# MATH DAY 2023

## INSTRUCTIONS:

### Exercises

JUNIOR: Exercises 1 to 12 included.
INTERMEDIATE: Exercises 5 to 16 included.
SENIOR: Exercises 11 to 16 included and the three Proof Exercises.
**WARNING** Exercises beyond the requested ones will not be counted!

### Grading

The exercises are multiple choice and graded as follows: 3 points for the correct answer, 0 points for the wrong answer, 1 point for no answer. Only one answer per question is correct.
Only for Senior: The proof exercises are graded with 9 points each. Written solutions have to be provided.

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*Please fill in the information below. Please write your answers to the multiple choice exercises into the table on the following page.*

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## FIRST NAME

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## SURNAME

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## CATEGORY

(Junior/Intermediate/Senior)

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| Exercise | Category | Answer (A,B,C,D,or E) |
|---|---|---|
| 1 | J | |
| 2 | J | |
| 3 | J | |
| 4 | J | |
| 5 | J, I | |
| 6 | J, I | |
| 7 | J, I | |
| 8 | J, I | |
| 9 | J, I | |
| 10 | J, I | |
| 11 | J, I, S | |
| 12 | J, I, S | |
| 13 | I, S | |
| 14 | I, S | |
| 15 | I, S | |
| 16 | I, S | |

## Exercises

1. The twins Erik and Oskar have to share their favorite game for one year (January 1st to December 31st). The year is not a leap year. They decide that Erik will have the game on days which have an even number in the month (the 2nd of each month, the 4th of each month and so on) and Oskar on days which have an odd number in the month. However, they realize that there are more odd days than even days. In fact, how many more?

   A: 2

   B: 5

   C: 7

   D: 12

Answer: C.

Solution: The months with 28 or 30 days have the same number of even and odd days. The seven months with 31 days have one odd day more. So the answer is 7 days.

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2. Two gentlemen, Mr Adams and Mr Beckam, arrive at a door at the same time, and have to agree upon who enters first. At regular intervals, they make an attempt to find an agreement. At every attempt, they can speak (saying "You go first") or stay silent. If they both speak, nobody enters. If nobody speaks, nobody enters. If only one speaks, the silent gentleman enters.
   If they speak at the same time, Mr Adams will stay silent for at least one round, and Mr Beckam for at least two rounds. If nobody has spoken for at least 3 consecutive rounds, then Mr Beckam will speak in the next round. Neither gentleman knows the rules governing the other gentleman, and they do not agree on a strategy. Must there eventually come a round where one gentleman enters the door?

   A: Yes

   B: No

Answer: B.

Solution: For example, they might both speak on every fourth turn and stay both silent in between (silent for three turns).

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3. There are four statues of an elf which only differ in their size. The sizes are XL (Extra Large), L (Large), M (Medium), S (Small) respectively. They have different weights decreasing in this order, i.e. the largest statue is the heaviest, the second-largest the second-heaviest etc. Your friend weighed the statues in pairs, and the weight of the various pairs, in grams, are

$$18, 24, 30, 30, 36, 42\,.$$

What is the total weight in grams of the four statues?

A: 60

B: 120

C: 150

D: 180

Answer: A.

Solution: Sum the weights and divide by 3, or sum the smallest weight (the two lightest statues) and the largest weight (the two larger statues). One could also have summed the two intermediate weights, which must be XL + S and L + M (and it is a coincidence that those two weights are the same, any of the two could be larger). Or one could have summed the second weight (XL + M) and the penultimate weight (L + S). In any case, the answer is 60.

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4. A kid is playing with a car that can be remotely maneuvered. The car can only turn left or right at 90° angles. Moreover, it turns exactly once after each meter it has gone, and the kid can only decide whether it will be a left or a right turn. The kid is playing on a 3 meter × 4 meter rectangular carpet. Starting from one carpet corner parallel to one of the carpet sides, how many further carpet corners can be reached by the car?

A: 1

B: 2

C: 3

Answer: C.

