# Isogenies

Belval - 13-04-2023

We consider abelian varieties over a field $k$

often $k = \bar{k}$

First we look at morphisms

$f : X \to Y$ with $X$ an AV, $Y$ arbitrary variety $/ \bar{k}$

For $x \in X$ look at $C_x$ = connected component of the fibre over $f(x)$ containing $x$. We let $F_x$ be the underlying variety. Then $F_0$ is an abelian variety and

$$F_x = t_x(F_0) = x + F_0$$

This follows from the rigidity lemma

$X, Y, Z$ alg. varieties over $k$, $X$ complete;
$f : X \times Y \longrightarrow Z$ morphism s.t. for some $y \in Y(k)$
the fibre $X \times \{y\}$ is mapped to a point $z \in Z(k)$
Then $f$ factors through the projection on $Y$.

Consider $\varphi : X \times F_x \longrightarrow Y$

$\varphi = f \circ m$

Then $\varphi$ contracts the fibre $\{0\} \times F_x$, hence factors through projection on $X$

this implies $f(y - x + F_x) = f(y)$

Take
$$\left.\begin{array}{lll} y = z,\ x = 0 & & z + F_0 \subseteq F_z \\ y = 0,\ x = z & & -z + F_z \subseteq F_0 \end{array}\right) \Rightarrow F_z = z + F_0$$

$\Rightarrow F_0$ is subgroup scheme, hence (because connected) abelian subvariety.

This shows that morphisms of AV are severely restricted:
if $X$ does not have non-trivial abelian subvarieties (over $\bar{k}$)
then if $f: X \to Y$ is any morphism to an alg. variety
then $f$ is finite or constant.

We are interested in isogenies
$f: X \to Y$ homom. of abelian varieties

Equivalent
1) $f$ surjective and $\dim X = \dim Y$
2) $\ker(f)$ is a finite group scheme and $\dim X = \dim Y$
3) $f$ is finite flat and surjective morphism.

~~Def~~ $f: X \to Y$ homom. of AV is called an isogeny if $f$ satisfies 1), 2) and 3).

For an isogeny the degree of $f$ is

$$\begin{aligned} \deg(f) &= [k(X) : k(Y)] \\ &= \operatorname{rank}_{\mathcal{O}_Y} f_* \mathcal{O}_X \\ &= \operatorname{rank}(\ker(f)) \quad \text{as a finite group scheme} \end{aligned}$$

Let us first look at AV $/ \mathbb{C}$.

$X = V_1 / \Lambda_X$, $Y = V_2 / \Lambda_Y$. $\quad$ $\Lambda_X$, lattice in $V_1$, $\Lambda_Y$ " in $V_2$

$\dim V_1 = V_2$

isogeny $f: X \to Y$ can be lifted to $\tilde{f}: V_1 \simeq V_2$

Then $\deg f = [\Lambda_Y : f(\Lambda_X)]$

Simplify even more : dim = 1.

$X = E_1 \xrightarrow{f} Y = E_2$

$\tilde{f} = \mathrm{id}_{\mathbb{C}} \qquad \Lambda_1 \subset \Lambda_2$ of index $\overset{n}{\cancel{d}} = \deg(f)$

$\Lambda_1$ is given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Lambda_2 \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2\times 2, \mathbb{Z})$

$GL(2, \mathbb{Q})$

$\det = \cancel{d} . n$

But we may change bases of $\Lambda_1, \Lambda_2$:

$f$ is hence given by a double coset

$\Gamma_1 \gamma \Gamma_1 \qquad \Gamma_1 = SL(2, \mathbb{Z})$

Fix a basis of $\Lambda_1$. Then we get all possibilities for $\Lambda_2$ by decomposing the double coset $\Gamma_1 \gamma \Gamma_1$ as a sum of right cosets:

$$\Gamma_1 \gamma \Gamma_1 = \sum_{i=1}^{r} \Gamma_1 \alpha_i$$

In fact can reduce to $\alpha_i = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \qquad ad = n, \quad 0 \leq b \leq n-1$

$$\underbrace{\Lambda_2 \overset{n}{\supseteq} \Lambda_1 \overset{n}{\supset} n\Lambda_2}_{\times n}$$

corresponds to subgroups $\mathbb{Z}/n\mathbb{Z}$ of $\Lambda_2 / n\Lambda_2 \cong (\mathbb{Z}/n\mathbb{Z})^2$

In particular, for $n=p$ prime we get $p+1 = \#\mathbb{P}^1(\mathbb{F}_p)$ possibilities.

If $f, g$ are isogenies $X \xrightarrow{f} Y \xrightarrow{g} Z$ then

$$\deg(gf) = \deg(g)\deg(f).$$

Prop For $n \neq 0$ $n_X : X \to X$ is an isogeny of degree $n^{2g}$

Proof 1) For $AV / \mathbb{C}$ this is clear: $\#[\Lambda : n\Lambda] = n^{2g}$

then use: every AV can be lifted to char 0.

