widetilde{dS}^2$.

Let $K = Ad^{-1}(SO(2))$, where $SO(2)$ is a maximal compact subgroup of $Ad(G) \cong O^0(1,2)$. Let $Z \cong \mathbf{Z}$ be the torsion-free factor of the center $Z(G)$. Let $\overline{K}$ be a compact fundamental domain in K for the Z-action with $\overline{K} = \overline{K}^{-1}$; for example, identifying $SO(2)$ with S^1 and K with $\mathbf{R}$, we can take $\overline{K} = [-1/2, 1/2]$. Let A be a maximal $\mathbf{R}$-split torus in G. We have $G = KAK = Z\overline{G}$, where $\overline{G} = \overline{K}A\overline{K}$. For any $g \in \overline{G}$,

$$g\overline{K} \cap \overline{K}A \neq \emptyset$$

The translation number helps to sift the Z-action from the $\overline{G}$-action, to say that any Γ acting cocompactly has infinitely-many elements with uniformly bounded projection in H and G-projection intersecting $\overline{G}$ in an infinite subset, thus contradicting properness.

There is an isomorphism $G \cong \widetilde{SL}_2(\mathbf{R})$, so G acts on the real line, with Z acting by integral translations. The *translation number*

$$\tau \ : \ G \to \mathbf{Z} \cong Z$$
$$\tau \ : \ g \mapsto \lim_{n\to\infty} \frac{g^n(0)}{n}$$

is a *quasi-morphism* (see [**Gh**]): there exists $D > 0$ such that

$$|\tau(gg') - \tau(g) - \tau(g')| < D \qquad \text{for all } g, g' \in G$$

We can choose $\overline{K}$ and A so that $\tau(\overline{K}) = [-1/2, 1/2]$, and $\tau(A) = 0$. Therefore if $g \in \overline{G}$, then $|\tau(g)| \leq 2D + 1$. Also note that for $n \in \mathbf{Z} \cong Z$ and $g \in G$, then $\tau(ng) = n + \tau(g)$.

Now suppose that $\overline{C} \subset \widetilde{dS}^2 \times H$ is a compact fundamental domain for Γ. Denote by ρ_1 and ρ_2 the projections onto $\widetilde{dS}^2$ and H, respectively. We may assume that the identity of H is in $\rho_2(\overline{C}) = U$. For $n \in Z$, let

$$S_n = \{(g,h) \in \Gamma \ : \ g \in n\overline{G}, \ hU \cap U \neq \emptyset\}$$

For a subset $L \subseteq G$, denote by $[L]$ its image in $\widetilde{dS}_2$. Note that, for any $\gamma \in S_n$, the intersection

$$\gamma([\overline{K}] \times U) \cap ([n\overline{K}] \times U) \neq \emptyset$$

Therefore, if Γ acts properly discontinuously, then $|S_n| < \infty$ for each $n \in Z$.

On the other hand, we have $[\overline{G}] \times U \subset \Gamma \cdot \overline{C}$. Let C be a compact lift of $\rho_1(\overline{C})$ to G. Then we have

$$\overline{G} \times U \subset \Gamma \cdot (CA \times U)$$

The restriction of $|\tau|$ to $\overline{G}CA$ is bounded, so, for $|n|$ sufficiently large,

$$n\overline{G}CA \cap \overline{G} = \emptyset$$

It follows that $\overline{G} \times U$ is contained in the union of finitely many $S_n \cdot (CA \times U)$, which is a union of finitely many translates $\gamma \cdot (CA \times U)$, which is impossible, because $[\overline{G}] \times U$ is not compact. q.e.d.

The Minkowski light cone.

A component of the light cone minus the origin in Minkowski space $\mathbf{R}^{1,k-1}$ is a degenerate orbit of $O^0(1, k-1)$, which we will momentarily denote by G^0. The stabilizer of an isotropic vector is isomorphic to

$M \ltimes U$, where $M, U \subset P$ are as above. We will show that no subgroup of G^0 acts properly discontinuously and cocompactly on this orbit.

Suppose that y is a point in the light cone and $\Gamma \subset G^0$ is a discrete subgroup such that $\Gamma \backslash G^0 / G^0(y)$ is a compact manifold. Then $\Gamma \backslash G^0 / U$ is also compact; we may assume it is orientable. Because U is unimodular, the homogeneous space G^0/U has a G^0-invariant volume form (see [**R**] I.1.4). This form descends to $\Gamma \backslash G^0 / U$, where it has finite total volume. The subgroup $A \cong \mathbf{R}^*$ of P normalizes U, with generator a acting by $Ad(a)(Y) = e^2 Y$ for all $Y \in \mathfrak{u}$. Then a acts on $\Gamma \backslash G^0 / U$ and scales the volume form by $1/e^{2(k-2)}$ at every point, which is impossible for a diffeomorphism of a compact manifold.

