MathDay 2022 Senior

First part (6 questions without proof)

1. You arrive on an island that is inhabited by 7 dwarves. A dwarf can either be a truth-teller or a liar. The truth-tellers always speak the truth and the liars always lie. All dwarves queue in a straight line to greet you. They all look into your direction.
   The first dwarf in the line says: "All dwarves behind me are liars."
   All other dwarves say: "The dwarf right in front of me is a liar."
   How many dwarves are liars?

   Correct answer: 4
   Solution: Dwarves just behind liars are truth-tellers, hence the first dwarf must be a liar. Dwarves just behind truth-tellers are liars, hence liars and truth-tellers alternate, giving a total of 4 liars.

2. You are in a video call with some friends who are native speakers of the Combish language. You only remember four different words in this language: Xix, Yiy, Ziz, Wiw. Exactly one of them is extremely funny. You know that if you send some of these words to any of your friends, then that friend will start laughing immediately if and only if the funny word is among the words you chose.

   You can write exactly one message with some Combish words to each of your friends in the call. You can choose how many words and which words to write. You can send different individual messages, but all messages are sent at the same time. You are then able to check in the video call who is laughing. What is the minimum number of friends that you need in the call so that you are able to determine without doubt the funny word by using the above method?

   Correct answer: 2
   Solution: One friend is not enough because from testing only one message you cannot determine the funny word in all possible cases. Two friends are enough: you send the first word only to the first friend, the second word only to the second, the third word to both, and the fourth word to none.

3. Amy and Ben play the Candy Game. At the beginning of the game, there are 10 candies. Amy and Ben take turns making moves. A move consists of removing either 2 or 3 candies. The first player that cannot make a move (because there are less than 2 candies left) looses. Amy makes the first move. If both Amy and Ben

aim to win and play according to the best possible strategy, who wins the game? Answer 1 for Amy and 2 for Ben.

Correct answer: 2
Solution: Consider the number of remaining candies. When you see 0 or 1 candies on the table, you lose. When there are 2,3 or 4 candies, you win (for 4 candies take 3 candies, for 3 candies take 2 or 3 candies, for 2 candies take 2 candies). When there are 5 or 6 candies, you lose (taking either 2 or 3 candies puts the other player in a winning situation). When there are 7,8 or 9 candies you win (taking either 2 or 3 candies puts the other player in a losing situation). When there are 10 candies, you lose. (taking either 2 or 3 candies puts the other player in a winning situation). So the second player always wins, i.e. Ben.

4. A very modern museum of very modern art has two floors, one on top of the other. Each floor consists of four corridors connected in the form of a square; it is possible to go from each corridor to the two neighbouring ones on the same floor; at the end of each corridor there is a staircase connecting vertically the two floors; the only entrance is also the only exit and it is located at one corner on the ground floor. You want to walk along each corridor exactly once (the direction doesn't matter to you). You can walk along different tours, according to the order in which you visit the corridors. How many different tours are there, assuming that you use the stairs exactly twice?

   Correct answer: 16
   Solution: You must use the same staircase for going up and down. Then what you can choose is: the staircase (4 possibilities) and the direction of your visit on the lower floor (2 possibilities), and the direction of your visit on the upper floor (2 possibilities). This gives a total of $4 \times 2 \times 2 = 16$ tours.

5. You have a coin that when being thrown comes up heads more often than tails. You and a friend play the following game: You toss the coin twice. If the result is twice the same, you win. If the results of the tosses are different, your friend wins. Who is more likely to win? Answer 1 if you have a higher probability to win, answer 2 if your friend has a higher probability to win, answer 3 if you and your friend have the same probability to win.
   Correct answer: 1
   Solution: Let h be the probability of coming up heads and $1-h$ the one of coming up tails. Getting the same result happens with probability $h^2+(1-h)^2 = 1+2h^2-2h$ while getting different results happens with probability $2h(1-h) = 2h - 2h^2$. The difference is $(1+2h^2-2h)-(2h-2h^2) = 1-4h+4h^2 = (1-2h)^2$. This is strictly positive, unless $h = 1/2$, which we excluded. So getting the same result is more likely than getting two different results.

6. You own a bag of letter-shaped pasta for children. There are 26 different letters. If you take 99 pieces of pasta from the bag, what is the largest integer number n

such that you can be sure to have at least n pieces representing the same letter?

Correct answer: 4

Solution: Pigeonhole principle. With 99 objects of 26 types, there are at least $\lceil 99/26 \rceil = 4$ pieces of pasta of the same kind.

Second part (3 problems with proof)

1. Prove that the square of a non-negative integer never leaves remainder 2 or 3 on division by 4.

   An even number is divisible by 2, so its square is divisible by 4 hence leaves remainder 0 on division by 4. An odd number is of the form $n = 2t+1$, where t is an integer. So its square is $n^2 = (2t+1)^2 = 4t^2+4t+1 = 4(t^2+t)+1$. Thus the remainder on division by 4 is 1. As the possible remainders are only $0, 1$, the remainders $2, 3$ are impossible.

2. Let $ABCD$ be a rectangle made of paper. Folding the piece of paper in two along some line and opening it again makes the line appear as a crease on the paper as if one had drawn it.

   (a) Fold the paper in two such that A lies on B and D lies on C.
   Call the resulting line L.

   (b) Fold the paper in two such that the segment BC lies on L.
   Call the resulting line L'.

   (c) Suppose that you can fold the paper in two such that A lies on L' and such that the resulting line, which we call L'', passes through D and through the intersection of AB and L.

   What is the ratio between the largest and the smallest side of the rectangle $ABCD$?

   Let $|AB| = x$, $|BC| = y$. Let M be the midpoint of A and B and N the midpoint of M and B. Notice that L and L' are parallel to AD and BC and go through M and N respectively.

   Let P be the point on L' such that $|AM| = |MP|$. We have $\tan(\measuredangle AMD) = \frac{y}{\frac{1}{2}x}$.
   Then, as $\cos(\measuredangle NMP) = \frac{\frac{1}{4}x}{\frac{1}{2}x} = \frac{1}{2}$, we deduce $\measuredangle NMP = \frac{\pi}{3}$.
   Hence, $\measuredangle AMD = \frac{1}{2}\measuredangle AMP = \frac{1}{2}(\pi - \frac{\pi}{3}) = \frac{\pi}{3}$. This implies $\frac{x}{y} = \frac{2}{\tan(\measuredangle AMD)} = \frac{2}{\sqrt{3}}$.

3. (a) Let a be a strictly positive real. Show that

$$a + \frac{1}{a} \geq 2.$$

   (b) Let x, y, z be strictly positive reals such that:

$$\begin{cases} a = x + y - z > 0 \\ b = x - y + z > 0 \\ c = -x + y + z > 0. \end{cases}$$

Prove the following inequality:

$$x + y + z - \frac{x^2 + y^2 + z^2 - 2xy - 2yz - 2xz}{(x + y - z)(x - y + z)(-x + y + z)} \geqslant 6\,.$$

Observe that $a + b + c = x + y + z$ and that

$$x^2 + y^2 + z^2 - 2xy - 2yz - 2xz = -ab - ac - bc\,.$$

It follows

$$\begin{aligned}
&x + y + z - \frac{x^2 + y^2 + z^2 - 2xy - 2yz - 2xz}{(x + y - z)(x - y + z)(-x + y + z)} \\
&= a + b + c + \frac{ab + ac + bc}{abc} \\
&= a + \frac{1}{a} + b + \frac{1}{b} + c + \frac{1}{c} \\
&\geq 2 + 2 + 2 = 6.
\end{aligned}$$