Derived Categories I:
t-structures on triangulated categories and their hearts.

Motivation: The category of perverse sheaves is the core of the perverse t-structure on the derived category of sheaves on an algebraic variety over a finite field (l-adic sheaves setting).

1) Derived categories

Let A be an abelian category.
Replace obj. of A by complexes and invert maps between those complexes if they induce an iso. in cohomology $\Rightarrow \mathcal{D}(A)$ derived cat.

$$A \longrightarrow \underset{\text{(co) chain compl.}}{\tilde{C}(A)} \longrightarrow \underset{\text{homotopy.}}{\tilde{K}(A)} \longrightarrow \underset{\text{derived}}{\tilde{\mathcal{D}}(A)}$$

$K(A)$: obj. same as in $C(A)$

morphisms: $\mathrm{Hom}_{K(A)}(X,Y) = \mathrm{Hom}_{C(A)}(X,Y)/\mathcal{I}$

$\mathcal{I}$: ideal of null homotopic maps between X and Y

Fact: Let $\psi, \varphi : X \to Y$.

If $\psi, \varphi : H^n(X) \overset{\cong}{\to} H^n(Y)$ induce iso on cohom. then $\psi - \varphi$ are null homotopic.

We call maps inducing iso on cohomology **quasi isomorphisms**.

Def. Take S = all quasi isomorphisms in $K(A)$, then we invert them formally (i.e. localize $K(A)$ at S) and get $\boxed{\mathcal{D}(A) = K(A)[S^{-1}]}$

More explicitly: Since S forms a multiplicative system, we can indeed localize at S, getting:

ob$(K(A)[S^{-1}])$ = ob $(K(A))$

morph $(K(A)[S^{-1}])$ = $A \underset{\substack{f \\ \in S}}{\overset{\simeq}{\longleftarrow}} B \underset{\substack{g \\ \in \text{ morph}(K(A))}}{\longrightarrow} C$

$$\text{morph}(K(A)) \hookrightarrow \text{morph}(K(A)[S^{-1}])$$

$$(A \xrightarrow{f} B) \longmapsto (A \xleftarrow{id} A \xrightarrow{f} B)$$

2) Derived cat. of abelian cat. are triangulated

Observation: If $\mathcal{A}$ is abelian, $\mathcal{D}(\mathcal{A})$ and $K(\mathcal{A})$ is additive.

– On $C(\mathcal{A})$: abelian struct.
  (example: exact sequences)

– On $K(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$: additive structure
  + some extra structure

$\downarrow$

axiomatization of this structure gives triangulated category

Observation: $K(A)$ is triangulated that is:

Let • $a: X \longrightarrow Y$ be a map of cochain complexes,

• $\Sigma: K(A) \to K(A)$ be a degree shift $\Sigma: X \mapsto X[1]$

s.th. $X[k]^n = X^{n+k}$

$\Sigma: X \mapsto X[1]$ is

$H^n(X) = H^{n-1}(X[1])$

$$\begin{array}{ccccccc}
\to & \overset{n}{X^{n}} & \to & \overset{n+1}{X^{n+1}} & \to & \overset{n+2}{X^{n+2}} & \to \\
 & \downarrow & & \downarrow & & \downarrow & \\
\to & \overset{n}{X^{n+1}} & \to & \overset{n+1}{X^{n+2}} & \to & \overset{n+2}{X^{n+3}} & \to
\end{array}$$

• C_a = mapping cone of a, with

$C_a^n = X^{n+1} \sqcup Y^n$ and differential $\begin{pmatrix} -d_X^{n+1} & 0 \\ a^{n+1} & d_Y^n \end{pmatrix}$

$\rightsquigarrow$ Distingushed triangle or mapping cone sequence:

$$X \xrightarrow{a} Y \xrightarrow{\beta} C_a \xrightarrow{\gamma} \overbrace{\Sigma X}^{X[1]},$$