In the next section, we will show that, if $im(\Phi)$ is reductive, then $\widetilde{Y}$ is related, by proper $\widetilde{G}^0$-equivariant maps, to $\widetilde{dS}^2 \times H$, where H is a connected Lie group. In case $im(\Phi)$ is unimodular, we will show that there is a proper G^0-equivariant map $(O(1, k-1)/U) \times G_2 \to Y$, where U is the unipotent radical of the minimal parabolic of $O(1, k-1)$, and G_2 is a local factor of G^0. In both the reductive and unimodular cases, no subgroup of G^0 can act properly discontinuously and cocompactly on Y. Both cases involve studying the representation Φ and applying dynamical results from Section 4.1.

An element B of $\mathfrak{g}$ is called *nilpotent* if $\mathrm{ad}(B)$ is nilpotent. An element B is *semisimple* if $\mathrm{ad}(B)$ is diagonalizable over $\mathbf{C}$, and B is $\mathbf{R}$-*split* if $\mathrm{ad}(B)$ is diagonalizable over $\mathbf{R}$.

6.3.3. Reductive case. In this case, $im(\Phi) \subset A \times M$. Because $G^0(y)$ is noncompact and Φ is proper, $im(\Phi)$ is not contained in M. The image is fully reducible on $T_y X$; it decomposes as a product $A' \times M'$, where M' is compact, A' is one-dimensional, and A' has nontrivial character on V_0. Note also that the exponential map is onto $(A')^0$ because it is onto both A^0 and M^0 (see [**Kn**] 1.104 and 4.48). Let $\widehat{A} \times \widehat{M}$ be the corresponding decomposition of $G^0(y)$. Properness of Φ implies $\widehat{M}$ is compact. Continuity implies $\mathrm{Ad}(a^n) \to \infty$ for all nontrivial $a \in \widehat{A}^0$: if $\mathrm{Ad}(a^n)$ were bounded, then $\overline{\mathrm{Ad}}(a^n)$ would be bounded, so $\Phi(a^n)$ would be bounded on V, a contradiction. Note the exponential map is onto $\widehat{A}^0$ because it is for $(A')^0$.

Proposition 6.5.

1) *The restriction of the metric to Y is Lorentzian, so $V_1 \neq V$.*
2) *There is an $\overline{\mathrm{ad}}$-invariant decomposition*

$$\overline{\mathfrak{s}}_0(y) \oplus \overline{\mathfrak{s}}_1(y) \oplus \overline{\mathfrak{s}}_2(y)$$

of $\mathfrak{g}/\mathfrak{g}(y)$ corresponding to the filtration in Proposition 6.2.

3) *The stabilizer subalgebra $\mathfrak{g}(y)$ contains no elements nilpotent in $\mathfrak{g}$; in particular, there are no root vectors of $\mathfrak{g}$ in $\mathfrak{g}(y)$.*

Proof:

1) Let $B \in \widehat{\mathfrak{a}}$, and let λ be the nonzero eigenvalue of $\varphi(B)$ on V_0, which we assume is positive. If V is degenerate, then the trace of $\varphi(B)$ on $V_1 = V$ is positive, so the trace of $\overline{\mathrm{ad}}(B)$ on $\mathfrak{g}/\mathfrak{g}(y)$ is positive. Now $B \in \mathfrak{z}(\mathfrak{g}(y))$, so the trace of $\mathrm{ad}(B)$ on $\mathfrak{g}(y)$ is 0. Then the trace of $\mathrm{ad}(B)$ on $\mathfrak{g}$ is positive, contradicting unimodularity of $\mathfrak{g}$.
2) Let $\mathfrak{s}(y) \subset \mathfrak{t}(y) \subset \mathfrak{g}$ be the $\mathfrak{g}(y)$-invariant subpaces in Proposition 6.2. Let $\overline{\mathfrak{s}}_0(y)$ be the projection of $\mathfrak{s}(y)$ to $\overline{\mathfrak{g}} = \mathfrak{g}/\mathfrak{g}(y)$. Let $\overline{\mathfrak{t}}(y)$ be the projection of $\mathfrak{t}(y)$. Because $\overline{\mathrm{ad}}(\mathfrak{g}(y))$ is fully reducible, there is an invariant complement $\overline{\mathfrak{s}}_1(y)$ to $\overline{\mathfrak{s}}_0(y)$ in $\overline{\mathfrak{t}}(y)$. Let $\overline{\mathfrak{s}}_2(y)$ be an invariant complement to $\overline{\mathfrak{t}}(y)$ in $\overline{\mathfrak{g}}$.
3) Suppose that $X \in \mathfrak{g}(y)$ is nilpotent. Then $\overline{\mathrm{ad}}(X)$ is nilpotent, so $\varphi(X)$ restricted to V is nilpotent. Because $im(\varphi)$ contains no nilpotent elements, $\varphi(X)$ is trivial on V. By (1), the inner product on $V^\perp \subset R^{1,n-1}$ is positive definite, so $\varphi(X)$ is skew-symmetric and generates a precompact subgroup of $O(1, n-1)$. Because Φ is proper, X should generate a precompact subgroup of G^0, a contradiction unless $X = \mathbf{0}$.

q.e.d.

