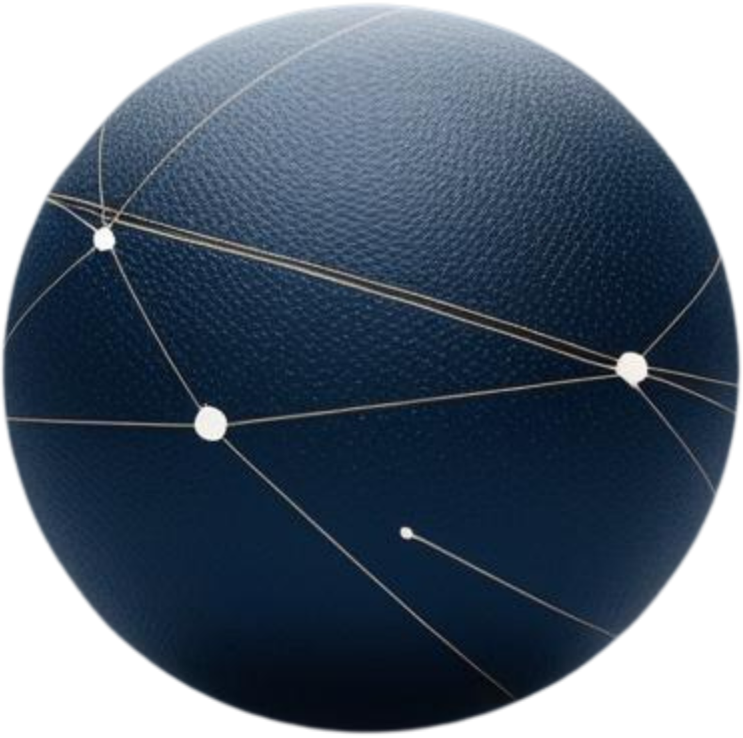
University of Luxembourg
10-12th September, 2025

Advances in Directional Statistics
Program and book of abstracts



ADISTA

LUXEMBOURG

10-12 SEPTEMBER

ADISTA: Advances in Directional Statistics

University of Luxembourg
10-12th September, 2025
Program and Book of Abstracts

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Presentation

Welcome to the 2025 International Workshop on Advances in Directional Statistics – a leading event dedicated to promoting progress in the field of directional statistics. Since our first meeting in 2014, a vibrant and expanding community of researchers has come together, united by a shared enthusiasm for both the theoretical foundations and practical applications of this dynamic discipline.

This year, we are privileged to celebrate the remarkable career of Professor Kanti Mardia, whose groundbreaking work has profoundly influenced several disciplines within statistics, and particularly, directional statistics. In honor of Professor Mardia's 90th birthday, this workshop seeks to stimulate research, encourage innovative methodologies, and foster enduring collaborations among experts in directional statistics.

The scientific program of the ADISTA 2025 spans three days and comprises a series of 25-minute invited presentations by leading experts in the field, jointly with poster flash presentations. Structured into eleven sessions, the program covers recent advances in the theory, methodology, and applications of directional statistics. The schedule is designed to facilitate in-depth scientific exchange, with dedicated time for discussion, as well as informal interactions during coffee breaks and lunches, thereby promoting collaborative research within the community.

Finally, we would like to extend our sincere gratitude to our sponsors for their generous support — in particular, the Department of Mathematics at the University of Luxembourg and the Luxembourg Statistical Society.

The Organizing Committee

UNIVERSITY OF LUXEMBOURG
Department of Mathematics

LUXEMBOURG STATISTICAL SOCIETY

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Abstracts: invited speakers

The projected normal signal-to-noise ratio distribution with applications to biomedical signal analysis

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Abstract. This paper is motivated by a cutting-edge application in Biomedical Signal Analysis which deals with analyzing biomedical signals. We concentrate on the electroencephalogram (EEG), which traces the electrical activity from the brain. The phase angles of segments of the EEG has been used in the detection of evoked responses to rhythmic stimulation. One of the measure of detection is called the component synchrony measure (CSM) [1] which in terms of Directional Statistics, CSM is simply the well known squared mean resultant length $\bar{R}$ of the phase angles. Its value depends on the signal to the noise ratio of the EEG trace which has been proposed to model through normal errors for the spectrum [6]. We show that the model leads to a specific projected normal distribution ([2]) of the phase angles which we call projected normal signal-to-noise ratio (PNS) distribution. We give its several proprieties including the trigonometric moments. We study the sampling distribution of the resultant R from the PNS distribution but this distribution is complicated but for the von Mises case, it is well studied (see, for example, [5]). Hence, we give two approximations to the PNS distribution and we give its two plausible approximations to the von Mises distribution: one based on the moment based approximation and the second approximation uses the new approach based on Score Matching Approximation proposed in [3] and [4]. We show that these lead to a good approximation to the distribution of R from the PNS distribution. We study inference problems for the PNS distribution bearing in mind its application to the EEG traces. In particular, it is found that the maximum likelihood of the concentration parameter for the PNS distribution is intractable and we give some plausible approximations. We apply our methodology to analyse an EEG data. It is to be noted that the smaller the signal-to-noise ratio the longer the stimulation period should be in order to make the responses evident since a long period for a suitable detection may result in damage to the patient. This constitutes a limitation in monitored surgeries and we use our PNS distribution to give some strategies.

Keywords: component synchrony measure; electroencephalogram; phase angle; score matching estimation; von Mises distribution.

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AlphaFold's roots in directional statistics and probability kinematics

Thomas Hamelryck[1,*] and Kanti v. Mardia[2]

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Abstract. In 2024, John Jumper and Demis Hassabis received the shared Nobel Prize in Chemistry for their work on AlphaFold (AF), now routinely used for protein structure prediction. The seminal breakthrough of AF1 [8] in protein structure prediction by deep learning relied on a learned potential energy function, in contrast to the later end-to-end architectures of AF2 and AF3. While AF1's potential was originally justified in a heuristic way by analogy to physical *potentials of mean force* (PMFs), we reinterpret AF1's potential as an instance of *probability kinematics* (PK) [1], a form of generalized Bayesian updating. We build on the work done prior to AF by us and our collaborators [4, 7, 9, 3, 6], and refer to our recently posted preprint [5]. PK, a principled but underrecognised generalization of conventional Bayesian updating, accommodates uncertain or *soft* evidence in the form of updated probabilities over a partition. In the case of AF1 this partition is induced by evidence on pairwise distances between amino acids. This perspective reveals AF1's potential as a form of generalized Bayesian updating rather than a thermodynamic potential [4, 7, 3]: a *directional empirical prior* concerning dihedral angles is updated with evidence on distances, both parameterized by deep models. To confirm our probabilistic framework's scope and precision, we analyze a synthetic 2D model [5] in which a directional prior – an angular random walk prior based on the von Mises distribution – is updated with evidence on distances via PK, mirroring AF1's approach. Our theoretical and experimental analyses [5] connect AF1 to a broader class of well-justified methods [1, 4, 9, 3] that allow precise quantification, surpassing merely qualitative heuristics based on PMFs. We argue that PK offers a principled, quantative foundation for building *compositional* deep probabilistic models based on generalized Bayesian updating.

Keywords: AlphaFold; probability kinematics; potentials; deep learning; Bayesian updating.

