SEPARATING SIMPLE CLOSED GEODESICS AND SHORT HOMOLOGY BASES ON RIEMANN SURFACES

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Abstract. This article is dedicated to a careful analysis of distances and lengths of separating simple closed geodesics on a hyperbolic Riemann surface. One of the main results is a sharp upper-bound on the length of the shortest separating curve on a surface of genus 2.

1. Introduction and preliminaries

1.1. **Introduction.** Hyperbolic Riemann surfaces are very beautiful and central objects in mathematics, and the study of their geometry has been a subject of active research since the works of Fricke and Klein. More recently, a considerable amount of work has focussed on the study of *systoles* or *systolic loops*, i.e., the shortest non-trivial closed curve of a surface (see for instance [1], [2], [16]). The main objective is to find a sharp upper-bound on the systole length in function of the topology of the surface. Generally the problem is attacked by trying to find the surfaces on which the bound is attained (and the bound is always attained by at least one surface). In the compact case, the sharp bound is only known in the case of genus 2 [9], although there is a very promising candidate in the case of genus 3 [18] and in the non-compact case (with cusp boundary), one could resume the state of knowledge by saying that the systoles of surfaces arising from the quotient of principal congruence subgroups of $\mathrm{PSL}(2,\mathbb{Z})$ generally do the trick [17]. The extremal surfaces for systole length that we know always seem to have other remarkable properties such as being arithmetic and with many self isometries. The systole problem in general is essentially the search for an upper-bound on the length of the shortest non-trivial *non-separating* simple closed curve. By taking a surface constructed by gluing pairs of pants with very short boundary curves, its not too difficult to see that no such bound exists for *separating* simple closed curves. A bound on the length of shortest separating curves necessarily depends on both the topology and the systole length, thus corresponds to limiting oneself to a *fat* part of Moduli space, i.e., the portion of Moduli space consisting of surfaces with systole length bounded from below. The underlying theme of this article is to further explore what type of bounds one can

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find on separating curves, in function of the topology and in function of a lower-bound on systole length.

In the first part of the article (section 2) we study and find maximal distance between boundary geodesics on hyperbolic surfaces of signature $(1, 2)$ and $(0, 4)$ (surfaces homeomorphic to a twice-punctured torus, resp. to a four times punctured sphere, with totally geodesic boundary). If one thinks of Riemann surfaces in terms of their Fenchel-Nielsen coordinates, one is immediately confronted of the problem of the *non-homogeneity* of the coordinates, i.e., lengths and twists. Surfaces of signature $(1, 2)$ and $(0, 4)$ can be seen as building blocks for other surfaces which already take into account twist parameters. As mentioned above, it is necessary to impose a lower bound on the (interior) systole of such a surface. Within the set of all surfaces with such a bound and of given signature (the fat part of the associated Moduli space), the surfaces with maximal distance between boundary curves are given (Theorems 2.8, 2.9 and 2.10). As a corollary, we obtain a sharp upper bound on the length of a shortest separating geodesic on surfaces of signature $(1, 2)$ and $(0, 4)$.

The second part of the article (section 3) is devoted to the relationship between lengths of separating simple closed geodesics and lengths of homology bases. We call a *canonical homology basis* $\mathcal{B}(S)$ on a surface S of genus g is a set of simple closed geodesics $\{(\alpha_1, \beta_1), \ldots, (\alpha_g, \beta_g)\}$ with the following properties.

(1) $\mathrm{int}(\alpha_i, \alpha_j) = \mathrm{int}(\beta_i, \beta_j) = 0$.
(2) $\mathrm{int}(\alpha_i, \beta_j) = \delta_{ij}$.

We consider them as a set of non-oriented simple closed geodesics and the intersection number $\mathrm{int}(\cdot, \cdot)$ is geometric. (To obtain a true basis of $H_1(S, \mathbb{Z})$ one would orient them and consider algebraic intersection number instead.) These bases were first studied from a geometric point of view by P. Buser and M. Seppälä in three distinct articles ([4], [5] and [6]). The original motivation was their usage for the calculation of period matrices of surfaces, which in turn are used in explicit numerical uniformization. The idea of the articles was to prove that for a given surface, one can chose a canonical homology basis with bounded length (in the articles the length ℓ of such a basis is defined as the length of the longest geodesic in the basis). To be more precise, the length ℓ of such a basis can be chosen bounded by the genus and the systole of the surface. Furthermore, in [6] the bound is shown to be asymptotically optimal by example. Note that studying the relationship between simple closed geodesics and homology has other uses: for instance, in the case of the once punctured torus, G. McShane and I. Rivin [14], [13] used a norm on homology to calculate the asymptotic growth of simple closed geodesics in function of length.

