k field, X variety over k, $\bar{k}$ algebraic closure

$$\begin{tikzcd} X_{\bar{k}} := X \otimes_k \bar{k} \arrow[d, "p"] \\ X \end{tikzcd} \qquad g \in Gal(\bar{k}/k) \qquad \begin{tikzcd} X_{\bar{k}} \arrow[rr, "\mathrm{id}_X \otimes g =: \tilde{g}"] \arrow[dr, "p"'] & & X_{\bar{k}} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

$\mathcal{F}$ étale sheaf $/X$ $\qquad p^* \mathcal{F}$ étale sheaf $/X_{\bar{k}}$

$$\tilde{g}^* : H^i(X_{\bar{k}}, p^*\mathcal{F}) \to H^i(X_{\bar{k}}, \underbrace{\tilde{g}^* p^* \mathcal{F}}_{\shortparallel \atop (p\tilde{g})^* \mathcal{F} = p^*\mathcal{F}}) = H^i(X_{\bar{k}}, p^*\mathcal{F})$$

This defines an action of $Gal(\bar{k}/k)$ on $H^i(X_{\bar{k}}, p^*\mathcal{F})$

<u>Continuity</u>: X smooth proper$/k$ $\qquad$ Smooth / proper base change

$$\begin{tikzcd} X \otimes_k \bar{k} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}\,\bar{k} \arrow[r, "\bar{x}"] & \mathrm{Spec}\,k \end{tikzcd}$$

$\mathcal{F}$ loc. const. constructible sheaf$/X$ $\qquad$ " LCC

$$H^i_{ét}(X_{\bar{k}}, f^*\mathcal{F}) = (R^i f_* \mathcal{F})_{\bar{x}}$$

<u>Thm</u>: $f: Y \to Z$ smooth proper map of k-schemes
If $\mathcal{F}$ LCC sheaf$/_Y$ then $R^i f_* \mathcal{F}$ LCC$/Z$.

In our situation, get that $R^i f_* \mathcal{F}$ is a LCC sheaf / Spec k.

Bryan's talk: $\{$LCC sheaves$/_{\mathrm{Spec}\,k}\} \longleftrightarrow \{$fin. continuous G_k - module$\}$

/

Any such sheaf has the form

$X \mapsto \mathrm{Hom}_{\mathrm{Spec}(k)}(X, \mathrm{Spec}(L))$

for some finite ext. L/k.

The cohom. of such a sheaf carries an action $\mathrm{Gal}(\bar{k}/k)$ that is trivial on $\mathrm{Gal}(\bar{k}/L)$.

If $\mathcal{F} = \mathbb{Z}/\ell^n\mathbb{Z}$, get $H^i(X_{\bar{k}}, \mathbb{Z}/\ell^n\mathbb{Z})$ cont's $\mathrm{Gal}(\bar{k}/k)$-modules

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$k = \mathbb{F}_q$ then $\mathrm{Gal}(\bar{k}/k) \simeq \hat{\mathbb{Z}}$ top. gen. by geom. Frobenius

$$\overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q, \quad x \mapsto x^{-q}$$

$X/\mathbb{F}_q$

Get a <u>frob. automorphism</u> $F_{X_{\overline{\mathbb{F}}_q}} : X_{\overline{\mathbb{F}}_q} \to X_{\overline{\mathbb{F}}_q}$

<u>Frob. endomorphism</u> of $X_{\overline{\mathbb{F}}_q}$.

Define a frob. endom. $\sigma_X : X \to X$ as - the identity on top. spaces

- the map $x \mapsto x^q$ on structure sheaves

Same definition of $\sigma_{X_{\overline{\mathbb{F}}_q}} : X_{\overline{\mathbb{F}}_q} \to X_{\overline{\mathbb{F}}_q}$

is a map of $\mathbb{F}_q$-schemes

$$\begin{tikzcd}
X_{\overline{\mathbb{F}}_q} \arrow[r, dashed] \arrow[rr, bend left, "\sigma_{X_{\overline{\mathbb{F}}_q}}"] \arrow[dr, bend right] & \sigma_X^* X_{\overline{\mathbb{F}}_q} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X_{\overline{\mathbb{F}}_q} \arrow[d] \\
 & X \arrow[r, "\sigma_X"] & X
\end{tikzcd}$$

the induced map is the frob. endom. of $X_{\overline{\mathbb{F}}_q}$

$Fr_{X_{\overline{\mathbb{F}}_q}}$

Rem.: $F_{X_{\overline{\mathbb{F}}_q}}$ and $\mathrm{Fr}_{X_{\overline{\mathbb{F}}_q}}$ have same action on $\overline{\mathbb{F}}_q$-points

Grothendieck's trace formula

$X/\mathbb{F}_q$ variety, $\mathcal{F}$ $\overline{\mathbb{Q}}_\ell$-sheaf over X, $\mathcal{F}_{\overline{\mathbb{F}}_q}$ = pullbacks of $\mathcal{F}$ along $X_{\overline{\mathbb{F}}_q} \to X$.

x closed point of X, $\bar{x}$ any geom'c pt of X above x.

$|X|$ = set of closed points of X

$\mathcal{F}_x$ = stalk of $\mathcal{F}$ at x is a $\overline{\mathbb{Q}}_\ell$-vector space of finite dim.

$F_{X_{\overline{\mathbb{F}}_q}} : X_{\overline{\mathbb{F}}_q} \to X_{\overline{\mathbb{F}}_q}$ Frob. autom. induces a $\overline{\mathbb{Q}}_\ell$-linear autom. F_x of the stalk $\mathcal{F}_{\bar{x}}$. Fact: this autom. only depends on x, not the choice of $\bar{x}$

Def.: The L-function of $\mathcal{F}$ is

degree of x

$$L(X, \mathcal{F}, t) = \prod_{x \in |X|} \det\Big(\underbrace{1 - t^{d(x)} F_x \mid \mathcal{F}_{\bar{x}}}_{\text{Char. polyn. of } F_x \text{ as an endom. of } \mathcal{F}_{\bar{x}} \text{ (in the variable } t^{d(x)})}\Big)^{(-1)}$$

Thm.: (Grothendieck's trace formula)

$$L(X, \mathcal{F}, t) = \prod_{i=0}^{2\dim(X)} \det\Big(1 - tF \mid H^i_c\big(X_{\overline{\mathbb{F}}_q}, \mathcal{F}_{\overline{\mathbb{F}}_q}\big)\Big)^{(-1)^{i+1}}$$

$\overline{\mathbb{Q}}_\ell$-vect. space (because $\mathcal{F}$ is a $\overline{\mathbb{Q}}_\ell$-sheaf)

Frob. endom.

To define F: $Fr_{X_{\overline{\mathbb{F}}_q}}$ is proper $\Rightarrow$ induces

$$\begin{aligned}
Fr^*_{X_{\overline{\mathbb{F}}_q}} : H^i_c\left(X_{\overline{\mathbb{F}}_q}, \mathcal{Y}_{\overline{\mathbb{F}}_q}\right) &\to H^i_c\left(X_{\overline{\mathbb{F}}_q}, Fr^*_{X_{\overline{\mathbb{F}}_q}} \mathcal{Y}_{\overline{\mathbb{F}}_q}\right) \\
&\qquad \| \quad \text{(see KW, 1.1)} \\
\xrightarrow{F}\ & H^i_c\left(X_{\overline{\mathbb{F}}_q}, \mathcal{Y}_{\overline{\mathbb{F}}_q}\right)
\end{aligned}$$