C^1 DEFORMATIONS OF ALMOST GRASSMANNIAN STRUCTURES WITH STRONGLY ESSENTIAL SYMMETRY

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ABSTRACT. We construct a family of $(2, n)$-almost Grassmannian structures of regularity C^1, each admitting a one-parameter group of strongly essential automorphisms, and each not flat on any neighborhood of the higher-order fixed point. This shows that Theorem 1.3 of [9] does not hold assuming only C^1 regularity of the structure (see also [2, Prop 3.5]).

1. INTRODUCTION

Almost-Grassmannian structures belong to the class of *irreducible parabolic geometries* (also called *almost-Hermitian symmetric* structures), which include projective and conformal structures, among many others. An (m, n)-*almost-Grassmannian structure* on an mn-dimensional manifold M comprises a vector bundle isomorphism of TM with $\mathcal{E}^* \otimes \mathcal{F}$, where $\mathcal{E}$ and $\mathcal{F}$ are vector bundles over M of respective ranks m and n, together with an isomorphism $\wedge^m \mathcal{E} \cong \wedge^n \mathcal{F}$; the latter corresponds to a volume form compatible with the tensor product. Denote $\mathrm{Gr}(m, n)$ the real Grassmannian variety of m-planes in $\mathbf{R}^{m+n}$, by $\mathcal{E}$ its tautological m-plane bundle, and by $\mathcal{F}$ the rank-n anti-tautological bundle. An (m, n)-almost-Grassmannian structure mimics the isomorphism of $T\mathrm{Gr}(m, n)$ with $\mathcal{E}^* \otimes \mathcal{F}$.

Almost-Grassmannian structures have been studied under the guise of *Segré structures*. The *Segré cone* $S(m, n)$ is the variety in $\mathbf{R}^{mn}$ comprising the rank-one elements under the identification with $\mathrm{Hom}(\mathbf{R}^m, \mathbf{R}^n)$. An (m, n)-Segré structure on M^{mn} is a bundle of Segré cones $S_x(m, n) \subset T_x M$. It is essentially equivalent to an (m, n)-almost Grassmannian structure (see [1]).

In the special case $m = n = 2$, when $\dim M = 4$, an almost-Grassmannian structure is equivalent to a conformal spin structure of split signature $(2, 2)$. In fact, $(2, n)$-almost-Grassmannian structures in many respects can be viewed as higher-dimensional analogs of signature-$(2, 2)$ conformal geometry, which is one of the reasons for the interest in them. Torsion-free $(2, n)$-almost-Grassmannian structures correspond to (anti-)self-dual conformal structures.

There is a close relation between $(2, n)$-almost Grassmannian structures and almost quaternionic structures (see [11]). They can be viewed as different real forms of the same complex geometry.

Finally, $(2, n)$-almost-Grassmannian structures are connected to projective structures by twistor theory via *path geometries*. The latter are parabolic geometries that model systems of second-order ODEs. More precisely, they correspond to collections of unparametrized curves $\mathcal{C}$ in a manifold X obtained as the solutions of such a system; these lift to a foliation $\widetilde{\mathcal{C}}$ of the projectivized tangent bundle $\mathbf{P}(TX)$. For one special class of path geometries, $\mathcal{C}$ are the unparametrized geodesics of an affine connection on M—that is, a projective structure ,—while for another class, the *path space* $\mathbf{P}(TX)/\widetilde{\mathcal{C}}$ locally inherits a $(2, n)$-almost Grassmannian structure . The intersection of the two classes is the *flat* path geometry, for which $\mathcal{C}$ comprises projective lines in $X = \mathbf{RP}^{n+1}$ (see [6, Secs 2 and 3]).

Irreducible parabolic geometries can admit certain very special automorphisms which fix a point and have trivial derivative at that point, which is then called a *higher-order fixed point*. Note that a semi-Riemannian metric or an affine connection does not admit nontrivial automorphisms with higher-order fixed points. These *strongly essential automorphisms* occur in abundance on the homogeneous model spaces for each parabolic geometry. A structure that is locally equivalent to this model is said to be *flat*. (See Sections 2.2 and 2.3 below).

Many rigidity results say that a strongly essential flow can occur only in the presence of flatness. Let x_0 be a higher-order fixed point.

- Nagano and Ochiai [10] proved for a torsion-free connection that existence of a strongly essential projective flow implies projective flatness of the connection on a neighborhood of x_0.
- The second author and Neusser proved the analogous result for almost-c-projective structures and almost-quaternionic structures in

[9, Thms 4.4, 1.2]. (See also [2, Thm 3.7] for a precursor result on almost-quaternionic structures.)

In conformal Lorentzian geometry, Frances smoothly deformed the Minkowski metric in a neighorhood of a point x_0 so that it retains a conformal flow with x_0 as higher-order fixed point. The resulting C^∞ metric is conformally flat inside the light cone of x_0, but nonflat outside [4, Sec 6]. Then came the following rigidity results:

- In semi-Riemannian geometry, Frances and the second author proved in [5] that existence of a strongly essential conformal flow implies conformal flatness on an open set U with $x_0 \in \overline{U}$.
- In [9, Thm 1.3], the second author and Neusser proved that a $(2,n)$-almost-Grassmannian structure admitting a strongly essential flow is flat on an open set U with $x_0 \in \overline{U}$ (see also [2, Prop 3.5] for a partial result).

Kruglikov and The exhibited a C^ω homogeneous path geometry which is not flat and admits a strongly essential flow in [8, Prop 5.3.2]. Path geometries are not irreducible. The local path space in their example admits a $(2,n)$-almost-Grassmannian structure. The flow descends, but it is not strongly essential on the quotient.

The proofs of the rigidity theorems cited above, as well as the construction of [8], make use of the *Cartan geometry* canonically associated to the parabolic geometric structures in question. This association is only possible with sufficiently high regularity; the minimal order required depends on the structure.

1.1. **Our examples.** In [12], the first author described the infinitesimal automorphisms and deformations of a parabolic geometry intrinsically in terms of the associated Cartan geometry, using the twisted de-Rham sequence of differential froms with coefficients in the adjoint tractor bundle and the corresponding BGG sequence of invariant differential operators. Motivated by this description of infinitesimal deformations, we explicitly construct a family, locally on $\mathrm{Gr}(2,n)$, that is invariant by a strongly essential flow and integrates to a family of deformed structures, all admitting this flow as automorphisms. These show that Theorem 1.3 of [9] does not hold assuming only C^1 regularity.

An almost-Grassmannian structure is said to be C^k if M is at least C^{k+1}; $\mathcal{E}$, $\mathcal{F}$, and the isomorphism $\wedge^m \mathcal{E} \cong \wedge^n \mathcal{F}$ are at least C^{k+1}; and the isomorphisms $TM \cong \mathcal{E}^* \otimes \mathcal{F}$ are C^k. Such structures are said to be equivalent if they are C^k equivalent (see Section 2.3 below).

Theorem 1.1. *Let $n \geq 3$ and $x_0 \in Gr(2,n)$. There are a dense, open neighborhood U of x_0; a strongly essential flow $\{z^t\} < Aut\ Gr(2,n)$ with x_0 as higher order fixed point; and an $(n-1)$-parameter family of C^1 almost-Grassmannian structures of type $(2,n)$ on U, of which each:*

- *contains $\{z^t\}$ in its automorphism group;*
- *is not locally equivalent to $Gr(2,n)$ on any open set V with $x_0 \in \overline{V}$;*

The deformations are given in Section 4.2, and the precise claims about them are in Proposition 4.1.

Remark 1.2. *In fact, none of these deformed structures are locally equivalent to the path space of a path geometry; the harmonic torsion is the full obstruction to this property (see [13, Props 4.4.3, 4.4.45]).*

2. BACKGROUND

2.1. **Almost-Grassmannian structures as first-order G-structures.** For almost-Grassmannian structures of low regularity, as we construct below, the description as Cartan geometries is not available. Thus we start by reviewing various descriptions of such structures with a special emphasis on the requirements on regularity.

Let us fix integers $m, n \geq 2$ as above, with the case $m = 2$, $n > 2$ being of primary interest. An almost-Grassmannian structure as defined above can be equivalently defined as a (first-order) G-structure for the Lie group

$$G_0 := \{(A, B) \in \mathrm{GL}(m, \mathbf{R}) \times \mathrm{GL}(n, \mathbf{R}) : \det(A)\det(B) = 1\}.$$

Under the identification $\mathbf{R}^{mn} \cong \mathrm{Hom}(\mathbf{R}^m, \mathbf{R}^n)$, the natural representation of G_0 on $\mathbf{R}^{mn}$ is $\rho(A, B) \cdot X := BXA^{-1}$. Observe that the resulting homomorphism $G_0 \to \mathrm{GL}(mn, \mathbf{R})$ has two element kernel $\{(\mathrm{Id}, \mathrm{Id}), (-\mathrm{Id}, -\mathrm{Id})\}$ and thus is infinitesimally injective, so this indeed defines a type of first-order G-structures on manifolds of dimension mn.

Such a structure is given by a principal bundle $p_0 : \mathcal{G}_0 \to M$ with structure group G_0 together with a ρ-equivariant bundle morphism to the first order

frame bundle $\mathcal{P}M$ of M. The structure is C^k provided $\mathcal{G}_0$ is a C^{k+1} principal bundle and the morphism to $\mathcal{P}M$ is C^k.