Now let $b \in \widehat{A}^0$, so $\mathrm{Ad}(b^n) \to \infty$. By Proposition 4.1, there exists an $\mathbf{R}$-split element B of $\mathfrak{g}$ and a root system such that

$$\bigoplus_{\alpha(B)>0} \mathfrak{g}_\alpha$$

is isotropic for the pullback inner product $<,>_y$ on $\mathfrak{g}$. By Proposition 6.5 (3), this sum of root spaces does not meet $\mathfrak{g}(y)$. Therefore,

$$\dim \left(\bigoplus_{\alpha(B)>0} \mathfrak{g}_\alpha \right) = 1$$

Then there is exactly one root α with $\alpha(B) > 0$. Let X_α and $X_{-\alpha}$ be root vectors spanning $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$, respectively. Together with B, they generate a direct factor, say $\mathfrak{g}_1$, of $\mathfrak{g}$, isomorphic to $\mathfrak{sl}_2(\mathbf{R})$.

Denote by $\mathcal{L}$ the null cone in $\mathfrak{g}/\mathfrak{g}(y) \cong V$. For $b \in \widehat{A}^0$, the sequence $\overline{\mathrm{Ad}}(b^n)$ has unique distinct attracting and repelling fixed points, p^+ and p^-, respectively, in the projectivization $\mathbf{P}(\mathcal{L})$; these correspond to the nontrivial eigenvectors of $\overline{\mathrm{Ad}}(b)$. For $i = 1, \ldots, l$, denote by $\overline{\mathfrak{g}}_i$ the image of $\mathfrak{g}_i$ modulo $\mathfrak{g}(y)$; each such subspace is $\overline{\mathrm{Ad}}(G^0(y))$-invariant. Similarly for $X \in \mathfrak{g}$, denote by $\overline{X}$ its image in $\mathfrak{g}/\mathfrak{g}(y)$.

Because X_α is isotropic for $\langle , \rangle_y$, the projection $\overline{X}_\alpha \in \mathcal{L} \cap \overline{\mathfrak{g}}_1$. Then either the projectivization $[\overline{X}_\alpha] = p^-$, or $[\overline{\mathrm{Ad}}(b^n)(\overline{X}_\alpha)] \to p^+$. In either case, one of p^-, p^+ is in $[\overline{\mathfrak{g}}_1]$, and $\overline{\mathrm{Ad}}(b)$ has an eigenvector with nontrivial real eigenvalue in $\overline{\mathfrak{g}}_1$.

Denote by b_1 the projection of b on G_1. It belongs to a 1-parameter subgroup e^{tZ} because the exponential map is onto $\widehat{A}^0$, so $\mathrm{Ad}(b_1)$ fixes Z. Now $\mathrm{Ad}(b_1)$ has eigenvalues $\lambda \neq 1$ and 1 on $\mathfrak{g}_1$, so λ^{-1} is also an eigenvalue, and b_1 is $\mathbf{R}$-split. The eigenvectors are nilpotent elements, so each has nontrivial projection modulo $\mathfrak{g}_1(y)$. Then both p^+ and p^- belong to $[\overline{\mathfrak{g}}_1]$, and they are the images of nilpotent elements of $\mathfrak{g}_1$.

Now let Y be any nilpotent in $\mathfrak{g}_i$ for $i > 1$. By 6.5 (3), $\overline{Y} \neq \mathbf{0}$. If $\overline{Y} \notin \overline{\mathfrak{s}}_1(y)$, then $[\overline{\mathrm{Ad}}(b^n)(\overline{Y})]$ converges in $\mathbf{P}(V)$ to p^+, or $[\overline{\mathrm{Ad}}(b^{-n})(\overline{Y})]$ converges to p^-; we assume the former. Then p^+ would be the image of a nilpotent element from $\mathfrak{g}_1$ and another from $\mathfrak{g}_i$. In the span of these two would be a nilpotent element of $\mathfrak{g}(y)$, a contradiction. Therefore, $\overline{\mathfrak{g}}_i \subseteq \overline{\mathfrak{s}}_1(y)$, and $\overline{\mathrm{Ad}}(b^n)$ is bounded on $\overline{\mathfrak{g}}_i$ for all $i > 1$.

Since $b_1 \neq 1$ and $\widehat{A}^0$ is 1-dimensional, the intersection $G_i \cap \widehat{A}^0 = 1$ for all $i > 1$. It follows that $\mathfrak{g}_i(y) \subseteq \mathfrak{m}'$, so it is definite for the restriction of the Killing form κ_i of $\mathfrak{g}_i$, and $\mathrm{Ad}(G^0(y))$ is bounded on $\mathfrak{g}_i(y)$. The orthogonal of $\mathfrak{g}_i(y)$ is an $\mathrm{Ad}(G^0(y))$-invariant complement in $\mathfrak{g}_i$, which projects equivariantly and isomorphically to $\overline{\mathfrak{g}}_i$. Now for all $i > 1$, the adjoint $\mathrm{Ad}(G^0(y))$ is bounded on $\mathfrak{g}_i$, which implies that $\mathrm{Ad}(\pi_i(G^0(y)))$ is precompact.