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A versatile trivariate wrapped Cauchy copula with applications to toroidal and cylindrical data

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Abstract. In this talk, we propose a new flexible distribution for data on the three-dimensional torus which we call a trivariate wrapped Cauchy copula. Our trivariate copula has several attractive properties. It has a simple form of density and desirable modality properties. Its parameters allow for an adjustable degree of dependence between every pair of variables and these can be easily estimated. The conditional distributions of the model are well studied bivariate wrapped Cauchy distributions. Furthermore, the distribution can be easily simulated. Parameter estimation via maximum likelihood for the distribution is given and we highlight the simple implementation procedure to obtain these estimates. We compare our model to its competitors for analyzing trivariate data and provide some evidence of its advantages. Another interesting feature of this model is that it can be extended to a cylindrical copula. We illustrate our trivariate wrapped Cauchy copula on data from protein bioinformatics of conformational angles, and our cylindrical copula on climate data related to buoy in the Adriatic Sea. The talk is motivated by these real trivariate datasets, but we indicate how the model can be extended to multivariate copulas.

Keywords: angular data; copula; directional statistics; flexible modeling; wrapped Cauchy distribution.

Acknowledgments. The authors gratefully acknowledge JSPS KAKENHI Grant Number 20K03759 which supported SK and the Luxembourg National Research Fund which supported SL via the grant PRIDE/21/16747448/MATHCODA.

Expectation-Maximization for circular Cauchy distributions: a Compact representation

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Abstract. We present a straightforward representation of the Expectation-Maximization (EM) algorithm for wrapped Cauchy-type circular distributions, where the density function is expressed using a reciprocal formulation. In addition, we explore a distinct family of skew-symmetric circular distributions, differing from sine-skewed models, and derive a simplified form of the EM algorithm for this class. Furthermore, we extend our approach to finite mixture models, providing efficient EM algorithms for parameter estimation. Finally, we illustrate the effectiveness of these methods through applications to circular data sets.

Keywords: directional statistics; EM algorithm; sine-skewed circular distributions; vector epsilon algorithm.

Acknowledgments. This work was supported in part by Grant-in-Aid for Scientific Research (C) Grant Number 25K15031 and Grant Number 24K06849.

Optimal symmetry test on the torus

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Abstract. Several complex real-world data can be viewed as points on the hyper-torus, which is the cartesian product of circles. Over the past few years, this has motivated new proposals of distributions on the torus, both (pointwise) symmetric and sine-skewed asymmetric. In practice, it is relevant to know whether one should use the simpler symmetric models or the more convoluted yet more general asymmetric ones. So far, only parametric likelihood ratio tests have been defined to distinguish between a symmetric density and its sine-skewed counterpart. A new semi-parametric test is presented, a test which is valid not only under a given parametric hypothesis but also under a very broad class of symmetric distributions. A description of its construction and asymptotic properties under the null and alternative hypotheses will be presented. Using Stein's method, bounds for the rate of convergence of the test statistic are derived, and finite sample behaviour (through Monte Carlo simulations) will be given, as well as an application of the test on protein data.

Keywords: asymptotic theory; Le Cam; Stein's method.

Acknowledgments. This research was funded in whole, or in part, by the Luxembourg National Research Fund (FNR), grant reference PRIDE/21/16747448/MATHCODA.

FMM-based modeling of peak width and wave duration in oscillatory signals over the circular domain

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Abstract. Accurate estimation of peak width and wave duration (WD) remains a central challenge in the analysis of oscillatory signals. Conventional methods such as Full Width at Half Maximum (FWHM) often perform inadequately in the presence of asymmetric peaks, signal overlap, noise, and, critically, when data arise from multi-channel or phase-dependent sources. This paper introduces a novel framework grounded in directional statistics and periodic signal modeling, based on the Frequency Modulated Möbius (FMM) approach [1, 2, 3]. Specifically, we derive a parametric formulation of FWHM and introduce a new WD measure, defined over the circular domain. The proposed methodology respects the intrinsic periodicity of oscillatory signals, enabling the joint modeling of phase and morphology with a directional interpretation. It offers robustness to overlapping structures, interpretable parameterization, and adaptability to multi-channel contexts. We demonstrate the method's utility through two application domains: (1) ECG analysis, where the WD measure effectively delineates segments of cardiac waveforms that correspond to physiologically meaningful phases of the cardiac cycle; and (2) spectroscopy, where the FWHM estimator provides a precise quantification of peak spread, a key feature used to infer structural or compositional properties of the analyzed material. Extensive simulation and empirical evaluations confirm the method's precision, robustness, and superior performance over classical width estimators, particularly in multivariate contexts.

Keywords: FMM model; peak estimation; FWHM; ECG segments; XPS.

Acknowledgments. The authors gratefully acknowledge the financial support received by the AEI of the Spanish Ministry of Science, Innovation and Universities PID2023-147839OB-I00.

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Robust spherical regression in an extrinsic semiparametric framework

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Abstract. Robust regression for spherical data with Euclidean covariates is an important yet underexplored problem relevant to many fields. This talk will discuss an approach described in [4] that provides a flexible and efficient single-index framework (see [2]) for semiparametric regression in an extrinsic setting, enabling robust estimation through a suitable choice of loss function. Large-sample properties of the resulting estimators are established in a general setting. Moreover, analyzing robustness is complicated by the spherical geometry and semiparametric structure, where traditional tools such as the influence function become inadequate. We thus assess robustness using the standardized influence function; see [1], [3] and [5]. Special attention is given to the exponential squared loss (ESL), see [6], whose application to spherical regression appears to be new. We demonstrate that the ESL offers comparable efficiency and superior robustness over the least squares loss under high concentration. Theoretical analysis of its tuning parameter reveals a trade-off between efficiency and robustness, guiding its optimal choice. Simulations and applications to geochemical data validate the computational efficiency and robustness of our methods.

Keywords: directional statistics; exponential squared loss; extrinsic regression; M-estimation; single-index model; standardized influence function.

Acknowledgments. This research was supported by Australian Research Council grant DP2202102232.

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Quantiles and quantile regression on Riemannian manifolds: a measure-transportation-based approach

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Abstract. Increased attention has been given recently to the statistical analysis of variables with values on nonlinear manifolds. A natural but nontrivial problem in that context is the definition of quantile concepts. We are proposing a solution for compact Riemannian manifolds without boundaries; typical examples are polyspheres, hyperspheres, and toroïdal manifolds equipped with their Riemannian metrics. Our concept of quantile function comes along with a concept of distribution function and, in the empirical case, ranks and signs. The absence of a canonical ordering is offset by resorting to the data-driven ordering induced by optimal transports. Theoretical properties, such as the uniform convergence of the empirical distribution and conditional (and unconditional) quantile functions and distribution-freeness of ranks and signs, are established. Statistical inference applications, from goodness-of-fit to distribution-free rank-based testing, are without number. Of particular importance is the case of quantile regression with directional or toroïdal multiple output, which is given special attention. Extensive simulations are carried out to illustrate these novel concepts.

Keywords: optimal transportation; Riemannian manifolds; quantile regression.

