MATH DAY 2024

Exercises

1. In ancient Egypt, a rich person had many ushabtis (small statues representing immortal servants). They had “normal ushabtis”, one for every day of the year, plus they had one “leader ushabti” for every full group of 10 normal ushabtis. What was the total amount of ushabtis?
   Comment: By year we mean 52 weeks and 1 day.

   A: 400

   B: 401

   C: 410

   D: 411

Answer: B.

Solution: There were 365 normal ushabtis and 36 leader ushabtis, which makes a total of 401 ushabtis.

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2. If six 3D printers print six mugs in six minutes, how many 3D printers are required to print twelve mugs in twelve minutes?
   Comment: The printers are all alike, the mugs are all alike.

   A: 24

   B: 12

   C: 6

   D: 3

Answer: C.

Solution: The given six printers print twelve mugs (twice the amount) in twelve minutes (twice the time).

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3. A clown is alternatingly telling one true fact and one lie. The clown says, in order, the following sentences:
– I like mushrooms.
– I don't like mushrooms.
– I don't like mushrooms and I don't like onions.
– I like either mushrooms or onions.
Which of the following assertions is correct?

A: The clown likes onions and mushrooms.

B: The clown likes onions but not mushrooms.

C: The clown likes mushrooms but not onions.

D: The clown likes neither onions nor mushrooms.

Answer: B.

Solution: It's not possible that the first and third assertion are both true (the first says that the clown likes mushrooms, the third says in particular that the clown does not like mushrooms). So it's the second and fourth assertions that are true. Then the clown likes either mushrooms or onions, and he does not like mushrooms. So the clown likes onions but not mushrooms.

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4. Consider a regular hexagon and a rhombus with the same side-length, and such that two angles in the rhombus are the same as the hexagon's angles. What is the ratio between the area of the hexagon and the area of the rhombus?

A: 1

B: 2

C: 3

D: 4

Answer: C.

Solution: Consider the usual partition of a regular hexagon into six equilateral triangles (each triangle has as side one of the hexagon's sides, and the opposite vertex is the center of the hexagon). Then two adjacent triangles form the given rhombus. The area ratio is then 6:2=3.

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5. Four friends Amy, Ben, Chi, Dan have chosen cups in four distinct colors, namely red, green, blue, yellow. You have the following information:

- The two cups of Amy and Chi are yellow and green.

- The two cups of Amy and Dan are red and green.

What is the color of Ben's cup?

A: Red

B: Green

C: Blue

D: Yellow

Answer: C.

Solution: Combining the two given statements, Amy has the green cup. Thus Chi has the yellow cup and Dan has the red cup. We deduce that Ben has the blue cup.

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6. Consider a code of the form $ABBA$, knowing that A and B are distinct numbers from 0 to 9 that satisfy: $A + B = B$ and $A + B = B \times B$. How many possibilities are there for this code?

A: 1

B: 2

C: 3

D: 4

Answer: A.

Solution: The first relation says that $A = 0$. The second relation then becomes $B = B \times B$. This is possible for $B = 0$ and for $B = 1$. However, as A and B must be distinct, we deduce that $B = 1$. So there is only one possibility for the code, namely $ABBA = 0110$.

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7. Elisa and Nicole have some candies. Elisa gives 1/3 of her candies to Nicole and, at the same time, Nicole gives 1/3 of her candies to Elisa. In the end, the number of candies that they have differs by 6. By how much did the number of candies differ before the exchange?

A: 0

B: 6

C: 12

D: 18

Answer: D.

Solution: Call E the number of candies of Elisa, and N the number of candies of Nicole (before the exchange). After the exchange, Elisa has $\frac{2}{3}E + \frac{1}{3}N$ candies, while Nicole has $\frac{2}{3}N + \frac{1}{3}E$ candies. Suppose without loss of generality that Elisa has more candies than Nicole in the end. Then we have $(\frac{2}{3}E + \frac{1}{3}N) = (\frac{2}{3}N + \frac{1}{3}E) + 6$. This gives $\frac{1}{3}E = \frac{1}{3}N + 6$. Thus $E = N + 18$ and the number of candies before the exchange differed by 18.

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8. Your grandparents' cake recipe for 10 people uses 6 eggs and 600 grams of flour. To be faithful to the recipe, it is important to preserve the ratio between eggs and flour. You need to use an integer amount of eggs. How many grams of flour do you use for making a cake that suffices for 8 people?

   A: 400

   B: 450

   C: 480

   D: 500

Answer: D.

Solution: For 8 people you should use $6 \times \frac{8}{10} = 4.8$ eggs, so you take 5 eggs. As you are using $\frac{5}{6}$ of the eggs, then you need to use $\frac{5}{6}$ of the flour, namely 500 grams.

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9. We call a number *supereven* if all of its digits are even numbers. What is the number of supereven numbers from 0 to 1000?

   A: 100

   B: 125

   C: 250

   D: 500

Answer: B.

Solution: We need to consider all non-negative integers with one to three digits (because 1000 is not supereven). To unify the cases, we consider instead an ordered triple of even digits, allowing any of them to be zero. The three digits can be any of the numbers $0, 2, 4, 6, 8$ and they can be chosen independently. So we have three independent choices with 5 possibilities each, leading to $5^3 = 125$ supereven numbers within the given range.