Proposition 2.1. *On a smooth manifold of dimension* mn, *a* C^k *first-order* G_0-*structure is equivalent to a* C^k *almost-Grassmannian structure of type* (m, n).

The proof is standard; we give the main points. The bundles $\mathcal{E}^*$ and $\mathcal{F}$ are associated bundles to $\mathcal{G}_0$, while conversely $\mathcal{G}_0$ is obtained as a subbundle of the fibered product of the frame bundles of $\mathcal{E}^*$ and $\mathcal{F}$. This shows that a ρ-equivariant bundle morphism from $\mathcal{G}_0$ to $\mathcal{P}M$ is equivalent to an isomorphism $TM \xrightarrow{\sim} \mathcal{E}^* \otimes \mathcal{F}$, and the correspondence respects C^k regularity.

A ρ-equivariant bundle morphism $\mathcal{G}_0 \to \mathcal{P}M$ can be equivalently encoded as a one-form $\theta \in \Omega^1(\mathcal{G}_0, \mathbf{R}^{mn})$ which is G_0–equivariant and strictly horizontal. Denoting by $r^g : \mathcal{G}_0 \to \mathcal{G}_0$ the principal action of $g \in G_0$, equivariance means $(r^g)^*\theta = \rho(g)^{-1} \circ \theta$. The second condition says that for each point $u \in \mathcal{G}_0$, the kernel of $\theta(u) : T_u\mathcal{G}_0 \to \mathbf{R}^{mn}$ is the vertical subspace in $T_u\mathcal{G}_0$. In this picture, C^k regularity means θ is C^k, in the sense that for each C^k vector field $\xi \in \mathfrak{X}(\mathcal{G}_0)$, the function $\theta(\xi) : \mathcal{G}_0 \to \mathbf{R}^{mn}$ is C^k.

2.2. The homogeneous model—the Grassmann variety. In this section we describe the (m, n)-almost-Grassmannian structure on $\mathrm{Gr}(m, n)$.

The group G_0 can be realized as the subgroup of $G := \mathrm{SL}(m+n, \mathbf{R})$ respecting the decomposition $\mathbf{R}^{m+n} \cong \mathbf{R}^m \oplus \mathbf{R}^n$. The Lie algebra $\mathfrak{g}_0$ is identified with the corresponding block diagonal subalgebra of $\mathfrak{g}$. Its adjoint action on $\mathfrak{g}$ preserves a decomposition $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are the subalgebras with nonzero entries only in the lower-left and upper-right blocks, respectively. The decomposition satisfies $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, where we set $\mathfrak{g}_k = \{0\}$ for $|k| > 1$. Note that $\mathfrak{g}_{-1} \cong \mathrm{Hom}(\mathbf{R}^m, \mathbf{R}^n)$, and the restriction of the adjoint action of elements of G_0 to $\mathfrak{g}_{-1}$ is the representation ρ from Section 2.1.

Next, let $P < G$ comprise the block-upper-triangular matrices, with Lie algebra $\mathfrak{p} := \mathfrak{g}_0 \ltimes \mathfrak{g}_1 \subset \mathfrak{g}$. It is the stabilizer of $\mathbf{R}^m \subset \mathbf{R}^{m+n}$, so G/P can be identified with $\mathrm{Gr}(m, n)$. As is well known, the tangent bundle $T(G/P)$ is the associated bundle $G \times_P (\mathfrak{g}/\mathfrak{p})$, where P acts via the adjoint representation, which factors on the quotient $\mathfrak{g}/\mathfrak{p}$ through projection to G_0. The vector space $\mathfrak{g}/\mathfrak{p}$ is moreover G_0-equivariantly isomorphic to $\mathfrak{g}_{-1}$. Consequently, $\mathrm{Gr}(m, n)$ carries an almost Grassmannian structure. Note that the

auxiliarly bundles $\mathcal{E} \cong G \times_P \mathbf{R}^m$ and $\mathcal{F} \cong G \times_P (\mathbf{R}^{m+n}/\mathbf{R}^m)$ for this structure are exactly the tautological and the anti-tautological bundles.

2.3. **Automorphisms and flatness.** Let M be a C^{k+1} manifold of dimension mn with a C^k almost-Grassmannian structure comprising

- C^{k+1} vector bundles $\mathcal{E}$ and $\mathcal{F}$ of ranks m and n, respectively
- $\Theta : TM \overset{\sim}{\to} \mathcal{E}^* \otimes \mathcal{F}$ of regularity C^k
- $\nu : \Lambda^m \mathcal{E} \overset{\sim}{\to} \Lambda^n \mathcal{F}$ of regularity C^k

Definition 2.2. *An* automorphism *of the C^k almost-Grassmannian structure above is a C^{k+1} diffeomorphism h of M together with*

- *lifts $h_\mathcal{E}^*$ and $h_\mathcal{F}$ of h to automorphisms of $\mathcal{E}^*$ and $\mathcal{F}$, respectively,*
- *such that $h^*\Theta = (h_\mathcal{E}^* \otimes h_\mathcal{F}) \circ \Theta$, and*
- *such that $\nu \circ \Lambda^m h_\mathcal{E} = \Lambda^n h_\mathcal{F} \circ \nu$*

In the G-structure framework, $h \in \mathrm{Diff}^{k+1} M$ is an automorphism if it lifts to a principal bundle automorphism of $\mathcal{G}_0$ which is semi-conjugate via the ρ-equivariant bundle morphism $\mathcal{G}_0 \to \mathcal{P}M$ to the natural lift of h to $\mathcal{P}M$.

Isomorphisms of almost-Grassmannian structures are defined by the obvious extension of Definition 2.2. Local isomorphisms are isomorphisms between connected open subsets, with their restricted structures.

Consider $g \in G$ as a diffeomorphism of $\mathrm{Gr}(m,n) \cong G/P$. It naturally acts by automorphisms $g_\mathcal{E}^*$ and $g_\mathcal{F}$ of the vector bundles $\mathcal{E}^* \cong G \times_P \mathbf{R}^{m*}$ and $\mathcal{F} \cong G \times_P (\mathbf{R}^{m+n}/\mathbf{R}^m)$, respectively. The P-equivariant isomorphism of $\mathrm{Hom}(\mathbf{R}^m, \mathbf{R}^{m+n}/\mathbf{R}^m)$ with $\mathfrak{g}/\mathfrak{p}$ gives a G-equivariant isomorphism of $T\mathrm{Gr}(m,n) \cong G \times_P \mathfrak{g}/\mathfrak{p}$ with $\mathcal{E}^* \otimes \mathcal{F}$, on which g acts by $g_\mathcal{E}^* \otimes g_\mathcal{F}$. Any $g \in G$ is thus an automorphism of the almost-Grassmannian structure on $\mathrm{Gr}(m,n)$.

Let P_+ be the connected, unipotent, normal subgroup of P with Lie algebra $\mathfrak{g}_1$. For $g \in P_+$, the linear isomorphisms $(g_\mathcal{E}^*)_{[\mathrm{Id}_G]}$ and $(g_\mathcal{F})_{[\mathrm{Id}_G]}$ are trivial, so the derivative of g on $T_{[\mathrm{Id}_G]}(G/P)$ is trivial. The G-conjugates of P_+ furnish nontrivial strongly essential automorphisms at every point of $\mathrm{Gr}(m,n)$.

Each $g \in G\backslash\{\pm\mathrm{Id}_{\mathbf{R}^{m+n}}\}$ is a nontrivial transformation of G/P, because there is no larger G-normal subgroup in P. Thanks to the canonical Cartan connection associated to an almost-Grassmannian structure, presented in Section 2.6 below, the automorphism group of $\mathrm{Gr}(m,n)$ is precisely $G/\{\pm\mathrm{Id}\}$. On any almost-Grassmannian manifold (of sufficient regularity), the Cartan

connection underlies the fact that the automorphism group is a Lie group of dimension at most $(m+n)^2 - 1 = \dim G$, with equality only if the structure is locally isomorphic to $\mathrm{Gr}(m,n)$, in which case it is said to be *flat*. Deciding whether an almost Grassmannian structure is flat thus is a fundamental question in the theory.

2.4. **The harmonic torsion.** The description as a G_0-structure $(p_0 : \mathcal{G}_0 \to M, \theta)$ directly leads to the first fundamental invariants of almost Grassmannian structures. We first choose a C^k principal connection $\gamma \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_0)$. If θ is at least C^1, we can define the *torsion* of γ as the covariant exterior derivative $d^\gamma \theta \in \Omega^2(\mathcal{G}_0, \mathbf{R}^{mn})$; explicitly, for $\xi, \eta \in \mathfrak{X}(\mathcal{G}_0)$,

(1) $$d^\gamma \theta(\xi, \eta) = d\theta(\xi, \eta) + \gamma(\xi)(\theta(\eta)) - \gamma(\eta)(\theta(\xi)).$$

If θ, ξ and η are C^k, then the above is a C^{k-1} function. From the properties of θ, it follows readily that $d^\gamma \theta$ is horizontal and G_0-equivariant and thus descends to a form $T^\gamma \in \Omega^2(M, TM)$, which is the usual interpretation of the torsion.