2) More honestly: choose a line bundle $\mathcal{L}$ which is ample and symmetric

Then $$n^*\mathcal{L} = \mathcal{L}^{n(n+1)/2} \otimes (-1)^*\mathcal{L}^{\otimes n(n-1)/2}$$

$$= \mathcal{L}^{\otimes n^2} \quad \text{for symmetric } \mathcal{L}$$

More generally we have $f, g, h : Y \to X$ morphism, $X$ AV

$$(f+g+h)^*\mathcal{L} \otimes (f+g)^*\mathcal{L}^{-1} \otimes (f+h)^*\mathcal{L}^{-1} \otimes (g+h)^*\mathcal{L}^{-1}$$
$$\otimes f^*\mathcal{L} \otimes g^*\mathcal{L} \otimes h^*\mathcal{L} \cong \text{trivial}.$$

(Thm of Cube)

Restriction of $n^*\mathcal{L}$ to $\ker f_n$ is trivial and ample $\Rightarrow$ $\ker n_X$ finite $\overset{n_X}{\Rightarrow}$ $n_X$ isogeny

$$(\deg n_X) D^g = \deg(n_X^* D)^g = (n^2 D)^g = n^{2g} D^g$$
for $\mathcal{L} = \mathcal{O}_X(D)$.

Prop. $f: X \to Y$ isogeny of degree $d$. Then there exists an isogeny $g: Y \to X$ with $g \circ f = d_X$ and $f \circ g = d_Y$.

Proof. $\ker(f)$ is finite commutative group scheme of rank $d$. By a thm of Deligne such a group scheme is annihilated by $d$.

Hence $d_X$ factors through $f \Rightarrow \exists\, g: Y \to X$ s.t. $g \circ f = d_X$.

Then $d_X\, g = g \circ f g = g\,(f \circ g)$

$\quad \parallel$

$g\, d_Y$

$\Rightarrow \quad g\,(d_Y - f \circ g) = 0 \quad \Rightarrow$

$d_Y - f \circ g$ factors through $\ker(g)$

$X$ connected, reduced $\Rightarrow$ hence through $\ker(g)^0_{red}$

$\parallel$

trivial.

$\Rightarrow \quad d_Y = f \circ g$

Corollary: Define $X \sim_k Y \iff \exists$ isogeny $X \to Y$

Then this is an equivalence relation

Frobenius in pos. characteristic is an example of an isogeny

Prop $X$ AV in char $p > 0$. Then the relative Frobenius $F_{X/k}: X \to X^{(p)}$ is a purely inseparable isogeny of degree $p^g$.

Pf local ring of $X$ at $0$: $A = k[x_1,\dots,x_r]/(f_1,\dots,f_n)$

$X^{(p)}$ at $0$: $A^{(p)} = k[x_1,\dots,x_r]/(f_1^{(p)},\dots,f_n^{(p)})$

$X[F] = \ker(F)$ given by $k[x_1,\dots,x_r]/(x_1^p,\dots,x_r^p, f_1,\dots,f_n)$

$X \longrightarrow X^{(p)}$ given by $A^{(p)} \longrightarrow A$

$x_i \longrightarrow x_i^p$

choose basis $x_1,\dots,x_g$ of $m/m^2 = T_{X,0}^\vee$

$\hat{A}$ complete local ring $\cong k[[t_1,\dots,t_g]]$

$$B = A/(x_1^p,\dots,x_r^p)A \cong \hat{A}/(x_1^p,\dots,x_g^p)\hat{A}$$

$(x_1^p,\dots,x_r^p) \in m$ $\qquad \cong k[[t_1,\dots,t_g]]/(t_1^p,\dots,t_g^p)$

$\cong k[t_1,\dots,t_g]/(t_1^p,\dots,t_g^p)$.

About inseparability

$G$ finite group scheme of ~~rank~~ finite rk over $k$

Prop $G$ is an extension of an étale gp scheme by a local ('connected') group scheme.

$$1 \to G^0 \to G \to G_{\acute{e}t} \to 1.$$

If $k$ perfect this splits $\Rightarrow$ semi-direct product.

Corollary. If $f$ is an isogeny then $f$ can be factored

$f: X \xrightarrow{g} Z \xrightarrow{h} Y$ with $g$ inseparable, $h$ separable

unique up to isomorphism of $Z$

<u>Prop</u> If $(n, \mathrm{char}(k)) = 1$ then $n_X$ is separable and

$$X[n](k_s) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$$

Proof: it is étale, hence a abelian group of order $n^{2g}$ killed by $n$.

$\sim$

Consider now. $p_X : X \to X$ multipl. by $p$ in char $p > 0$.

This factors through

$$X \xrightarrow{F_{X/k}} X^{(p)} \xrightarrow{V} X$$

Verschiebung can be separable or not

p-rank of $X$ = $g$ − degree of inseparability

$\parallel$

$f$. $\# X[p](\bar{k}) = p^f$.

$$0 \leq f \leq g$$

The dual abelian variety.