Here we further explore the problem of finding a bound on length of canonical homology bases. To begin, a new definition of length of such bases is introduced. The geodesics of a basis are defined in pairs, and each pair $\{\alpha_j, \beta_j\}$ is contained in a unique surface of signature $(1, 1)$ with boundary geodesic the commutator γ_j

of the two elements in the pair. These boundary geodesics do not intersect, and by cutting along these geodesics, one obtains a surface of signature $(0, g)$. We now define the length ℓ' of a basis as the length of the longest boundary geodesic of the pieces of signature $(1, 1)$ constructed as above. This new definition of length takes into account both geodesics in a pair, and, using the results known for ℓ, a basis can be clearly be chosen such that its length ℓ' is bounded. The idea is to find extremal surfaces for this length, meaning surfaces of a given genus and given systole, such that the length of a shortest canonical homology basis is maximal.

In section 3 addresses the following problems. By carefully analysing one holed tori, we show how the two definitions of length (ℓ and ℓ') are related. Among other things, it is shown that a bound on ℓ' is "stronger" than a bound on ℓ. The main result concerns surfaces of genus 2. Using results in section 2, the following theorem is proven.

Theorem 1.1. *Let S be a surface of genus 2 with systole σ. Then there is a separating simple closed geodesic γ with*

$$\ell(\gamma) \leq 2 \operatorname{arccosh} \frac{2\cosh^3\frac{\sigma}{2} + 3\cosh^2\frac{\sigma}{2} - \cosh\frac{\sigma}{2}}{\cosh\frac{\sigma}{2} - 1}.$$

This bound is sharp.

By definition, this theorem of course implies that there is canonical homology basis $\mathcal{B}$ on S whose length ℓ' satisfies the same inequality. In section **??**, fundamental properties of extremal surfaces for ℓ' in all genus are shown and surfaces conjectured to be extremal for given genus and systole are constructed.

Finally, note that the problem described in this article is related to the problem of finding surfaces with maximum size systoles. Some of the techniques used are similar to those used by P. Schmutz-Schaller in [16] and the resolution of the genus 2 case uses the corresponding result for maximum size systoles [9].

1.2. **Notations.** Here a *surface* will always be a compact Riemann surface, possibly with *geodesic* boundary, equipped with a metric of constant curvature -1. Such a surface is always locally isometric to the hyperbolic plane $\mathbb{H}$. A surface will generally be represented by S and distance on S (between points, curves or other subsets) by $d_S(\cdot,\cdot)$. The signature of such a surface will be denoted (g, n) (genus g with n boundary curves). All boundary curves will be simple closed geodesics. A surface of signature $(0, 3)$ is called a *Y-piece* or a *pair of pants* and will generally be represented by $\mathcal{Y}$ or $\mathcal{Y}_i$. A surface of signature $(1, 1)$ will often be referred to as a *one holed torus* or a *Q-piece*. A surface of signature $(0, 4)$ is referred to as an *X-piece*. The Teichmüller space for surfaces of signature (g, n) will be denoted denoted by $\mathcal{T}_{g,n}$, and note that boundary curve length is allowed to vary.

A curve, unless specifically mentioned, will always be non-oriented and primitive. The set of all free homotopy classes of closed curves of a surface S is denoted $\pi(S)$. A *non-trivial curve* on S is a curve which is not freely homotopic to a point. A closed curve on S is called *simple* if it has no self-intersections. Closed curves (geodesic or not) will generally be represented by greek letters (α, β, γ and γ_i etc.) whereas paths (geodesic or not) will generally be represented by lower case letters (a, b etc). The function that associates to a finite path or curve its length will be represented by $\ell(\cdot)$, although generally a path or a curve's name and its length will not be distinguished. This may cause some confusion: in the same statement "α" can denote both a simple closed geodesic and its length. However this is deemed necessary in order to avoid extremely heavy notation and the distinction between the two usages should be clear in a given context. The shortest simple closed geodesic of a surface is called a *systole* or *systolic loop*, and its length is referred to as *systolic length*. This is somewhat non-standard notation in the differential geometry context, but has become standard for hyperbolic Riemann surfaces. In general, both a systole and its length will be denoted σ.