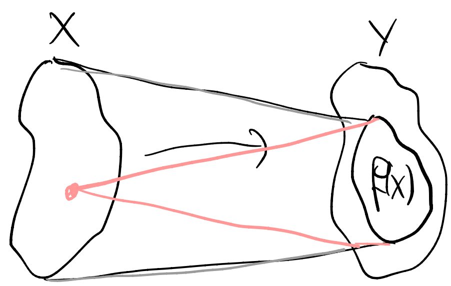
$$n: \quad X^n \xrightarrow{a} Y^n \xrightarrow{\begin{bmatrix} 0 \\ id \end{bmatrix}} X^{n+1} \sqcup Y^n \xrightarrow{\begin{bmatrix} -id & 0 \end{bmatrix}} X^{n+1}$$

"Mapping cone":

$\Updownarrow$

Chain complexes

($\sim$ singular chains on top. spaces)



Observation:

If $f: A \to B$ is a quasi isomorphism then $\exists$ exact triangle in $K(A)$

$$A \xrightarrow{f} B \to C_f \to A[1]$$

s.th. C_f is acyclic ($H^n(C_f) = 0$ for all $n \in \mathbb{N}$).

Facts: 1) In $K(A)$ a triangle is exact or distinguished if it is isomorphic to a mapping cone sequence above.

It turns out that additionally $\boxed{\text{Axioms TR1) - TR4)}}$ for triangulated categories are satisfied (We will see them in a moment).

2) $D(A)$ is also triangulated, exact triangles in $\mathcal{D}(A)$ are those isomorphic to images of exact triangles in $K(A)$.
(This is so since the class of maps at which we localize $K(A)$ is compatible with its triangulated structure).

3) [Very important, will see later]

The canonical functor $F: A \to K(A) \to D(A)$ maps <u>short exact sequences</u>

$$0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$$

to <u>distinguished / exact triangles</u>

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} X[1]$$

in $\mathcal{D}(A)$. Actually, β induces a quasi iso. between Z and $M\alpha$.
mapping cone of α

3) Triangulated categories

Let D be an additive category with $\Sigma X = X[1]$ an additive automorphism $\Sigma: D \to D$, $X \mapsto X[1]$, which is called the translation functor.

A *triangle* in D is a sequence (α, β, γ) of maps $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} X[1]$

and a morphism between triangles (α, β, γ) and $(\alpha', \beta', \gamma')$ is a triple $(\varphi_1, \varphi_2, \varphi_3)$ s.th. the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\varphi_1"] & Y \arrow[r, "\beta"] \arrow[d, "\varphi_2"] & Z \arrow[r, "\gamma"] \arrow[d, "\varphi_3"] & X[1] \arrow[d, "\varphi_1[1]"] \\
X' \arrow[r, "\alpha'"] & Y' \arrow[r, "\beta'"] & Z' \arrow[r, "\gamma'"] & X'[1]
\end{tikzcd}$$

D called triangulated if it is equipped with a class of distinguished / exact triangles satisfying TR1 – TR4 and is called pre-triangulated if only T1–T3 are satisfied.

- TR4) Octahedral axiom

TR1) a) A triangle isomorphic to an exact triangle is exact.

b) $0 \to X \xrightarrow{id} X \to 0$ is exact triangle

c) Each map $X \xrightarrow{\alpha} Y$ fits into an exact triangle (think of the mapping cone).

TR2) $T := X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} X[1]$ exact

$\Leftrightarrow$ $\mathrm{rot}(T) := \underbrace{Z[-1]}_{\substack{X^n \to X^{n-1} \\ n \quad\;\; n}} \xrightarrow{-\gamma[-1]} X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ exact

TR3) For a pair of exact triangles (α, β, γ) and $(\alpha', \beta', \gamma')$

if we have φ_1, φ_2 s.th. $\alpha' \varphi_1 = \varphi_2 \alpha$

then $\exists\ \varphi_3$ s.th. $(\varphi_1, \varphi_2, \varphi_3)$ is a morphism of exact triangles.