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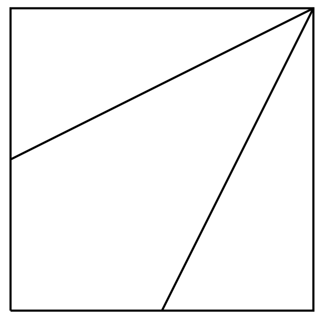
10. Consider a square. Connect one square vertex to the two middle points of the two non adjacent sides. This subdivides the square into a quadrilateral and two triangles. What is the area of the quadrilateral, computed in percentage with respect to the area of the whole square?



A: 60%
B: 55%
C: 54%
D: 50%

Answer: D.

Solution: The two triangles are right triangles: one leg is the side-length of the square, the other leg is half of the side-length of the square. We deduce that each triangle has area 25% of the area of the square. Since the two triangles together cover 50% of the square, the quadrilateral covers the remaining 50% of the square.

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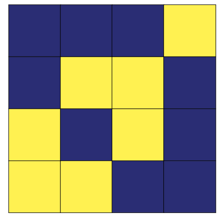
11. Consider the 16 fields of a 4×4 square board. At most how many fields can be chosen so that no four midpoints of the chosen fields form the vertices of a rectangle with sides parallel to the square board?

A: 7
B: 8
C: 9
D: 10

Answer: C.

Solution: If we choose all 4 fields from one row, then we can take at most 1 field in any different row, so in total at most 7 fields.
If we choose 3 fields from a row, then we can choose at most 2 fields in each of the remaining 3 rows. This gives at most 9 fields. And we may choose precisely 9 fields in the following configuration:



If we choose at most 2 fields from each row, we get at most 8 fields. So the largest possible number of fields is 9.

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12. Five teenagers at a dinner share one round table. The designated places are, in clockwise order: Amy, Ben, Cheng, Dan, Ed. However, they totally dislike this arrangement. Each teenager does not want to have as neighbor any of the two foreseen neighbours. Moreover, Amy wishes to keep her designated seat. In how many ways can they rearrange themselves so that their wishes are all fulfilled?

    A: 0

    B: 1

    C: 2

    D: 3

Answer: C.

Solution: The new neighbors for Amy are Cheng and Dan and there are two possibilities, namely Cheng on the left and Dan on the right of Amy or conversely. Dan's new neighbors are Amy and Ben, so Ben sits next to Dan. Finally, Ed sits between Ben and Cheng. Then there are only two possible configurations: Amy, Dan, Ben, Ed, Cheng (naming them either clockwise or countercockwise).

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13. We have some red and blue balls in an urn. The probability of drawing a red ball is 1/3. If we draw a red ball (and we don't put this ball back in the urn), then the probability of drawing another red ball is 1/4. If we draw two red balls, what is the probability of drawing a third red ball?

    A: 1/5

    B: 1/6

    C: 1/7

    D: 1/8

Answer: C.

Solution: Let n be the number of red balls, so there are $2n$ blue balls. After drawing a red ball, there remain $n-1$ red balls, and there are three times as many blue balls, so $2n = 3(n-1)$ yielding $n = 3$. Thus, after drawing two red balls, there remain 6 blue balls and just one red ball. Hence, the probability of drawing a third red ball is $1/7$.

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14. Hannah and Trevor are playing a game of luck. They toss a fair coin multiple times. Hannah wins as soon as HEADS has come out 4 times, while Trevor wins as soon as TAILS has come out 4 times.
They have now tossed the coin 5 times: HEADS has come out 3 times and TAILS has come out 2 times. What is the probability for Hannah to win the game?

   A: 25%

   B: 50%

   C: 75%

   D: 100%

Answer: C.

Solution: Hannah only needs HEADS one more time to win. So, she only loses if two tails come out next in which case Trevor wins. This has probability 25%. Thus, the probability for Hannah's victory is 75%.

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15. Charles is taking part in a 10 km marathon with the aim of achieving an average speed of 12 kilometers per hour. However, after having run for 5 kilometers, he checks his watch and finds out that his average speed so far has only been 10 kilometers per hour. What average speed, in kilometers per hour, must Charles have in the second half of the marathon to meet his goal?

   A: 14

   B: 15

   C: 16

   D: 17

Answer: B.

Solution: By running at 12 kilometers per hour one completes the 10 km marathon in 50 minutes, and this is Charles' goal. It took him 30 minutes to complete the first half of the marathon, so he has 20 minutes to complete the second half. Running 5 km in 20 minutes gives an average speed of 15 km per hour. The answer is then 15 kilometers per hour.

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16. In a rectangle $ABCD$, we have $\overline{AB} = 4\overline{AD}$. The vertices E and F of a parallelogram $ABEF$ are on the line CD and the angle $\widehat{FAB}$ is $30°$. What is the ratio of the perimeters of the parallelogram and the rectangle?

    A: 1,2

    B: 1,5

    C: 1,8

    D: 2

Answer: A.