Because G^0 preserves a Lorentz metric on $Y \cong G^0/G^0(y)$, any element of $Z(G^0) \cap G^0(y)$ would have trivial derivative along Y at y. Then $\Phi(Z(G^0) \cap G^0(y))$ is precompact. By properness of Φ, this group is finite. Therefore, $\pi_i(G^0(y))$ is precompact; let K_i denote the compact closure. Let $K = K_2 \times \cdots \times K_l$ for $i > 1$.

We have already established that the projection $\pi_1(\widehat{A})$ contains a nontrivial $\mathbf{R}$-split element. Because any other element of $\pi_1(G^0(y))$ must centralize this one, we conclude that $\pi_1(G^0(y))$ is isogenous to a maximal $\mathbf{R}$-split subgroup A_1 of G_1.

Therefore, $G^0(y) \subseteq A_1 \times K$, and there is a G^0-equivariant proper map

$$Y \cong G^0/G^0(y) \to G^0/(A_1 \times K)$$

so Γ_0 acts properly and cocompactly on both spaces. Then Γ_0 acts properly and cocompactly on $G_1/A_1 \times H$, where $H \cong G_2 \times \cdots \times G_l$. But now the universal cover of G_1/A is homothetic to $\widetilde{dS}^2$, so Proposition 6.4 applies, giving a contradiction.

6.3.4. Unimodular case. Now assume $im(\Phi) = M' \ltimes U'$ with M' compact and U' unipotent. First we collect some algebraic facts for this case.

Proposition 6.6. *Let $B \in \mathfrak{g}(y)$.*

1) *B is not $\mathbf{R}$-split.*
2) *If $\varphi(B)$ is nilpotent, then B is nilpotent.*
3) *If B is nilpotent, then on the filtration in Proposition 6.2, $\overline{ad}(B)$ carries each subspace to the next. In other words, $\overline{ad}(B)$ is trivial on each factor of the associated graded space.*

4) *If $\varphi(B)$ is nilpotent and $\mathfrak{g} \neq \mathfrak{k}(y)$, then $\overline{ad}(B)$ has nilpotence order 3.*

Proof:

1) Suppose B is $\mathbf{R}$-split. Let α be a root with $\alpha(B) \neq 0$ and $X_\alpha, X_{-\alpha}$ nonzero elements of the corresponding root spaces generating a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbf{R})$. Because $\varphi(B)$ can have no eigenvectors with nonzero real eigenvalue, X_α and $X_{-\alpha}$ are both contained in $\mathfrak{g}(y)$. Then $\mathfrak{g}(y) \subset \mathfrak{p}$ contains a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbf{R})$, a contradiction.
2) If $\varphi(B)$ is nilpotent, then $\varphi(B) \in \mathfrak{u}'$. Then $\overline{\mathrm{ad}}(B)$ is nilpotent and $\mathrm{ad}(B)$ is nilpotent on $\mathfrak{g}(y)$, which implies nilpotence of B.
3) If B is nilpotent, then $\mathrm{ad}(B)$ is trivial on both $\mathfrak{g}/\mathfrak{k}(y)$ and $\mathfrak{s}(y)/\mathfrak{g}(y)$, because they are both at most one-dimensional. Because $\mathrm{ad}(B)$ is skew-symmetric and nilpotent on $\mathfrak{k}(y)/\mathfrak{s}(y)$, this representation is also trivial: indeed, for any $k \geq 1$,

$$(\mathrm{ad}B)^{2k} = (-1)^k (\mathrm{ad}B^t \circ \mathrm{ad}B)^k$$

which is zero if and only if $B = 0$.
4) If $\varphi(B)$ is nilpotent and $\mathfrak{g} \neq \mathfrak{k}(y)$, then V is Lorentzian and the inner product on $V^\perp$ is positive definite, so $\varphi(B)$ is trivial on $V^\perp$. By injectivity of φ, the restriction of $\varphi(B)$ to V is nontrivial, so $\overline{\mathrm{ad}}(B)$ is nontrivial. By item (3), $\overline{\mathrm{ad}}(B)$ has nilpotence order at most 3. Let $W \in \mathfrak{g} \backslash \mathfrak{k}(y)$. We will show that $\mathrm{ad}^2(B)(W) \notin \mathfrak{g}(y)$.

   Denote by $<,>$ the pullback of the inner product from T_yX to $\mathfrak{g}$. For any $W, Z \in \mathfrak{g}$

$$< \mathrm{ad}(B)(W), Z > + < W, \mathrm{ad}(B)(Z) >= 0$$

   First we will show that $\mathrm{ad}(B)(W) \notin \mathfrak{s}(y)$. Suppose it is. For any $Z \in \mathfrak{s}(y) \backslash \mathfrak{g}(y)$, the inner product $< W, Z > \neq 0$. The identity

$$< \mathrm{ad}(B)(W), W > + < W, \mathrm{ad}(B)(W) >= 2 < \mathrm{ad}(B)(W), W >= 0$$

   implies $\mathrm{ad}(B)(W) \in \mathfrak{g}(y)$. Now $\mathrm{ad}(B)(\mathfrak{k}(y)) \subseteq \mathfrak{g}(y)$ would imply $\overline{\mathrm{ad}}(B)$ is trivial, which cannot be. Then there must be some $Z \in \mathfrak{k}(y)$ such that $\mathrm{ad}(B)(Z) \in \mathfrak{s}(y) \backslash \mathfrak{g}(y)$. Then

$$< \mathrm{ad}(B)(W), Z >= - < W, \mathrm{ad}(B)(Z) > \neq 0$$

   But the left side above is zero if $\mathrm{ad}(B)(W) \in \mathfrak{g}(y)$, contradicting the original assumption that $\mathrm{ad}(B)(W) \in \mathfrak{s}(y)$.