Spherical Oja means

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Abstract. We propose a new class of location functionals for probability measures P supported on the unit hypersphere $\mathcal{S}^{d-1}$. The main idea is to minimize with respect to $\mu \in \mathcal{S}^{d-1}$ the expected (squared) ℓ-measure of the simplex with vertices $X_1, \ldots, X_\ell$ and μ, where the X_i's form a random sample from P. Here, $\ell \in \{1, \ldots, d\}$. For $\ell = 1$, it is easy to see that this provides the spherical mean functional, whereas for $\ell = d$, the resulting location functional is somehow related to the Oja median from [2]. Since the construction is based on an L_2 loss rather than an L_1 loss, this is actually closer to some of the location functionals introduced in [1] for Euclidean data. We leverage the methods introduced in [1] and [3] to study the properties of these new location functionals. One of the main issues is that, unlike in these earlier Euclidean works, the objective function to be minimized fails to be convex. Quite fortunately, it is possible to explicitly solve the non-convex optimization problem at hand, which is a key asset not only to study the theoretical properties of these functionals but also to evaluate these in the sample case.

Keywords: directional statistics; location measures; non-convex optimization problems; random simplices; spherical means.

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Hypothesis testing around the uniform distribution on spheres

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Abstract. One of the most classical problems in the field of directional statistics is considered in this talk, specifically the problem of testing uniformity on the unit hypersphere. Unlike approaches that limit themselves to detecting only specific types of alternatives, this talk explores a variety of tests that enable omnibus testing of uniformity. We delve into both their null and non-null behaviors, providing a comprehensive understanding of their applications and limitations. In the second part of the talk, we illustrate the fact that uniformity tests can be adapted and used to develop tests for other statistical problems.

Keywords: directional data; hypothesis testing, asymptotic inference.

Wrapped Cauchy copulas for bivariate directional densities

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Abstract. The density [1] is exploited to introduce a new bivariate copula that fulfills the periodicity conditions that are typically required in directional statistics. The copula depends on a single correlation parameter that modulates the distribution of the copula density on a torus. The proposal can be exploited to specify bivariate distributions that include one or two directional components. Maximum likelihood estimation and simulation of the proposal is straightforward, making it an excellent candidate as a building block of more complex models. The copula is illustrated in two environmental case studies involving cylindrical data [2] and mixed circular and axial data [3].

Keywords: copula; inference functions for margins; Kato-Pewsey model; mixture model.

Acknowledgments. This work has been supported by MIUR, grant number 2022XRHT8R - The SMILE project: Statistical Modelling and Inference for Living the Environment.

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On Jeffreys's cardioid distribution

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Abstract. Despite being one of the classical models for circular data, the cardioid distribution has been somewhat hard done by both in terms of the treatment it has received in the literature as well as the functionality supporting it in R. We summarize and extend published results on the model, and identify the scope and limitations of the existing support for it in R. The results from Monte Carlo experiments into the performance of trigonometric moment and maximum likelihood based approaches to point and interval estimation of the model's location and concentration parameters provide insight into which of the methods should be used in practice. A suite of reliable R functions for the model's practical application is available from the author.

Keywords: interval estimation; parametric bootstrap ; point estimation; R software; random number generation.

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Method for constructing tractable distributions on the cylinder and its application to time series modelling

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Abstract. Cylindrical data, which consists of linear and circular observations, appears in various scientific fields, such as biology, ecology, environmental science, sports science, and so on. However, there are few distributions for modelling cylindrical data. Some examples are listed in [1] and [2]. In this talk, a method for constructing cylindrical distributions is proposed. The construction is based on two univariate distributions, which become linear and circular marginal distributions of the constructed cylindrical distribution. The advantage of the method is its tractability, owing to the lack of extra normalizing constants and complicated distribution functions. The properties of the Fisher information matrix and estimation via the EM algorithm are also provided. Finally, the application of the constructed distribution to the hidden Markov model is considered, and its utility is shown by fitting to time series data.

Keywords: cylindrical distribution; EM algorithm; Fisher information; hidden Markov model.

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Adaptive estimation for nonparametric circular regression with errors in variables

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Abstract. We consider the problem of circular regression with errors in variables. We observe the i.i.d dataset

$$(Z_1, \Theta_1), \ldots, (Z_n, \Theta_n)$$

where

$$\Theta_l = m(X_l) + \zeta_l,$$

and

$$Z_l = X_l + \varepsilon_l,$$

with the responses Θ_l and the regression errors ζ_l lying $\in \mathbb{S}^1$, the unit circle of $\mathbb{R}^2$. We shall consider two scenarii: the first one when the covariates X_l belong to $\mathbb{S}^1$ (circular predictor case) and a second one when the X_l are in $[0,1]$ (linear predictor case). The predictors X_l are latent and the covariates errors ε_l are i.i.d unobservable random variables having a density f_ε, which characteristic function is assumed to be known (while the distribution of ζ_l need not to be known). The ε_l are independent of the Y_l and the X_l. We aim at estimating nonparametrically the circular regression function m from the i.i.d dataset $(Z_1, \Theta_1), \ldots, (Z_n, \Theta_n)$. We propose a data-driven estimation procedure based on a Goldenshluger-Lepski rule to select the resolution level of our estimators. To study the performances of our adaptive methodology (meaning that it does not need the specification of unknown smoothness parameters), we establish upper bounds for the pointwise risk when the regression function belongs to Hölder or Sobolev classes. The obtained rates reveal the specific nature of regression for circular responses and corrupted measurements which involve a deconvolution problem. Finally, a numerical study is conducted, illustrating the good performances of our approach.

Keywords: circular data; nonparametric regression; errors-in-variables models; deconvolution; adaptive estimation.

Extreme behavior in the frequency modulated Möbius periodic regression model

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Abstract.

The frequency modulated Möbius periodic regression model is an elegant and tractable model to describe how a real-valued response depends on a periodic explanatory variable, e.g. time with a daily cycle. The underlying idea is first to warp time using a Möbius transformation, and then to model the expected response as a cosine wave of the warped time. This talk contains two contributions for a deeper appreciation of the model.
First, the model contains two parameters to describe the shape of the response function, but their effects on shape are not very intuitive. Therefore, two new summary measures, *wave skewness* and *wave breadth*, are introduced to give a clearer appreciation of the range of possible shapes.
Second, it is possible to extend the regression model to include a *limiting* version (and a *beyond-the-limit* version). The price to be paid is that the response function has a singularity, either at a single time point (or an interval of time points, respectively). However, when fitting the extended regression model to a data set, any singularities will typically lie in between the observed time points, and hence be invisible to the user. Hence it is entirely feasible for the best-fitting model for a given data set to be at or beyond the limit.

Keywords: periodic regression; Möbius transformation; time warping; wave skewness; wave breadth.

Acknowledgments. Norah Almasoud thanks King Saud University for PhD sponsorship.

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Nonparametric density estimation for polyspherical data

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Abstract. In this talk, a nonparametric density estimator for data on the polysphere $\mathbb{S}^{d_1} \times \cdots \times \mathbb{S}^{d_r}$, with $r, d_1, \ldots, d_r \geq 1$, is presented. We derive the main asymptotic properties of the estimator, including mean square error, normality, and optimal bandwidths. We extend the kernel theory of the estimator beyond the von Mises-Fisher kernel by introducing more efficient alternatives, and we examine their normalizing constants, moments, and sampling methods. Plug-in and cross-validated bandwidth selectors are also obtained. As an application of the kernel density estimator, we propose a nonparametric k-sample test based on the Jensen–Shannon divergence, which is consistent against alternatives with non-homogeneous densities. Numerical experiments illustrate the superior performance of the k-sample test with respect to parametric alternatives in certain scenarios. Our smoothing methodology is applied to analyze the morphology of a sample of infant hippocampi represented via skeletal models (s-reps) embedded in the high-dimensional polysphere $(\mathbb{S}^2)^{168}$.

