SEMINAR ON "PERVERSE SHEAVES AND LAUMON'S PROOF OF THE WEIL CONJECTURES"

We will meet on **Thursdays** between **2 and 3:30pm** in rooms:

- **MNO 1.050** until October 21,
- **MNO 1.020** from October 28 to November 4,
- **MNO 1.010** from November 11 to December 16.

The link for attending online, as well as notes and other material for the seminar, will be published on math.uni.lu/nt.

For any questions or comments, you can contact andrea.conti@uni.lu or fabio.larosa@uni.lu.

Here are some useful references that can be used to complement those given in the talk descriptions.

- On ℓ-adic cohomology and Deligne's proof of the Weil conjectures: the original papers [Del74; Del80], the books [FK88], [Tam94], [Del77], [Kah18], the notes [Mus11] and [Gon18], the lecture notes and videos [Lit20].
- On perverse sheaves: the seminal reference [Bei+82], the book [KW01], the lecture notes [Cha18].
- On Laumon's proof of the "Riemann hypothesis": the book [KW01], Laumon's papers [Lau87].

Let us know if you have any you would like to add.

Below is a tentative program. Feel free to suggest improvements or to volunteer for a talk!

1. Introductory talk: from ℓ-adic cohomology to the Weil conjectures, minus the Riemann hypothesis
Speaker, date: Fabio La Rosa, 11/10/21

The exposition is inspired by Mustaţă's notes [Mus11], though some proofs are incomplete in the reference.

2. Étale cohomology I: Construction of the étale site and définition of the étale cohomology of a sheaf

Speaker, date: Sebastiano Tronto, 21/10/21

Explain why we need ℓ-adic coefficients: one cannot hope for a Weil cohomology theory with $\mathbb{Q}$-coefficients (example of supersingular elliptic curves [FK88, p. I.1]).
Define étale ring extensions and étale coverings of schemes ([FK88, pp. I.1.1–5], [Tam94, pp. 85.87]).
Define sheaves on the étale site and their cohomology [FK88, pp. I.1.9–13].
State that étale cohomology groups can be computed via Čech cohomology [FK88, pp. II.2.3–9].

3. Étale cohomology II: Some computations

Speaker, date: Bryan Advocaat, 28/10/21

Show that the étale cohomology of a sheaf on $\mathrm{Spec}(K)$, K a field, recovers Galois cohomology of its $\overline{K}$-sections ([FK88, pp. I.1.14–16],[Tam94, p. II.2]).
Give some basic results on the étale cohomology of a sheaf over a curve, for instance [FK88, Propositions I.5.1, I.5.3] (without proofs).
Define direct and inverse direct image (f_*, f^*) [FK88, pp. 27-28]. Define the extension by zero functor $f_!$ [FK88, p. I.8.1] and étale cohomology groups with compact support [FK88, p. I.8.6].
Define constructible sheaves and state without proof some of their key properties ([FK88, Section I.4], [Tam94, p. II.9.3]).

4. Étale cohomology III: ℓ-adic sheaves and Grothendieck's trace formula

Speaker, date: ?, 4/11/21

Define ℓ-adic sheaves [FK88, Definition I.12.6].
State that ℓ-adic cohomology is a Weil cohomology theory (it satisfies the properties given in Talk 1) and the comparison theorem with Betti cohomology [FK88, p. I.11.6].
Define the Galois, and in particular Frobenius, action on ℓ-adic sheaves [KW01, Section 1.1].

Define the L-function of an ℓ-adic constructible sheaf on a variety over a finite field and state Grothendieck's trace formula [FK88, Section II.4]. If time allows give a sketch of its proof.

5. Derived Categories I: t-structures on triangulated categories and their hearts

Speaker, date: ?, 11/11/21

We collect some ideas and facts from triangulated and derived categories. The notion of **truncation structure** and its **heart** or **core** are crucial for the definition of perverse sheaves. We follow the presentation of the first four sections of [KW01, Chapter II]. Sections 2 and 4 of [Cha18] are another possible reference for this material.
After recalling the definition of a triangulated category and stating Theorem 1.3 [KW01, Section II.1], briefly introduce the derived category of an abelian category (for example, Appendix A.II. in [FK88]).
Introduce truncation (or in short t-) structures and mention Lemmata 2.2, 2.3, 2.4 [KW01, Sections II.2-II.3]. State that the heart of a t-structure is an abelian category and explain the relation between exactness of sequences in the heart and distinguished triangles (Theorem 3.1).

