Perverse sheaves I

Recall $D^b_c(X, \bar{\mathbb{Q}}_\ell)$: derived category of bounded complexes of $\bar{\mathbb{Q}}_\ell$-sheaves with constructible cohomology sheaves on X

k: finite field or separably closed field
$\ell \neq \mathrm{char}(k)$
$X_{/k}$ scheme

$$D^b_c(X, \bar{\mathbb{Q}}_\ell) = \text{"}\varinjlim_{E/\mathbb{Q}_\ell \text{ finite}}\text{"} D^b_c(X, E)$$

$D^b_c(X, E)$: localization of $D^b_c(X, \mathcal{O})$ where $\mathcal{O} = \mathcal{O}_E$

$D^b_c(X, \mathcal{O}) = \text{"}\varprojlim\text{"} D^b_{ct,f}(X, \mathcal{O}_r)$

$D^b_{ct,f}(X, \mathcal{O}_r)$: category of complexes of étale constructible $\mathcal{O}_r$-sheaves quasi-isom to a bounded $\mathcal{O}_r$-flat complex.

Let $(K^\bullet_r)_{r \geq 1} \in D^b_c(X, \mathcal{O})$

By definition, for all $\nu \in \mathbb{Z}$, $\mathcal{H}^\nu K^\bullet = (\mathcal{H}^\nu(K^\bullet_r))_{r \geq 1}$ and this are A-R π-adic sheaves.

Up to quasi-isomorphism, we can assume that:

* $\exists p < q \mid \forall r \geq 1$, $K^\bullet_r \in D^{[p,q]}_c(X, \mathcal{O}_r))$ (uniform boundedness)
* $\forall m \in \mathbb{Z}$, $\forall r \geq 1$, K^m_r is $\mathcal{O}_r$-flat and constructible ($\Rightarrow K^\bullet_r$ is perfect)
* transition maps $K^\bullet_{r+1} \to K^\bullet_r$ are complex maps.

We defined the standard t-structure on $D^b_c(X, \mathcal{O})$:

$$D^{\leq 0}(X, \mathcal{O}) = \{ K^\bullet \in D^b_c(X, \mathcal{O}) \mid \mathcal{H}^\nu(K^\bullet) = 0 \text{ for all } \nu > 0 \}$$
$$D^{\geq 0}(X, \mathcal{O}) = \{ K^\bullet \text{ —— } \nu < 0 \}$$

and $\text{core(standard)} := D^{\geq 0} \cap D^{\leq 0} \xrightarrow{\sim} A\text{-}R$ equivalence of categories.
$$K^\bullet \longmapsto \mathcal{H}^0(K^\bullet)$$

Overview of the talk

We define the (mixed middle) perverse t-structure on $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ by:

$$^pD^{\leq 0}(X) := \{ B \in D^b_c(X, \overline{\mathbb{Q}}_\ell) \mid \forall i \in \mathbb{Z},\ \dim \operatorname{supp}(\mathcal{H}^{-i} B) \leq i \}$$

$$^pD^{\geq 0}(X) := \{ B \in D^b_c(X, \overline{\mathbb{Q}}_\ell) \mid \forall i \in \mathbb{Z},\ \dim \operatorname{supp}(\mathcal{H}^{-i}(\underbrace{\mathbb{D}(B)}_{X})) \leq i \}$$

1 dualizing complex of B, to be explained

2 Check that this indeed defines a t-structure.

Pointwise characterization:

$x \in X$, let $Y = \bar{x}^{red} \xhookrightarrow{i_x} X$, let $d(x) = \dim Y$.

Then:

$$B \in {}^pD^{\leq 0}(X) \text{ iff. } \forall x \in X, \forall \nu > -d(x),\ \mathcal{H}^\nu(i_x^* B) = 0$$

$$B \in {}^pD^{\geq 0}(X) \text{ iff } \forall x \in X, \forall \nu < -d(x),\ \mathcal{H}^\nu(\underbrace{i_x^!}_{} B) = 0$$

proper inverse image to be explained

Relative duality for singular morphisms

Thm (Deligne, SGA4)
Let

$$\begin{tikzcd} & P \arrow[dr, "\text{proper}"] & \\ X \arrow[ur, "\text{open immersion}"] \arrow[rr] & & S \end{tikzcd}$$

$f : X \longrightarrow S$

be a <u>compactifiable</u> morphism between f.g. schemes over k.