We next compute the dependence of T^γ on γ. First note that, at a point $u \in \mathcal{G}_0$, (1) depends only on $\gamma_u : T_u\mathcal{G}_0 \to \mathfrak{g}_0$. As discussed in 2.2, we can view θ as having values in $\mathfrak{g}_{-1}$. For any other principal connection $\hat{\gamma}$, the difference $\hat{\gamma}_u - \gamma_u$ is given by $f_u \circ \theta_u$ for some linear map $f_u : \mathfrak{g}_{-1} \to \mathfrak{g}_0$. The first differential in the cochain complex of $\mathfrak{g}_{-1}$ with coefficients in $\mathfrak{g}$ restricts on $\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0$ to the following G_0-equivariant linear map:

(2) $\partial_1 : \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0 \to \Lambda^2 \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$ $(\partial_1 f)(w, v) = f(w)v - f(v)w$

For all u,

(3) $$T_u^{\hat{\gamma}} - T_u^\gamma = (\partial_1 f_u) \circ \theta_u.$$

The image of ∂_1 determines a smooth subbundle $\mathcal{A} \subset \Lambda^2 T^*M \otimes TM$. The projection of T^γ to $(\Lambda^2 T^*M \otimes TM)/\mathcal{A}$ is thus independent of the choice of connection (see [13, Secs 3.1.10–3.1.13]). This invariant of the almost Grassmannian structure is called the *intrinsic torsion* or the *harmonic torsion*.

For $\mathrm{Gr}(m,n)$ it is easy to see that locally there always are torsion-free connections preserving the structure, so the intrinsic torsion of the homogeneous model vanishes identically. Thus nonzero intrinsic torsion is an obstruction to local isomorphism of a given almost Grassmannian structure to $\mathrm{Gr}(m,n)$. For a C^1-structure, it is an obstruction to local C^1-isomorphism to $\mathrm{Gr}(m,n)$

(for which the corresponding map between the underlying manifolds would be a local C^2-diffeomorphism).

We now explicitly describe the subbundle $\mathcal{A} \subset \Lambda^2 T^*M \otimes TM$ as a $\mathfrak{g}_0$ representation when $m = 2$. Recall that the representation corresponding to the tangent bundle TM is $\mathfrak{g}_{-1} \cong \mathbf{R}^{2*} \boxtimes \mathbf{R}^n$, where the exterior tensor product corresponds to the direct sum decomposition of $\mathfrak{g}_0$. Next we have the decomposition into irreducible components

$$(4) \qquad \Lambda^2(\mathfrak{g}_{-1}^*) \cong (\Lambda^2 \mathbf{R}^2 \boxtimes S^2 \mathbf{R}^{n*}) \oplus (S^2 \mathbf{R}^2 \boxtimes \Lambda^2 \mathbf{R}^{n*}).$$

We tensor these with $\mathfrak{g}_{-1}$ and decompose into irreducibles. For $k \geq 2$, the representation $S^2 \mathbf{R}^k \otimes \mathbf{R}^{k*}$ splits into a trace-free component, denoted $(S^2 \mathbf{R}^k \otimes \mathbf{R}^{k*})_0$, and a trace component, isomorphic to $\mathbf{R}^k$, and similarly for the dual. There is an analogous decomposition of $\Lambda^2 \mathbf{R}^k \otimes \mathbf{R}^{k^*}$, but here the trace-free part is trivial when $k = 2$.

For $m = n = 2$, the map ∂_1 from (2) is surjective, so no intrinsic torsion is available. In our case when $m = 2$, $n > 2$,

$$(\Lambda^2 \mathbf{R}^2 \otimes \mathbf{R}^{2*}) \boxtimes (S^2 \mathbf{R}^{n*} \otimes \mathbf{R}^n) \subset \mathrm{Im}(\partial_1).$$

The intersection of $\mathrm{Im}(\partial_1)$ with the other irreducible components of $\Lambda^2(\mathfrak{g}_{-1}^*) \otimes \mathfrak{g}_{-1}$ is the trace component, which can be written

$$(5) \qquad \mathbf{R}^2 \boxtimes (\Lambda^2 \mathbf{R}^{n*} \otimes \mathbf{R}^n) + (S^2 \mathbf{R}^2 \otimes \mathbf{R}^{2*}) \boxtimes \mathbf{R}^{n*},$$

where the factors $\mathbf{R}^2$ and $\mathbf{R}^{n*}$ are embedded via a tensor product with Id followed by a symmetrization and an alternation, respectively. Hence the harmonic torsion corresponds to a section of the bundle associated to the remaining irreducible component

$$\mathbb{T} = (S^2 \mathbf{R}^2 \otimes \mathbf{R}^{2*})_0 \boxtimes (\Lambda^2 \mathbf{R}^{n^*} \otimes \mathbf{R}^n)_0.$$

To verify non-vanishing harmonic torsion in the example we are going to construct, we need the following result.

Lemma 2.3. *Let $\xi, \eta \in Hom(\mathbf{R}^2, \mathbf{R}^n)$ both have kernel spanned by $0 \neq v \in \mathbf{R}^2$, and let $T \in \mathrm{Im}(\partial_1)$. Then $T(\xi, \eta) : \mathbf{R}^2 \to \mathbf{R}^n$ maps v into the span of the images of ξ and η.*

Proof: Take $0 \neq \alpha \in \mathbf{R}^{2*}$ with $\alpha(v) = 0$, so we can write $\xi = \alpha \otimes w_1$ and $\eta = \alpha \otimes w_2$ for elements $w_1, w_2 \in \mathbf{R}^n$. Let $T = T_1 + T_2$ be the decomposition of T corresponding to (4). Clearly $T_1(\xi, \eta) = 0$. Now decompose T_2 according to (5) as $T_{21} + T_{22}$ (in a non-unique way). Given $\tilde{v} \in \mathbf{R}^2$, embedded

in the trace component of $S^2\mathbf{R}^2 \otimes \mathbf{R}^{2*} \cong \text{Hom}(\mathbf{R}^2, S^2\mathbf{R}^2)$, it sends v to a multiple of the symmetric product $v \odot \tilde{v}$. Since $\xi(v) = \eta(v) = 0$, we conclude that $T_1(\xi, \eta)(v) = 0$. On the other hand, $\tau \in \mathbf{R}^{n^*}$, embedded in the trace component of $\Lambda^2\mathbf{R}^{n*} \otimes \mathbf{R}^n \cong \text{Hom}(\Lambda^2\mathbf{R}^n, \mathbf{R}^n)$, sends (w_1, w_2) to a multiple of $\tau(w_1)w_2 - \tau(w_2)w_1$, so all values of $T_2(\xi, \eta)$ belong to the span of the images of ξ and η. The desired conclusion follows. $\diamondsuit$

2.5. **Deformations of almost-Grassmannian structures.** Given an almost-Grassmannian structure with $\theta : TM \xrightarrow{\sim} \mathcal{E}^* \otimes \mathcal{F}$, we will construct deformations by post-composing with a continuous family $\{\Phi_t\}$ of linear automorphisms of $\mathcal{E}^* \otimes \mathcal{F}$. To construct this family, we will first construct endomorphisms, that is, a section Φ of $\text{End } \mathcal{E}^* \otimes \text{End } \mathcal{F}$ and then show that this exponentiates to a one-parameter family of automorphisms.

At a given point $x \in M$, write $\mathcal{E}_x \cong E$ and $\mathcal{F}_x \cong F$. The vector space automorphisms of $E^* \otimes F$ respecting the tensor product are those of the form $\Psi_{E^*} \otimes \Psi_F$, for $\Psi_{E^*} \in \text{Aut } E^*$ and $\Psi_F \in \text{Aut } F$. Given a one-parameter group of such automorphisms $\Psi_{E^*}^t \otimes \Psi_F^t$, the generating endomorphism has the form $\psi_{E^*} \otimes \text{Id}_F + \text{Id}_{E^*} \otimes \psi_F$ for $\psi_{E^*} \in \text{End } E^*$ and $\psi_F \in \text{End } F$. The condition $\det \Psi_{E^*}^t \cdot \det \Psi_F^t \equiv 1$ is equivalent to $\text{tr } \psi_{E^*} + \text{tr } \psi_F = 0$.

The sections of $\text{Aut}(\mathcal{E}^* \otimes \mathcal{F})$ arising from automorphisms of the almost-Grassmannian structure are those of the form $\Psi_{\mathcal{E}^*} \otimes \Psi_{\mathcal{F}}$, for $\Psi_{\mathcal{E}^*} \in \text{Aut } \mathcal{E}^*$ and $\Psi_{\mathcal{F}} \in \text{Aut } \mathcal{F}$, with $\nu \circ \Lambda^m \Psi_{\mathcal{E}^*} = \Lambda^n \Psi_{\mathcal{F}} \circ \nu$. The generator of a nontrivial deformation is thus nontrivial modulo $\text{End } \mathcal{E}^* \otimes \text{Id}_{\mathcal{F}} + \text{Id}_{\mathcal{E}^*} \otimes \text{End } \mathcal{F}$. A pointwise complementary subbundle is given by the tensor product of trace-free endomorphisms $\text{End}_0 \mathcal{E}^* \otimes \text{End}_0 \mathcal{F}$. We will construct a section of this bundle in Sections 4.1 and 4.2 below.

The results of [12] apply to almost-Grassmannian structures of sufficient regularity to define a Cartan connection (see Section 2.6). Here infinitesimal automorphisms and deformations are described as the kernel and cokernel, respectively, of BGG operators acting on sections of the adjoint tractor bundle, with a certain "twisted" linear connection. The infinitesimal change of harmonic torsion and harmonic curvature (for the latter, see Section 2.6) resulting from a given infinitesimal deformation can also be described in general from these operators and this connection. This point of view was the inspiration for the concrete deformations we construct below.

