Sets

A **set** is a collection of elements which are all distinct: two sets are called equal when they have the same elements (we write $x \in S$ if x is an element of the set S, and $x \notin S$ otherwise). The **cardinality** of a set is the number of its elements (we write $\#S$ for the cardinality of the set S). The set with no element is the empty set $\emptyset$. A set can be finite (if there are finitely many elements), or infinite.
You can describe a set by listing its elements, or by distinguishing them with a property:

$$S = \{5{,}10{,}15{,}\ldots{,}90{,}95\} = \{x : 1 < x < 100 \text{ and } x \text{ is divisible by 5}\}$$

One can repeat elements because this does not count:

$$\{1{,}2{,}2\} = \{1{,}2\}$$

A **subset** of a set S is a set T such that each element of T is also an element of S (we write $T \subseteq S$ or $S \supseteq T$). Subsets of subsets of S are again subsets of S. If $T \subseteq S$ and $S \subseteq T$, then we have $S = T$.

How many subsets does a set have? The empty set counts as a subset, the whole set as well, and there are usually many possibilities. . . (See Problem 2).

Intersection

When we require several properties we are intersecting sets:

$$\{\text{even number from 5 to 8}\} = \{\text{even numbers}\} \cap \{\text{numbers from 5 to 8}\}$$

Formally:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

Of course we have $A \cap A = A$ and $A \cap \emptyset = \emptyset$. If $A \subseteq B$, then clearly $A \cap B = A$:

$$\{\text{numbers from 1 to 5}\} \cap \{\text{numbers from 1 to 7}\} = \{\text{numbers from 1 to 5}\}$$

We can intersect several sets, the ordering of the sets plays no role. For example:

$$A \cap B \cap C = \{x : x \in A \text{ and } x \in B \text{ and } x \in C\}\,.$$

This means doing more times the intersection of two sets (can you prove this?):

$$A \cap B \cap C = (A \cap B) \cap C = A \cap (B \cap C)\,.$$

The intersection is a subset of each of the given sets (can you prove this?).

Two sets are called **disjoint** if their intersection is the empty set. For more sets, we can speak of **pairwise disjoint** sets. This not only means that the intersection of all sets is empty, but also the stronger condition that the intersection of any two of the sets is empty!
The following sets are not pairwise disjoint, although their intersection is empty:

$$\{1,2\} \cap \{2,3\} \cap \{3,1\} = \emptyset$$

Union

When we allow several possibilities we are taking the union of sets:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Of course, we have $A \cup A = A \cup \emptyset = A$. If $A \subseteq B$, then we clearly have $A \cup B = B$:

$$\{\text{numbers from 1 to 5}\} \cup \{\text{numbers from 1 to 7}\} = \{\text{numbers from 1 to 7}\}\,.$$

We can take the union of several sets, the ordering of the sets plays no role. This means doing more times the union of two sets:

$$A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C)\,.$$

Difference

When we want to exclude some elements, we are building the difference of sets: The complement of a set B in a set A consists of all elements in A that are not in B:

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$$

Of course we have $A \setminus \emptyset = A$ and $A \setminus A = \emptyset$.

Now the ordering of the sets or the parentheses make a difference, as for the subtraction of numbers: $3 - 5 \neq 5 - 3$ and $5 - (3 - 2) \neq (5 - 3) - 2$.

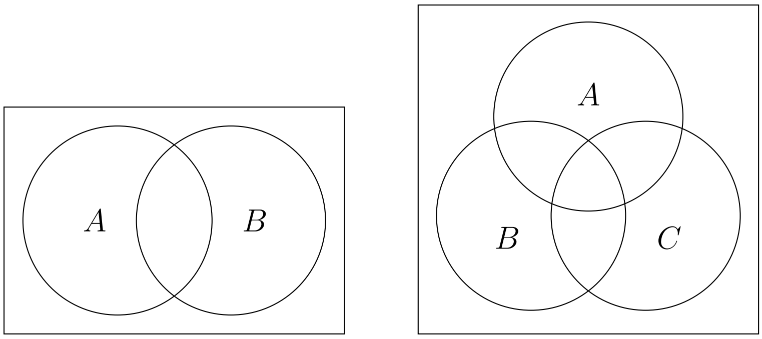
• If $A \subseteq B$, then $A \cup (B \setminus A) = B$ but in general we only have $A \cup (B \setminus A) \supseteq B$.

• The intersection $A \cap (B \setminus A)$ is always empty and we have

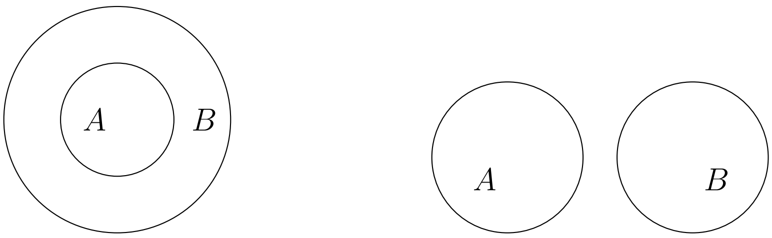
$$(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$$

Venn diagrams

With 2 sets A, B you have in general $4 = 2^2$ possibilities for an element: only in A; only in B; in A and in B; not in A and not in B. You can draw a diagram with 4 zones to see this. For 3 sets A, B, C you have $8 = 2^3$ possibilities. So you need 8 zones. For n sets you need 2^n zones, which means you have 2^n possibilities. Alternatively, you can write down a table and list all cases...



