MathDay 2022 Intermediate

Questions (12 questions without proof)

1. A fairground booth boasts a game involving the wheel of fortune. The numbers 1 to 5 are written on the wheel, and when it is spun, every number occurs with equal probability.

   To play the game, you place a bet on any number between 2 and 10 (inclusive) of your choice. The wheel is then spun twice and if the sum of the two resulting numbers equals the number you bet on, you win a teddy-bear. Otherwise, you don't win anything. What number should you bet on to maximize your chance of winning the teddy-bear?

   Correct answer: 6
   Solution: Call x the result of the first spin, and y the result from the second spin (both x and y are integers from 1 to 5). When you count all the possible sums the pair (x, y) can add up to, 6 occurs five times, while all other numbers occur less often.

2. What is the maximum number of queens you can put on a 5×5 chessboard, so that no two queens are in the same diagonal, row or column?

   Correct answer: 5
   Solution: For example, put them on a4, b2, c5, d3, e1. It cannot be more than 5, because there are 5 columns and you can put at most one queen per column.

3. Sixty children received an invitation to a birthday party, but not all of them came. At the party, the children played a game for teams of 12 players and no child was left without a team. They also played a game for teams of 5 players; one child was left without a team and became referee for this game. How many children were at the party?

   Correct answer: 36.
   Solution: We look for a number from 0 to 60 which is a multiple of 12 and which leaves remainder 1 on division by 5. The only such number is 36.

4. You have two identical apples, two identical oranges, and one banana. You have to give them to five children, so that each child receives exactly one piece of fruit. In how many different ways can you distribute the pieces of fruit to the children?

   Correct answer: 30
   Solution: You only have to choose whom to give the banana to (5 choices) and

whom the two oranges. Discarding the child who took the banana, there are four children left and you have to select a pair of children for the oranges (12 choices, as there are 4 choices for the first child, 3 choices for the second child). As the ordering does not matter, you divide by 2 and are left with 6 possibilities. Hence there are 30 possibilities in total.

5. In a foreign land there is a currency called AUR. There are golden coins with values 1AUR, 3AUR and 9AUR. What is the smallest number of coins you need to be able to pay any bill in the range from 1AUR to 107AUR? You can choose the coins freely, but you have to choose them before knowing the amount of the bill.

   Correct answer: 15.
   Solution: You can take 11 9AUR coins, 2 3AUR coins, 2 1AUR coin. This is optimal because with only 10 or fewer coins of 9AUR, you would need at least 17 coins to pay 107 AUR (10 x 9AUR + 5 x 3AUR + 1 x 1AUR), and with 11 coins of 9 AUR you need at least 4 more coins to pay 107 AUR. The given coins allow to pay any bill in the range: using the 9AUR coins one is left to pay an amount between 1AUR and 8AUR, that can be paid with the given 3AUR or 1AUR coins.

6. Alice and Zoe, when they run alone, always run at their usual constant speed. Alice runs 1 kilometer in 4:10 (4 minutes 10 seconds), while Zoe runs 1 kilometer in 5:00. They planned to run together on a 11 km long straight path along a river. But they just texted each other and found that, due to a misunderstanding, they are at opposite ends of the path. Now they start running towards each other. After how much time will they meet? Give your answers in minutes.

   Correct answer: 25.
   Solution: Alice runs 1km in 250 seconds, Zoe in 300 seconds, so Alice's speed is 6/5 of Zoe's speed. Hence they will meet after Alice made 6/11 of the total distance and Zoe 5/11 of the total distance. Thus Alice has run 6km, which will take 25 minutes.

7. You arrive on an island that is inhabited by 7 dwarves. A dwarf can either be a truth-teller or a liar. The truth-tellers always speak the truth and the liars always lie. All dwarves queue in a straight line to greet you. They all look into your direction.
   The first dwarf in the line says: "All dwarves behind me are liars."
   All other dwarves say: "The dwarf right in front of me is a liar."
   How many dwarves are liars?