$X$ AV. $H^1(X, \mathcal{O}_X^*)$ group of $\cong$ classes of line bundles on $X$.

for families of AV, say AV/S $X \to S$ we look at the functor of $\cong$ classes of line bundles on $X$ modulo those coming from $S$ by pull back. This functor is represented by a group scheme $\mathrm{Pic}_{X/k}$ for $X$ AV/$k$.

We look at the connected component $\mathrm{Pic}^0_{X/k}$. This is an AV, the dual of $X$.

The tangent space of $\mathrm{Pic}^0_{X/k}$ is $H^1(X, \mathcal{O}_X)$.

$\sim$

$\mathcal{L}$ line bundle on $X$. This determines a map

$$\varphi_{\mathcal{L}} : X \longrightarrow \mathrm{Pic}^0_{X/k} \qquad x \to [t_x^* \mathcal{L} \otimes \mathcal{L}^{-1}]$$

$\varphi_{\mathcal{L}}$ is a homomorphism (Thm of the Square)

$$X^t := \mathrm{Pic}^0_{X/k} \qquad \text{dual AV.}$$

Relation with $X$? Given $\mathcal{L}$.

Look at $X \times X$. Here we have the Mumford line bundle

$$\Lambda(\mathcal{L}) = m^* \mathcal{L} \otimes p_1^* \mathcal{L}^{-1} \otimes p_2^* \mathcal{L}^{-1}$$

View this as a family of line bundles on $X$ (first factor) parametrized by $X$ (second factor)

Define $K(\mathcal{L})$ = max. subscheme of $X$ s.t.
$\Lambda(\mathcal{L})|_{X \times K(\mathcal{L})}$ is trivial ~~over~~
trivial family over $K(\mathcal{L})$.

<u>Lemma</u> $\mathcal{L}$ ample then $K(\mathcal{L})$ is finite gp scheme.

Pf $K(\mathcal{L})^{\circ}_{red}$ is AV, say $Y \subset X$.

$\Lambda(\mathcal{L})|_{Y \times Y}$ is trivial.

$$\begin{tikzcd} Y \times Y \\ Y \arrow[u, "{(1,-1)}"'] \end{tikzcd}$$

pull back is ample ($\mathcal{L} \otimes (-1)^{*}\mathcal{L}$) and trivial

$\Rightarrow$ $K(\mathcal{L})$ finite

$X^t \cong X / K(\mathcal{L})$ for an <u>ample $\mathcal{L}$</u>

$\varphi_{\mathcal{L}} : X \to X^t$

so $\varphi_{\mathcal{L}}$ is an isogeny : defined by polarization $\mathcal{L}$.

<u>Thm</u> $f : X \to Y$ isogeny of AV. Then $f^t : Y^t \to X^t$ (defined by pull back of line bundles) is again an isogeny and we have a canonical isom. of gp schemes

$$\ker(f)^D = (\ker(f^t))$$

For a finite comm. gp scheme $G$, the dual gp scheme is defined by $\mathrm{Hom}_{gpsch}(G, \mathbb{G}_m)$. Can be explained in terms of Hopf algebras. 'Cartier dual'

<u>Idea</u>: $f^t : \mathrm{Pic}^0(Y) \to \mathrm{Pic}^0(X)$ obtained by pulling back line bundles on $Y$ to $X$.

Suppose $M \in \ker(f^t)$; then $f^*M$ is trivial on $X$ but admits an action of $\ker(f) = G$.
This action defines a 'character' of the group scheme $G$.

For elliptic curves $f : E_1 \to E_2$

$$f^t : E_2^t \to E_1^t$$

$$\begin{array}{c} \| \\ E_2 \to E_1 \end{array}$$

so $f^t \cdot f = \deg(f)_{E_1}$

In char $p>0$.

$g=1$. The correspondence in $A_1 \times A_1$ (over $\mathbb{Z}$) determining all isogenies of degree $n$ (i.e. all pairs $E_1 \xrightarrow{\varphi} E_2$ with $\deg \varphi = n$) has degree $p+1$ for $n=$ prime $p$.

In char. $p$ we have the isogenies $F: E \to E^{(p)}$ and other isogenies. In fact, the correspondence breaks up

$$T(p) = (F + F^t) \qquad p+1 = p+1$$

For higher $g$ we often want to preserve the polarization

So ~~one E~~ we want $f: X \to Y$ that $f^* L_Y = L_X^{\otimes n}$

In particular we now look at double cosets

$$\Gamma_g \gamma \Gamma_g \qquad \gamma \in GSp(2g, \mathbb{Q})$$

$$= \{ \gamma \in GL(2g, \mathbb{Q}),\ \gamma J \gamma^t = n \cdot J \}$$

$$J = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix} \text{ — (or } J = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix})$$

Decomposing $\Gamma_g \gamma \Gamma_g = \sum \Gamma_g \alpha_i \qquad \alpha_i \in GSp(2g, \mathbb{Q})$

gives all the lattices $L' \subset L$ of index $n$ in the symplectic lattice $L$ of rank $2g$