If some of the zones are empty (e.g. if $A \subseteq B$ or if A and B have no common elements, or if one only considers elements from A or B), then the diagrams can also be drawn in a simplified version:



There are many formulas on the internet about sets, for example:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

You should not try to guess the formulas, but you can convince yourself that they are true (slowly and stepwise) with the help of a Venn diagram i.e. showing that the two sets cover the same zones in the diagram. This mathematically means considering all possible cases! You can also visualise an inclusion of sets with a Venn diagram because a subset will cover the same zones, or less.
A Venn diagram also helps you showing that two sets are not equal (or that you do not have an inclusion) because you can detect the problematic zones and exploit them to build a counterexample!

Partitions

If you try to cut a set into parts as if you would cut a cake, you obtain a partition:
A **partition** of a set $S \neq \emptyset$ *consists of non-empty subsets of* S *that are pairwise disjoint and such that their union is* S.
In other words: Every subset is non-empty and every element from S belongs to exactly one subset of the partition.

Example: If T is a non-empty subset of S distinct from S, then T and $S \setminus T$ give a partition of S.

With two partitions you call the second *finer* if every set of the second partition is contained in a set of the first partition. In other words: we are cutting the set even more. It can also be that two partitions have nothing to do with each other (none is finer).

Example: The blood types $A, B, AB, 0$ give a partition of humans. The Rhesus antigen $+, -$ give another partition, and none of the two is finer.

We can combine partitions by intersecting their sets (in this way we get a partition which is finer than both). In the above example we get the blood types:

$$A+, A-, B+, B-, AB+, AB-, 0+, 0- \ .$$

The Inclusion-Exclusion principle

When we count the elements of A and then the elements of B, then we counted the elements of $A \cap B$ twice. For this reason we have the formula:

$$\#(A \cup B) = \#A + \#B - \#(A \cap B)$$

There is a generalisation to three sets (and in fact there is also a version for n sets):

$$\#(A_1 \cup A_2 \cup A_3) = \#A_1 + \#A_2 + \#A_3 - \#(A_1 \cap A_2) - \#(A_1 \cap A_3) - \#(A_2 \cap A_3) + \#(A_1 \cap A_2 \cap A_3)$$

In general, the cardinality of the union is less than or equal to the sum of the cardinalities:

$$\#(A_1 \cup A_2 \cup \ldots \cup A_n) \leq \#A_1 + \#A_2 + \ldots + \#A_n$$

We have equality if the sets are *pairwise* disjoint. For this reason it is better to work with pairwise disjoint sets, and this is why partitions are very practical!

Problems around Sets

1. **Play with logic:** The intersection of sets has to do with the word ‘and’, the union with the word ‘or’ and the complement of a set with the word ‘not’. Can you understand in this way the following De Morgan’s laws?
   Here we write $\complement$ to mean the complement inside a fixed given set:

   $$A^\complement \cup B^\complement = (A \cap B)^\complement$$

   $$A^\complement \cap B^\complement = (A \cup B)^\complement$$

   Recall (from the Logic Theme) that a negation exchanges ‘and’ and ‘or’!

2. **Play with induction:** How many subsets does a set with $1, 2$ or 3 elements have? Can you guess a general formula for the number of subsets for a set with n elements?

   Afterwards, you can try to prove your formula with the induction principle... Hint: for going from n to $n+1$ consider a set of $n+1$ elements and mark one element X. You can first count the subsets without X, and then similarly those with X.

   Recall (from the Induction Theme) how a proof by induction works!

3. **Venn diagram SOS:** Show the following with the help of a Venn diagram (analysing the four zones): “If $A \subseteq B$, then $A \cup (B \setminus A) = B$.”
   Also find two general sets that do not satisfy the equality.

4. **Small dilemmas:** Let A, B be sets. Do we always have $\#A = \#(A \cap B) + \#(A \setminus B)$? Do we always have $\#A = \#B + \#(A \setminus B)$?

5. **Pick up the right one:** Which of the following is a partition of the set $S = \{a, b, c, d\}$, and why?

   (1) $T_1 = \{a\}$, $T_2 = \{b, d\}$, $T_3 = \{c, c\}$

   (2) $T_1 = \{a, b\}$, $T_2 = \{b, c\}$, $T_3 = \{d\}$

   (3) $T_1 = \{a, b\}$, $T_2 = \{d\}$

6. **The antigen count:** By a blood analysis over 300 people we had the following results: 150 had the antigen A, 120 the antigen B, and 100 had none. How many had both?

7. **Aerobics, Ballet or Cricket?** During a sports day 64 pupils could choose to try the following activities: Aerobics (A), Ballet (B) or Cricket (C). We have the following information:

2 tried nothing;

34 tried (A), 30 tried (B), 33 tried (C);

11 tried (A) and (B), 15 tried (B) and (C), 17 tried (A) and (C);

19 tried exactly two among (A),(B),(C).

How many pupils tried all (A),(B),(C)?
How many tried only (A)?
How many pupils tried (C) and (B), but not (A)?

ANY QUESTION? JUST ASK!