Keywords: nonparametric statistics; polyspherical data; skeletal representation; smoothing.

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Parametrically guided kernel density estimation on the sphere

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Abstract. We investigate hyperspherical kernel density estimators (KDEs) guided by a parametric distribution, with the goal of improving density estimation on the unit hypersphere ([3]). The key motivation is to reduce the bias of traditional non-guided KDEs ([1, 2]) by leveraging a suitable guiding distribution, while retaining comparable variance. Our methodology extends classical KDE frameworks by incorporating a von Mises-Fisher distribution as a guiding kernel. This approach exhibits robustness: even when the guiding distribution deviates significantly from the true underlying density, the proposed estimator performs on par with the classical KDE.

This phenomenon is particularly noteworthy in the hyperspherical context due to the compactness of the support, distinguishing it from analogous approaches for Euclidean data, where guidance can degrade performance under model misspecification. In addition, we address the crucial issue of data-driven bandwidth selection, which ensures practical applicability and optimal smoothing.

Through extensive simulations and real-data applications, we demonstrate that our guided estimator achieves superior finite-sample performance compared to standard KDEs and other recent hyperspherical density estimators. These results underscore the potential of guided KDEs in directional data analysis, especially when a reasonably informative parametric model is available.

Keywords: density estimation; kernel smoothing; parametric guide.

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Mathematical morphology on directional data

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Abstract. Mathematical morphology combines non-linear image processing and filtering techniques based on ideas from random set theory, topology, and stochastic geometry. It is widely used to process and analyse images of spatial structures, for example in the fields of geology, biology, and materials science.

We will concentrate on materials science applications and study fibre composites, which are nowadays used in a wide range of applications. Prominent examples include glass or carbon fibre-reinforced polymers which are used in the production of cars and airplanes. Furthermore, the development of fibre-reinforced high-performance concrete is an active area of research in civil engineering. When analysing the fiber direction distribution of such composite materials based on 3D image data, it is common to compute the local fiber direction in each fiber pixel. This results in images whose pixel values are unit vectors.

Transferring classical morphological operations such as erosion and dilation to this setting requires an ordering relation for unit vectors. This can be obtained by using depth functions which provide a centre-outward ordering with respect to a specified centre vector. Using a suitable depth function, we define morphological operators for directional images and apply them to synthetic image data. We then compare the results with those obtained using classical morphological operators for grey-scale images. Additionally, we present application examples from materials science.

Keywords: depth function; erosion; dilation; micro-computed tomography imaging; fiber directions.

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Non-euclidean statistics for learning structural biology

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Abstract. We address two problems of predicting ribonucleic acid (RNA) structure geometry on suite level (from one base to one of its adjacent bases) based on experimental maps of electron density. From such electron densities, at high resolution all atomic centers can be detected, except for hydrogen atom positions - as they give away their electron. For this and other reasons, reconstructed RNA geometry can feature clashes, i.e. placing atoms at positions that are chemically not possible. Further, at low resolution, location of the phosphate groups and the glyocosidic bonds can be determined from electron density maps but all other backbone atom positions cannot. The first problem addresses clash removal, the second prediction of high detail structure (all atomic locations).

We propose two learning methods, CLEAN [3] for clash removal and RNAprecis [4] for high detail prediction. This requires creating ground truth gold standard data bases of manually corrected suites. From these, structure clusters are identified by MINT-AGE [2], an adaptive iterative linkage preclustering method on metric spaces (AGE), mapping each precluster to a (stratified) sphere. Then dimension is reduced by torus PCA [1] combined with mode hunting (using the variant WiZer of the SiZer) on the resulting one-dimensional torus (MINT). We illustrate the power of both methods and observe, in particular, that MINT-AGE clusters are in high agreement with curated gold standard suite conformers, sometimes even finer.

Keywords: adaptive iterative clustering; torus PCA; shape spaces; procrustes means.

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Functional convergence of geometric functionals of Gaussian fields

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Abstract. We will consider random measures generated by smooth Gaussian fields (like e.g. the volume of level sets, or excursion areas) and establish *functional central limit theorems* for these objects, in the sense of the Skorokhod topology. One crucial example concerns *Berry's random wave model*, for which only partial results are available, because of a fundamental analytical obstacle, known as *Berry's cancellation phenomenon*. We will see that our results are tightly connected to a certain intepretation of *Berry's random wave conjecture* in quantum chaos theory, as well as to the high-resolution analysis of random fields on manifolds. Based in ongoing joint work with L. Gass (Luxembourg).

Keywords: Gaussian random fields; functional convergence; high-energy limit.

Reproducing kernel approach to low-dose tomographic data

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Abstract. Can we reliably estimate the mean and covariance structure of brain tumor patients by pooling data from low-dose radiographs, thereby reducing radiation exposure? In cryogenic electron microscopy (cryo-EM), can we recover biomolecular structures of i.i.d. specimens from a limited number of tilt series to improve individual resolution? These questions give rise to *heterogeneous* inverse problems, where the aim is to infer population-level statistics from indirect, noisy, and irregularly sampled observations. In this talk, we consider the tomographic operator as an operator between reproducing kernel Hilbert spaces (RKHS) and establish representer theorems to address the problem of mean and covariance estimation. We also present the uniform rates of convergence of our estimators with respect to our observation scheme, evaluating efficiency through simulation results across various tomographic configurations.

Keywords: RKHS; inverse problems; low-dose tomography.

Diffusions on the torus with exact likelihood inference

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Abstract. We provide a class of diffusion processes for continuous time-varying multivariate angular data with explicit transition probability densities, enabling exact likelihood inference. The presented diffusions are time-reversible and can be constructed for any pre-specified stationary distribution on the torus, including highly-multimodal mixtures. We give results on asymptotic likelihood theory allowing one-sample inference and tests of linear hypotheses for k groups of diffusions, including homogeneity. We show that exact and direct diffusion bridge simulation is possible too. A class of circular jump processes with similar properties is also proposed. Several numerical experiments illustrate the methodology for the circular and two-dimensional torus cases. The new family of diffusions is applied (*i*) to test several homogeneity hypotheses on the movement of ants and (*ii*) to simulate bridges between the three-dimensional backbones of two related proteins.

Keywords: angular data; diffusion bridges; stochastic processes; torus.

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Regression for spherical responses with linear and spherical covariates using a scaled link function

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Abstract.

We propose a regression model in which the responses are spherical variables and the covariates include both linear and spherical variables. A novel link function is introduced by extending the Möbius transformation on the sphere [3]. This link function is an anisotropic mapping that enables scale control along each axis of the spherical covariates and for each linear covariate. It generalizes the link functions proposed by Downs and Mardia [1] for circular covariates and by Fisher and Lee [2] for linear covariates, among others. Each parameter of the link function is clearly interpretable. For the error distribution, we adopt a general class of elliptically symmetric distributions, which includes the Kent distribution [4], the elliptically symmetric angular Gaussian distribution [5], and the scaled von Mises–Fisher distribution [6]. Orthonormal bases of the error distribution can be defined using parallel transport. Maximum likelihood estimation is feasible via reparameterization of the proposed model. Moreover, the parameters of the link function and the shape/scale parameters of the error distribution are orthogonal in the sense of the Fisher information matrix. The proposed regression model is illustrated using a real dataset.

