MathDay 2022 Senior

First part (6 questions without proof)

Each question is worth 3 points. The correct answer gives 3 points, the wrong answer gives 0 points, not answering gives 1 point.

1. You arrive on an island that is inhabited by 7 dwarves. A dwarf can either be a truth-teller or a liar. The truth-tellers always speak the truth and the liars always lie. All dwarves queue in a straight line to greet you. They all look into your direction.
   The first dwarf in the line says: "All dwarves behind me are liars."
   All other dwarves say: "The dwarf right in front of me is a liar."
   How many dwarves are liars?

2. You are in a video call with some friends who are native speakers of the Combish language. You only remember four different words in this language: Xix, Yiy, Ziz, Wiw. Exactly one of them is extremely funny. You know that if you send some of these words to any of your friends, then that friend will start laughing immediately if and only if the funny word is among the words you chose.

   You can write exactly one message with some Combish words to each of your friends in the call. You can choose how many words and which words to write. You can send different individual messages, but all messages are sent at the same time. You are then able to check in the video call who is laughing. What is the minimum number of friends that you need in the call so that you are able to determine without doubt the funny word by using the above method?

3. Amy and Ben play the Candy Game. At the beginning of the game, there are 10 candies. Amy and Ben take turns making moves. A move consists of removing either 2 or 3 candies. The first player that cannot make a move (because there are less than 2 candies left) looses. Amy makes the first move. If both Amy and Ben aim to win and play according to the best possible strategy, who wins the game? Answer 1 for Amy and 2 for Ben.

4. A very modern museum of very modern art has two floors, one on top of the other. Each floor consists of four corridors connected in the form of a square; it is possible to go from each corridor to the two neighbouring ones on the same floor; at the end of each corridor there is a staircase connecting vertically the two floors; the only entrance is also the only exit and it is located at one corner on the ground

floor. You want to walk along each corridor exactly once (the direction doesn't matter to you). You can walk along different tours, according to the order in which you visit the corridors. How many different tours are there, assuming that you use the stairs exactly twice?

5. You have a coin that when being thrown comes up heads more often than tails. You and a friend play the following game: You toss the coin twice. If the result is twice the same, you win. If the results of the tosses are different, your friend wins. Who is more likely to win? Answer 1 if you have a higher probability to win, answer 2 if your friend has a higher probability to win, answer 3 if you and your friend have the same probability to win.

6. You own a bag of letter-shaped pasta for children. There are 26 different letters. If you take 99 pieces of pasta from the bag, what is the largest integer number n such that you can be sure to have at least n pieces representing the same letter?

Second part (3 problems with proof)

1. (3 Points) Prove that the square of a non-negative integer never leaves remainder 2 or 3 on division by 4.

3. (6 Points) Let $ABCD$ be a rectangle made of paper. Folding the piece of paper in two along some line and opening it again makes the line appear as a crease on the paper as if one had drawn it.

   (a) Fold the paper in two such that A lies on B and D lies on C.
   Call the resulting line L.

   (b) Fold the paper in two such that the segment BC lies on L.
   Call the resulting line L'.

   (c) Suppose that you can fold the paper in two such that A lies on L' and such that the resulting line, which we call L'', passes through D and through the intersection of AB and L.

   What is the ratio between the largest and the smallest side of the rectangle $ABCD$?

3. (6 Points) (a) Let a be a strictly positive real. Show that

$$a + \frac{1}{a} \geq 2.$$

   (b) Let x, y, z be strictly positive reals such that:

$$\begin{cases} a = x + y - z > 0 \\ b = x - y + z > 0 \\ c = -x + y + z > 0. \end{cases}$$

   Prove the following inequality:

$$x + y + z - \frac{x^2 + y^2 + z^2 - 2xy - 2yz - 2xz}{(x + y - z)(x - y + z)(-x + y + z)} \geqslant 6\,.$$