The geometric intersection number between two distinct curves α and β will be denoted $\text{int}(\alpha,\beta)$. As a noun, the term *geodesic* will sometimes be used in place of a simple closed geodesic curve. A *non-separating closed curve* is a closed curve γ such that the set $S\setminus\gamma$ is connected. Otherwise, a closed curve is called *separating*. Twist parameters are defined as in [3]. A pasting with half twist corresponds to when the twist parameter is exactly $\frac{1}{2}$.

A *geodesic length function* is an application that associates length to geodesics according to homotopy class under the action of a continuous transformation of a surface. In [11], S. Kerckhoff proves that geodesic length functions are convex along *earthquake paths*, i.e., convex along twist paths. In the article we shall only use the fact that they are convex along twists along simple closed geodesics.

The remainder of the article is organized as follows. Section 2 is dedicated to the study of surfaces of signature $(1,2)$ and $(0,4)$. Section 3 then builds on the previous section to prove Theorem 1.1 and the relationship between different lengths of canonical homology bases and lengths of separating simple closed geodesics. Finally, for convenience, a list of well-known trigonometric formulae for hyperbolic polygons is given in the Appendix.

2. Maximal distances between boundary curves on surfaces of signature $(1,2)$ and $(0,4)$

We shall be considering surfaces of signature $(1,2)$ and $(0,4)$ with fixed boundary length. We can think of them lying in a slice of the Moduli space of surfaces of given signature. We begin by identifying, in a given slice, the surfaces of signature $(1,2)$ with maximal distance between the two boundary curves. The tools used are generally explicit computation using hyperbolic trigonometry, the convexity of geodesic length functions along earthquake paths and simple topological arguments.

2.1. **Maximal lengths on surfaces of signature** $(1,2)$**.** The set of all surfaces of signature $(1,2)$ with boundary geodesics α and β and with all simple closed geodesics in the interior of S of length $\geq \sigma$ shall be denoted $F_{\alpha,\beta,\sigma}$. For fixed α and β and if σ is sufficiently large, this set is empty. We shall always consider σ to be such that this is not the case. Note the following: if $S \in F_{\alpha,\beta,\sigma}$, then there is not necessarily a closed geodesic of length σ on S. In other words the lower bound on the systole can not be broken, but is not necessarily reached. This implies that for $x < x'$ then $F_{\alpha,\beta,x'} \subset F_{\alpha,\beta,x}$. Also notice that either α or β (or both) can have length inferior to σ. Furthermore, notice that we have not imposed that $\alpha, \beta \geq \sigma$.

With these notations, the first problem treated is the following: for which $S \in F_{\alpha,\beta,\sigma}$ is $d_S(\alpha,\beta)$ maximal? Finding this S_{max} gives an explicit upper bound on $d_S(\alpha,\beta)$. The main result is the following. Take two Y-pieces $(\alpha,\gamma,\tilde{\gamma})$ and $(\beta,\gamma,\tilde{\gamma})$ with $\gamma = \tilde{\gamma} = \sigma$. Paste them along γ and $\tilde{\gamma}$ with half twists. The surface obtained is S_{max}.

To begin, we will deal with some basic properties of compact surfaces of signature $(1,2)$ and make a few remarks concerning the problem.

In this subsection, a *maximal surface* will mean a surface belonging to $F_{\alpha,\beta,\sigma}$ with maximal distance between α and β. A minimizing path on S is a shortest path between α and β. Notice that a minimizing path is always a simple geodesic path. The first step is to prove that the distance between two boundary geodesics has an upper bound under the assumption that the length of all interior geodesics on S has a lower bound. The existence of an upper bound is the object of the next lemma.

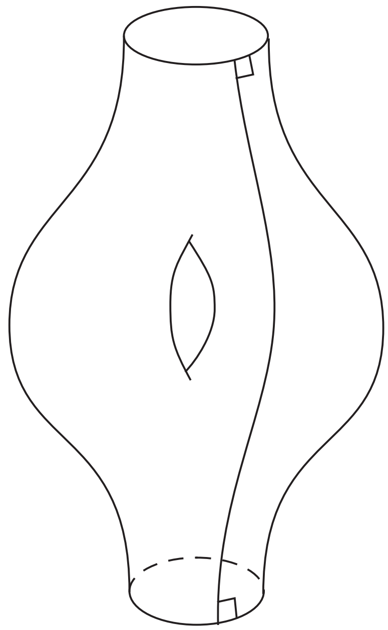


FIGURE 1. A piece of signature $(1,2)$ and a minimizing path

Lemma 2.1. *The length of a minimizing path on S can not be greater than $\frac{2\pi}{\sinh\frac{\sigma}{4}}$.*