Keywords: anisotropic mapping; elliptically symmetric distribution; link function; Möbius transformation; scaled von Mises–Fisher distribution.

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Regularized directional regression models: A simulation study

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Abstract. Circular regression models are widely used in several areas, including meteorology, biology, and geology. The issue of regularization in circular regression models remains an intriguing subject. Least-squares estimation has been the predominant approach for determining regression parameters in circular data. However, this approach does not produce accurate results in the presence of multicollinearity. This study focuses on response prediction and regularized circular regression models. Regularization techniques are extensively researched in linear regression but are comparatively underexamined in directional models. This study aims to examine the application and efficacy of regularization techniques within a directional regression framework, with the objective of enhancing the predictive performance. A simulation study is conducted to assess the proposed models.

Keywords: circular regression; circular-linear regression; machine learning; multicolinearity; shrinkage.

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A semiparametric Framework for Circular Regression with Application to Bird Orientation

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Abstract. We present a new semiparametric regression framework for modeling circular response variables influenced by linear or circular covariates. The model relies on a flexible parametric family of circular densities that accommodates asymmetry and varying peakedness around the modal direction. The modal direction and the concentration are allowed to change smoothly with the covariate through local polynomial modeling and kernel weighting. We establish asymptotic normality for the estimators of the modal direction and concentration, and provide an expression for the optimal bandwidth, together with a data-driven selection procedure. The proposed methodology is illustrated with an ecological application, where we analyze how migratory bird orientation responds to changes in altitude and wind direction.

Keywords: directional statistics; flexible modeling; local likelihood; modal direction regression.

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Generalized Laplace regression for cylindrical responses with an application to road traffic accidents

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Abstract. We propose a regression model for cylindrical response variables that arise in the analysis of events occurring randomly over time. We investigate about 134,000 road traffic accidents from the Fatality Analysis Reporting System (FARS) of the U.S. National Highway Traffic Safety Administration, between 2005 and 2020. We define the circular component as the time of the day at which the accident occurs, and the linear component as the gravity of the event. The latter comprises an overall score for the severity of the injuries experienced by the persons involved in the accidents, along with the median age of the victims. Our analysis shows that the timing and severity of the accidents are influenced by temporal and geographical factors.

We use the multivariate generalized Laplace (GL) distribution, whose parameters make this model more flexible than—as well as a generalization of—the multivariate Gaussian model. The GL has density

$$f_{GL}(\boldsymbol{y}) = \frac{2\exp\{\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{y}-\boldsymbol{\delta})\}}{(2\pi)^{d/2}\Gamma(\alpha)|\boldsymbol{\Sigma}|^{1/2}} \left(\frac{Q(\boldsymbol{y},\boldsymbol{\delta})}{P(\boldsymbol{\Sigma},\boldsymbol{\mu})}\right)^{\omega} \mathcal{K}_{\omega}\left(Q(\boldsymbol{y},\boldsymbol{\delta})P(\boldsymbol{\Sigma},\boldsymbol{\mu})\right), \tag{1}$$

where $\boldsymbol{y} \in \mathbb{R}^d$ and $\boldsymbol{y} \neq \boldsymbol{\delta}$, $\boldsymbol{\delta} \equiv (\delta_1,\ldots,\delta_d)^\top \in \mathbb{R}^d$ is the location parameter, $\boldsymbol{\Sigma}$ is a positive-definite $d \times d$ matrix related to the scale of the distribution, $\boldsymbol{\mu} \equiv (\mu_1,\ldots,\mu_d)^\top \in \mathbb{R}^d$ is the skewness parameter, $\alpha > 0$ controls the behavior of the tails and kurtosis, $\omega = \alpha - \frac{d}{2}$, $\mathcal{K}_u$ is the modified Bessel function of the third kind with index u, $Q(\boldsymbol{y},\boldsymbol{\delta}) = \sqrt{(\boldsymbol{y}-\boldsymbol{\delta})^\top\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}-\boldsymbol{\delta})}$, and $P(\boldsymbol{\Sigma},\boldsymbol{\mu}) = \sqrt{2+\boldsymbol{\mu}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}$. Let $\boldsymbol{W} \equiv (W_1,\ldots,W_K)^\top \in \mathbb{R}^K$ be a random vector of K continuous variables measuring the intensity of the event—these variables may represent, for example, human, economic, and environmental losses. Also, let $\Theta \in (-\pi,\pi]$ be the circular random variable representing the timing of such an event. We assume

$$\begin{aligned} &\boldsymbol{Y} = \boldsymbol{\delta} + \boldsymbol{\varepsilon}, \qquad \boldsymbol{\varepsilon} \sim GL_{K+2}(\boldsymbol{0},\boldsymbol{\Sigma},\boldsymbol{\mu},\exp(\tau)), \qquad \boldsymbol{\delta} = \boldsymbol{X}^{(\delta)}\boldsymbol{\beta}_{\delta}, \\ &\boldsymbol{\mu} = \boldsymbol{X}^{(\mu)}\begin{bmatrix}\boldsymbol{\beta}_{\mu}\\ \boldsymbol{0}_q\\ \boldsymbol{0}_q\end{bmatrix}, \qquad \tau = \boldsymbol{z}^\top\boldsymbol{\beta}_{\tau}, \end{aligned} \tag{2}$$

where $\boldsymbol{Y} = (\boldsymbol{W}^\top,\boldsymbol{S}^\top)^\top \in \mathbb{R}^{K+2}$, $\boldsymbol{S} = (R\cos\Theta, R\sin\Theta)^\top$, and $R = \|\boldsymbol{S}\|$. We take $\boldsymbol{X}^{(\delta)} = \boldsymbol{I}_{K+2} \otimes \boldsymbol{x}^\top$ and $\boldsymbol{X}^{(\mu)} = \boldsymbol{I}_{K+2} \otimes \boldsymbol{z}^\top$, where $\boldsymbol{x}$ is a p-dimensional vector with first element equal to 1 and $\boldsymbol{z}$ is a q-dimensional vector with first element equal to 1, with $q < p$. For estimation purposes, we re-write (2) using the scale-mixture representation of the GL, that is

$$\boldsymbol{Y} = \boldsymbol{\delta} + V\boldsymbol{\mu} + \sqrt{V}\boldsymbol{U}, \tag{3}$$

where $V \sim G(\alpha, 1)$ is standard gamma with shape $\alpha = \exp(\tau)$ and $\boldsymbol{U} \sim N_{K+2}(\mathbf{0}, \boldsymbol{\Sigma})$ is multivariate normal. After marginalizing the distribution of $\boldsymbol{Y}|V$ with respect to R (which yields the projected normal), we apply a maximum likelihood estimator with Gaussian quadrature over the distribution of V.

Keywords: Gaussian quadrature; generalized hyperbolic; projected normal; scale-mixture of normals.

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Reducing boundary bias on curved support via domain transformation

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Abstract. Often, random variables take values within a bounded support. Local density estimation methods suffer from severe bias near the support boundaries due to the well-known *density overflow*. A standard solution in the unidimensional case is the reflection method (see [2]). The problem of boundary bias becomes much more challenging in the multivariate case, as the support of the population models can exhibit highly varied and complex shapes, and the issue becomes much harder as the dimensionality increases.