   Correct answer: 4
   Solution: Dwarves just behind liars are truth-tellers, hence the first dwarf must be a liar. Dwarves just behind truth-tellers are liars, hence liars and truth-tellers alternate, giving a total of 4 liars.

8. You are in a video call with some friends who are native speakers of the Combish language. You only remember four different words in this language: Xix, Yiy, Ziz, Wiw. Exactly one of them is extremely funny. You know that if you send some of these words to any of your friends, then that friend will start laughing immediately if and only if the funny word is among the words you chose.

   You can write exactly one message with some Combish words to each of your friends in the call. You can choose how many words and which words to write. You can send different individual messages, but all messages are sent at the same time. You are then able to check in the video call who is laughing. What is the minimum number of friends that you need in the call so that you are able to determine without doubt the funny word by using the above method?

   Correct answer: 2
   Solution: One friend is not enough because from testing only one message you cannot determine the funny word in all possible cases. Two friends are enough: you send the first word only to the first friend, the second word only to the second, the third word to both, and the fourth word to none.

9. Amy and Ben play the Candy Game. At the beginning of the game, there are 10 candies. Amy and Ben take turns making moves. A move consists of removing either 2 or 3 candies. The first player that cannot make a move (because there are less than 2 candies left) looses. Amy makes the first move. If both Amy and Ben aim to win and play according to the best possible strategy, who wins the game? Answer 1 for Amy and 2 for Ben.

   Correct answer: 2
   Solution: Consider the number of remaining candies. When you see 0 or 1 candies on the table, you lose. When there are 2,3 or 4 candies, you win (for 4 candies take 3 candies, for 3 candies take 2 or 3 candies, for 2 candies take 2 candies). When there are 5 or 6 candies, you lose (taking either 2 or 3 candies puts the other player in a winning situation). When there are 7,8 or 9 candies you win (taking either 2 or 3 candies puts the other player in a losing situation). When there are 10 candies, you lose. (taking either 2 or 3 candies puts the other player in a winning situation). So the second player always wins, i.e. Ben.

10. A very modern museum of very modern art has two floors, one on top of the other. Each floor consists of four corridors connected in the form of a square; it is possible to go from each corridor to the two neighbouring ones on the same floor; at the end of each corridor there is a staircase connecting vertically the two floors; the only entrance is also the only exit and it is located at one corner on the ground floor. You want to walk along each corridor exactly once (the direction doesn't matter to you). You can walk along different tours, according to the order in which you visit the corridors. How many different tours are there, assuming that you use the stairs exactly twice?

Correct answer: 16
Solution: You must use the same staircase for going up and down. Then what you can choose is: the staircase (4 possibilities) and the direction of your visit on the lower floor (2 possibilities), and the direction of your visit on the upper floor (2 possibilities). This gives a total of $4 \times 2 \times 2 = 16$ tours.

11. You have a coin that when being thrown comes up heads more often than tails. You and a friend play the following game: You toss the coin twice. If the result is twice the same, you win. If the results of the tosses are different, your friend wins. Who is more likely to win? Answer 1 if you have a higher probability to win, answer 2 if your friend has a higher probability to win, answer 3 if you and your friend have the same probability to win.
Correct answer: 1
Solution: Let h be the probability of coming up heads and $1-h$ the one of coming up tails. Getting the same result happens with probability $h^2+(1-h)^2 = 1+2h^2-2h$ while getting different results happens with probability $2h(1-h) = 2h-2h^2$. The difference is $(1+2h^2-2h)-(2h-2h^2) = 1-4h+4h^2 = (1-2h)^2$. This is strictly positive, unless $h = 1/2$, which we excluded. So getting the same result is more likely than getting two different results.

12. You own a bag of letter-shaped pasta for children. There are 26 different letters. If you take 99 pieces of pasta from the bag, what is the largest integer number n such that you can be sure to have at least n pieces representing the same letter?

Correct answer: 4
Solution: Pigeonhole principle. With 99 objects of 26 types, there are at least $\lceil 99/26 \rceil = 4$ pieces of pasta of the same kind.