2.6. **Prolongation and the canonical Cartan connection.** We will verify in Section 4.4 that the results of [2] apply to C^k almost-Grassmannian structures with $k \geq 2$, so any example of this regularity admitting a strongly essential flow by automorphisms has vanishing harmonic curvature on an open set containing the higher-order fixed point in its closure. We explain in this section that $(2, n)$-almost Grassmanian structures of regularity C^k with $k \geq 2$ determine a canonical C^0 Cartan connection as well as C^0 harmonic curvature. In low regularity, general existence results do not apply, so we briefly sketch the explicit constructions, following [14].

2.6.1. *Construction of the prolongation.* Given a C^k almost Grassmannian structure $(p_0 : \mathcal{G}_0 \to M, \theta)$ of type $(2, n)$, we will prolong $\mathcal{G}_0$ to a C^{k-1} principal P-bundle $\mathcal{G} \to M$. To this end, we view $\mathfrak{g}_0$ as a subalgebra of End $\mathfrak{g}_{-1}$. The kernel $\ker(\partial_1)$ of the differential from (2) is the subspace of $\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0 \subset \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$ of elements symmetric in $\mathfrak{g}_{-1}^*$, which is precisely the *first prolongation* of $\mathfrak{g}_0$ (see [7, I.1]). It is isomorphic to $\mathfrak{g}_1$, embedded into $\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0$ via the adjoint representation.

The bundle $\mathcal{G}$ is constructed as a $\mathfrak{g}_1$-bundle over $\mathcal{G}_0$. Note that $P \cong G_0 \ltimes \mathfrak{g}_1$. Given $u_0 \in \mathcal{G}_0$, denote by $V_{u_0}\mathcal{G} \subset T_{u_0}\mathcal{G}_0$ the vertical subspace, so θ_{u_0} defines a linear isomorphism $T_{u_0}\mathcal{G}_0 / V_{u_0}\mathcal{G}_0 \to \mathbf{R}^{2n}$. Recall from Section 2.4 that (1) depends at u_0 only on the value γ_{u_0} of a chosen principal connection. Let $u : T_{u_0}\mathcal{G}_0 \to \mathfrak{g}_0$ be any linear map recognizing fundamental vector fields— that is $u(\zeta_A(u_0)) = A \in \mathfrak{g}_0$ for $\zeta_A(u_0) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 u_0.e^{tA}$. Now $d\theta_{u_0} + [u, \theta_{u_0}]$ vanishes when either input is in $V_{u_0}\mathcal{G}_0$, so it equals $\theta_{u_0}^* T_u$ for a unique map $T_u \in \Lambda^2 \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$.

Varying the choice of u yields, as in Section 2.4, the affine subspace $T_u +$ Im(∂_1); moreover, the trace-free subspace $\mathbb{T} = (S^2 \mathbf{R}^{2*} \otimes \mathbf{R}^2)_0 \boxtimes (\Lambda^2 \mathbf{R}^n \otimes \mathbf{R}^{n*})_0$ is a G_0-invariant complement to Im(∂_1). Thus for each point $u_0 \in \mathcal{G}_0$, the linear map u can be chosen such that T_u is totally trace-free. For a fixed u_0 and $T \in \mathbb{T}$, the space of linear maps u giving rise via θ_{u_0} to T is, according to (3), an affine space modeled on $\ker \partial_1 \cong \mathfrak{g}_1$. Explicitly, any two such maps differ according to $\hat{u} - u = \mathrm{ad}_Z \circ \theta_{u_0} \in \mathfrak{g}_0$, for a unique element $Z \in \mathfrak{g}_1$.

Now $\mathcal{G}$ is the family of maps u as above for which $T_u \in \mathbb{T}$, as u_0 varies over $\mathcal{G}_0$. Denote q the projection $\mathcal{G} \to \mathcal{G}_0$. We can realize $\mathcal{G}$ as a subspace of the vector bundle $T^*\mathcal{G}_0 \otimes \mathfrak{g}_0 \to \mathcal{G}_0$ defined as above in terms of θ and $d\theta$, which is C^{k-1}; it follows that $\mathcal{G}$ is a C^{k-1} submanifold here. Elementary representation

theory gives a G_0-equivariant linear map $S : \Lambda^2 \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_{-1} \to \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_0$ which vanishes on $\mathbb{T}$ and such that $\partial_1 \circ S$ is the projection to $\mathrm{Im}(\partial_1)$. From a C^{k-1} principal connection on $\mathcal{G}_0$, one can use S to modify it to a C^{k-1} principal connection γ with $\gamma_{u_0} \in \mathcal{G}$ for all $u_0 \in \mathcal{G}_0$; connections of this type correspond to local C^{k-1} sections of q. Now $q : \mathcal{G} \to \mathcal{G}_0$ is a C^{k-1} principal P_+-bundle, and $p := p_0 \circ q : \mathcal{G} \to M$ is a C^{k-1} principal P-bundle (see [14] for more details).

2.6.2. *Harmonic curvature and Cartan connection.* Construction of the Cartan connection on $\mathcal{G}$ entails, by analogy with the prolongation process of the previous section, finding canonical $\mathfrak{g}_1$-valued one-forms on $\mathcal{G}$, which turn out to be unique. There are tautological forms $\theta_{-1} + \theta_0$, where $\theta_{-1} := q^*\theta$ and $(\theta_0)_u := q^*u$, viewing $u \in \mathrm{Hom}(T_{q(u)}\mathcal{G}_0, \mathfrak{g}_0)$. It is easy to see that $(\theta_0)_u(\zeta_A) = A$ for all $A \in \mathfrak{g}_0$, and that $\theta_{-1} + \theta_0$ is P-equivariant once $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ is identified with $\mathfrak{g}/\mathfrak{g}_1$.

Assuming that θ is at least C^2, the form θ_0 is at least C^1, so we can form its exterior derivative $d\theta_0 \in \Omega^2(\mathcal{G}, \mathfrak{g}_0)$. Let $\phi : T_u\mathcal{G} \to \mathfrak{g}_1$ be a linear map satisfying $\phi(\zeta_{A+Z}(u)) = Z$ on the fundamental vector fields for $A + Z \in \mathfrak{g}_0 \ltimes \mathfrak{g}_1 \cong \mathfrak{p}$. As before,

$$(6) \qquad (d\theta_0)_u + \frac{1}{2}[(\theta_0)_u, (\theta_0)_u] + [\phi, (\theta_{-1})_u]$$

vanishes if either input is in $V_u\mathcal{G}$ (the vertical bundle for $p : \mathcal{G} \to M$), so it equals $(\theta_{-1})^*K_\phi$ for a linear map $K_\phi : \Lambda^2\mathfrak{g}_{-1} \to \mathfrak{g}_0$.

For another choice $\hat{\phi}$, the difference is $\hat{\phi} - \phi = f \circ (\theta_{-1})_u$ for some linear map $f \in \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_1$. Another grading component of our Lie algebra differential, $\partial_2 : \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_1 \to \Lambda^2\mathfrak{g}^*_{-1} \otimes \mathfrak{g}_0$, is a G_0–equivariant linear map for which $K_{\hat{\phi}} - K_\phi = \partial_2(f)$. Projection modulo the subbundle $\mathcal{B} \subset \Lambda^2T^*\mathcal{G} \otimes \mathfrak{g}_0$ determined by $\mathrm{Im}(\partial_2)$ yields another invariant: given a local principal connection γ on $\mathcal{G}_0$ for which $\gamma_{u_0} \in \mathcal{G}$ for all u_0 in the domain, and a locally defined $\phi \in \Omega^1(\mathcal{G}, \mathfrak{g}_1)$, smooth of class C^{k-1}, one obtains a C^{k-1} section of $(\Lambda^2T^*\mathcal{G} \otimes \mathfrak{g}_0)/\mathcal{B}$ called the *harmonic curvature* of the geometry. This can be recovered as a component of the curvature of any adapted connection.

Now, it can be shown that ∂_2 is injective, and that there is a natural G_0–invariant complement $\mathbb{K}$ to $\mathrm{Im}(\partial_2)$. Hence for each $u \in \mathcal{G}$, there is a unique ϕ such that $K_\phi \in \mathbb{K}$. We obtain $\theta_1 \in \Omega^1(\mathcal{G}, \mathfrak{g}_1)$ of class C^{k-2}, and a C^{k-2} Cartan connection $\omega = \theta_{-1} \oplus \theta_0 \oplus \theta_1 \in \Omega^1(\mathcal{G}, \mathfrak{g})$. Note that when $k \geq 3$, the Cartan curvature can be defined by $K = d\omega + \frac{1}{2}[\omega, \omega] \in \Omega^2(\mathcal{G}, \mathfrak{g})$, and

the harmonic torsion and the harmonic curvature are components of K. For the homogeneous model, ω is the Maurer-Cartan form, so K vanishes identically, as does the harmonic curvature (see [14] for more details).