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\varphi_1"] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[r] \arrow[d, "\varphi_2"] \arrow[dr, phantom, "\circlearrowleft"] & Z \arrow[r] \arrow[d, "\exists\, \varphi_3"] \arrow[dr, phantom, "\circlearrowleft"] & X[1] \arrow[d, "\varphi_1[1]"] \\
X' \arrow[r, "\alpha'"'] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

TR4) <u>Octahedral axiom</u>

Theorem [<u>Long exact Hom-sequence</u>]

Let $U \xrightarrow{\alpha} V \xrightarrow{\beta} W \xrightarrow{\gamma} \Sigma U$ be an exact triangle in a (pre) triang. categ. D. Then for each $X \in D$

$$\mathrm{Hom}_D(X,-) \quad \text{and} \quad \mathrm{Hom}_D(-,X)$$

give long exact sequences of abelian groups

$$\cdots \to \mathrm{Hom}(X, W[-1]) \to \mathrm{Hom}(X,U) \xrightarrow{\alpha_*} \mathrm{Hom}(X,V) \xrightarrow{\beta_*} \mathrm{Hom}(X,W) \to \cdots \quad (\Delta)$$

and

$$\cdots \longrightarrow \mathrm{Hom}(\Sigma U, X) \to \mathrm{Hom}(W,X) \to \mathrm{Hom}(V,X) \to \mathrm{Hom}(U,X) \to \cdots$$

<u>Proof</u>: (For (Δ)) If $U \to V \to W \to \Sigma U$ is exact triangle, so is $W[-1] \to U \to V \to W$, hence it is enough to check the exactness of (Δ) at any spot: take the exact triangle $X \to X \to 0 \to \Sigma X$

$$\rightsquigarrow \begin{tikzcd}
U \arrow[r, "id"] \arrow[d, "id"] & U \arrow[r] \arrow[d, "\alpha"] & 0 \arrow[r] \arrow[d, dashed] & \Sigma X \arrow[d, dashed] \\
U \arrow[r, "\alpha"'] & V \arrow[r, "\beta"'] & W \arrow[r] & \Sigma U
\end{tikzcd}$$

by TR3

$$\rightsquigarrow \beta \circ \alpha = 0$$

$\rightsquigarrow$ at each step $\mathrm{im}\, \alpha_* \subset \ker \beta_*$

$$\begin{tikzcd}
X \arrow[r, "id"] \arrow[d, "\varphi'"'] & X \arrow[r] \arrow[d, "\varphi"] & 0 \arrow[r] \arrow[d] & \Sigma X \arrow[r, "\Sigma id"] \arrow[d, dashed] & \Sigma X \arrow[d, dashed] \\
U \arrow[r, "\alpha"'] & V \arrow[r, "\beta"'] & W \arrow[r] & \Sigma U \arrow[r, "\Sigma\alpha"'] & \Sigma V
\end{tikzcd}$$

by rotation axiom TR2)

by TR3

$\rightsquigarrow \beta \circ \varphi = 0$

$\Rightarrow \exists\, \varphi'$ s.th. $\alpha \varphi' = \varphi$

$\rightsquigarrow \ker \beta_* \subset \mathrm{im}\, \alpha_*$

Corollary 1) If $(\varphi_1, \varphi_2, \varphi_3)$ is a map of exact triangles, s.th. φ_1, φ_2 are isos then so is φ_3 (use 5 lemma).

Corollary 2) Using (id, id, φ) as above, we conclude that all exact triangles of the form $X \to Y \to Z \to \Sigma X$ have isomorphic Z (non canonic.).