   Now $\mathrm{ad}(B)(W)$ must be in $\mathfrak{k}(y) \backslash \mathfrak{s}(y)$, so

$$< \mathrm{ad}(B)^2(W), W >= - < \mathrm{ad}(B)(W), \mathrm{ad}(B)(W) > \neq 0$$

   which implies $\mathrm{ad}(B)^2(W) \notin \mathfrak{g}(y)$, as desired.

q.e.d.

Let $\widehat{M} \ltimes \widehat{U}$ be the decomposition of $G^0(y)$ corresponding to $im(\Phi) = M' \ltimes U'$. Again, because Φ is proper, $\widehat{M}$ is compact. Let $\widehat{\mathfrak{m}}$ and $\widehat{\mathfrak{u}}$ be the corresponding subalgebras of $\mathfrak{g}$. From item (2) above, $\widehat{u}$ consists of nilpotent elements. Let J be the set of i such that $\pi_i(\widehat{\mathfrak{u}}) \neq \mathbf{0}$.

We will show by induction that there exists $X \in \widehat{\mathfrak{u}}$ such that $\pi_i(X) \neq \mathbf{0}$ if and only if $i \in J$. Let $i_1, \ldots, i_k$ be some order on the elements of J. Clearly, there is some $X_1 \in \widehat{\mathfrak{u}}$ such that $\pi_{i_1}(X_1) \neq \mathbf{0}$. Suppose $X_m \in \widehat{\mathfrak{u}}$ is such that $\pi_{i_j}(X_m) \neq \mathbf{0}$ for all $j \leq m$. There exists $Y_m \in \widehat{\mathfrak{u}}$ such that $\pi_{i_{m+1}}(Y_m) \neq \mathbf{0}$. For some real number c, the element $X_{m+1} = X_m + cY_m$ will have $\pi_{i_j}(X_{m+1}) \neq \mathbf{0}$ for all $j \leq m+1$. Write this element $X = \sum_{i \in J} X_i$ with $X_i \in \mathfrak{g}_i$. Note that nilpotence of X implies nilpotence of each X_i.

The Jacobson-Morozov theorem (see [**H**] IX.7.4) yields, for each $i \in J$, an $\mathbf{R}$-split element $A_i \in \mathfrak{g}_i$ and a nilpotent element $Y_i \in \mathfrak{g}_i$ such that

$$[A_i, X_i] = 2X_i, \qquad [A_i, Y_i] = -2Y_i, \qquad \text{and } [X_i, Y_i] = A_i$$

Then the elements $A = \sum_i A_i$ and $Y = \sum_i Y_i$ satisfy

$$[A, X] = 2X, \qquad [A, Y] = -2Y, \qquad \text{and } [X, Y] = A$$

The subalgebra generated by X, A, and Y is isomorphic to $\mathfrak{sl}_2$. Let L be the corresponding subgroup of G^0. The adjoint of $\mathfrak{g}(y)$ is trivial on $\mathfrak{s}(y)/\mathfrak{g}(y)$ by 6.6 (3), so G^0 preserves a vector field tangent to the invariant isotropic line field along Y. Now Proposition 4.1 and Remark 4.3 for $g_n = e^{nX}$ give some $k \in L$ such that, for $A' = \mathrm{Ad}(k)(A)$,

$$\bigoplus_{\alpha(A')>0} \mathfrak{g}_\alpha \subset \mathfrak{s}(y)$$

Let $X' = \mathrm{Ad}(k)(X) \in \mathfrak{s}(y)$. We may assume $k \in PSL_2(\mathbf{R})$. For

$$k = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

the bracket

$$[X, X'] = \begin{pmatrix} -\sin^2\theta & 2\cos\theta\sin\theta \\ 0 & \sin^2\theta \end{pmatrix}$$

