Roots of polynomials

1. A *polynomial* is a function sending any real number x to a sum of non-negative powers of x multiplied by constants (they will be called *coefficients*). For instance, the following is a polynomial : $x \mapsto x^3 + 5x^2 - \frac{1}{7}x + \pi$.

   The *degree* of a non-zero polynomial P is the integer d such that x^d is the highest power arising in P. It will be denoted $\deg P$. By definition, $\deg 0 = -\infty$. If P and Q are both polynomials, one can easily check that $\deg(P + Q) \leq \max(\deg P, \deg Q)$ and $\deg(P \times Q) = \deg P + \deg Q$.

   The *leading coefficient* of a polynomial P is the coefficient in front of $x^{\deg P}$.

   An *integer valued polynomial* is a polynomial P such that for any integer k, $P(k)$ is an integer.

   If P is a polynomial, a real number a is a *root* of P if $P(a) = 0$. In other words, a root is a solution to the equation $P(x) = 0$ with the variable x.

2. There is a general method to find the roots of polynomials of degree 2. Consider the following polynomial $P : x \mapsto ax^2 + bx + c$ where a, b, c are real numbers and $a \neq 0$. A root x of P will verify $ax^2 + bx + c = 0$. You can write a polynomial in a *canonical form* as follows:

$$ax^2 + bx + c = ax^2 + a \times 2\frac{b}{2a}x + c = a\left(x^2 + 2\frac{b}{2a}x\right) + c = a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right) + c$$

   Finally, the equation is equivalent to $(x + \frac{b}{2a})^2 = \frac{b^2-4ac}{4a^2}$. There are three different cases depending of the sign of $\Delta := b^2 - 4ac$. This is called the *discriminant* of P.

   - If $\Delta < 0$, then there is no solution because a square is non-negative.
   - If $\Delta = 0$, then $x = -\frac{b}{2a}$ is the unique solution.
   - If $\Delta > 0$, then $\frac{b^2-4ac}{4a^2} > 0$ and

$$x_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{\Delta}}{2a},$$

     are the two solutions of the equation.

3. Let P be a polynomial and let x_0 be a root of P. Then there exists another polynomial Q such that $P(x) = (x - x_0)Q(x)$. Furthermore, computing the coefficients of

$Q(x)$ is easy. Indeed if $P(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0$ and $Q(x) = b_{d-1}x^{d-1} + \cdots + b_0$ then we can write

$$a_d x^d + \cdots + a_0 = (x - x_0)(b_{d-1}x^{d-1} + \cdots + b_0)$$
$$= b_{d-1}x^d + (b_{d-2} - x_0 b_{d-1})x^{d-1} + \cdots + (b_0 - x_0 b_1)x - x_0 b_0$$

Since the coefficients in front of the powers of x need to be equal on each side, we can compute b_{d-1} from the equation $a_d = b_{d-1}$. Then we can compute b_{d-2} from the equation $a_{d-1} = b_{d-2} - x_0 b_{d-1}$, etc., and finally b_0 from the equation $a_1 = b_0 - x_0 b_1$. The remaining equation $a_0 = -x_0 b_0$ should be true (if it is not, it means that you have a mistake in your computation). For instance, knowing that 1 is a root of $x^3 - 2x^2 + 3x - 2$, could you factorize it?[1]

Because of this nice factorization property, non-zero polynomials cannot have more roots than their degree. If a polynomial P of degree d has d distinct roots $x_1, \ldots, x_d$ with leading coefficient a_d then

$$P(x) = a_d(x - x_1) \times \cdots \times (x - x_d).$$

4. Finding the roots of a polynomial is a very complicated problem in general. A very efficient tool is the Eisenstein trick. Let $P(x) = a_n x^n + \cdots + a_0 x^0$ be a polynomial with integer coefficients. If the integer z is a root of P, then

$$0 = P(z) = a_n z^n + \cdots + a_1 z + a_0 \text{ whence } z(a_n z^{n-1} + \cdots + a_1) = -a_0,$$

therefore z divides $-a_0 = -P(0)$. Thus, whenever you look for an integer root of P, you only need to check the divisors of $\pm a_0$.