Curved surfaces hinder the use of the reflection method. They lack a simple reflection symmetry, so reflecting points across a curved surface would distort the local structure of the data, leading to inaccurate density estimates. However, simple domain transformations could favor its application.

Among the possible curved surfaces, we focus on a disc of radius R, say $\mathbb{D}$, centred on $(0,0)$ and parallel to xy plane, i.e. $\mathbb{D} = \left\{(r^{\mathbb{D}}, \theta^{\mathbb{D}}) \mid 0 \leq r^{\mathbb{D}} \leq R, 0 \leq \theta^{\mathbb{D}} < 2\pi\right\}$, with $\theta^{\mathbb{D}}$ and $r^{\mathbb{D}}$ being, respectively, the angle and the radial distance representing a generic observation on $\mathbb{D}$. On the basis of the homeomorphism between a disk and the surface of a hemisphere, we transfer data onto a cup of a sphere of radius R and we employ the equator-reflected density estimator. This transformation resembles, in spirit, the principle adopted by [1]. The last step consists on of back-transforming the obtained density estimate onto the disk. We discuss the asymptotic properties of the estimator and present some numerical examples.

Keywords: bounded support; curved surface; density overflow; reflection.

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High volatility flexible families of probability distributions for directional Data

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Abstract. Modern data in many areas of applied sciences are characterized by high volatility, e.g. long tails a well as high kurtosis or peakedness. They also exhibit asymmetry and multimodality. Classical families of distributions with well-defined probability density functions are inadequate to model such data. On the other hand, such families as may be good contenders for modelling such data consist of members which do not in general admit of probability density functions on the Euclidean manifold. But these families often can be mapped onto non-Euclidean manifolds for which the densities can have elegant Fourier series representations. This connection also immensely helps in statistical inference for the associated parameters. Also, on their own, these families provide ample flexibilities in modelling data on non-Euclidean manifolds. We first develop the related theories for deriving these highly flexible families. Next, optimal inference procedures are outlined. Finally, emerging real-life examples from directional data are presented to illustrate the results.

Keywords: Fourier series; high volatility; probability distributions on manifolds.

Abstracts: poster presenters

Copula-based hidden semi-Markov models for cylindrical time series

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Abstract. Bivariate sequences of angles and intensities are often referred to as cylindrical time series, because they can be represented as series of points on a cylinder. Traditionally, their analysis is hampered by the wrap-around geometry of the circle and by the scarce availability of parametric models that capture circular-linear dependence. Here, we introduce a new, highly flexible family of cylindrical distributions, building upon a recently proposed circular copula for axial-circular data [2]. The proposed copula relies on the bivariate wrapped Cauchy distribution [3] and yields densities that (i) satisfy the required periodicity, (ii) admit any choice of linear and circular marginals, (iii) are indexed by a single correlation parameter that varies smoothly on $[-1, 1]$. As it is, the copula is able to fit cylindrical data when they are in the form of independent and identically distributed observations. However, sequences of cylindrical data are often heterogeneous with a distribution that dynamically varies across different regimes, introducing multi-modality and complex auto-correlation structures. In particular, time series of wave heights and directions, which motivated this study, provide a typical example of cylindrical observations where not only the data distribution dynamically varies across different latent sea regimes, but the duration of these regimes (dwell times) also may vary according to time-varying weather conditions. We therefore integrate the proposed copula-based cylindrical densities within a hidden semi-Markov model [1], extending previous approaches based on cylindrical hidden Markov models [4, 5]. Under the hidden semi-Markov framework, the data are modeled as a mixture of cylindrical distributions whose parameters evolve according to the states of a latent chain. Unlike standard hidden Markov models, here the evolution of the latent chain depends on its history since the last transition, allowing for flexible, non-geometric dwell time distributions. This flexibility enables the model to capture multi-modal behavior, non-geometric persistence, and complex circular–linear dependence within a unified framework, providing a principled tool for analyzing heterogeneous cylindrical data across diverse application fields.

Keywords: copula; hidden semi-Markov models; cylindrical data; sea regimes.

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Bias reduced kernel estimator with contaminated angles

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Abstract. We aim to estimate circular probability densities when angles are affected by measurement errors, typically arising from instrument degradation, suboptimal calibration, and intrinsic limitations in the precision of data acquisition methods (see [1] and [2] for more details). As a consequence, in this so-called errors-in-variables problem, nonparametric methods need to account for an additional source of bias, beyond the usual bias typical of local methods. To manage this challenge, we propose a deconvolution-based estimator that employs lower bias obtained starting from a second-sin order kernel. The asymptotic behaviour of the proposed estimator is examined, and simulation experiments are presented to evaluate the performance of the proposed methods. As a real case study we analyse a dataset about ant movement directions, which has been typically used as a benchmark in previous research on circular density estimation under measurement error.

Keywords: ant movement direction; circular deconvolution; Fourier coefficient; lower bias kernel.

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The dynamic torus: with an application to wind and wave direction in the Adriatic

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Abstract. This paper develops a novel score-driven time series framework for modelling bivariate circular data lying on the torus. Extending the dynamic univariate circular model of [1], the proposed model accommodates time-varying location, concentration, and association parameters within a bivariate von Mises distribution—specifically the sine formulation of [2]. The dynamic updating is driven by the conditional score, ensuring robust use of the information contained in the data. The methodology is illustrated using half-hourly data on wind and wave direction collected from a the Adriatic Sea. Empirical results reveal complex and evolving patterns of co-movement between wind and wave directions, emphasizing the value of dynamic modelling in environmental applications. The model is flexible and offers a foundation for further developments in the analysis of multivariate directional time series data.

Keywords: bivariate circular data; circular correlation; directional statistics; score-driven model; heteroscedasticity; von Mises distribution.

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A Hidden Markov Model with the Weibull-extended sine-skewed von Mises distributions as emission densities

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Abstract. Bivariate data such as wind direction and wind speed consist of an angular component and a non-negative linear component, and are commonly referred to as cylindrical data. These data often exhibit temporal dependence, and their underlying distributional structure may vary over time. Several modeling approaches for such data have been proposed, notably by [1]. However, the marginal angular distributions in such models can, in some cases, exhibit substantial skewness. When such skewness is present, maximum likelihood estimates based on existing models may lie near the boundary of the parameter space, indicating that the models fail to capture the skewness adequately. To address this limitation, we propose a hidden Markov model whose emission densities are defined on the cylinder and possess marginal angular distributions capable of accommodating stronger skewness. In particular, we consider the following cylindrical distribution as the emission density:

$$f^{(q)}_{\text{WeiESSvM}}(\theta,x) = \frac{\alpha\beta^{\alpha}}{\pi\cosh(\kappa)} G_q(\lambda\sin(\theta-\mu))\, x^{\alpha-1} \exp\left\{-(\beta x)^{\alpha}\left(1-\tanh(\kappa)\cos(\theta-\mu)\right)\right\}, \qquad (-\pi \le \theta < \pi, \quad x \ge 0)$$

where the parameters satisfy $\alpha > 0$, $\beta > 0$, $\kappa > 0$, $-\pi \le \mu < \pi$, and $-1 \le \lambda \le 1$. q is a hyperparameter and the function $G_q(x)$ is defined as the cumulative distribution function of the probability density

$$g_q(x) = \frac{\Gamma(2(q+1))}{2^{2q+1}\Gamma(q+1)^2}(1-x^2)^q, \qquad (-1 \le x \le 1),$$

where $\Gamma(\cdot)$ stands for the Gamma function. We demonstrate the effectiveness of the proposed hidden Markov model by applying it to real wind direction and speed data observed near the coast of Japan. The performance is evaluated and compared with that of existing models using information criteria.