Proposition 2.4. *For $k \geq 2$, a C^k almost-Grassmannian structure of type $(2, n)$, $n \geq 3$, on M determines a C^{k-1} principal P-bundle $\mathcal{G} \to M$ equipped with a C^{k-2} Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. A C^k morphism between two such almost-Grassmannian structures canonically lifts to a morphism of the associated Cartan geometries.*

Proof: It remains only to verify the last statement. Let h be a local C^{k+1} diffeomorphism between open subsets of M and $\widetilde{M}$, lifting to a G_0-equivariant local C^k-diffeomorphism $\Phi_0 : \mathcal{G}_0 \to \widetilde{\mathcal{G}}_0$ with $\Phi_0^* \tilde{\theta} = \theta$. Given $u \in q^{-1}(u_0) \subset \mathcal{G}$, let $\Phi(u) := (\Phi_0^{-1})^* u$. It is easy to check that $T_{\Phi(u)} = T_u$, so $\Phi(u) \in (\tilde{q})^{-1}(\Phi_0(u_0)) \subset \widetilde{\mathcal{G}}$. This evidently defines a P-equivariant C^{k-1} lift $\Phi : \mathcal{G} \to \widetilde{\mathcal{G}}$ of Φ_0. The construction implies that $\Phi^* \tilde{\theta}_i = \theta_i$ for $i = -1, 0$. Then repeating this argument with the harmonic curvature allows us to conclude that $\Phi^* \tilde{\omega} = \omega$, so Φ is a morphism of Cartan geometries. $\Diamond$

As a corollary, we note that for a structure of class at least C^2, nonvanishing harmonic curvature is an obstruction to local isomorphism to $\mathrm{Gr}(2, n)$.

3. Description of the strongly essential flow in coordinates

By homogeneity of $\mathrm{Gr}(2, n)$, we may assume the point x_0 in Theorem 1.1 is the standard 2-plane spanned by the first two coordinate vectors in $\mathbf{R}^{2+n}$. The deformation will be constructed on the open subset U comprising the orbit of x_0 under all transformations $\mathrm{Id} + X$, with $X \in \mathrm{Hom}(\mathbf{R}^2, \mathbf{R}^n)$. This set is the domain of an affine chart in the Plücker coordinates on $\mathrm{Gr}(2, n)$, and equals the top cell in the standard Schubert decomposition. Identify U with $\mathrm{Hom}(\mathbf{R}^2, \mathbf{R}^n)$, and represent an element of U in coordinates

$$X = (x_{ij})_{i=1, j=1}^{i=n, j=2}$$

The principal P-bundle $G \to \mathrm{Gr}(2, n) \cong G/P$ restricted to U is smoothly equivalent to the trivial bundle $U \times P$. The quotient bundle G/G_1 restricted to U is $\mathcal{G}_0|_U$, which is smoothly equivalent to $U \times G_0$.

We will use the following explicit trivializations over U of the tautological and anti-tautological bundles, together with their duals. Denote by $\{e_1, \ldots, e_{n+2}\}$ the standard basis of $\mathbf{R}^{2+n}$ with dual basis $\{e^1, \ldots, e^{n+2}\}$.

For $j' = 1', 2'$, define a section of $\mathcal{E}|_U$ by $E_{j'}(X) = (\mathrm{Id} + X)e_{j'}$, and let $E^{j'}$ be the sections $e^{j'}$ of $\mathcal{E}^*$. Next let $E_i(X)$ equal the image of e_{i+2} in $\mathbf{R}^{2+n}/X$ for $i = 1, \ldots, n$, and $E^i(X) = e^{i+2} - x_{i1}e^1 - x_{i2}e^2$, which are well-defined on $\mathbf{R}^{2+n}/X$. We will henceforth denote the restrictions of these bundles to U simply by $\mathcal{E}$, $\mathcal{E}^*$, $\mathcal{F}$, and $\mathcal{F}^*$.

The restriction of the standard flat Grassmannian structure on $\mathrm{Gr}(2, n)$ to U is given by the obvious isomorphism from $TU \cong U \times \mathrm{Hom}(\mathbf{R}^2, \mathbf{R}^n)$ with $\mathcal{E}^* \otimes \mathcal{F}$. It sends the coordinate vector fields $\partial_i^{j'} := \partial/\partial x_{ij'}$ on U to the sections $E^{j'} \otimes E_i$ of $\mathcal{E}^* \otimes \mathcal{F}$.

3.1. The strongly essential flow.

Let Z be a rank-one element of $\mathfrak{g}_1 \cong \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^2)$. Let $\{z^t\}$ be the one-parameter subgroup of $P < G$ generated by Z; it is just the group of matrices $\{\mathrm{Id} + tZ\} < \mathrm{SL}(2 + n, \mathbf{R})$. Theorem 1.1 applies to any strongly essential flow generated by a rank-one element of $\mathfrak{g}_1$. After conjugation in G, we may assume $Z = e^1 \otimes e_{1'}$, where now $\{e^1, \ldots, e^n\}$ is the standard basis of $\mathbf{R}^{n*}$ and $\{e_{1'}, e_{2'}\}$ the standard basis of $\mathbf{R}^2$. Then $\mathrm{im}\, Z = \mathbf{R}e_{1'}$, and $\ker Z = \mathrm{span}\{e_2, \ldots, e_n\}$.

For $X \in U \cong \mathfrak{g}_{-1}$, denote e^X the corresponding lower-triangular unipotent matrix in G. Then compute the image in $U \times P$

$$
z^t.e^X = \begin{pmatrix} \mathrm{Id}_2 + tZX & tZ \\ X & \mathrm{Id}_n \end{pmatrix}
$$
$$
= \begin{pmatrix} \mathrm{Id}_2 & 0 \\ X(\mathrm{Id}_2 + tZX)^{-1} & \mathrm{Id}_n \end{pmatrix} \begin{pmatrix} \mathrm{Id}_2 + tZX & tZ \\ 0 & \mathrm{Id}_n - X(\mathrm{Id}_2 + tZX)^{-1}tZ \end{pmatrix},
$$

assuming that $\mathrm{Id}_2 + tZX$ is invertible, which for fixed t holds on an open neighborhood of 0.

The following two subspaces are fixed by $\{z^t\}$:

$$
F_1 = \{X \ : \ XZ = 0\}
$$

and

$$
F_2 = \{X \ : \ ZX = 0\}
$$

The intersection $F_1 \cap F_2$ will be called the *strongly fixed set* and denoted SF. Note that $X \in SF$ if and only if $[X, Z] = 0$ and, in coordinates,

$$
SF = \{X \ : \ x_{12} = 0 = x_{i1} \text{ for all } i = 1, \ldots, n\}
$$

Let $H_0 = \{X \; : \; x_{11} = 0\}$. For $X \notin H_0$, decompose X as

$$
X_f + X_d = \begin{pmatrix} 0 & 0 \\ & x_{22} - \frac{x_{12}}{x_{11}} \cdot x_{21} \\ \vdots & \vdots \\ 0 & x_{n2} - \frac{x_{12}}{x_{11}} \cdot x_{n1} \end{pmatrix} + \begin{pmatrix} x_{11} & \frac{x_{12}}{x_{11}} \cdot x_{11} \\ \\ \vdots & \vdots \\ x_{n1} & \frac{x_{12}}{x_{11}} \cdot x_{n1} \end{pmatrix}
$$

with $X_f \in SF$ and X_d rank 1. If $X \notin H_0$, then $z^t.X = X_f + z^t.X_d$, which equals

$$
(7) \qquad\qquad X_f + \begin{pmatrix} \frac{x_{11}}{1+tx_{11}} & \frac{\frac{x_{12}}{x_{11}} \cdot x_{11}}{1+tx_{11}} \\ \vdots & \vdots \\ \frac{x_{n1}}{1+tx_{11}} & \frac{\frac{x_{12}}{x_{11}} \cdot x_{n1}}{1+tx_{11}} \end{pmatrix}
$$

Let $H_+ = \{X \in U \; : \; x_{11} > 0\}$ and $H_- = \{X \in U \; : \; x_{11} < 0\}$. The formula (7) above yields $z^t.X \to X_f$ as $t \to \pm\infty$ for $X \in H_\pm$, respectively.

Note that if $X \in H_0$, then $ZXZ = 0$, and $(\mathrm{Id} + tZX)^{-1} = \mathrm{Id} - tZX$. Then $z^t X = X(\mathrm{Id} - tZX)$ which equals X as t varies if and only if $XZX = 0$; the latter holds only when ZX or $XZ = 0$. We conclude that $F_1 \cup F_2$ equals the fixed set of z^t in U.

3.2. Action on associated vector bundles. The matrix in P

$$
p_t(X) = \begin{pmatrix} \mathrm{Id}_2 + tZX & tZ \\ 0 & \mathrm{Id}_n - X(\mathrm{Id}_2 + tZX)^{-1}tZ \end{pmatrix}
$$

from above encodes the action of z^t on $\mathcal{E}$ and $\mathcal{F}$, and, in turn, on TU. For $X \notin H_0$,

$$
(8) \quad p_t(X) = p_t(X_d) = \begin{pmatrix} 1 + tx_{11} & tx_{12} & t & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & \frac{1}{1+tx_{11}} & & \\ & & \frac{-tx_{21}}{1+tx_{11}} & 1 & \\ & & \vdots & & \ddots \\ & & \frac{-tx_{n1}}{1+tx_{11}} & & 1 \end{pmatrix}
$$

For $X \in H_0$, straightforward calculation gives the formula (8), with $x_{11} = 0$.