Solution: The answer is 3, as the car can reach all further corners. By symmetry, it does not matter at which corner we start. With coordinates, let (0,0) be the starting corner, and let (4,3) be the opposite corner. Some paths going to the further 3 corners are:

$$(0,0),(1,0),(1,1),(2,1),(2,2),(3,2),(3,3),(4,3)$$

$$(0,0),(1,0),(1,1),(2,1),(2,2),(3,2),(3,1),(4,1),(4,0)$$

$$(0,0),(1,0),(1,1),(0,1),(0,2),(1,2),(1,3),(0,3)$$

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5. There is a world similar to ours, but where people are either liars or truth-tellers. Liars always lie and truth-tellers always tell the truth. A child says: "Everyone in my family is a liar". Is this assertion:

   A: True

   B: False

   C: Forcibly neither true nor false

Answer: B.

Solution: The assertion cannot be true because, if true, the child would be a liar (as part of the family) telling the truth. The assertion can be false, for example if only half of the family members are liars.

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6. You are preparing a yoga session where the participants will be seated on chairs arranged in a row. The chairs are next to one another with no space in between. The participants must however be able to stretch their arms sideways without touching one another, therefore between any two occupied seats, there must be two empty seats. After setting up the row of chairs by taking all the available space, you decide to remove two chairs from one side because doing this does not decrease the number of seats available to the participants. What is the remainder after division by 3 of the number of chairs in the row, after having removed the two chairs?

   A: 0

   B: 1

   C: 2

Answer: B.

Solution: It is optimal to start the row with an occupied seat. Adding two chairs does not make a difference if and only if the last seat is also occupied, which means that the remainder after division by 3 equals 1.

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7. You are allowed to draw $N$ balls from an urn containing many balls of each of the following colors: blue, green, yellow, red. You win once you draw 1 blue ball, or 2 green balls, or 3 yellow balls, or 4 red balls. What is the least possible value of $N$ for which you are guaranteed to win?

   A: 4

   B: 5

   C: 6

   D: 7

Answer: D.

Solution: If you draw 1 green ball and 2 yellow balls and 3 red balls, then you lose (this is the most unfortunate situation). Once you draw any additional ball, you are guaranteed to win. Hence $N = 7$.

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8. In a shop, they sell triangular tiles, where each tile has the side lengths 8,12, and 18 centimeters. You like the shape, but you need a larger tile. The triangle you want is similar to the ones sold at the shop and the side lengths, in centimeters, are again integers. Moreover, two of its side lengths are 12, and 18 centimeters. What is the third side length, in centimeters?

   A: 24

   B: 27

   C: 30

   D: 36

Answer: B.

Solution: Call $X$ the missing side length. We must have $8/12 = 12/18 = 18/X$ hence $X = 27$.

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9. Alice takes 4 hours to paint a fence while Bob takes 12 hours for the same task. How many hours will it take if they work together on the task?

   A: 1

   B: 2

   C: 3

   D: 4

Answer: C.

Solution: If Alice and Bob work for $X$ hours, the fraction of the complete task they get done is $\frac{1}{4}X + \frac{1}{12}X$. The task gets completed when this fraction equals 1, that is, if $X = 3$.

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10. You are organizing a movie day at your school. You know that 50% of the pupils have watched "Aliens and Alligators" (movie $A$) and 40% of the pupils have watched "Basketball and Biscuits" (movie $B$). You know that 20% of the pupils who watched movie $A$ also watched movie $B$. What percentage of the pupils who watched movie $B$ also watched movie $A$?

    A: 5%

    B: 10%

    C: 25%

    D: 50%

Answer: C.

Solution: Say (without loss of generality) that we have 100 pupils in total. Then 10 pupils watched movies $A$ and $B$, which corresponds to 25% of the pupils who watched movie $B$.

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11. In your favorite restaurant, you can buy tofu nuggets to go. They are sold in packs of 3, 5, or 7. It is thus impossible to order, for instance, precisely 2 tofu nuggets. How many integers $n \geq 1$ exist for which you cannot buy precisely $n$ nuggets?