Proof. The hyperbolic area of S is exactly 4π. Let c be a minimizing path between α and β. Let $T_r(c) = \{p \in S \mid d_S(p,c) < r\}$ be the tubular neighborhood around c of radius r. Using Fermi coordinates (see [12]) one can show that if $T_r(c)$ is simply connected then its area is $2c\sinh r$. Here we must prove that $T_{\sigma/4}(c)$ is simply connected. Suppose $T_{\sigma/4}(c)$ is not simply connected. Then there are two points p,q on c that are joined a geodesic arc of length inferior to $\sigma/2$. The geodesic segment of c that joins p and q has length inferior to $\sigma/2$ as well, otherwise c is not minimal. In this case, the simple closed geodesic in the homotopy class of the reunion of the two segments would be of length inferior σ). This gives $2c\sinh\frac{\sigma}{4} \leq 4\pi$ and the result follows. □

The set of surfaces $F_{\alpha,\beta,\sigma}$ is a bounded subset in the underlying moduli space [7], and because the length of a minimizing path is bounded, it follows that there is at least one surface for which the lowest upper bound is reached. The surface attaining this property will be given explicitly in what follows.

On a given surface the distance between the boundary geodesics is given by the length of the shortest path between them. There can be several minimizing paths, but these paths do not intersect. (If they did in a point p, then both paths c and d are necessarily of equal distance from α to p and from p to β. We can construct a new path using half of c and half of d which also joins the two boundary geodesics and has the same length. In the neighborhood of p this new path is not length minimizing and can be shortened. With the same arguments, such a path is necessarily simple.) Thus we need to know how many simple disjoint topologically different paths can go from boundary to boundary. From now on, a *seam* will be a simple path joining the two (distinct) boundary geodesics. Two seams will be called non-homotopic if they are not in the same free homotopy class of paths with end points moving along the boundary geodesics.

Lemma 2.2. *There are at most* 4 *simple disjoint non-homotopic seams on a topological surface of signature* $(1, 2)$.

Proof. Let c_1 be such a path. Cutting along this path we obtain a one holed torus. Let c_2 be a path joining what were formerly the two boundary curves such as in figure 2.

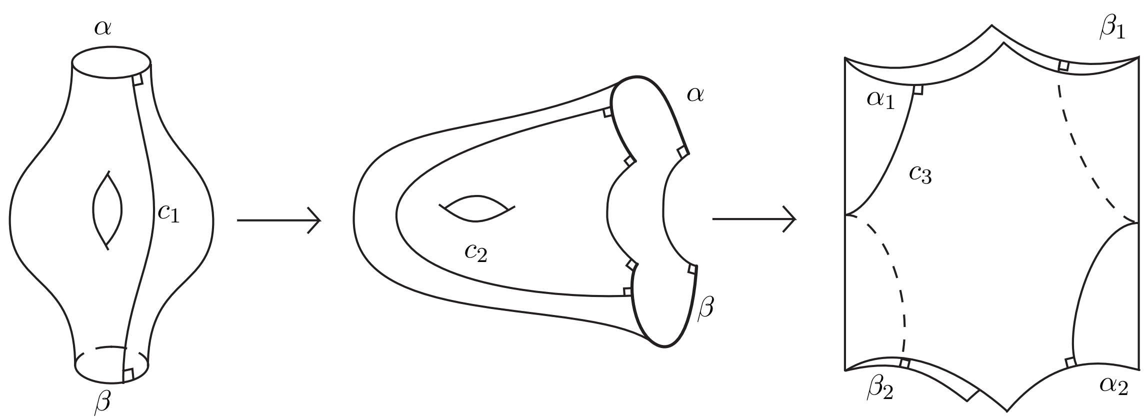


FIGURE 2. No cuts, one cut, two cuts

Cutting along c_2 we obtain a cylinder such as in figure 2.

