On the Selmer group and rank of a family of elliptic curves and curves of genus one violating the Hasse principle

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For every $m, n \in \mathbb{Z}$ we consider the discriminants $D = 4m^3 - 27n^2$ which are squarefree. It is a known result that in this case the 3-rank $r_3(D)$ of the ideal class group of $K_D = \mathbb{Q}(\sqrt{D})$ is non-zero. We define the family of elliptic curves $E_D : y^2 = x^3 + 16D$ and their λ-isogenous curves $E_{D'} : y^2 = x^3 - 27 \cdot 16D$, where $D' = -3D$. In this paper we compute the exact values of the rank of the Selmer groups of E_D and $E_{D'}$, relative to the isogenies λ and λ'. Furthermore, we show that when an unramified cubic extension of K_D is defined by a monic polynomial, then this polynomial gives the elliptic curve $E_{D'}$ a rational point and we show that the set of these points span all of $E_{D'}(\mathbb{Q})/\lambda(E_D(\mathbb{Q}))$. Combining this result with the theory of binary cubic forms, we show that a genus-1 covering of E_D has rational points if and only if it corresponds to a monic binary cubic form that represents an unramified cubic extension of K_D. We give an explicit example with non-trivial $Ш(E_D)[\lambda]$ and we present three genus-1 curves in $\mathbb{P}^2$ which violate the local-global principle. Finally, by combining our results with a known result on the parity of $Ш(E_D)$ emerging from the Cassels-Tate pairing, we can describe the behaviour of and provide improved lower bounds for the rank of the elliptic curves E_D.