4) Abstract truncations: t-structure and heart (core) of a triangulated category

4) A) t-structure

Motivating example: Let $\mathcal{A}$ abelian, $D(\mathcal{A})$ it's derived cat. (hence triangulated)

- $D(\mathcal{A})^{\leq 0} := \{X \in D(\mathcal{A}) : H^i(X) = 0 \text{ for } i > 0\}$
- $D(\mathcal{A})^{\geq 0} := \{X \in D(\mathcal{A}) : H^i(X) = 0 \text{ for } i < 0\}$
- $D(\mathcal{A})^{\heartsuit} := \{X \in D(\mathcal{A}) : H^i(X) = 0 \text{ for } i \neq 0\} \cong \mathcal{A}$

"heart" or "core" $\qquad$ $\mathcal{A} \hookrightarrow Com(\mathcal{A})$

Now define a s.e.s. of cochain complexes in $Kom(\mathcal{A})$:

$$0 \to \tau^{\leq 0} C \to C \to \tau^{\geq 1} C \to 0$$

(becomes a distinguished triangle in $D(\mathcal{A})$)

$\tau^{\leq 0} C := (\cdots \to C^{-2} \to C^{-1} \xrightarrow{d^0} \ker d^0 \to 0 \to 0 \cdots)$

$\tau^{\geq 1} C := (\cdots \to 0 \to C^0/\ker d^0 \to C^1 \to \cdots)$

<u>Fact:</u> $A \underset{\sim}{\rightarrow} D(A)^{\heartsuit} = D^{\leq 0}(A) \cap D^{\geq 0}(A)$
equiv. of abelian cat.

- In general, for a triang. cat. $\mathcal{T}$:
- Core $(\mathcal{T})$ is an abelian category,
- still unclear: $\mathcal{T} \underset{??}{\rightarrow} \left(D(\text{Core}(\mathcal{T}))\right)_t$ ← t-struct.

$\boxed{\text{Def}}$ A <u>t-structure</u> on a triangulated category D consists of two strict subcategories, $D^{\leq 0}$ and $D^{\geq 0}$, s.th. using that $D^{\geq n} = D^{\geq 0}[-n]$ and $D^{\leq n} = D^{\leq 0}[+n]$ we have

t1) $\mathrm{Hom}_D(D^{\leq 0}, D^{\geq 1}) = 0$

t2) $D^{\leq 0} \subseteq D^{\leq 1}$ and $D^{\geq 1} \subseteq D^{\geq 0}$

t3) For every E in D there exists one distinguished triangle (A, E, B) s.th. $A \in D^{\leq 0}$ and $B \in D^{\geq 1}$.

- D is bounded wrt. the t-structure if every object of D is contained in some $D^{\geq a}$ and $D^{\leq b}$, $a, b \in \mathbb{N}$.

<u>Important observation:</u> t3) gives a unique distinguished triangle

$$\tau_{\leq 0}(E) \xrightarrow{\mathrm{ad}_{\leq 0}} E \xrightarrow{\mathrm{ad}_{\geq 1}} \tau_{\geq 1}(E) \rightarrow \tau_{\leq 0}(E)[1]$$

Why?

$$\overset{\in D^{\le 0}}{A} \xrightarrow{a} E \xrightarrow{b} \overset{\in D^{\ge 1}}{B} \longrightarrow \Sigma A$$

Consider $i_1 : D^{\le 0} \hookrightarrow D$

$i_2 : D^{\ge 1} \hookrightarrow D$

Now take $X = i_1(\overset{\in D^{\le 0}}{X})$ and $Y = i_2(\overset{\in D^{\ge 1}}{Y})$

Take $Hom(X, -)$ and $Hom(-, Y)$

by t1): $Hom(X, B) = 0$ $\qquad$ $Hom(A, Y) = 0$

$\Rightarrow$ • $Hom(X, \underbrace{A}_{\tau_{\le 0}(E)}) \overset{a_*}{\cong} Hom(i_1 X, E)$