There are exactly two non-homotopic disjoint seams possible and this because the original boundary curves are now disjoint and separated on the boundary of the cylinder. All further paths are either homotopic to one of the four paths, or intersect them. □

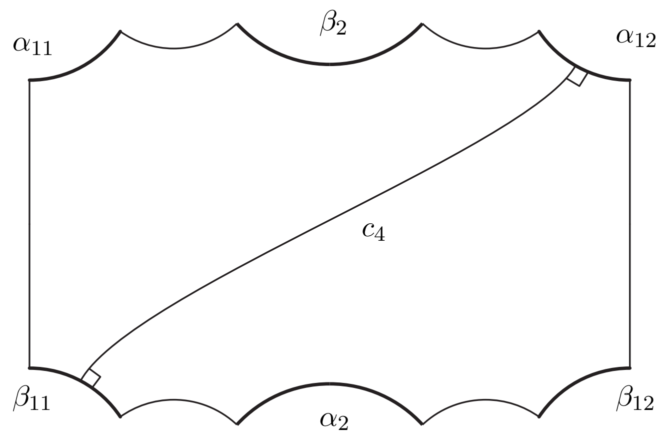


FIGURE 3. Three cuts

From this lemma it follows that there at most four distinct minimizing paths on a given surface. In fact, we will see that on a maximal surface there are exactly four such paths.

We will need a certain number of lemmas in order to find a maximal surface. If γ is a simple closed geodesic on S, and c is a geodesic seam with $\text{int}(\gamma, c) = 1$ there is a natural way to view the twist parameter along γ. (Notice that γ is necessarily non-separating in this case). There is only one simple closed geodesic $\tilde{\gamma}$ that does not intersect either γ or c. γ and $\tilde{\gamma}$ form a pants decomposition of S (cutting along both of them separates S into two Y-pieces). The twist parameters are now naturally defined.

Let S be a surface of signature $(1, 2)$ obtained by pasting two Y-pieces $(\alpha, \gamma, \tilde{\gamma})$ and $(\beta, \gamma, \tilde{\gamma})$ along γ and $\tilde{\gamma}$. A *zero twist* along γ will mean that the perpendicular between α and γ intersects the same point on γ as the perpendicular between β and γ. A *half twist* along γ corresponds to when the perpendicular between α and γ intersects the same point on γ as the perpendicular between γ and $\tilde{\gamma}$.

The idea behind the forthcoming lemmas is the following. Let S be a maximal surface. In order to prove properties of such surfaces, two types of transformations will be performed. Let γ be a closed geodesic on S. The verb *lengthening* γ corresponds to replacing S by a surface defined with the same Fenchel-Nielsen parameters, except with γ of greater length. As previously defined in the preliminaries, the verb *twisting* along γ is replacing S with a surface defined with the same parameters, only a different twist parameter along γ. Both operations act continuously on geodesic length functions. Continuous twisting is moving along an earthquake path in $T_{1,2}$. We recall that the geodesic length function of a geodesic that intersects γ is strictly convex, and the geodesic length function of a geodesic that does not cross γ is constant along these paths.

Lemma 2.3. *Let c be a minimizing path on a maximal surface S. Let γ be a simple closed geodesic on S with $\gamma \cap c \neq \emptyset$. Then at least one of the following statements is true.*

(1) *There is a path d between α and β such that $c \cap d = \emptyset$, $\ell(c) = \ell(d)$ and $d \cap \gamma \neq \emptyset$.*
(2) *γ crosses a systole.*

Proof. Performing a continuous twist along γ only affects paths and closed geodesics that intersect γ. The convexity of geodesic length functions along earthquake paths implies that either in at least one of the two possible twist directions the length of c is increased. Since S is maximal that means such a twist must have caused one of the two following events:

1. There is now a path shorter than c was originally. This of course cannot be the image of c as the image of c is longer.

2. There is now a closed geodesic on S with length $< \sigma$.

The twist applied to γ can be as small as possible. Thus, before twisting, either there was a closed geodesic of length σ or a second geodesic seam d with the same length as c. □

Lemma 2.4. *Let c be a minimizing path on a maximal surface S. Let γ_1 and γ_2 be non-intersecting simple closed geodesics on S with $\gamma_2 \cap c = \emptyset$ and $\mathrm{int}(\gamma_1, c) = 1$. Then at least one of the following statements is true.*

(1) *There is a minimizing path that crosses γ_2.*
(2) *γ_2 crosses a systole.*

Proof. The length of c can be given, using hyperbolic trigonometry, by the lengths of γ_1, γ_2 and the twist parameter along γ_1. If one lengthens γ_2, one necessarily lengthens c. As in the previous lemma, because S is supposed to be maximal, one of two events must have occurred.

(1) There is now a path shorter than c was originally. This of course cannot be the image of c as the image of c is longer.
(2) There is now a closed geodesic on S with length $< \sigma$.

As lengthening γ_2 can be performed continuously and acts continuously on geodesic length functions, the lemma follows. □

The following lemma deals with asymmetric situations.