The functor

$$Rf_! : D^b_c(X, \overline{\mathbb{Q}}_\ell) \longrightarrow D^b_c(S, \overline{\mathbb{Q}}_\ell)$$

admits a right adjoint triangulated functor

$$f^! : D^b_c(S, \overline{\mathbb{Q}}_\ell) \longrightarrow D^b_c(X, \overline{\mathbb{Q}}_\ell)$$

i.e., $\exists$ functorial isomorphism

$$\boxed{\mathrm{Hom}(K, f^!(L)) \xrightarrow{\sim} \mathrm{Hom}(Rf_!(K), L)}$$

Rem · Hom's = Hom's in the categories $D^b_c(-, \overline{\mathbb{Q}}_\ell)$
· $f^!$ is not the right derived functor of some functor $\underline{Sh}_S \to \underline{Sh}_X$.
but some people write $Rf^!$ for $f^!$.

Thm (Relative Poincaré duality)
$\exists$ functorial isomorphism

$$\boxed{Rf_* R\mathcal{H}om(K, f^! L) \xrightarrow{\sim} R\mathcal{H}om(Rf_!(K), L)}$$

$R\mathcal{H}om(A, B)$ is a complex of sheaves whose ν-th cohom. is $\mathcal{E}xt^\nu(A, B)$

Def ① $X \xrightarrow{f} S = \mathrm{Spec}(k)$. The dualizing complex of X is

$$K_X := f^!(\overline{\mathbb{Q}}_{\ell,S}) \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$$

② The (contravariant) dualizing functor of X

$$\mathbb{D} = \mathbb{D}_X : \begin{cases} D^b_c(X, \overline{\mathbb{Q}}_\ell) \to D^b_c(X, \overline{\mathbb{Q}}_\ell) \\ \quad L \longmapsto R\mathcal{H}om(L, K_X) \end{cases}$$

coro (Poincaré duality)
Under the assumptions of Deligne's theorem,

$$\forall L \in D^b_c(X, \overline{\mathbb{Q}}_\ell), \qquad Rf_*(\mathbb{D}_X(L)) = \mathbb{D}_S(Rf_!(L))$$

pf. easy, $f^! K_S = K_X$.

Thm (Biduality, Deligne SGA $4\frac{1}{2}$)
We have

$$\mathbb{D}_X \circ \mathbb{D}_X = \mathrm{id}.$$

Therefore,

$$\mathbb{D}_X : D^b_c(X, \overline{\mathbb{Q}}_\ell) \to D^b_c(X, \overline{\mathbb{Q}}_\ell)$$

is an anti-equivalence of categories, and

$$\mathrm{Hom}(\mathbb{D}_X(L), \mathbb{D}_X(K)) = \mathrm{Hom}(K, L).$$

coro Collection of formulae

ⓐ $\mathbb{D} \circ \mathbb{D} = \mathrm{id}$

ⓑ $\mathbb{D} \circ Rf_! = Rf_* \circ \mathbb{D}$

ⓒ $\mathbb{D} \circ Rf_* = Rf_! \circ \mathbb{D}$

ⓓ $\mathbb{D} \circ f^* = f^! \circ \mathbb{D}$

ⓔ $\mathbb{D} \circ f^! = f^* \circ \mathbb{D}$

ⓕ $R\mathcal{H}om(A, B) = \mathbb{D}(A \overset{L}{\otimes} \mathbb{D}(B))$

↑ derived tensor product, whose ν-th cohomology $= \mathrm{Tor}^\nu(A, B)$

(g) $Rf_!(A \overset{L}{\otimes} f^* B) = Rf_! A \overset{L}{\otimes} B$ (Künneth-type formula)

(h) $f^! R\mathcal{H}om(A, B) = R\mathcal{H}om(f^* A, f^! B)$.

Absolute smooth case

Assume $f : X \to S = \mathrm{Spec}(k)$, X smooth, equidimension of dimension d.

Then $K_X \cong \underbrace{\overline{\mathbb{Q}}_{\ell X}[2d]}_{\text{constant sheaf placed in degree } -2d.}(d)$ ← Tate twist

$(\Rightarrow \forall L \in D^b_c(S, \overline{\mathbb{Q}}_\ell),\ \mathcal{H}^\nu(Rf^* L) = 0$ for all $\nu > 2d.)$

Perverse t-structure.