On $\mathcal{E} = (G \times_P \mathbf{R}^2)\big|_U \cong U \times \mathbf{R}^2$ the action of $\{z^t\}$ is

$$
(9) \qquad (z_{\mathcal{E}}^t)_X = \begin{pmatrix} 1 + tx_{11} & tx_{12} \\ 0 & 1 \end{pmatrix} \quad \text{with respect to } \{E_{1'}, E_{2'}\}
$$

and on $\mathcal{E}^*$,

$$(10) \quad (z_{\mathcal{E}}^{-t})^*_{z^t.X} = \begin{pmatrix} \frac{1}{1+tx_{11}} & \frac{-tx_{12}}{1+tx_{11}} \\ 0 & 1 \end{pmatrix} \text{ with respect to } \{E^{1'}, E^{2'}\}$$

On $\mathcal{F}$, the action is

$$(11) \quad (z_{\mathcal{F}}^t)_X = \begin{pmatrix} \frac{1}{1+tx_{11}} & & & \\ \frac{-tx_{21}}{1+tx_{11}} & 1 & & \\ \vdots & & \ddots & \\ \frac{-tx_{n1}}{1+tx_{11}} & & & 1 \end{pmatrix} \text{ w.r.t. } \{E_1, \ldots, E_n\}$$

and, finally, on $\mathcal{F}^*$,

$$(12) \quad (z_{\mathcal{F}}^{-t})^*_{z^t.X} = \begin{pmatrix} 1+tx_{11} & & & \\ tx_{21} & 1 & & \\ \vdots & & \ddots & \\ tx_{n1} & & & 1 \end{pmatrix} \text{ w.r.t. } \{E^1, \ldots, E^n\}$$

4. The invariant deformation and non-flatness

Our deformation is constructed in Sections 4.1 and 4.2 below. Then in Section 4.3 we compute terms of the harmonic torsion which are nonzero on an open, dense subset of any neighborhood of x_0. It follows that our C^1, deformed structures are not flat on any open set V with $x_0 \in \overline{V}$, and thus that the C^1 version of [9, Thm 1.3] does not hold. The proof of vanishing harmonic torsion in [9] requires several degrees of differentiability of the structure, so it remains open whether C^1 is the maximal regularity of such a counterexample. Our result [2, Prop 3.5], on the other hand, says that the harmonic curvature must vanish on an open set V with $x_0 \in \overline{V}$, in the presence of a flow by strongly essential autmorphisms, and we explain in Section 4.4 below that it applies to C^2 structures. In Section 4.5, we verify that the harmonic curvature of our deformations restricted to their common smooth locus is nonzero. Our deformations are thus in some sense structures of maximal regularity for which the conclusion of [2, Prop 3.5] does not hold.

4.1. **Eigen-sections of associated bundles.** Here we define sections of $\mathcal{E}$ and $\mathcal{F}$ and of the dual bundles, which are invariant by z^t up to multiplication by a function on U. For any X where the decompositions

$$(13) \quad \mathcal{E}_X = \ker X_d \oplus \operatorname{im} Z \quad \text{and} \quad \mathcal{F}_X = \ker Z \oplus \operatorname{im} X_d$$

are valid, each section will have values in one factor or its dual. The sections together will span the fibers over X in $\mathcal{E}, \mathcal{F}$, or their duals.

Define

$$v(X) = -x_{12}E_{1'} + x_{11}E_{2'} \qquad \iota(X) = E_{1'}$$
$$\tilde{v}(X) = E^{2'} \qquad \tilde{\iota}(X) = x_{11}E^{1'} + x_{12}E^{2'}$$

These pairs of sections are smooth on U and independent on $U \setminus H_0$. We compute from (9) and (10)

$$z_{\mathcal{E}}^t(v(X)) = (1 + tx_{11}) \cdot v(z^t.X) \qquad z_{\mathcal{E}}^t(\iota(X)) = (1 + tx_{11}) \cdot \iota(z^t.X)$$
$$(z_{\mathcal{E}}^{-t})^*(\tilde{v}(X)) = \tilde{v}(z^t.X) \qquad (z_{\mathcal{E}}^{-t})^*(\tilde{\iota}(X)) = \tilde{\iota}(z^t.X)$$

Now define sections of $\mathcal{F}$ and $\mathcal{F}^*$ for $i = 2, \ldots, n$ by

$$w(X) = x_{11}E_1 + \cdots + x_{n1}E_n \qquad \kappa_i(X) = E_i$$
$$\tilde{w}(X) = E^1 \qquad \tilde{\kappa}^i(X) = x_{11}E^i - x_{i1}E^1$$

These sections transform, for $i = 2, \ldots, n$, according to (11) and (12) by

$$z_{\mathcal{F}}^t(w(X)) = w(z^t.X) \qquad z_{\mathcal{F}}^t(\kappa_i(X)) = \kappa_i(z^t.X)$$
$$(z_{\mathcal{F}}^{-t})^*(\tilde{w}(X)) = (1 + tx_{11}) \cdot \tilde{w}(z^t.X) \qquad (z_{\mathcal{F}}^{-t})^*(\tilde{\kappa}^i(X)) = (1 + tx_{11}) \cdot \tilde{\kappa}^i(z^t.X)$$

4.2. **Invariant section of the endomorphism bundle.** Now consider the sections

$$\varphi' = v \otimes \tilde{\iota} \qquad \text{and} \qquad \varphi_i = \tilde{\kappa}^i \otimes w$$

of End $\mathcal{E}^*$ and End $\mathcal{F}$, respectively, for $i = 2, \ldots, n$. These preserve the decompositions (13). They are each nilpotent endomorphisms of order two for any X: $(\varphi'_X)^2 = 0$ and $((\varphi_i)_X)^2 = 0$ for all i; in particular, they are trace-free.

The tensor $\varphi' \otimes \varphi_i$ is a section of the subbundle $\text{End}_0 \, \mathcal{E}^* \otimes \text{End}_0 \, \mathcal{F} \subset \text{End} \, TU$, as in Section 2.5, corresponding to nontrivial deformations of the structure. The flow acts on this section by

$$(z^t)_*(\varphi'(X) \otimes \varphi_i(X)) = (1 + tx_{11})^2 \cdot \varphi'(z^t.X) \otimes \varphi_i(z^t.X)$$

Define $q(X) = x_{12}^2 + x_{11}^2 + \cdots + x_{n1}^2$. Note that $q(z^t.X) = (1 + tx_{11})^{-2} \cdot q(X)$. Then the section

$$\Phi_i = \frac{1}{q} \varphi' \otimes \varphi_i$$

is z^t-invariant. The coefficients of the components of Φ are rational functions in the variables $x_{12}, x_{11}, \ldots, x_{n1}$ with numerator degree four and denominator degree two. They are smooth on $U \backslash SF$ and C^1 on SF, in particular at the origin.

Denote by $E^{i'}_{j'}$ the elementary endomorphism of $\mathcal{E}^*$ sending $E^{j'}$ to $E_{i'}$, and by E^i_j the elementary endomorphism of $\mathcal{F}$ sending E_i to E_j. The coefficients of Φ_i are given by

$$\frac{1}{q(X)} \cdot \left(-x_{11}x_{12}E^{1'}_{1'} - x^2_{12}E^{2'}_{1'} + x^2_{11}E^{1'}_{2'} + x_{11}x_{12}E^{2'}_{2'} \right)$$

$$\otimes \left(\sum_{k=1}^n x_{11}x_{k1}E^i_k - \sum_{k=1}^n x_{i1}x_{k1}E^1_k \right)$$

which expands further, denoting $E^{i'}_{j'} \otimes E^\ell_k$ by $E^{i'\ell}_{j'k}$, as

$$\sum_k \frac{x_{12}x_{11}x_{i1}x_{k1}}{q(X)} E^{1'1}_{1'k} + \sum_k \frac{-x_{12}x^2_{11}x_{k1}}{q(X)} E^{1'i}_{1'k} + \sum_k \frac{x^2_{12}x_{i1}x_{k1}}{q(X)} E^{2'1}_{1'k}$$

$$+ \sum_k \frac{-x^2_{12}x_{11}x_{k1}}{q(X)} E^{2'i}_{1'k} + \sum_k \frac{-x^2_{11}x_{i1}x_{k1}}{q(X)} E^{1'1}_{2'k} + \sum_k \frac{x^3_{11}x_{k1}}{q(X)} E^{1'i}_{2'k}$$

$$+ \sum_k \frac{-x_{12}x_{11}x_{i1}x_{k1}}{q(X)} E^{2'1}_{2'k} + \sum_k \frac{x_{12}x^2_{11}x_{k1}}{q(X)} E^{2'i}_{2'k}$$

Of course, for any constant $\mathbf{c} = (c_2, \ldots, c_n)$, the endomorphism field

$$\Phi = \Phi_{\mathbf{c}} = \sum_{i=2}^n c_i \Phi_i$$

will be z^t-invariant and C^1.

Proposition 4.1. *Let $\theta : TU \xrightarrow{\sim} Hom(\mathbf{R}^2, \mathbf{R}^n)$ be the flat almost-Grassmannian structure on U. Then for any $\mathbf{c} \neq \mathbf{0}$, $(Id + \Phi_{\mathbf{c}}) \circ \theta$ is a $\{z^t\}$-invariant, C^1 almost-Grassmannian structure on U not C^1 equivalent to θ.*

Fix $\mathbf{c} \neq \mathbf{0}$ and denote $\Phi = \Phi_{\mathbf{c}}$. We first show that $(Id + \Phi) \circ \theta$ is an almost-Grassmannian structure on U. Recall from above that $(\varphi')^2 = 0$ and $(\varphi_i)^2 = 0$ for all $i = 2, \ldots, n$; note that moreover, $\varphi_i \circ \varphi_j = 0$ for all $i, j = 2, \ldots, n$, so that $c_2\varphi_2 + \cdots + c_n\varphi_n$ is also nilpotent of order two. It follows that Φ_X is a nilpotent endomorphism of $(\mathcal{E}^* \otimes \mathcal{F})_X$ of order two for all $X \in U$, so the matrix exponential of Φ_X in $SL(\mathcal{E}^* \otimes \mathcal{F})_X$ is simply $Id + \Phi_X$. We conclude that $Id + \Phi_X$ is an isomorphism for all $X \in U$, so

$(\mathrm{Id} + \Phi) \circ \theta$ is an almost-Grassmannian structure on U. The $\{z^t\}$-invariance holds by construction.