Lemma 2.5. *Let S be the surface obtained by gluing two Y-pieces $(\alpha, \gamma, \tilde{\gamma})$ and $(\beta, \gamma, \tilde{\gamma})$ along γ and $\tilde{\gamma}$ with half twists. Let c be a shortest seam that intersects γ (and does not intersect $\tilde{\gamma}$) and $\tilde{c}$ be a shortest seam that intersects $\tilde{\gamma}$ (and does not intersect γ). If $\ell(\gamma) \geq \ell(\tilde{\gamma})$, then $\ell(c) \leq \ell(\tilde{c})$.*

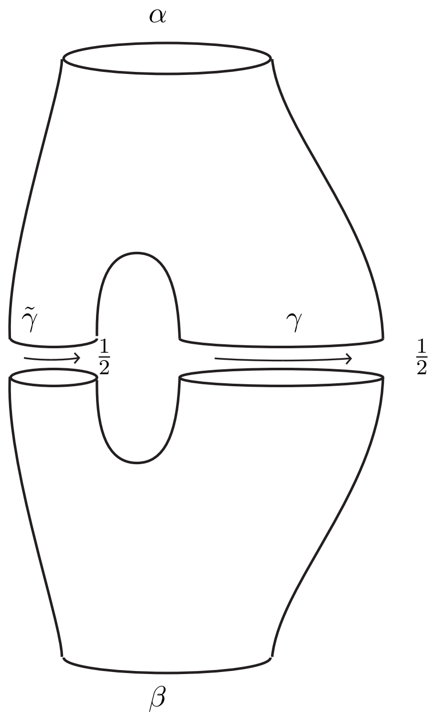


FIGURE 4. An asymmetric situation

Proof. First notice that $\text{int}(c, \gamma) = 1$ and $\text{int}(\tilde{c}, \tilde{\gamma}) = 1$. The proof is a calculation, which here is shown when $\ell(\alpha) = \ell(\beta) = 2a$. The general case works the same way, but the method is less apparent. Looking at the two following figures, we can deduce the following.

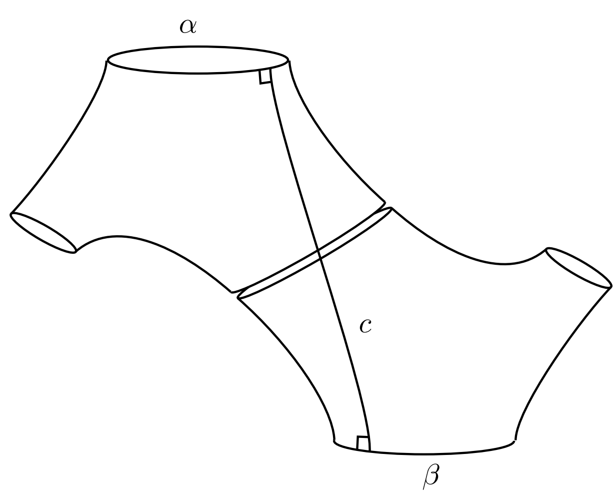


FIGURE 5. Pasting with half-twists

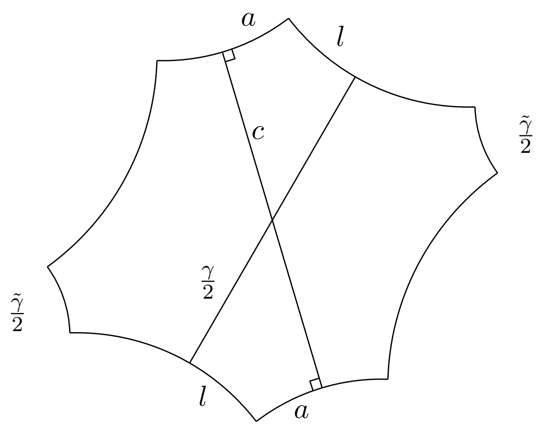


Figure 6. The corresponding figure in the hyperbolic plane

First of all neither c nor $\tilde{c}$ are unique. There are two possibilities for both. Using hyperbolic trigonometry (proposition 4.3) we can calculate the length of the side of the non-convex hexagon labeled l as in figure 7.

$$\cosh l = \frac{\cosh \frac{\tilde{\gamma}}{2} + \cosh a \cosh \frac{\gamma}{2}}{\sinh a \sinh \frac{\gamma}{2}}.$$