Solution: If $\overline{AD}$ is 1 unit, then the perimeter of the rectangle $ABCD$ is 10 units. The triangle DAF is half of an equilateral triangle and therefore $\overline{AF} = 2\overline{AD}$. Thus, the perimeter of the parallelogram $ABEF$ is 12 units. Therefore, the ratio of the perimeters is $12/10 = 1.2$.

Solutions to the Proof Exercises

SOLUTION TO PROBLEM 1: Call h the height of the octagon, meaning the distance between two parallel opposite sides. Then the area of the square is h^2. Call s the side-length of the octagon. The complement of the octagon in the square consists of four triangles that are each the half of a (smaller) square with side $\frac{s}{\sqrt{2}}$. So we have

$$h = s + 2 \times \frac{s}{\sqrt{2}} = s\left(1+\sqrt{2}\right)$$

and hence

$$h^2 = s^2\left(1+\sqrt{2}\right)^2 = s^2\left(3+2\sqrt{2}\right).$$

The area of the octagon is then

$$h^2 - 2 \times \left(\frac{s}{\sqrt{2}}\right)^2 = s^2\left(2+2\sqrt{2}\right).$$

The requested ratio is then

$$\frac{3+2\sqrt{2}}{2+2\sqrt{2}} = \frac{1+\sqrt{2}}{2}.$$

An alternative argument is if instead of s we use the side-length t of the four little triangles in the corners. Also, we can assume that the square is a unit square (i.e. $h = 1$). Then the side-length of the octagon is $t\sqrt{2}$, so the side-length of the big square is $1 = (2+\sqrt{2})t$. This gives $t = 1 - \frac{1}{\sqrt{2}}$. The area of the octagon is

$$1 - 2t^2 = 1 - 2(1 - \sqrt{2} + \frac{1}{2}) = 2\sqrt{2} - 2.$$

Thus, the requested ratio is

$$\frac{1}{2\sqrt{2}-2} = \frac{1+\sqrt{2}}{2}. \qquad \square$$

SOLUTION TO PROBLEM 2: The condition means the equation

$$\binom{A}{2} + \binom{B}{2} = AB.$$

So we have $\frac{A(A-1)}{2} + \frac{B(B-1)}{2} = AB$. After ordering according to the powers of B, we obtain

$$B^2 - (2A+1)B + A^2 - A = 0.$$

By the quadratic formula,

$$B = \frac{2A+1 \pm \sqrt{(2A+1)^2 - 4(A^2-A)}}{2} = \frac{2A+1 \pm \sqrt{8A+1}}{2}.$$

The solution with the plus sign clearly satisfies $B \geq A$ but we must discard the solution with the minus sign. Indeed, we have

$$A > \frac{2A + 1 - \sqrt{8A+1}}{2}$$

because this inequality is equivalent to

$$\sqrt{8A+1} > 1\,.$$

Let $8A + 1 = v^2$ (thus, $v \geq 3$ and v is odd). Then we have

$$A = \frac{v^2 - 1}{8} = \frac{\frac{(v+1)}{2}\frac{(v-1)}{2}}{2} = \binom{\frac{v+1}{2}}{2}$$

and

$$B = \frac{2A + 1 + \sqrt{8A+1}}{2} = \frac{\frac{v^2-1}{4} + 1 + v}{2} = \frac{v^2 + 4v + 3}{8} = \frac{\frac{(v+3)}{2}\frac{(v+1)}{2}}{2} = \binom{\frac{v+3}{2}}{2}.$$

Thus $A = \binom{n}{2}$ and $B = \binom{n+1}{2}$, where $n = \frac{v+1}{2} \geq 2$ (since v is odd, n is an integer). So we have

$$A + B = \frac{n(n-1)}{2} + \frac{n(n+1)}{2} = n^2\,.$$

An easy calculation shows that these values of n with the corresponding values for A and B satisfy the condition. Indeed, we just have to split up the contestants into two groups as suggested above:

$$A = \binom{n}{2} = \frac{n(n-1)}{2} \qquad B = \binom{n+1}{2} = \frac{n(n+1)}{2}\,.$$

So on the one hand we have

$$AB = \frac{n^2(n+1)(n-1)}{4}\,.$$

On the other hand we have

$$\binom{B}{2} = \frac{\frac{n(n+1)}{2}\left(\frac{n(n+1)}{2} - 1\right)}{2} = \frac{\frac{n(n+1)}{2}\frac{n(n+1)-2}{2}}{2} = \frac{n(n+1)(n-1)(n+2)}{8}$$

and similarly

$$\binom{A}{2} = \frac{n(n-1)(n+1)(n-2)}{8}\,.$$

We deduce that

$$\binom{A}{2} + \binom{B}{2} = \frac{n(n+1)(n-1)(2n)}{8} = \frac{n^2(n+1)(n-1)}{4}\,.$$

So we have proven that the number of contestants must be of the form n^2 for $n \geq 2$ and we have also proven that each of these numbers, for a suitable choice of A and B, satisfies the requested condition that

$$\binom{A}{2} + \binom{B}{2} = AB. \qquad \square$$