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Unimodular hunting

Gaëtan Chenevier

Laboratoire de math. Orsay, CNRS

gaetan.chenevier@math.cnrs.fr

I. Introduction

Pb: Classify the integral unimodular lattices $L \subset \mathbb{R}^n$ modulo isometries. ($\forall x, y \in L,\ x \cdot y \in \mathbb{Z}$ and covol $L = 1$)

$$\rightsquigarrow \quad X_n = O(\mathbb{R}^n) \backslash \mathcal{L}_n$$

$\leftarrow$ set of all i.u.l. in $\mathbb{R}^n$

$$\simeq \{ M \in M_n(\mathbb{Z}) \text{ sym. def. pos; } \det M = 1 \} / GL_n(\mathbb{Z})$$

Examples: $I_n = \mathbb{Z}^n$, E_8, Leech...

General: X_n finite, known "mass" $\sum_{L \in X_n} 1/|O(L)|$

Representatives of X_n are known up to $n = 25$ by works of Lagrange, Gauss, Mordell, Witt, Kneser, Niemeier, Conway & Sloane, Borcherds

n	≤ 7	≤ 11	≤ 13	14	15	16	17	18	19	20	21	22	23	24	25
$\# X_n$	1	2	3	4	5	8	9	13	16	18	40	68	117	297	665

<u>Theorem A</u> $\# X_{26} = 2566$, $\# X_{27} = 17059$, $\# X_{28} = 374\,062$

(with Bill Allombert)

with concrete representatives in each case

$\longrightarrow$ see my homepage for lists!

. Pf relies of course on computer calculations (PARI/GP) parallel GP

CPU time: about 1 month $n=26$, 1 year $n=27$, 72 y. $n=28$!

. Fortunately, much easier and shorter to check that the lists are complete a posteriori: 5h., 40 h. & 27 d.!

• All i.u.l. in rank $n \leq 28$ turn out to be distinguished by their config. of vectors v of "norm" $v.v \leq 3$.

• For each such L, we may and do compute $|O(L)|$ using Plesken-Souvignier algorithm.

• Conclude checking $\sum_i \frac{1}{|O(L_i)|} = \text{mass}(\mathcal{L}_n)$

II Kneser neighbors

All our lattices will be given as d-neighbors of I_n.
Idea to do so already present in former work of Bacher, Venkov.

Definition: Let $L \in \mathcal{L}_n$ and $d \geq 1$. A (cyclic) d-neighbor of L is an $N \in \mathcal{L}_n$ with $L/_{L \cap N} \simeq \mathbb{Z}/d$

Fact: Easy to construct and parameterise in terms of the finite quadric $C_L(\mathbb{Z}/d) = \{ \ell \subset L \otimes \mathbb{Z}/d \mid \ell \cdot \ell \equiv 0 \ (ed) \}$

$\ell \simeq \mathbb{Z}/d$

$$e = \begin{cases} 1 & d \text{ odd} \\ 2 & d \text{ even} \end{cases}$$

We have an $e:1$ surjective map $\left\{ \begin{matrix} d\text{-neighb.} \\ \text{of } L \end{matrix} \right\} \xrightarrow{\ell} C_L(\mathbb{Z}/d)$

given by $\ell(N) := dN/dL$.

<u>Case $L = I_n$</u> $\quad x \in \mathbb{Z}^n$, $\sum_{i=1}^{n} x_i^2 \equiv 0 \ (ed)$, d-prim.

The d-neighbors of I_n with line $\langle x \rangle \subset I_n \otimes \mathbb{Z}/d$

are the $\quad N_d(x) = M_d(x) + \mathbb{Z}\dfrac{\tilde{x}}{d}$

where $\begin{cases} \tilde{x} \in \mathbb{Z}^n \text{ arbitrary s.t. } \tilde{x} \equiv x \ (d) \text{ and } \tilde{x}\cdot\tilde{x} \equiv 0 \ (d^2) \\ M_d(x) = \left\{ v \in \mathbb{Z}^n \mid \sum_i v_i x_i \equiv 0 \ (d) \right\} = I_n \cap N_d(x) \end{cases}$

$\longrightarrow$ very compact, often beautiful, def. of i.u.l.

<u>Ex:</u> $E_8 \cong N_2(1^8)$, $N_2(1^n)$ for $n \equiv 0 \mod 4$

$\text{Leech} \cong N_{94}(1,3,5,\dots,47)$ (Thompson, $1^2+2^2+3^2+\dots+23^2 \equiv 0 \mod 47$)

<u>Explanation</u>: Leech is the unique $L \subset \mathbb{R}^{24}$, up to isometry, with L <u>even</u> and $R(L) := \{v \in L \mid v.v = 2\}$ empty.

($x.x \in 2\mathbb{Z}\ \forall x \in L$)

"roots", always ADE root system

<u>exercise</u> $N_d(x)$ even $\Longleftrightarrow$ x_i odd $\forall i$ and $\sum x_i^2 \equiv 0 \mod 4d$ if $d \equiv 2\ (4)$

Roots of $N_d(x)$? We always have the inclusion

$$R(N_d(x)) \supset R(M_d(x)) = \{ \pm\varepsilon_i \pm \varepsilon_j \mid \pm x_i \pm x_j \equiv 0 \ (d) \}_{i \neq j}$$

So $N_d(x) \simeq \text{Leech} \Rightarrow x_i \not\equiv \pm x_j \ (d) \ \forall i \neq j \Rightarrow d \geq 94$

and for $d = 94$ unique poss. $\{\pm 1, \pm 3, \pm 5, \ldots, \pm 47\}$

this possibility actually works ! (magic ?)

