Proposed schedule for the seminar “The spectral halo for GL_2”
University of Luxembourg, Summer Semester 2021

Organisers: Andrea Conti and Alexandre Maksoud

The seminar will take place on Tuesdays from 10:30 to 12:30 in room MNO 1.010.

The main goal of the seminar is to understand the construction of the “adic” eigencurve for $\mathrm{GL}_{2/\mathbb{Q}}$ as presented in the paper [AIP18] (in French) by Andreatta, Iovita and Pilloni. Such a curve lives over a compactified weight space that admits some points of characteristic p; the fiber of the eigencurve at these points describes the Hecke eigensystems of the “overconvergent mod p modular forms” constructed in [AIP18], and the existence of this extended eigencurve is expected to shed some light on the geometry of the usual eigencurve at the boundary of the weight space. The references [AIP16; JN19] (in English) treat more general constructions, but some parts of them can also be helpful for understanding the case of $\mathrm{GL}_{2/\mathbb{Q}}$.

Topics that require two talks can be split up between two speakers. If you wish to give a particular talk, please write to `andrea.conti@uni.lu` or `alexandre.maksoud@uni.lu`!

1. Overview [?, 16/2]
Give a brief review of the rigid analytic eigencurve, and the motivation for the seminar: studying the geometry of the eigencurve at the boundary by constructing an extended eigencurve over a compactified weight space, via the theory of adic spaces. Mention some interesting consequences of the construction. One can follow [AIP18, Section 1] and [JN19, Introduction].

2. The classical theory of p-adic families of modular forms [?, 23/2; probably two talks]
Recall the basic results about the interpolation of Hecke eigensystems of classical modular forms in Hida and Coleman families. Without proofs, state the existence and properties of the rigid analytic eigencurve constructed by Coleman–Mazur and Buzzard, of its maps to the spectral curve and to the weight space, and of the sheaf of Galois (pseudo-)representations on it. One can extract from [Buz07] an overview of the construction of the eigencurve, and the most elementary construction of the sheaf of Galois representations from [Che04]. A good reference for the construction of the modular sheaves is [Pil13]; it would be good to see this in some detail as a motivation for the construction in the adic setting. There are of course various other classical references on this material, such as [CM98].

3. Reminders on adic spaces [?, 2/3; could be two talks]
Recall the construction of adic spaces, without details but maybe with some examples, such as the open adic unit disc that plays a crucial role in the rest of the seminar. One finds in [Sch12] a compact introduction to the topic, that can be complemented with some material from last term’s seminar; some elements from [Con18; Wei17] could be helpful. Mention what the analytic points of an adic space are.

4. Construction of the compactified weight space [?, 9/3]
Following [AIP18, Section 2], construct the adic version of the weight space; this is the adic space $\mathfrak{W} = \mathrm{Spa}(\Lambda, \Lambda)$ where $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$, and is a kind of “compactification” of the usual rigid analytic weight space. Introduce the open subspace $\mathcal{W}$ of analytic points of $\mathfrak{W}$, the boundary of $\mathcal{W}$, and the universal character over $\mathcal{W}$, giving its analyticity properties.

5-6. Formal modular curves and partial Igusa towers (two talks) [?, 16/3]

Define formal neighborhoods of the ordinary locus of formal modular curves, indexed by their radius of overconvergence. Define the partial Igusa tower over such a neighborhood and prove some of its properties. One can follow the structure of [AIP18, Section 3], but it may be helpful to begin by reviewing the formal constructions, including the theory of the canonical subgroup, in more detail.

7-8. Overconvergent mod p modular forms (two talks) [?, 30/3 and 6/4]

Define the overconvergent Igusa tower, and use it to define overconvergent modular forms modulo p. The material is essentially contained in [AIP18, Section 4], but again one can present some elements in more detail than what is done there (for instance, the action of the spherical Hecke algebra).

9-10. Modular sheaves over the interior of the weight space (two talks) [?, 13/4 and 20/4]

For every annulus in the characteristic 0 part of the weight space, construct a sheaf over the product of the annulus with a certain neighborhood of the ordinary locus in an adic modular curve. Show that after taking generic fibers these sheaves coincide with the usual sheaves of p-adic overconvergent modular forms. All of this is done in [AIP18, Section 6].

11-12. Extending the sheaves to the boundary (two talks) [?, 27/4 and 4/5]

Extend the sheaves defined in the previous talk to the boundary of the weight space, corresponding to its characteristic p points. Show that the fibers of these sheaves at the boundary give the forms of characteristic p constructed earlier in the seminar. Construct the adic modular curve over the compactified weight space, by plugging the sections of the modular sheaves into the eigenvariety machine. The material is mostly contained in [AIP18, Section 6], but one can refer to [Lud20] for a detailed presentation of the adic eigenvariety machine.

13. Generalizations and applications [?, 11/5]

It would be interesting to say something about the conjectural applications of the construction to the study of the geometry of the eigencurve at the boundary of the weight space (in the spirit of the Coleman–Mazur–Buzzard–Kilford conjecture). There is a discussion about this in both [AIP18, Section 1] and [JN19, Introduction].

One could also give an overview of the constructions of adic eigenvarieties for other groups than $\mathrm{GL}_{2/\mathbb{Q}}$: in the Hilbert case this was done by Andreatta, Iovita and Pilloni [AIP16], and for more general reductive groups by Johansson and Newton [JN19] via overconvergent cohomology and an adic version of the eigenvariety machine. The construction of the adic eigenvariety machine is presented in detail in the notes [Lud20] by Ludwig.

References

[AIP16] Fabrizio Andreatta, Adrian Iovita, and Vincent Pilloni. "The adic, cuspidal, Hilbert eigenvarieties". In: *Res. Math. Sci.* 3 (2016), Paper No. 34, 36.

[AIP18] Fabrizio Andreatta, Adrian Iovita, and Vincent Pilloni. "Le halo spectral". In: *Ann. Sci. Éc. Norm. Supér. (4)* 51.3 (2018), pp. 603–655.

[Buz07] Kevin Buzzard. "Eigenvarieties". In: *L-functions and Galois representations. Proceedings of the Durham Conference 2004*. Vol. 320. London Math. Soc. Lecture Note Series. Cambridge Univ. Press, 2007, pp. 59–120.

[Che04] Gaëtan Chenevier. "Familles p-adiques de formes automorphes pour $\mathrm{GL}(n)$". In: *J. Reine Angew. Math.* (2004), pp. 143–217.

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[Con18] Brian Conrad. *A brief introduction to adic spaces*. 2018. URL: http://math.stanford.edu/~conrad/papers/Adicnotes.pdf.

[JN19] Christian Johansson and James Newton. "Extended eigenvarieties for overconvergent cohomology". In: *Algebra Number Theory* 13.1 (2019), pp. 93–158.

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