This bracket belongs to $\mathfrak{g}(y)$, which contains no $\mathbf{R}$-split elements (6.6 (3) and (1)). Then $\sin\theta$ must be 0, so $\mathrm{Ad}(k)$ is trivial, and

$$\bigoplus_{\alpha(A)>0} \mathfrak{g}_\alpha \subset \mathfrak{s}(y)$$

Now we will show that this sum of root spaces is in fact contained in $\mathfrak{g}(y)$. For Y the negative root vector as above, $\mathrm{ad}^2(X)(Y) \in \mathfrak{g}(y)$, so $\overline{\mathrm{ad}}(X)$ has order less than 3 on the corresponding element of $\mathfrak{g}/\mathfrak{g}(y)$. Then $Y \in \mathfrak{t}(y)$ by Proposition 6.6 (4). Then $[X, Y] = A \in \mathfrak{s}(y)$ by 6.6 (3), but A cannot be in $\mathfrak{g}(y)$ by 6.6 (1), so

$$\mathfrak{s}(y) = \mathbf{R}A + \mathfrak{g}(y)$$

Now suppose $\alpha(A) > 0$ and let X' be an arbitrary element of $\mathfrak{g}_\alpha \subseteq \mathfrak{s}(y)$. Since $[\mathfrak{s}(y), \mathfrak{s}(y)] \subseteq \mathfrak{g}(y)$, the bracket $[A, X'] = \alpha(A)X' \in \mathfrak{g}(y)$. Therefore, $\oplus_{\alpha(A)>0}\mathfrak{g}_\alpha \subseteq \mathfrak{g}(y)$, as desired; in particular, $X_i \in \mathfrak{g}(y)$ for all $i \in J$.

Next we will show that $|J| = 1$. As above, Proposition 6.6 implies $Y_i \in \mathfrak{t}(y)$ and $A_i \in \mathfrak{s}(y)$ for all $i \in J$. If $|J| > 1$, then, for one $i \in J$ and some nonzero $c \in \mathbf{R}$, the difference $cA - A_i$ is a nontrivial $\mathbf{R}$-split element of $\mathfrak{g}(y)$, contradicting 6.6 (1).

Now $G^0(y) \cong \widehat{M} \ltimes \widehat{U}$ with $\pi_i(\hat{u}) = 0$ for all i except, say, 1. Then $G^0(y)$ has precompact projection on all local factors except G_1. By Kowalsky's Theorem ([**K1**]), $\mathfrak{g}_1 \cong \mathfrak{o}(2, k)$, for some $k \geq 3$, or $\mathfrak{o}(1, k)$, for some $k \geq 2$. We will deduce that $\mathfrak{g}_1$ must be the latter, and that $G_1(y)$ is as in the Minkowski light cone. The subspaces $\mathfrak{s}_1(y)$ and $\mathfrak{t}_1(y)$ will denote the intersections $\mathfrak{g}_1 \cap \mathfrak{s}(y)$ and $\mathfrak{g}_1 \cap \mathfrak{t}(y)$, respectively, below.

Step 1: $\mathfrak{g}_1 \cong \mathfrak{o}(1, 2)$ implies $G_1 y$ degenerate.

If $\mathfrak{g}_1 \cong \mathfrak{o}(1, 2)$, then it is generated by X, A, and Y from above. Recall $X \in \mathfrak{g}_1(y)$; $A \in \mathfrak{s}_1(y)$; and $Y \in \mathfrak{t}_1(y)$. Then $\mathfrak{g}_1/\mathfrak{g}_1(y)$ is 2-dimensional and degenerate with respect to the inner product pulled back from $T_y X$; therefore, the orbit $G_1 y$ is also degenerate.

Step 2: $G_1 y$ is degenerate in general.

Assume that $\mathfrak{g}_1$ is not isomorphic to $\mathfrak{o}(1, 2)$, and suppose that the orbit $G_1 y \subseteq Y$ is of Lorentzian type. Then Theorem 1.5 of [**ADZ**] gives that $G_1 y$ is equivariantly homothetic, up to covers, to dS^k or AdS^k for some $k \geq 3$; in either case, $\mathfrak{g}_1(y)$ would be semisimple, a contradiction.

Step 3: Case $\mathfrak{g}_1 \cong \mathfrak{o}(2, k)$.

Now suppose $\mathfrak{g}_1 \cong \mathfrak{o}(2, k)$ for some $k \geq 3$. Let Δ be a root system of $\mathfrak{g}$ as above. Let $A \in \mathfrak{s}(y)$ be as above. Let $\alpha \in \Delta$ be such that $\alpha(A) = 2$. Let $X \in \mathfrak{g}_\alpha \cap \mathfrak{g}(y)$. The root system of $\mathfrak{o}(2, k)$ is generated by two simple roots, β and γ. The root spaces for β and γ are each $(k-2)$-dimensional. The other positive roots are $\beta - \gamma$ and $\beta + \gamma$, with one-dimensional root spaces.

First suppose $\alpha = \beta$, so $X \in \mathfrak{g}_\beta \subset \mathfrak{g}(y)$. Let L be a generator of $\mathfrak{g}_{-\beta-\gamma}$. For any such X and L, the adjoint $\mathrm{ad}^2(X)(L) \neq \mathbf{0}$. Since the orbit $G_1 y$ is degenerate, $L \in \mathfrak{t}(y)$, and $\mathrm{ad}(X)(L) \in \mathfrak{s}(y)$. Let $W = \mathrm{ad}(X)(L) \in \mathfrak{g}_{-\gamma}$. Any nilpotent subalgebra of $\mathfrak{g}_1(y) \subseteq \mathfrak{p}$ is abelian, so $W \in \mathfrak{s}(y) \backslash \mathfrak{g}(y)$. Then $cW - A \in \mathfrak{g}_1(y)$ for some nonzero $c \in \mathbf{R}$. But now $L \in \mathfrak{t}(y) \backslash \mathfrak{s}(y)$ would be an eigenvector for this element with nonzero real eigenvalue. Then $\varphi(cW - A)$ would have a nonzero real eigenvalue on V, contradicting Lemma 6.3.