6. Derived Categories II: Cohomological functors and the category $\mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$

Speaker, date: Flavio Perissinotto, 18/11/21

We complete the discussion of triangulated categories by introducing **cohomological functors**, then we introduce the category for which we will develop the theory of perverse sheaves: the triangulated category $\mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$. This is not a derived category in the usual sense and it is constructed as a projective limit of certain triangulated categories.
For a finite extension $E/\mathbb{Q}_\ell$, let $\mathfrak{o}$ denote its valuation ring, π a generator of $\mathfrak{o}$ and, for $r \geq 1$, set $\mathfrak{o}_r := \mathfrak{o}/\pi^r\mathfrak{o}$. Denote by $\mathrm{D}^b(X, \mathfrak{o}_r)$ the bounded derived category of étale $\mathfrak{o}_r$-modules sheaves. Let $\mathrm{D}^b_c(X, \mathfrak{o}_r)$ be the triangulated subcategory of $\mathrm{D}^b(X, \mathfrak{o}_r)$ consisting of those objects with constructible cohomology sheaves. Finally, consider the subcategory $\mathrm{D}_{\mathrm{ctf}}(X, \mathfrak{o}_r)$ of $\mathrm{D}^b_c(X, \mathfrak{o}_r)$ consisting of those complexes which are quasi-isomorphic to a bounded $\mathfrak{o}_r$-flat complex.
First construct $\mathrm{D}^b_c(X, \mathfrak{o})$ as a limit of the system $\{\mathrm{D}^b_{\mathrm{ctf}}(X, \mathfrak{o}_r)\}_{r\geq 1}$ and define a t-structure on it. Once this is achieved, construct in the natural way the category $\mathrm{D}^b_c(X, E)$ and obtain $\mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$ as the inductive limit over the system $\{\mathrm{D}(X, E)\}_{E\subset\overline{\mathbb{Q}}_\ell}$. Introduce cohomological functors for a triangulated category with t-structure [KW01, Section II.4], state Theorem 4.3 and 4.4.
The construction of $\mathrm{D}^b_c(X, \mathfrak{o})$ begins at page 96 of [KW01]. The pages 94-96 explain the finiteness assumption that we need to make sense of $\mathrm{D}^b_c(X, \mathfrak{o})$ as a limit of triangulated categories in a natural way. The preceding pages of Section II.5 are of rather technical nature and a thorough understanding of the details is not essential.
The last important point of the talk is the construction of the t-structure on $\mathrm{D}^b_c(X, \mathfrak{o})$, covered in Section II.6. The construction is first carried over a point and then extended to a general X. Again, the details of every step of the construction are not essential, the key point is obtaining a good understanding of Definition 6.3 and Theorem 6.4. Finally, define $\mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$ as in Appendix A, pages 330-331, is defined.

7. Perverse Sheaves I

Speaker, date: ?, 25/11/21

All the references for this talk are to be found in [KW01, Chapters II and III].
Given a scheme X of finite type over a finite field or over an algebraically closed field, we can naturally introduce the so-called **perverse t-structure** on $\mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$; its heart (or core) is the category of **perverse sheaves** we will be working with. The difficulty is in showing that the given prescription actually defines a t-structure. The proof of this fact goes via a **gluing** procedure.
Consider an open subscheme $j\colon U \hookrightarrow X$ and its closed complement $i\colon Y \hookrightarrow X$. Let $T(U)$ and $T(Y)$ be full triangulated subcategories of $\mathrm{D}^b_c(U, \overline{\mathbb{Q}}_\ell)$ and $\mathrm{D}^b_c(Y, \overline{\mathbb{Q}}_\ell)$, respectively, and assume that $T(U)$ and $T(Y)$ are equipped with some t-structure. If a technical condition is satisfied, then it is possible to glue these t-structure to a t-structure on $\mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$.
The definition of the perverse t-structure requires introducing the **dualizing functor** $D_X\colon \mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$ and, to this purpose, the functor $f^!$. Explain Theorem 7.1, Definition 7.2, Corollary 7.3 and Theorem 7.4 from Section II.7. The formulae in Corollary 7.5 are very helpful, but can be cited as needed later on.