• $Hom(\underbrace{B}_{\tau_{\ge 1}(E)}, Y) \overset{b^*}{\cong} Hom(E, i_2 Y)$

— Want: $i : D^{\le n} \hookrightarrow E : \underline{\tau_{\le n}}$ $\quad$ <u>for each n</u>

$\rightsquigarrow Hom(i(\overset{\in D^{\le n}}{X}), E) \cong Hom(X, \tau_{\le n}(E))$

$\wr\!\wr$

$Hom(\underbrace{i(X)[+n]}_{D^{\le 0}}, E[+n]) \cong Hom(X[+n], \tau_{\le 0}(E[+n]))$

$\wr\!\wr$

$Hom(X, \tau_{\le 0}(E[+n])[-n])$

$\Rightarrow \tau_{\le n}(E) = \tau_{\le 0}(E[n])[-n]$

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or $(\tau_{\le n}(E))[1] = \tau_{\le n-1}(E[1])$

<u>Lemma (Orthogonality)</u> For $E \in D$ equivalent are: i) $E \in D^{\geq n+1}$ ii) $\mathrm{Hom}(D^{\leq n}, E) = 0$ | dual 1) $E \in D^{\leq n}$ 2) $\mathrm{Hom}(E, D^{\geq n+1}) = 0$

Proof: Wlog. $n = 0$:

i) $\Rightarrow$ ii) is T1)

ii) $\Rightarrow$ i) $\tau_{\leq 0}(E) = 0$

$\rightsquigarrow$ triangles

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r] & \tau_{\geq 1}(E) \arrow[r] & 0 \\
0 \arrow[r] \arrow[u, "\simeq"] & E \arrow[r] \arrow[u, "\simeq"] & E \arrow[r] \arrow[u, "\simeq"] & 0 \arrow[u, "\simeq"]
\end{tikzcd}$$

<u>Lemma (Extensions)</u> Let (X, Y, Z) be a disting. triang.

1) $X, Z \in D^{\leq n} \Rightarrow Y \in D^{\leq n}$

2) $Y \in D^{\leq n}, X \in D^{\leq n+1} \Rightarrow Z \in D^{\leq n}$

<u>Proof</u>:

1) $X \to Y \to Z \to \Sigma X$

$\rightsquigarrow \cdots \to \mathrm{Hom}(X, E) \to \mathrm{Hom}(Y, E) \to \mathrm{Hom}(Z, E) \to \cdots$

$\forall E \in D^{\geq n+1}$: $\mathrm{Hom}(Y, E) = 0$ $\rightsquigarrow$ dual of Orthog. lemma

<u>Lemma (Compatibility)</u>

i) $\tau_{\geq m}(D^{\leq n}) \subset D^{\leq n}$ $(\neq 0$ for $m \leq n)$ "$[m, n]$"

ii) $\tau_{\leq n}(D^{\geq m}) \subset D^{\geq m}$ — " —

iii) For $m > n$: $\tau_{\geq m}(D^{\leq n}) = \tau_{\leq n}(D^{\geq m}) = 0$

Pf: Use Extension lemma

4) B) Core of a t-structure

Theorem: $\mathrm{Core}(D) = D^{\leq 0} \cap D^{\geq 0}$ is abelian

Pf: • $\mathcal{D}$ additive $\Rightarrow$ Hom's in $\mathrm{Core}(\mathcal{D})$ are abelian groups with bilinear composition

Remains: 1) finite direct sums in $\mathrm{Core}(D)$:

Show: $X \to X \oplus Y \to Y \to \Sigma X$

is exact (using TR3) + long exact Hom + 5 Lemma)

$\Rightarrow$ If $X, Y \in \mathrm{Core}(D)$ so is $X \oplus Y$ by Extension lemma

2) $\forall\, f: A \to B$: $\ker(f), \mathrm{coker}(f) \in \mathrm{Core}(D)$

and $\underbrace{\mathrm{coker}(\ker(f)) \cong \ker(\mathrm{coker}(f))}_{\text{using octahedral axiom}} \in \heartsuit$

