The pigeonhole principle

Pigeonhole principle / Principe des tiroirs / Schubfachprinzip

The pigeonhole principle is an intuitive and yet powerful statement about set theory. Elements are visualized as objects (pigeons, socks…) and sets as containers (holes, drawers…). The principle simply states: **if there are more objects than containers, then at least one container has more than one object** (to be precise: provided that there is at least one container, and let us stick to finitely many objects and containers).

If you have seven dices then at least two of them show the same number. Here the pigeonhole principle applies because you have 7 objects (the dices) and 6 containers (the numbers on a dice), and 7 is bigger than 6. If you had only 6 dices then the pigeonhole principle would not have helped at all.

You can also apply the pigeonhole principle to answer this question:
If you have socks of five different colours (and it is dark), what is the minimal number of socks you have to pick in order to be sure to have a pair of the same colour?
Considering the socks as objects and the colours as drawers, six socks suffice…

Suppose that you have six socks which are either black or white. By the pigeonhole principle you have at least two socks of the same colour. In fact, at least three of them have the same colour. And if you have five socks which are either black or white, again at least three of them have the same colour.

The **quantitative version of the Pigeonhole principle** says that there is at least one container with at least N objects, where the number N is obtained as follows:
divide the number of objects minus 1 by the number of containers;
if this ratio is not an integer, then round it down to an integer;
add 1.

For example, if you have 31 socks and 3 colours you get $N = 11$, and if you have 30 socks you get $N = 10$.

Some general facts if you want to apply the pigeonhole principle to solve a mathematical problem:

- You have to decide in the problem what to treat as objects and what as containers. Sometimes there is an obvious choice, and sometimes you need a clever one.
- It is of crucial importance to have strictly more objects than containers. There may still be the possibility of showing that some containers must be empty (and then reducing to a favourable situation with less containers than objects).
- The containers could be sets coming from geometry. *For example, consider seven people on Earth: at least four of them are in one same hemisphere.*
- The containers (as sets) need not to look alike or have the same number of elements. *For example, if you pick three numbers among* $17, 27, 37, 14, 24$ *then at least two of them have the same last digit. As containers you can choose the set* $\{17, 27, 37\}$ *and the set* $\{14, 24\}$*.*
- The containers may have to be constructed from the problem. *For example, if you pick three distinct numbers among* $2, 3, 4, 6$ *then you can multiply two of them and obtain* 12*. Simply choose as containers the sets* $\{2, 6\}$ *and* $\{3, 4\}$*.*
- Some problems have the pigeonhole principle only as part of their solution, and you have to treat other parts in a different way.
- Sometimes a problem has more than one solution, and using the pigeonhole principle is only one of the possibilities. Remember that in mathematics you can always choose the solution that you like best!

Problems around the pigeonhole principle

1. **Should we shake hands?** You are at a party...and in the end you may have shaken hands with nobody, somebody, or everybody else...
   (you cannot shake hands with yourself, and shaking hands more times with the same person does not count).
   Are there two distinct people at the party that have shaken hands with exactly the same number of people, and why? Is there someone else at the party that has shaken hands with exactly as many people as you did?

2. **Friends or strangers?** *Prove that in any group of six people there are either three mutual friends or three mutual strangers* (friendship is reciprocal: two people are either mutual friends or mutual strangers).

3. **Can you do slacklining?** If you want to become the next slackline champion, you really need to train. Fortunately, you can access a garden (which has the form of a square) and there are five trees, all of which would be suitable... Your slackline is some meters longer than half of the diagonal of the garden.
   Prove that your slackline is long enough!

4. **Dominoes on the chessboard?** What can you do if you have lost your set of chess pieces, and your chessboard has lost two diagonally opposite corner squares? Well, play with dominoes! *Try to cover the chessboard with dominoes that each cover exactly two squares... Is it possible at all?*

5. **Should we play Nine Dwarves?** In the famous card game “Nine Dwarves” you have cards representing 1 to 8 Dwarves and you win as soon as you have two cards whose sum is exactly 9 Dwarves. (All cards show a different number of Dwarves.) *Prove that having five cards in your hand is enough to win!*

You can also solve Problem 4 without the pigeonhole principle (nevertheless, you can ask yourself how the principle can be applied).

ANY QUESTION? JUST ASK!