The derivatives of the rational coefficients in Φ are undefined on SF, the zero set of q. The numerators are homogeneous polynomials in $x_{12}, x_{11}, \ldots, x_{n1}$ of degree five, with denominators all equal q^2. Such functions extend continuously to 0 on SF.

The final claim of the proposition is proved in the following section.

4.3. Calculation of nonzero harmonic torsion terms.

4.3. **Calculation of nonzero harmonic torsion terms.** Recall from Section 2.4 that the harmonic torsion can be computed from any principal connection $\gamma \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_0)$. Such a connection is equivalent to a pair of (volume-compatible) linear connections $\nabla_{\mathcal{E}^*}$ on $\mathcal{E}^*$ and $\nabla_{\mathcal{F}}$ on $\mathcal{F}$. These induce a connection on $\mathcal{E}^* \otimes \mathcal{F}$ of the form $\nabla_{\mathcal{E}^*} \otimes \mathrm{Id}_{\mathcal{F}} + \mathrm{Id}_{\mathcal{E}^*} \otimes \nabla_{\mathcal{F}}$. Via an almost-Grassmannian structure $\Psi : TU \xrightarrow{\sim} \mathcal{E}^* \otimes \mathcal{F}$ of class at least C^1, this connection can be pulled back to TU, and that pullback has a well-defined torsion. For the harmonic torsion, we map back to $\mathcal{E}^* \otimes \mathcal{F}$ via Ψ and project to the bundle associated to $(S^2 \mathbf{R}^2 \otimes \mathbf{R}^{2*})_0 \boxtimes (\Lambda^2 \mathbf{R}^{n*} \otimes \mathbf{R}^n)_0$, as in Section 2.4. Nonvanishing of the result is an obstruction to C^1 flatness.

As $\mathcal{E}^*$ and $\mathcal{F}$ are trivial bundles over U, we can use the trivial connections on each. By construction, the frame $\{E_j^{i'}\}$ of $\mathcal{E}^* \otimes \mathcal{F}$ is parallel. Denote by ∇ the pullback to TU via our deformed almost-Grassmannian structure corresponding to $\mathrm{Id} + \Phi$. The pullbacks of $\{E_j^{i'}\}$ comprise a framing of TU by parallel vector fields $\{\tilde{E}_j^{i'}\}$. The torsion is then determined by their brackets. We compute a specific component of the torsion and apply Lemma 2.3 to show that the harmonic torsion is nonzero.

Consider the sections $E_s^{2'}$ for $s > 1$ and $E_1^{2'}$ of $\mathcal{E}^* \otimes \mathcal{F}$. Both have rank one with kernel spanned by $E_{1'}$. Now $T(\tilde{E}_s^{2'}, \tilde{E}_1^{2'})$ by construction depends only on the component of T in the subbundle corresponding to the second summand in the decomposition (4). Consequently, Lemma 2.3 implies nonvanishing of the harmonic torsion provided $T(\tilde{E}_s^{2'}, \tilde{E}_1^{2'})(E_{1'})$ is not contained in the span of E_1 and E_s. From the explicit description of Φ from above, we obtain for $s > 1$

$$\tilde{E}_s^{2'} = \theta^{-1} \circ (\mathrm{Id} + \Phi)^{-1}(E_s^{2'}) = \partial_s^{2'} - c_s \cdot \sum_{p'=1, k=1}^{p'=2, k=n} \frac{x_{1p'} x_{11}^2 x_{k1}}{q(X)} \partial_k^{p'},$$

while

$$\tilde{E}_1^{2'} = \theta^{-1} \circ (\mathrm{Id} + \Phi)^{-1}(E_1^{2'}) = \partial_1^{2'} + \sum_{p'=1,k=1,i=2}^{p'=2,k=n,i=n} c_i \cdot \frac{x_{1p'}x_{11}x_{k1}x_{i1}}{q(X)} \partial_k^{p'}.$$

Since these fields are parallel, the torsion is given by

$$T(\tilde{E}_s^{2'}, \tilde{E}_1^{2'}) = -[\tilde{E}_s^{2'}, \tilde{E}_1^{2'}] = -\frac{c_s x_{11}^2}{q(X)} \cdot \sum_k \left(x_{k1}\partial_k^{2'} - \frac{2x_{k1}x_{12}}{q(X)} \sum_{p'} x_{1p'}\partial_k^{p'} \right).$$

Mapping this vector field $\tilde{D}$ to $\mathcal{E}^* \otimes \mathcal{F}$ via $(\mathrm{Id} + \Phi) \circ \theta$, we obtain, in order of increasing *net degree*—degree of numerator minus degree of denominator,

$$\begin{aligned} D := (\mathrm{Id} + \Phi) \circ \theta(\tilde{D}) = \quad & -\frac{c_s x_{11}^2}{q(X)} \cdot \sum_k \left(x_{k1}E_k^{2'} - \frac{2x_{k1}x_{12}}{q(X)}(x_{11}E_k^{1'} + x_{12}E_k^{2'}) \right. \\ + \quad & \left. x_{k1}\Phi(E_k^{2'}) - \frac{2x_{k1}x_{12}}{q(X)}\left((x_{11}\Phi(E_k^{1'}) + x_{12}\Phi(E_k^{2'})) \right) \right) \end{aligned}$$

This is a continuous section of $\mathcal{E}^* \otimes \mathcal{F}$. Now compute

$$D(E_{1'}) = \frac{c_s x_{11}^2}{q(X)} \cdot \sum_k \frac{-2x_{k1}x_{12}x_{11}}{q(X)} E_k + \text{ higher-order terms},$$

where the higher-order terms have net degree at least three. This has non-trivial projection modulo $\mathrm{span}\{E_1, E_s\}$ on an open, dense subset of any neighborhood of 0 in U, provided $c_s \neq 0$.

We conclude that the harmonic torsion of the deformed structure given by $\mathrm{Id} + \Phi$ is nontrivial on an open, dense subset of any neighborhood of 0. This structure is thus inequivalent to $\mathrm{Gr}(2, n)$ on any open subset containing 0 in its closure.

4.4. Vanishing of harmonic curvature for C^2 structures.

Let $(\mathcal{G}_0 \to M, \theta)$ be a $(2, n)$ almost-Grassmannian structure of regularity C^2 admitting a strongly essential flow $\{z^t\}$ with higher-order fixed point x_0. We verify here that the proof of [2, Thm 3.1] applies to such a structure, so it has vanishing harmonic curvature on an open set containing the higher order fixed point in its closure.

Let $(p : \mathcal{G} \to M, \omega)$ be the C^1 prinicpal P-bundle and C^0 Cartan connection, respectively, given by Proposition 2.4. There is a C^1 exponential map exp : $\mathcal{G} \times \mathfrak{g} \to \mathcal{G}$, giving C^1 exponential curves $\tilde{\gamma}_X(s) = \exp(u, sX)$ as in [2, Def 1.4]. This differentiability is sufficient to apply all of the holonomy calculations of [2, Sec 2].

Let ξ be the vector field generating $\{z^t\}$. For any $u \in p^{-1}(x_0)$, the value $\omega_u(\xi) \in \mathfrak{g}_1$ ([2, Sec 1.2]); let Z be the value for a particular choice of u. The rank of Z as an element of $\mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^2)$ can be two or one. In either case, Z defines a subset $\mathcal{T}(X) \subset \mathfrak{g}_{-1}$ comprising elements generating an $\mathfrak{sl}_2$-triple $\{X, A = [Z, X], Z\}$ (see [2, Def 2.11]). Along exponential curves $\tilde{\gamma}_X$ for $X \in \mathcal{T}(Z)$, the harmonic curvature and harmonic torsion must belong to the *stable subspaces* $\mathbb{K}^{[st]}$ and $\mathbb{T}^{[st]}$, respectively, determined by A ([2, Def 2.13]). This restriction appears in Corollary 2.14 (1) of [2], which in turn rests on Proposition 2.9 of the same; the required property here is that the harmonic curvature and torsion are given by *continuous* P-equivariant functions on $\mathcal{G}$.

When rk $Z = 2$, then $|\mathcal{T}(Z)| = 1$, and $\mathbb{K}^{[st]} = 0$. The harmonic curvature vanishes not only along the curve $\gamma_X = p \circ \tilde{\gamma}_X$, but on a neighborhood of $\gamma_X \backslash \{x_0\}$, as given by [2, Prop 3.3].

When rk $Z = 1$, as in the examples constructed above, then the *strongly stable subspace* $\mathbb{K}^{[ss]}$ determined by A is trivial. Together with other purely algebraic features of the $\mathfrak{sl}_2$-triple and its representation on $\mathbb{K}$, this property suffices to again prove vanishing of the harmonic curvature on a neighborhood of $\gamma_X \backslash \{x_0\}$, as shown in [2, Prop 3.5 (4a)].