Existence of kernels and cokernels in $\mathrm{Core}(\mathcal{D}) := \heartsuit$

• $\underset{\heartsuit}{\underset{\cap}{X}} \xrightarrow{f} \underset{\heartsuit}{\underset{\cap}{Y}} \longrightarrow \underset{\mathcal{D}}{\underset{\in}{Z}} \to \underset{\text{"}D^{-1}\text{"}}{\underset{\cap}{X[1]}}$ and by rotation

• $\underset{\text{"}D^{n}\text{"}}{\underset{\cap}{Z[-1]}} \to \underset{\heartsuit}{\underset{\cap}{X}} \xrightarrow{f} \underset{\heartsuit}{\underset{\cap}{Y}} \to Z$

$\overset{n}{Z} \to Z^{n-1}$

$\mathrm{coker}(f) := \tau_{\geq 0}(Z)$

$\ker(f) := \tau_{\leq -1}(Z)$
$= \tau_{\leq 0}(Z[-1])$

• $Z \in D^{\leq 0} \cap D^{\geq -1} \rightsquigarrow \tau_{\geq 0}(Z) \in \heartsuit$

$Z[-1] \in D^{\leq 1} \cap D^{\geq 0} \rightsquigarrow \tau_{\leq 0}(Z[-1]) \in \heartsuit$

• Take $A \in \heartsuit$, Apply $\mathrm{Hom}_D(A,-)$ and $\mathrm{Hom}_D(-,A)$

l.e.s. $\tau_{\leq -1}(X[1]) = \tau_{\leq 0}(X[+1][-1]) = \tau_{\leq 0}(X)$, ... $\tau_{\geq 0}(X[1]) = 0$

$$\to \mathrm{Hom}_D(\underbrace{X[1]}_{=0}, A) \to \mathrm{Hom}_D(Z, A) \to \mathrm{Hom}_D(Y, A) \to \mathrm{Hom}_D(X, A) \to$$

$X[1] \in D^{\leq -1}$

and

$$\to \mathrm{Hom}_D(A, \underbrace{Y[-1]}_{=0}) \to \mathrm{Hom}_D(A, Z[-1]) \to \mathrm{Hom}_D(A, X) \to \mathrm{Hom}_D(A, Y)$$

$Z[-1] \in D^{\geq 1}$

$$\rightsquigarrow \quad 0 \to \mathrm{Hom}_D(\tau_{\geq 0} Z, A) \to \mathrm{Hom}_D(Y, A) \xrightarrow{f^*} \mathrm{Hom}_D(X, A)$$

$$0 \to \mathrm{Hom}_D(A, \tau_{\leq 0} Z[-1]) \to \mathrm{Hom}_D(A, X) \xrightarrow{f_*} \mathrm{Hom}_D(A, Y)$$

Since this holds for any $A \in \heartsuit \rightsquigarrow$ universal prop. of kernel and cokernel verified.

<u>Theorem</u> Let $X, Y \in \heartsuit$.

If $0 \to X \xrightarrow{f} Y \to Z \to 0$ exact in $\heartsuit$, then

$0 \to X \xrightarrow{f} Y \to Z \to \Sigma X$ disting. triang. in D

<u>Pf</u>: Embed $X \xrightarrow{f} Y$ into triangle

$X \xrightarrow{f} Y \to W \to X[1]$

$\rightsquigarrow W \in D^{\leq -1} \cap D^{\geq 0}$, But since $\ker f = 0$

$= \tau_{\leq 0} W[-1] = \tau_{\leq -1} W \Rightarrow W \in \heartsuit$

and $W \cong \tau_{\geq 0} W = \operatorname{coker} f = Z$

$\Rightarrow X \to Y \to Z \to X[1]$ in D $\square$