    A: 1

    B: 2

    C: 3

    D: 4

    E: 5

Answer: C.

Solution: If the amount of nuggets is a multiple of 3 we can buy several 3-bags. If it is a multiple of 3 plus 2 (at least 5), we can buy a bag of 5 and possibly several 3-bags, if it is a multiple of 3 plus 1 (at least 7), then we can buy a 7-bag and possibly several 3-bags. So the only amounts of nuggets that we cannot buy are 1,2, and 4.

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12. Your class is preparing for an excursion and your teachers are filling lunch bags with sandwiches, one lunch bag for each pupil. The lunch bags and the sandwiches are all alike. Each sandwich can be cut into 2 or 3 equal parts. You know that with 14 sandwiches, one can fill 9 lunch bags but not 10 lunch bags. How many sandwiches were used to prepare lunch bags for the 24 pupils?

A: 24

B: 30

C: 32

D: 36

E: 48

Answer: D.

Solution: The amount of sandwich in each bag is in particular a rational number of the form $X/6$. From the information about 14 sandwiches, we know that $14/10 \leq X/6 \leq 14/9$. We deduce that $8 < X < 10$, hence $X = 9$. So each bag contains $9/6 = 3/2$ sandwiches, so for 24 bags one needs $24 \cdot 3/2 = 36$ sandwiches.

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13. In a bag there are balls, each of which is coloured red, yellow or green. We pick two balls in a row, without putting the first one back. The probability of picking two red balls is $\frac{1}{7}$ and the one of picking two yellow balls is $\frac{1}{5}$. What is the least possible number of balls in the bag to make this true?

A: 15

B: 35

C: 70

Answer: A.

Solution: Let $n$ be the number we are looking for. Let $r$ and $y$ be the number of red and yellow balls, respectively. We are given that

$$\frac{r(r-1)}{n(n-1)} = \frac{1}{7}, \qquad \frac{y(y-1)}{n(n-1)} = \frac{1}{5},$$

hence $7r(r-1) = n(n-1) = 5y(y-1)$. This shows that $n(n-1)$ is divisible by 5 and by 7, thus by 35. The least $n \geq 1$ for which this is the case is $n = 15$. To see that $n = 15$ is actually suitable, notice that $r = 6$ and $y = 7$ lead to the desired probabilities.

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14. The new Luxembourgish car plates consist of two letters followed by a 4-digit number. As the letter $O$ and the digit 0 look too similar on plates, one of the two symbols needs to be forbidden. Which symbol should we forbid to achieve the largest number of possible car plates? (The considered alphabet has 26 letters.)

    A: The letter $O$.

    B: The digit 0.

    C: The choice does not matter.

Answer: A.

Solution: Forbidding the letter $O$ leaves $25^2 \cdot 10^4$ allowed car plates, while forbidding the digit 0 only leaves $26^2 \cdot 9^4$. We conclude because $25 \cdot 10^2 > 26 \cdot 9^2$.

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15. How many integers $n \geq 1$ exist for which $16n^2 + 25$ is the square of a natural number?

    A: 0

    B: 1

    C: 2

    D: 5

Answer: B.

Solution: We have $(4n)^2 < 16n^2+25 < (4n+5)^2$, hence a perfect square of the form $16n^2 + 25$ must be among $(4n+1)^2$, $(4n+2)^2$, $(4n+3)^2$ and $(4n+4)^2$. Since $16n^2 + 25$ is odd, it is neither $(4n+2)^2$ nor $(4n+4)^2$. Putting $16n^2 + 25 = (4n+1)^2$ gives $n = 3$. Putting $16n^2 + 25 = (4n+3)^2$ yields $n = \frac{2}{3}$, which is not an integer. Hence only the value $n = 3$ is counted.

Alternative solution: Assume $16n^2+25 = x^2$, where $x$ is a positive integer. Then $25 = x^2 - 16n^2 = (x+4n)(x-4n)$, and since $25 = 5^2$ is the square of a prime, the only possibility is given by the equations $x + 4n = 25$ and $x - 4n = 1$, which give $n = 3$ and $x = 13$.