$^pD^{\leq 0}(X) := \{ B \in D^b_c(X, \overline{\mathbb{Q}}_\ell) \mid \forall i \in \mathbb{Z},\ \dim \mathrm{supp}(\mathcal{H}^{-i} B) \leq i \}$

$^pD^{\geq 0}(X) := \{ B \in D^b_c(X, \overline{\mathbb{Q}}_\ell) \mid \forall i \in \mathbb{Z},\ \dim \mathrm{supp}(\mathcal{H}^{-i}(\underbrace{\mathbb{D}(B)}_{X})) \leq i \}$

Let $x \in X$, $Y := \bar{x}^{red} \xhookrightarrow{i_x} X$, $d = \dim Y$, $B \in D^b_c(Y, \overline{\mathbb{Q}}_\ell)$

Claim $\exists\, U$ open smooth neighborhood of x on which B becomes a smooth complex (i.e. $\mathcal{H}^\nu(B)$ are smooth (= lcc) $\forall \nu \in \mathbb{Z}$)

$\Rightarrow K_U \cong \overline{\mathbb{Q}}_{\ell U}[2d](d)$

$\Rightarrow \forall \eta$ generic over x, $(\mathbb{D}_Y(B))_\eta \cong (B_\eta)^\vee [2d](d)$

(stalk complex) $(-)^\vee = \mathrm{Hom}_{\overline{\mathbb{Q}}_\ell}(-, \overline{\mathbb{Q}}_\ell)$

Now take $L \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$, and $B = i_x^* L$, or $i_x^! L$

then:

$$i_x^*(\mathbb{D}(L))_\eta = \mathbb{D}(i_x^! L)_\eta = i_x^!(L)_\eta[2d(x)](d(x))$$

where $d(x) = \dim Y$.

Therefore, we have

$B \in {}^pD^{\leq 0}(X)$ iff. $\forall x \in X, \forall \nu > -d(x)$, $\mathcal{H}^\nu(i_x^* B)_\eta = 0$

$B \in {}^pD^{\geq 0}(X)$ iff $\forall x \in X, \forall \nu < -d(x)$, $\mathcal{H}^\nu(i_x^! B)_\eta = 0$

Rem Assume X is essentially smooth (i.e. $(X \times \bar{k})_{red}$ smooth)

X is equidimensional

$L \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$ smooth

Then:

$L \in {}^pD^{\leq 0}(X)$ iff $\mathcal{H}^\nu B = 0$ $\forall \nu > -\dim X$

$L \in {}^pD^{\geq 0}(X)$ iff $\mathcal{H}^\nu B = 0$ $\forall \nu < -\dim X$

csq: it is the same as the standard t-structure, but shifted by $\dim X$.

Gluing

Consider $j : U \hookrightarrow X$ open immersion

$i : Y = X \setminus U \hookrightarrow X$ closed immersion

Take two full triangul. subcat. $T(U) \subset D^b_c(U, \overline{\mathbb{Q}}_\ell)$

$T(Y) \subset D^b_c(Y, \overline{\mathbb{Q}}_\ell)$

s.t. $A \in T(U) \Rightarrow i^* R j_* A \in T(Y)$

- Define $T(X, U) \subset D^b_c(X, \overline{\mathbb{Q}}_\ell)$:

$B \in T(X, U)$ iff $\begin{cases} j^* B \in T(U) \\ i^* B \in T(Y), \quad i^! B \in T(Y). \end{cases}$

Note that $i^! R j_* = 0$, so $B \in T(U) \Rightarrow R j_* B \in T(X, U)$.

- Given t-structures on $T(Y)$, and $T(U)$, we get a t-str. on $T(X, U)$:

$B \in T^{\leq 0}(X, U) \iff j^* B \in T^{\leq 0}(U)$ & $i^* B \in T^{\leq 0}(Y)$

$B \in T^{\geq 0}(X, U) \iff$ ———— ≥ 0 —— & $i^! B \in T^{\geq 0}(Y)$

Claim This indeed defines a t-stru.

Recall (i) $\mathrm{Hom}(T^{\leq 0}, T^{\geq 1}) = 0$

(ii) $T^{\leq 0} \subset T^{\leq 1}$, $T^{\geq 1} \subset T^{\geq 0}$

(iii) $\forall L \in T$, $\exists \Delta$ (A, L, B) with $A \in T^{\leq 0}$, $B \in T^{\geq 1}$

Check: $T(U) = D^b_c(U, \overline{\mathbb{Q}}_\ell)$ $\quad T^{\leq 0} = {}^p D^{\leq 0}$

$T(Y) = D^b_c(Y, \overline{\mathbb{Q}}_\ell)$ $\quad T^{\geq 0} = {}^p D^{\geq 0}$

$\Rightarrow T^{\leq 0}(X, U) = {}^p D^{\leq 0}(X, -)$

$T^{\geq 0}$ ———— ≥ 0 ——

Main theorem

The perverse t-structure on X is indeed a t-str.

pf : $U \hookrightarrow X$ open dense ess. smooth

$Y = X \setminus U$ $\dim Y < \dim X$

$$D^b_c(X, \overline{\mathbb{Q}}_\ell) = \bigcup_{\substack{U \subset X \\ \text{dense open ess.-sm.}}} T(X, U) .$$

$\forall E \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$, $\exists U \subset X$ ess. smooth open dense /

$j^! E$ smooth