The **perverse t-structure** on $\mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell)$ is defined in section III.1, page 135. State Lemma-Definition 1.1 and Definition 1.3 with the subsequent pointwise characterization of the perverse t-structure.
Explain the key points of the **gluing** construction from Section III.3, pp. 139-142, and how it is used in the proof of Lemma III.1.1 at the beginning of page 143. In order to explain the proof of Lemma III.1.1, give the definition of smooth complex and Remark 2.2 from Section II.2 (the rest of the material from Section II.2 will not be needed).

8. Perverse Sheaves II

Speaker, date: ?, 2/12/21

This talk is based on sections III.4 and III.5 of [KW01]. Having constructed the category of perverse sheaves on X, denoted $Perv(X)$, we begin the study of some of its most interesting features. On one hand, this will lead to the celebrated decomposition theorem of Gabber. On the other, it allows us to introduce the notion of Fourier transform that Laumon used to prove the generalized Weil conjectures.
Introduce the notions of t-right exact and t-left exact functor and state Lemma 4.1. Explain the restriction sequence and Lemma 4.2, state Lemma 4.3.
Introduce the construction of **intermediate extension**, Lemma-Definition 5.2, ignore the mixedness statement. Explain the characterization of the intermediate extension in terms of Lemma 5.1(2) and how it leads to Corollary 5.3. Explain the proof of the classification theorem of simple perverse sheaves (Corollary 5.5) and the proof of Corollary 5.7. Sketch the proofs of Corollaries 5.8, 5.11 and 5.14.

9. The ℓ-adic Fourier Transform

Speaker, date: ?, 9/12/21

This talk has two goals: introducing **mixed complexes** and explaining the construction and the main properties of the **ℓ-adic Fourier transform**. All references are to [KW01, Chapters II and III].
From Chapter II, state some basic definitions and facts about mixed complexes: Definition 12.1, Theorem 12.2, Definition 12.3 and Lemma-Definition 12.7, without proofs.
Moving on to Chapter III, explain how the t-structure on $\mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell)$ induces a t-structure on the category of mixed complexes (page 144). Explain that the intermediate extension functor, defined in talk 8, preserves mixedness (Lemma-Definition 5.2, page 148).
The construction of the **Fourier transform** begins on page 159 (the **Artin–Schreier Sheaf**, however, is defined on page 38). Explain that the transform preserves mixedness and sketch the proof of Theorem 8.1 (we don't need the direction of part (2) that assumes the Weil Conjectures). The proof of part 3 relies on the fact that if a morphism f is affine, then $R(f)_!$ is t-left exact. It also uses the t-exactness of $f^*[r]$ when f is smooth and equidimensional of relative dimension r. These facts are proved in Corollary 6.2 and Theorem 7.2 and can be black-boxed.
State and sketch the proof of Theorem 12.1. The proof of part 6 requires some ideas from Section III.11, that can be explained briefly or simply mentioned depending on time.

10. Gabber's Decomposition Theorem

Speaker, date: ?, 16/12/21

With this talk, we would like to understand the celebrated **Decomposition Theorem** of Gabber. For a variety X_0 over a finite field κ and its base change X to $\overline{\kappa}$, the theorem states that if $B_0 \in \mathrm{D}_c^b(X_0, \overline{\mathbb{Q}}_\ell)$ is τ-pure of weight w, then the base change of B_0 to $B \in \mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell)$ decomposes as a finite direct sum of translates of a simple perverse sheaf on X.
We follow Sections III.9 and III.10 of [KW01]. State Lemma I and explain the derivation of Corollary 9.2. State Lemmas II and III and explain in which sense the weight filtration is canonical.
State and prove Theorem 10.1, Corollaries 10.2, 10.4 and 10.5, and Theorem 10.6.

We will arrange for one or two talks in January to wrap up Laumon's proof of the generalized Weil conjectures, mostly following [KW01, Chapter I].

References

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