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16. There are five small statues of an elf, and you know their combined weight $W$. The statues differ in their sizes, which are XL (Extra Large), L (Large), M (Medium), S (Small), XS (Extra Small) and they have different weights decreasing in this order. Your friend weighed the statues in pairs, and wrote the list of values

$$W_1, W_2, \ldots, W_{10}$$

from the largest to the smallest. Knowing only $W_2 + W_{10}$, which is the statue of which you can find out the weight?

A: XL

B: L

C: M

D: S

E: XS

Answer: B.

Solution: Call the weight of a statue with its size, for example let $XL$ be the weight of the largest and heaviest statue. Clearly $W_1 = XL + L$ and, as the two weights we sum must be distinct, we have $W_2 = XL + M$. Similarly, we have $W_{10} = S + XS$. Thus $W_2 + W_{10} = XL + M + S + XS$. Considering that $L = W - (W_2 + W_{10})$, we can compute the weight of $L$.

## Proof Exercises, only for SENIOR

**Problem 1.** *Let $ABCD$ be a rectangle with $AB > BC$ and such that $BC = 1$. Let $E$ be the point different from $B$ such that $CE = BC$ and $\angle AEC = 90°$. Assume that $\angle DCE = 30°$. Find the length $AB$.*

***Answer:*** $AB = \sqrt{3}$.

*Proof.* Indeed, $\angle EAC = \angle BAC$ (as can be seen since $E$ is the reflection of $B$ with respect to $(AC)$, or also since $E, C, B, A$ are on a circle and the angle bisector of $\angle EAB$ and the perpendicular bisector of $[BE]$ meet in the same point $C$ of the circle). Thus

$$\alpha := \angle EAC = \angle BAC = \angle ACD$$

and hence $\angle DCE = 90° - 2\alpha$. If $\angle DCE = 30°$, then $\alpha = 30°$ and

$$AB = \frac{BC}{\tan(\alpha)} = \sqrt{3}.$$ □

**Problem 2.** *You are trying to crack a code of 8 digits. You know that the code represents the birthday of someone who was born between the year 1 (included) and today, in the format ddmmyyyy (for example, today's date would be 25022023). You also have the following information:*

- *The day is a prime number;*
- *the prime decomposition of the year is given by $p \cdot (p + 10) \cdot (2p + 3)$, where $p$ is a prime number;*
- *the person was born on a Wednesday.*

*What is the code?*

***Answer:*** *11012023.*

*Proof.* If you go through the primes in order, starting with 2, the first prime $p$ for which $p + 10$ and $2p + 3$ are also prime, is $p = 7$. In this case we get $7 \cdot 17 \cdot 17 = 2023$ as year. Larger primes result in years later than 2023 and thus can be discarded. So the day must have been in January or February 2023 ('before today, the 25 February'). One can count back from the date of the competition, and the only Wednesday in 2023 so far that fell on a prime day is 11 January 2023. So the code is 11012023. □

**Problem 3.** *The following is an integer; find its value:*

$$2\sqrt{\frac{2\sqrt{5} - 4}{\sqrt{5} + 1}} + \sqrt[3]{17\sqrt{5} - 38}.$$

***Answer:*** 1.

*Proof.* Denote the mentioned number by $x$. Then

$$\begin{aligned}
x &= 2\sqrt{\frac{(2\sqrt{5}-4)(-\sqrt{5}+1)}{(\sqrt{5}+1)(-\sqrt{5}+1)}} + \sqrt[3]{(\sqrt{5}-2)^3} \\
&= 2\sqrt{\frac{6\sqrt{5}-14}{-4}} + (\sqrt{5}-2) \\
&= \sqrt{-6\sqrt{5}+14} + \sqrt{5}-2 \\
&= \sqrt{(-\sqrt{5}+3)^2} + \sqrt{5}-2 \\
&= -\sqrt{5}+3+\sqrt{5}-2 \\
&= 1.
\end{aligned}$$

□