We conclude that X cannot be in $\mathfrak{g}_\beta$. The same argument shows X cannot be in $\mathfrak{g}_\gamma$; in fact, $\mathfrak{g}_1(y) \cap \mathfrak{g}_\omega$ must be $\mathbf{0}$ for $\omega = \pm\beta, \pm\gamma$.

Now suppose that $\alpha = \beta \pm \gamma$, so either $(\beta+\gamma)(A)$ or $(\beta-\gamma)(A)$ equals 2. Then one of $\beta(A)$ or $\gamma(A)$ is nonzero, which again implies that one of $\mathfrak{g}_{\pm\beta}, \mathfrak{g}_{\pm\gamma}$ is in $\mathfrak{g}(y)$, a contradiction.

$G_1 y$ is the Minkowski light cone

Now we have that $\mathfrak{g}_1 \cong \mathfrak{o}(1,k)$ for some $k \geq 2$. Let α be the positive root of $\mathfrak{g}_1$ with $\alpha(A) = 2$. From above, $\mathfrak{g}_\alpha \subset \widehat{\mathfrak{u}}$. Since this root space is a maximal abelian subalgebra of nilpotent elements in $\mathfrak{g}_1$, this containment is equality by 6.6 (2).

There is a proper equivariant map

$$G^0/\widehat{U} \to G^0/(\widehat{M} \ltimes \widehat{U}) \cong Y$$

so no subgroup of G^0 acts properly discontinuously and cocompactly on Y, as in Section 6.3.2.

7. Appendix: Non-split example

As promised in the introduction, the following example illustrates the necessity of the hypothesis of finite center in (2) **A** in order to conclude that M splits locally along G^0-orbits as a metric product. In the example, $Isom^0(X) \cong \widetilde{SL}(2,\mathbf{R})$, a noncompact semisimple group with infinite center; it acts properly on X; and the metric on $\widetilde{SL}(2,\mathbf{R})$-orbits varies among Riemannian, Lorentzian, and degenerate.

Consider the basis for $\mathfrak{sl}(2,\mathbf{R})$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The brackets among these generators are

$$[A,P] = 2K \qquad [A,K] = 2P \qquad [K,P] = 2A$$

Denote by B_λ the inner product on $\mathfrak{sl}(2,\mathbf{R})$ in which A, P, and K are mutually orthogonal,

$$B_\lambda(A,A) = 1 = B_\lambda(P,P), \qquad \text{and} \qquad B_\lambda(K,K) = \lambda$$

When $\lambda = -1$, then B_λ is a constant multiple of the Killing form. Denote also by A, P, K the corresponding left-invariant vector fields on $\widetilde{SL}(2,\mathbf{R})$. Note that B_λ determines a left-invariant inner product on $T\widetilde{SL}(2,\mathbf{R})$.

Let $X = \widetilde{SL}(2, \mathbf{R}) \times \mathbf{R}$ with the following Lorentz metric ν:

$$\begin{aligned}
\nu_{(x,t)}\big|_{span\{A,P,K\}} &= B_{\cos t} \\
\nu_{(x,t)}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) &= -\cos t \\
\nu_{(x,t)}\left(\frac{\partial}{\partial t}, A\right) &= 0 = \nu_{(x,t)}\left(\frac{\partial}{\partial t}, P\right) \\
\nu_{(x,t)}\left(\frac{\partial}{\partial t}, K\right) &= \sin t
\end{aligned}$$

The $\widetilde{SL}(2, \mathbf{R})$-fibers are Riemannian when $\cos t > 0$, degenerate when $\cos t = 0$, and Lorentzian when $\cos t < 0$. Obviously, $\widetilde{SL}(2, \mathbf{R}) \subseteq Isom^0(X)$.

It is straightforward to compute the following values for the Levi-Civita connection ∇:

$$\begin{aligned}
\nabla_A A &= 0 = \nabla_P P \\
\nabla_K K &= \frac{1}{2}\sin^2 t \cdot K - \frac{1}{2}\cos t \sin t \cdot T \\
\nabla_T T &= (1 - \frac{1}{2}\sin^2 t) \cdot K + \frac{1}{2}\sin t \cos t \cdot T \\
\nabla_A P &= K = -\nabla_P A \\
\nabla_P K &= \cos t \cdot A = \nabla_K P - 2A \\
\nabla_A K &= -\cos t \cdot P = \nabla_K A + 2P \\
\nabla_T A &= -\sin t = \nabla_A T \\
\nabla_T P &= \sin t = \nabla_P T \\
\nabla_T K &= -\frac{1}{2}\sin t \cos t \cdot K - \frac{1}{2}\sin^2 t \cdot T = \nabla_K T
\end{aligned}$$