Keywords: cylindrical distributions; hidden Markov Models; skew modeling; wrapped Cauchy distributions.

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Regularized directional regression models: An environmental data application

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Abstract. In the context of circular regression, existing literature often overlooks potential correlations between regressors and fails to address multicollinearity among the predictors. Application of existing methods in the presence of significant multicollinearity between regressors produces conceptually erroneous estimates and reduces the model's generalisation performance. To mitigate this problem, regularization techniques are proposed. Regularization techniques are well-studied in linear regression but underexplored in directional regression models. An application focussing on environmental data is considered to analyse the application and effectiveness of regularization methods within a directional regression framework, with the aim of improving prediction performance. Overall, this study contributes a more robust methodology for performing circular regression in the presence of multicollinearity.

Keywords: circular regression; circular-linear regression; environmental data; machine learning; multicollinearity.

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A likelihood ratio test for circular multimodality

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Abstract. The existence of multiple modes in a distribution is intimately related with the presence of different clusters or groups within the population. Thus, assessing the number of modes of a distribution provides useful information about the inner structure of the population. This idea has been shown to be useful in some practical applications. For example, [1] follow this approach to analyze how human activity alters fire seasonality.

For the particular case of circular densities, we address the problem of testing if, given an observed sample of a random angle, the underlying circular distribution model is multimodal. Our work is motivated by the analysis of migration patterns of birds previously performed by [3] the methodological proposal follows a novel approach based on likelihood ratio ideas, combined with the concept of critical bandwidth introduced by [4]. The behaviour in practice of the new test is illustrated with simulation examples and its performance is compared with its main competitor in the literature. Further simulation studies and theoretical details can be found in [2].

Keywords: animal orientation; circular data; critical bandwidth; likelihood ratio test; multimodality.

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Model-based clustering using a new mixture of circular regressions

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Abstract. Regression models with circular response variables arise frequently in fields such as biology, geology, and meteorology. These models typically assume that the conditional distribution of the response follows a von Mises distribution. However, this assumption falls short when the response is multimodal. To address this limitation, we propose a finite mixture of regression models for circular response data, incorporating both circular and/or linear covariates. Mixture models are particularly effective when the underlying population exhibits multimodality, yet their application in circular regression settings remains largely unexplored. This paper seeks to fill that gap by introducing a novel modeling framework. Estimation is carried out via maximum likelihood, using the Expectation-Maximization (EM) algorithm. We conduct a comprehensive simulation study to evaluate the performance and practical utility of the method. Additionally, the model serves as an effective model-based clustering approach. The proposed methodology is further illustrated using real wind direction data from a wind farm in South Africa.

Keywords: mixtures of von-Mises; circular data; model-based clustering; Expectation-Maximization algorithm; circular regression.

Bias correction for kernel density estimation with spherical data

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Abstract. Kernel density estimation on the sphere allows for flexible modeling of directional data, including rotationally symmetric, skewed, and multimodal distributions. However, standard Kernel density estimation methods, which are typically based on rotationally symmetric kernels such as the von Mises kernel, suffer from bias, and their mean integrated squared error (MISE) lacks root-n consistency, and increasing the dimension slows its convergence rate [1]. To address this, we propose a bias correction approach using generalized jackknifing techniques [2] derived from rotationally symmetric kernels such as the von Mises kernel [3]. We derive the asymptotic MISE of the proposed estimators and show that their convergence rates improve upon those of conventional methods. Simulation results confirm that the proposed estimators outperform standard the von Mises kernel density estimation, particularly for finite samples drawn from mixtures of von Mises distributions.

Keywords: spherical data; kernel density estimation; bias correction; nonparametric method.

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Testing uniformity on the sphere: from pairs to m-tuples

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Abstract. When uniformity on the sphere is tested, most classical tests employ a V-statistic of second degree investigating the relation between pairwise observations. In this work, we propose two new classes of uniformity tests based on U- and V-statistics of arbitrary degree m that investigate the interaction between m-tuples of observations. We prove that they extend the well-known Sobolev class of uniformity tests, arising for $m = 2$. The computation of the new class of V-statistics is shown to be manageable even for large degrees and sample sizes, and we provide closed-form circular test statistics that extend classical tests for $m = 2$. We study the asymptotic null distribution for these m-tests, its usability in practice, and its asymptotic behavior under local alternatives. Furthermore, we investigate the impact of m on the rotational invariance of the proposed test statistics and introduce a modified construction that enforces this invariance for all $m > 2$, with closed expressions and known asymptotic distribution in the circular case. Through simulations, we show that tests with $m > 2$ lead to increases in power, compared to $m = 2$, for certain scenarios.

Keywords: circular data; spherical data; uniformity tests.

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Insights into the construction of an alternative bivariate cardioid distribution

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Abstract.

Bivariate circular data arise across a range of scientific disciplines, from environmental sciences to molecular biology. Recent developments in cardioid-type models have expanded the flexibility of marginal behaviour and dependence structures in such settings [2]. Building on the mixture framework of [1], this work constructs a new class of bivariate circular distributions by replacing the classical cardioid kernel with the power-cardioid distribution [4]. Within this framework, dependence is induced through shared latent variables, with the concentration parameter effectively modelled probabilistically via beta and bivariate beta mixing distributions. These choices allow for tractable inference, hierarchical simulation, and meaningful extensions beyond two dimensions. We explore the impact of these mixing mechanisms on the resultant bivariate power-cardioid models, highlighting theoretical aspects and demonstrating performance through application to circadian gene expression data from mouse heart and liver tissue [3]. Numerical illustrations show that the proposed models can capture complex dependence structures effectively, with improved model fit compared to existing bivariate cardioid alternatives.

Keywords: bivariate circular data; power-cardioid distribution; mixture models; concentration mixture.

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Asymmetry Analysis of Bilateral Shapes

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Abstract. Many biological objects possess bilateral symmetry about a midline or midplane, up to a "noise" term. We use landmark-based methods to investigate departures from bilateral symmetry, especially for paired data and two-group problems, where one group is assumed to be more symmetric than the other one. Our work is formulated under the size-and-shape framework including registration under rigid-body motion. We then construct elementary asymmetry features at individual landmark coordinates for each object. There are two approaches for comparing bilateral symmetry: (1) combine-then-compare, for which the elementary asymmetry features are combined to a non-negative scalar, then standard univariate tests are used to test for asymmetries; (2) compare-then-combine, for which a univariate test statistic is constructed for each feature, then the maximum of these test statistics leads to an overall test statistic under the union intersection test framework. Important features can also be extracted from the second approach. We illustrate our methods using two spatio-temporal landmark datasets: (1) Smile Data 1, which consists of normal subjects and cleft subjects who have undergone surgery; (2) Smile Data 2, which contains data of the same subjects before and after the orthognathic surgery. Both datasets contains trajectories of 3D landmarks over the face for various subjects as they smile (from close lip to maximum open lip smile). Three frames are mainly used: first frame (close lip), middle frame (middle of the smile) and last frame (maximum open lip smile). The aim is to assess the success of these two kinds of surgeries using Smile Data 1 and Smile Data 2.