$\longrightarrow$ good moment to have a look at lists.

III. Why does it work (in theory!)

Consider more generally any spinor genus $\mathcal{G}$ of integral lattices in $\mathbb{R}^n$, with $n > 2$. Main examples here are $\mathcal{G} = \mathcal{L}_n^{odd}$ or $\mathcal{L}_n^{even}$

Theorem (Hsia-Jöchner 97) For all L and L' in $\mathcal{G}$, there exists ∞ many primes p s.t. L' is isometric to a p-neighbor of L.

Our second main result is:

Theorem B For all $L, L' \in \mathcal{G}$

$$\#\left\{ \begin{array}{l} N \ p\text{-neighbor of } L \\ \text{with } N \simeq L' \end{array} \right\} \Big/ \# C_L(\mathbb{Z}/p) \xrightarrow{p \to \infty} \frac{1/|O(L')|}{\text{mass}(\mathcal{G})}$$

$\simeq p^{n-2}$

Meaning: the probability to find the isometry class of a given odd L as a p-neighbor of I_n is proportional to

$$\text{mass}(L) := 1/|O(L)|$$

$\rightarrow$ lattices with small mass are harder to find / to construct

Proof: Use quite deep results from theory of automorphic forms to estimate Hecke eigenvalues on $\mathbb{Z}[O(\mathbb{R}^n)\backslash \mathcal{G}]$ (Arthur's endoscopic classification, Ramanujan like estimates)

Rmk: several variants!

IV Some indications on proof of theorem A

(i) Reduce to classify i.u.l with no norm 1 vector (induction on n)

(ii) Split the classification "root system by root system"

Use for this a key computation by King $\rightsquigarrow$ For all r.s. R we know the mass of the lattices L in $\mathcal{L}_n$ with $R(L) \simeq R$ for $n \leq 30$

n	12	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
#RS	1	1	1	1	1	4	3	12	12	28	49	149	327	1086	2796	4722
												156	368	1901	14493	374062

(iii) efficient enumeration of $O(I_n)$-orbits of isotropic lines in $I_n \otimes \mathbb{Z}/d$ (huge symmetry group for I_n, $\{\pm 1\}^n \rtimes S_n$)

$\Rightarrow$ enumerate the $x \in \mathbb{Z}^n$, $1 \leq x_1 \leq x_2 \leq \dots \leq x_n \leq d/2$

(generally assume $x_1 = 1$ and $\mathrm{mult}(x_i) \leq \mathrm{mult}(1)$ $\forall i$)

(iv) bet on sufficient invariants. We use some invariants of vectors of norm ≤ 3 inspired by works of Megarbané, Bacher-Venkov

(too time consuming to use P.-S. algorithm to check isom.)

(v) The systematic enumeration of all d-neighbors allows in practice to find most, but not all, lattices: would take ∞ many times. Need some clever constructions of lattices. (and case-by-case considerations...)

(vi) Key idea: biaise statistics by imposing visible root system:

$$D_n \simeq R(I_n) \supset R(M_d(x)) \subset R(N_d(x))$$

"visible root system"

easy to impose in terms of x

can't be equality in general, also $R(M_d(x))$ saturated in $M_d(x)$

V Example $n = 26$ root system $10A_1$

$2^{11} \cdot \text{mass} = 4424507 / 58060800$ ("reduced mass")

We show there are 7 i.u.l with this r.s. How?

exercise: safe visible root system is $8A_1$

$\longrightarrow$ we look for x with 8 pairs of x_i equal, all 10 other coordinates $\neq$ $\Rightarrow$ $d \geq 36$

$d = 36$ 98 / 216 orbits of iso. lines give neighb. with r.s. $10A_1$

find 4 $\neq$ lattices !   reduced masses $\frac{1}{32}, \frac{1}{48}, \frac{1}{48}, \frac{1}{320}$

$d = 37$ & 38 : $1243 / 2852$, no new lattices

$d = 39$: $820 / 11821$, 1 new lattice, red. mass $1/6144$

$40 \leq d \leq 49$: several millions of lines, no new lattices ...

remaining red. mass $17/58060800$

$\Rightarrow$ probability to find a new lattice $\leq 3 \cdot 10^{-6}$

Need ideas! one method, start with

$L = N_{35}\,(\underline{1\ 1}\ 2\ 3\ \underline{4\ 4}\ 5\ 6\ \underline{7\ 7}\ 8\ \underline{9\ 9}\ 10\ 11\ \underline{12\ 12}\ 13\ 14\ \underline{15\ 15}\ 16\ \underline{17\ 17})$

one of the "last" lattices in X_{25}, with root system $8A_1$

(visible r.s. $7A_1$)

Compute 2-neighbours of $L \oplus I_2$ with similar visible r.s.

$\rightarrow$ 3632 / 16384 , find 2 new lattices with r. mass

$d = 70$: $^1/_{3\,686\,400}$ and $^1/_{46\,448\,640}$

QED.

Thank you!