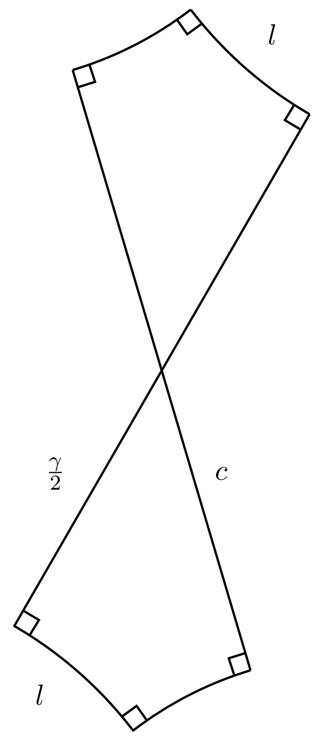


FIGURE 7. The hexagon linking l and c

This hexagon yields:

$$\cosh c = \cosh \frac{\gamma}{2} \sinh^2 l - \cosh^2 l.$$

Similarly for $\tilde{c}$ we obtain:

$$\cosh \tilde{c} = \cosh \frac{\tilde{\gamma}}{2} \sinh^2 \tilde{l} - \cosh^2 \tilde{l}$$

with

$$\cosh \tilde{l} = \frac{\cosh \frac{\gamma}{2} + \cosh a \cosh \frac{\tilde{\gamma}}{2}}{\sinh a \sinh \frac{\tilde{\gamma}}{2}}.$$

Once these formulas are obtained the length comparison between c and $\tilde{c}$ is relatively simple. □

Lemma 2.6. *Let $c_1, \dots, c_4$ be four minimizing seams. Let γ and $\tilde{\gamma}$ be two disjoint closed geodesics such that $(\gamma \cup \tilde{\gamma}, c_i) = 1$ for $i = 1, \dots, 4$. Then:*

(1) *$c_1, \dots, c_4$ are disjoint.*
(2) *Both γ and $\tilde{\gamma}$ intersect exactly two of the four paths.*
(3) *S is obtained by pasting $(\alpha, \gamma, \tilde{\gamma})$ and $(\beta, \gamma, \tilde{\gamma})$ along γ and $\tilde{\gamma}$ with half twists.*

Proof. 1. This has been previously proved in the paragraph that precedes lemma 2.2.

2. Both γ and $\tilde{\gamma}$ intersect at least two of the paths. If they do not intersect exactly two then this contradicts the hypotheses.

3. Let c and d be minimizing paths intersecting a closed geodesic once each. Then figure 8 holds.

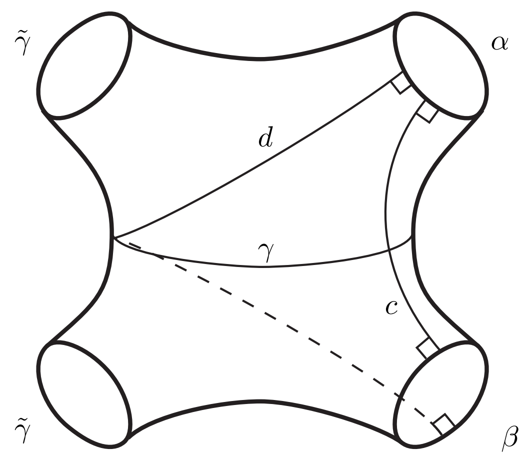


FIGURE 8. The X-piece obtained by cutting along $\tilde{\gamma}$

There are three possible paths from α to β that intersect γ only once. Two of these intersect. The only case where two that do not intersect have equal length is when the twist is a half. □

The next lemma deals with symmetric surfaces.

Lemma 2.7. *Let S_x be the surface obtained by pasting $(\alpha, \gamma, \tilde{\gamma})$ and $(\beta, \gamma, \tilde{\gamma})$ along γ and $\tilde{\gamma}$ with half twists with lengths of both γ and $\tilde{\gamma}$ equal to $x > 0$. The function $f(x) = \cosh(d_{S_x}(\alpha, \beta))$ is strictly convex.*

Proof. Let $l_{\alpha\gamma}$, $l_{\alpha\tilde{\gamma}}$, $l_{\beta\gamma}$ and $l_{\beta\tilde{\gamma}}$ be the perpendiculars as on figure 9.