Keywords: asymmetry measures; cleft lip surgery; shape analysis; size-and-shape analysis of smile.

Robust measures of circular dispersion with an outlier detection rule

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Abstract.

An important issue to consider when dealing with circular data is how to measure their dispersion. This work addresses such an issue from a robust statistical perspective. Particularly, it considers three well-worn robust measures of dispersion for the analysis of linear data (the Median Absolute Deviation, the Least Median Spread, and the Least Trimmed Spread), and it discusses their extension to the case of circular variables. Their properties are studied and their influence functions are computed. Exploiting their relationship with the concentration parameter κ of the von Mises distribution, they are utilized to define three new robust estimators of such a κ. Finally, the estimators are used in robust anomaly detection.

Keywords: relative bias; influence function; von Mises.

Estimation and classification for a folded directional distribution

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Abstract. When directional data are restricted to the positive orthant of the unit hypersphere, a folded directional distribution is preferable to a simple directional distribution. This type of data arises when compositional data are transformed into directional data using the square root transformation, for instance. When this transformation is applied, the signs of the vector components are unknown, therefore the resulting data can be modeled using a folded directional distribution. Since the Watson distribution is widely used for modeling axial data (see, for example, [2] and [1]), we consider a folded Watson distribution for such cases. First, we consider the maximum likelihood estimation for the folded Watson distribution. Due to the complexity of its probability density function, it is not possible to maximize the likelihood function directly. Instead, we apply an Expectation-Maximization (EM) algorithm, treating the signs of the vector components as missing values, following an approach similar to [3]. Through simulation, we analyze the properties of the concentration parameter estimator and conclude that it exhibits lower bias compared to the maximum likelihood estimator for the Watson distribution. Next, we develop a Bayes classification rule for the folded Watson distribution to assign observations to predefined groups. We assess the performance of this rule using data simulated from two folded Watson distributions, considering both known and unknown parameters across various hypersphere dimensions and parameter values. When the parameters are known, the results show that the folded Watson classification rule outperforms the Watson rule in all cases considered. Furthermore, when the parameters are unknown, estimating them via maximum likelihood does not substantially affect classification accuracy. Finally, we apply the classification rules to spherical data.

Keywords: Bayes classification rule; directional data; EM algorithm; folded distribution; Watson distribution.

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Unconstrained kernel density estimation on the sphere

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Abstract. All the existing kernel density estimators on the sphere are constrained, in the sense that they are based on an isotropic kernel indexed by a single smoothing parameter ([2, 3]). However, for Euclidean data, the use of unconstrained (or anisotropic) kernels has been shown to result in sensible gains in performance (see [4]). Here, we take advantage of recent advances regarding the anisotropic geodesic normal distribution on the sphere ([1]) to propose a novel unconstrained kernel density estimator for spherical data, and compare its performance against the classical constrained counterparts.

Keywords: anisotropic; geodesic normal distribution; kernel density estimation; spherical data.

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Nonparametric estimation for censored circular data

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Abstract. We are interested in a density estimation problem for censored circular data. Our censorship model was introduced by [1] is the following. Let X be a random variable with values in the circle $\mathbb{S}^1$ with probability density function f to estimate, (L,U) a couple of random variables with values in $\left(\mathbb{S}^1\right)^2$ such that X is independent from the couple (L,U). X is observed if $X \in [L,U]$ and we do not observe it otherwise (we choose an arbitrary value outside of $[0,2\pi[$ in those case, here we took $-\pi$). We write $\Delta = 1_{\{X\in[L,U]\}}$ and $X' = \Delta X - \pi(1-\Delta)$. Then our observations are a sample of n iid triplets (X_i', L_i, U_i). We manage to implement an adaptive estimator (meaning that it does not need the specification of the regularity parameter of f). Our adaptive estimation procedure is done by contrast minimization based on a weighted $\mathbb{L}^2$ scalar product (weighted by the function σ define by $\sigma : x \in \mathbb{S}^1 \mapsto \mathbb{P}(x \in [L,U])$) and upon trigonometric spaces. We consider the minimax setting to establish our theoretical results. First, for the MISE and Sobolev classes of order β, we show that our estimator achieves the classical nonparametric rate to estimate a density of regularity β in dimension 1. Moreover, we prove a matching lower bound which ensures that our data-driven estimator is minimax optimal in this censored circular setting. Finally, simulation illustrate the good performances of our estimation procedure.

Keywords: nonparametric estimation; circular data; censored data; adaptive estimator; lower bound.

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Stein's method of moments on the sphere

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Abstract. We use Stein characterizations to obtain new moment-type estimators for the parameters of three classical spherical distributions (namely the Fisher-Bingham, the von Mises-Fisher, and the Watson distributions) in the i.i.d. case. This leads to explicit estimators which have good asymptotic properties (close to efficiency) and therefore provide interesting alternatives to classical maximum likelihood methods or more recent score matching estimators.

Keywords: point estimation; Fisher-Bingham distribution; Stein's method.

Rolled Gaussian process models for curves on manifolds

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Abstract. Curves on manifolds arise as data in various applications, but with few available statistical models suited to them. The main obstacle is the nonlinearity of manifolds, which makes it difficult to specify useful models. We use the geometrical operations of "rolling" and "wrapping" paired with a Gaussian process model in Euclidean space to define a natural analogue of the Gaussian process model for curves on a manifold. The model is both generative and amenable to statistical inference given observed curve data. In this approach we are able to: (i) to use rolling and wrapping with respect to coordinate frames and specify how a Gaussian process model in $\mathbb{R}^d$ can be identified with a "rolled Gaussian process" model on M, and establish its invariances and equivariances to choice of frames; (ii) to establish conditions for the Fréchet mean of the rolled Gaussian process model to equal the rolling of the mean of the Gaussian process in $\mathbb{R}^d$; (iii) to prescribe a finite-dimensional parameterisation of the rolled Gaussian process model, with statistical estimators and tests for the model parameters, in the practically useful setting when sample curves are observed at a finite set of times; and (iv) to establish conditions under which estimators of the parameters defining the mean curve consistently estimate their population counterpart. We illustrate with examples on the unit sphere, symmetric positive-definite matrices, and with a robotics application involving 3D orientations.

Keywords: Fréchet mean; functional data; gaussian processes; parallel transport; manifolds.

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Testing uniformity of orbits of binary stars

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Abstract. If the planes of binary star orbits have a common alignment then this may be a signature of a common origin, a remnant of the angular momentum of the gas cloud where they formed. Alternatively, it may indicate a dynamical process which brings binary star systems slowly into common alignment. Uniformity of poles (axes normal to orbits) would support either a more chaotic origin within gas clouds, or a subsequent mixing of any initial orbital alignments among stars with different birthplaces. Observations are often limited by an ambiguity: the direction of an orbital pole cannot be distinguished from its reflection in the plane of the sky. Tests of uniformity are presented here that are modifications of the Sobolev tests [2] on the sphere. The tests are applied to data from a standard catalogue of orbits of visual binary stars. Despite the wide scattering of orbital poles, there is consistent evidence of a lack of uniformity, and some evidence of a common alignment of orbits of binaries that are more than 20 parsecs from the Sun. Details are given in [1].

Keywords: Riemannian manifolds; uniformity on sphere; variance bound.

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