The proof of [9, Thm 1.3] that $(2, n)$-almost-Grassmannian manifolds with strongly essential automorphisms have vanishing harmonic torsion on an open set containing the higher-order fixed point in its closure is more involved, and requires higher regularity; in particular, the full Cartan curvature and a *fundamental derivative* (see [9, Sec 3.4]) of it must be continuous. Thus it remains unclear whether a deformation with the properties in the conclusion of Theorem 1.1 could have higher regularity than C^1.

4.5. **Nonvanishing harmonic curvature on smooth locus.** Our C^1 deformed structures $(\mathrm{Id} + \Phi) \circ \theta$ do not have a well-defined harmonic curvature tensor everywhere, but we can prove that the harmonic curvature for these structures is nontrivial on the smooth locus $U \backslash SF$, for sufficiently small values of $\mathbf{c}$. The *infinitesimal change* of harmonic curvature produced by the infinitesimal deformation $\Phi_{\mathbf{c}}$ is the derivative at $t = 0$ of the change of harmonic curvature produced by the deformations $(\mathrm{Id} + t\Phi_{\mathbf{c}}) \circ \theta = (\mathrm{Id} + \Phi_{t\mathbf{c}}) \circ \theta$. For our aim, it suffices to show that the infinitesimal change of harmonic curvature caused by $\Phi_{\mathbf{c}}$ is nontrivial for $\mathbf{c} \neq \mathbf{0}$.

According to Theorem 3.6 of [12], the infinitesimal change of harmonic curvature induced by an infinitesimal deformation can be computed on the smooth locus with the BGG sequence constructed from a certain linear connection on the adjoint tractor bundle. The operators in that BGG sequence act between sections of bundles associated to Lie algebra homologies $H_*(\mathfrak{g}_1, \mathfrak{g})$, which are isomorphic to the Lie algebra cohomology spaces $H^*(\mathfrak{g}_{-1}, \mathfrak{g})$. These are representations of $\mathfrak{g}_0$, which can be computed explicitly using Kostant's version of the Bott-Borel-Weil Theorem. The specific calculations for the Grassmannian case can be found in Section 3.5 of [3] and in Section 4.1.3 Step (D) of [13].

In degree one, this representation is irreducible and isomorphic to $\mathfrak{sl}(\mathbf{R}^{2*}) \boxtimes \mathfrak{sl}(\mathbf{R}^n)$. In degree two, there are two irreducible components, one of which is the module $\mathbb{T}$ of Section 2.4 corresponding to the harmonic torsion. The other component $\mathbb{K} \subset \Lambda^2 \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0$ is the harmonic curvature module of Section 2.6.2. It is the component of maximal highest weight in

$$(\Lambda^2 \mathbf{R}^2 \boxtimes S^2 \mathbf{R}^{n*}) \otimes (\mathfrak{sl}(\mathbf{R}^{2*}) \oplus \mathfrak{sl}(\mathbf{R}^n))$$

(recall the decomposition in (4)), which turns out to be

$$\mathbb{K} \cong \Lambda^2 \mathbf{R}^2 \boxtimes (S^3 \mathbf{R}^{n*} \otimes \mathbf{R}^n)_0$$

The construction of BGG sequences provides an invariant differential operator D mapping sections of $\mathrm{End}_0(\mathcal{E}^*) \boxtimes \mathrm{End}_0(\mathcal{F})$, corresponding to infinitesimal deformations, to sections of the bundle associated to $\mathbb{K}$, yielding the infinitesimal change of harmonic curvature (see Section 3.6 of [12]). Any BGG operator admits a universal formula in terms of any *distinguished connection* of the structure, its curvature and torsion, and their covariant derivatives. In our case, the initial structure is the flat structure on the open set $U \backslash SF \subset \mathrm{Gr}(2, n)$, for which the flat connection ∇_0 induced by the trivial connections on $\mathcal{E}^*$ and $\mathcal{F}$ as in Section 4.3 is a distinguished connection.

Representation theory implies that D must be a second-order operator. Since ∇_0 is torsion-free and flat, the universal formula for D can only consist of applying two covariant derivatives, which automatically are symmetric, followed by tensorial operations induced by $\mathfrak{g}_0$-equivariant maps on the inducing representations. The latter is a $\mathfrak{g}_0$-equivariant map

$$\rho : S^2(\mathfrak{g}_{-1}^*) \otimes (\mathfrak{sl}(\mathbf{R}^{2*}) \boxtimes \mathfrak{sl}(\mathbf{R}^n)) \to \mathbb{K}.$$

Similarly as in (4), we can decompose

$$S^2(\mathfrak{g}_{-1}^*) \cong (S^2\mathbf{R}^2 \boxtimes S^2\mathbf{R}^{n*}) \oplus (\Lambda^2\mathbf{R}^2 \boxtimes \Lambda^2\mathbf{R}^{n*}),$$

It is easy to see from representation theory that ρ factors through the first summand. There is a unique homomorphism $S^2\mathbf{R}^2 \otimes (\mathbf{R}^2 \otimes \mathbf{R}^{2*})_0 \to \Lambda^2\mathbf{R}^2$ up to scale: the unique nonzero contraction with values in $\mathbf{R}^2 \otimes \mathbf{R}^2$, followed by an alternation. There is also a unique homomorphism

$$S^2\mathbf{R}^{n*} \otimes (\mathbf{R}^n \otimes \mathbf{R}^{n*})_0 \to (S^3\mathbf{R}^{n*} \otimes \mathbf{R}^n)_0$$

up to scale: symmetrization of the three $\mathbf{R}^{n*}$ components followed by projection on the module of trace-free elements.

Using similar index notation as before, we now form the second derivative $\{\nabla_{i'}^j \nabla_{\ell'}^m \Phi_{q'r}^{p'o}\}$, and then projection to $\mathbb{K}$ is achieved by

(1) contracting the indices p' and ℓ';
(2) skew-symmetrizing the indices i' and q';
(3) symmetrizing the indices j, m, and o; and
(4) removing the trace of r with (jmo)

Like in Section 4.3, it now suffice to apply the operations in the $\mathbf{R}^2$-part and then find a nonzero component which cannot be contained in the pure trace component. Namely, we now compute the term $\kappa_{2'1'r}^{111}$, for $r > 1$, which evidently has this property.

$$
\begin{aligned}
\kappa_{2'1'r}^{111} &= \frac{1}{2}\left(\nabla_{2'}^1\nabla_{1'}^1\Phi_{1'r}^{1'1} + (\nabla_{2'}^1)^2\Phi_{1'r}^{2'1} - (\nabla_{1'}^1)^2\Phi_{2'r}^{1'1} - \nabla_{1'}^1\nabla_{2'}^1\Phi_{2'r}^{2'1}\right) \\
&= \frac{1}{2}\left(2\nabla_{2'}^1\nabla_{1'}^1\Phi_{1'r}^{1'1} + (\nabla_{2'}^1)^2\Phi_{1'r}^{2'1} - (\nabla_{1'}^1)^2\Phi_{2'r}^{1'1}\right)
\end{aligned}
$$

using that $\nabla_{2'}^1\nabla_{1'}^1\Phi_{1'r}^{1'1} = -\nabla_{1'}^1\nabla_{2'}^1\Phi_{2'r}^{2'1}$ by trace-freeness in the primed indices and flatness of ∇. Recall from Section 4.2

$$\Phi_{1'r}^{1'1} = \sum_{i>1} c_i \frac{x_{12}x_{11}x_{i1}x_{r1}}{q(X)} \qquad \Phi_{1'r}^{2'1} = \sum_{i>1} c_i \frac{x_{12}^2 x_{i1}x_{r1}}{q(X)} \qquad \Phi_{2'r}^{1'1} = -\sum_{i>1} c_i \frac{x_{11}^2 x_{i1}x_{r1}}{q(X)}$$

Then compute, writing $q = q(X)$,

$$
\nabla^1_{2'}\nabla^1_{1'}\Phi^{1'1}_{1'r} = \sum_{i>1} c_i \left(\frac{x_{i1}x_{r1}}{q} - \frac{2(x^2_{11} + x^2_{12})x_{i1}x_{r1}}{q^2} + \frac{8x^2_{12}x^2_{11}x_{i1}x_{r1}}{q^3} \right)
$$

$$
(\nabla^1_{2'})^2\Phi^{2'1}_{1'r} = \sum_{i>1} c_i \left(\frac{2x_{i1}x_{r1}}{q} - \frac{10x^2_{12}x_{i1}x_{r1}}{q^2} + \frac{8x^4_{12}x_{i1}x_{r1}}{q^3} \right)
$$

$$
(\nabla^1_{1'})^2\Phi^{1'1}_{2'r} = -\sum_{i>1} c_i \left(\frac{2x_{i1}x_{r1}}{q} - \frac{10x^2_{11}x_{i1}x_{r1}}{q^2} + \frac{8x^4_{11}x_{i1}x_{r1}}{q^3} \right)
$$

Finally,

$$
\kappa^{111}_{2'1'r} = \sum_{i>1} c_i \left(\frac{3x_{i1}x_{r1}}{q} - \frac{7(x^2_{11} + x^2_{12})x_{i1}x_{r1}}{q^2} + \frac{4(x^2_{11} + x^2_{12})^2 x_{i1}x_{r1}}{q^3} \right) \neq 0.
$$

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