5. If P is a polynomial of degree 2, then we can write $P(x) = ax^2 + bx + c$. On the other hand, if P has two real roots x_1 and x_2, then $P(x) = a(x - x_1)(x - x_2)$. Developing this product, one can check that

$$x_1 x_2 = \frac{c}{a} \text{ and } x_1 + x_2 = -\frac{b}{a}.$$

If P is a polynomial of degree 3, then we can write $P(x) = ax^3 + bx^2 + cx + d$. On the other hand, if P has three real roots x_1, x_2 and x_3 then $P(x) = a(x - x_1)(x - x_2)(x - x_3)$. Developing this product, one can check that

$$x_1 x_2 x_3 = -\frac{d}{a},\ x_1 x_2 + x_1 x_3 + x_2 x_3 = \frac{c}{a} \text{ and } x_1 + x_2 + x_3 = -\frac{b}{a}.$$

These are called Viéta's formulas. They make a link between the roots of a polynomial and its coefficients. Viéta's formulas exist for polynomial of arbitrary degree. Can you guess the general formulas?

[1] Write $x^3 - 2x^2 + 3x - 2 = (x - 1)(ax^2 + bx + c)$ and develop the product : $ax^3 + (b - a)x^2 + (c - b)x - c$. Then we get $a = 1$, $b - a = -2$, $c - b = 3$ and $-c = -2$. Therefore $a = 1$, $b = -1$ and $c = 2$.

Problem 1. What is b if the difference of the roots of $x^2 + bx - 7$ is $5\sqrt{\frac{7}{2}}$?

Problem 2. What are the real roots of the polynomial $x^3 + 6x^2 + 15x + 14$?

Problem 3. Solve the following equation on the real numbers:

$$(x-6)^4 + (x-4)^4 = 512.$$

Problem 4. Prove that for any polynomial p with integer coefficients and $x \neq 0$, we have

$$|p(x)p\left(\frac{1}{x}\right)| \geq p(1)^2.$$

Problem 5. Is there a polynomial p with integer coefficients such that

1. $p(1) = 8$ and $p(4) = 15$?
2. $p(1) = 3$ and $p(k) = 2k$ where k is odd?
3. $p(1) = 3$ and $p(k) = 2k$?

Problem 6. Is it true that for a polynomial p with degree at most 3 that

1. if 6 divides $p(n)$ for every integer n, then p has integer coefficients ?
2. if 3 divides $p(n)$ for every integer n, then p has integer coefficients ?

Problem 7 (*)**.** Let $n \geq 1$. Count the number of polynomials P whose coefficients are either $0, 1, 2$ or 3 such that $P(2) = n$.

Problem 8 (*)**.** If $n \geq 1$, the number $n! := 1 \times 2 \times \cdots \times n$ is called *factorial* n. We define the n-th *Hermit polynomial* by $H_n(x) = \frac{x(x-1)\ldots(x-n+1)}{n!}$. By definition we have $H_0(x) = 1$.

1. Prove that $H_n(x) = H_n(x-1) + H_{n-1}(x-1)$.
2. Deduce from this that for any integer k, $H_n(k)$ is an integer.
3. Prove that any integer valued polynomial can be written as a sum of Hermit polynomials or their opposites.

Problem 9 (*, Benelux 2010)**.** Find all polynomials P such that for any real numbers a, b, c, we have :

$$P(a+b-2c) + P(a-2b+c) + P(-2a+b+c) = 3P(a-b) + 3P(b-c) + 3P(c-a).$$

Problem 10 (*, Canada 1970)**.** Let P be a polynomial with integer coefficients and a, b, c, d be distinct integers such that $P(a) = P(b) = P(c) = P(d) = 5$. Prove that $P(k)$ is never equal to 8 when k is an integer.

Note: $*$ means that this exercise is harder.

ANY QUESTION? JUST ASK!