Let

$$\begin{aligned}
X_1 &= A & X_3 &= \cos\frac{t}{2} \cdot K + \sin\frac{t}{2} \cdot T \\
X_2 &= P & X_4 &= \sin\frac{t}{2} \cdot K - \cos\frac{t}{2} \cdot T
\end{aligned}$$

These vector fields form a Lorentz framing of X—that is, with respect to the basis they form at (x, t), the metric takes the form

$$\nu_{(x,t)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The sectional curvatures of X with respect to this framing have the following values at (x,t):

$$\begin{array}{rcl} S(X_1,X_2) & = & -3\cos t - 4 \qquad S(X_3,X_4) \;=\; -\dfrac{1}{2}\cos t \\ S(X_1,X_3) & = & \cos^2\dfrac{t}{2} \;=\; S(X_2,X_3) \\ S(X_1,X_4) & = & -\sin^2\dfrac{t}{2} \;=\; S(X_2,X_4) \end{array}$$

Then the scalar curvature of X at (x,t) is $-3\cos t - 8$. The important point is that it is a nonconstant function of t. Then any isometric flow must preserve the $\widetilde{SL}(2,\mathbf{R})$-fibers of X. Now because $\widetilde{SL}(2,\mathbf{R}) \subseteq Isom^0(X)$ acts transitively on each fiber, if the stabilizer of each point were trivial, we could conclude that $Isom^0(X) \cong \widetilde{SL}(2,\mathbf{R})$. Unfortunately, the stabilizers are not quite trivial: they include $SO(2,\mathbf{R})$, rotating in the plane spanned by A and P.

It is thus necessary to perturb the inner products B_λ on $\mathfrak{sl}(2,\mathbf{R})$ to destroy this symmetry. This can be accomplished, for example, by defining a new inner product $B_{\epsilon,\lambda}$ in which A,P,K are mutually orthogonal, and

$$\begin{array}{rcl} B_{\epsilon,\lambda}(P,P) & = & 1-\epsilon \\ B_{\epsilon,\lambda}(A,A) & = & 1 \\ B_{\epsilon,\lambda}(K,K) & = & \lambda \end{array}$$

Then define a new metric $\nu'_{(x,t)}$ on X with $B_{\epsilon,\cos t}$ in place of $B_{\cos t}$. For the metric ν', it is still true that $\widetilde{SL}(2,\mathbf{R}) \subseteq Isom^0(X)$. Because ν' is close to ν for ϵ sufficiently small, the scalar curvature of ν' is still a nonconstant function of t. Thus any isometric flow on X preserves the $\widetilde{SL}(2,\mathbf{R})$-fibers. Let φ^s be such a flow. For any fixed $t_0 \notin (\mathbf{Z}+1/2)\pi$, the flow on the fiber over t_0 is an isometry of the pulled-back nondegenerate metric on $\widetilde{SL}(2,\mathbf{R})$. By post-composition with a path in $\widetilde{SL}(2,\mathbf{R})$, we may assume the restriction of φ^s fixes the identity 1. Now the differential φ^s_{*1} must preserve both the inner product $B_{\epsilon,\cos t_0}$ and the Ricci curvature form on $\mathfrak{sl}(2,\mathbf{R})$.

For example, when $\cos t_0 = \epsilon - 1$, for $0 < \epsilon < 1/2$, the inner product takes the form, with respect to the basis A,P,K,

$$\begin{pmatrix} 1 & & \\ & 1-\epsilon & \\ & & \epsilon-1 \end{pmatrix}$$

and the Ricci curvature is

$$\begin{pmatrix} \frac{-2}{(1-\epsilon)^3} & & \\ & \frac{-2+5\epsilon}{(1-\epsilon)^2} & \\ & & \frac{-2+5\epsilon}{(1-\epsilon)^2} \end{pmatrix}$$

For sufficiently small ϵ, the linear isometries of $\mathfrak{sl}(2,\mathbf{R})$ preserving both inner products can be identified with

$$g^{-1}O(2,1)g \cap h^{-1}O(3)h$$

where

$$g = \begin{pmatrix} 1 & & \\ & \sqrt{1-\epsilon} & \\ & & \sqrt{1-\epsilon} \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} \sqrt{\frac{2}{(1-\epsilon)^3}} & & \\ & \frac{\sqrt{2-5\epsilon}}{1-\epsilon} & \\ & & \frac{\sqrt{2-5\epsilon}}{1-\epsilon} \end{pmatrix}$$

It is left to the reader to verify that for small positive ϵ, the identity component of this intersection is trivial.

We conclude that φ^s is trivial when restricted to the fiber over t_0. Then $\varphi^s_{*(x_0,t_0)}$ is trivial on the span of A, P, K at (x_0, t_0). Then it must also fix the orthogonal direction, and so $\varphi^s_{*(x_0,t_0)}$ is trivial for all s. But any isometry of X fixing a point and having trivial derivative at that point is trivial. Finally, φ^s is trivial, and $Isom^0(X) \cong \widetilde{SL}(2,\mathbf{R})$.

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