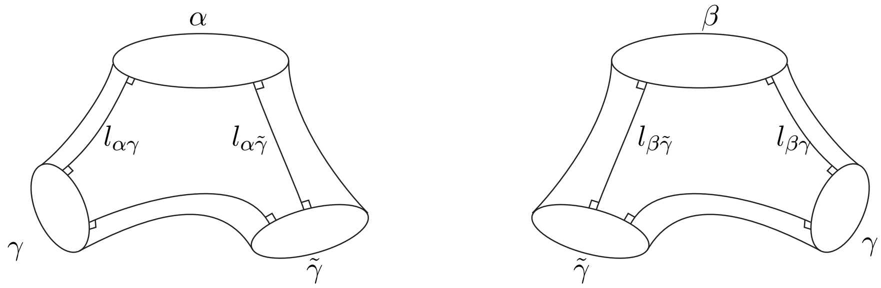
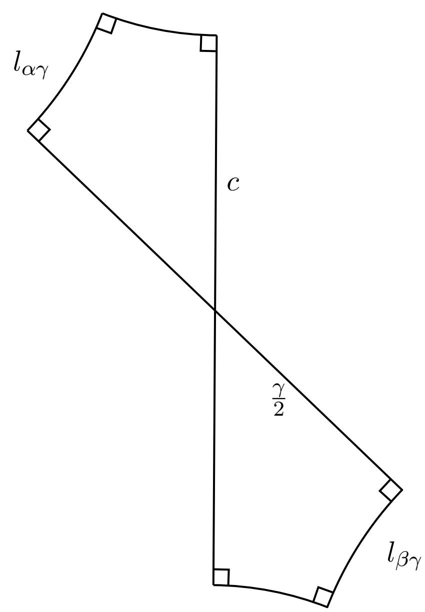


FIGURE 9. The two Y-pieces obtained by cutting along γ

There are four seams on S_x that have length $\cosh d_{S_x}(\alpha, \beta)$. Their length is the length of the path c in the following generalized right-angled hexagon as in the following figure.



Thus $f(x) = \sinh l_{\alpha\gamma} \sinh l_{\beta\gamma} \cosh \frac{x}{2} + \cosh l_{\alpha\gamma} \cosh l_{\beta\gamma}$. Considering that α and β are considered fixed, both $l_{\alpha\gamma}$ and $l_{\beta\gamma}$ only depend on x. Using the formula for the right-angled hexagon we can see that:

$$\begin{aligned} \cosh l_{\alpha\gamma} &= \frac{\cosh \frac{x}{2} + \cosh \frac{x}{2} \cosh \frac{\alpha}{2}}{\sinh \frac{x}{2} \sinh \frac{\alpha}{2}} \\ &= \coth \frac{x}{2} \frac{1 + \cosh \frac{\alpha}{2}}{\sinh \frac{\alpha}{2}} = A \coth \frac{x}{2} \end{aligned}$$

and

$$\cosh l_{\beta\gamma} = \coth\frac{x}{2}\frac{1+\cosh\frac{\beta}{2}}{\sinh\frac{\beta}{2}} = B\coth\frac{x}{2}$$

where $A, B > 1$ are constants. We can now rewrite f as

$$f(x) = \sqrt{(A^2\coth^2\frac{x}{2}-1)(B^2\coth^2\frac{x}{2}-1)}\cosh\frac{x}{2} + AB\coth^2\frac{x}{2}.$$

Let us call $g(x)$ the first part of the summand and $h(x)$ the second part. It easy to show the convexity of $h(x)$ by calculation.

$$h''(x) = \frac{AB}{2}(\coth^2\frac{x}{2}-1)(3\coth^2\frac{x}{2}-1) > 0.$$

For g the calculation is straightforward but extremely long and tedious and for this reason is not included. Both g and h are convex and thus so is f. □

If σ is sufficiently small, then two simple closed geodesics of length σ cannot intersect. For instance if $\sigma \leq 2\,\text{arcsinh}1$ this is always the case. There is a more precise condition on σ but it depends on α and β. In a sense, this is the more important case because it is in this case that seams are very long.

Theorem 2.8. *Let $F_{\alpha,\beta,\sigma}$ be the set of all surfaces of signature $(1,2)$ with boundary geodesics α and β and with all interior closed geodesics of length superior or equal to σ. Furthermore, let σ be such that two closed geodesics of length σ cannot cross. Then, among all elements of $F_{\alpha,\beta,\sigma}$, S_σ has maximal distance between α and β.*

Proof. Let S_{max} be a maximal surface in $F_{\alpha,\beta,\sigma}$. Let c be a minimizing path. By applying lemmas 2.3 and 2.4 repeatedly we can prove that there are four minimizing paths on S_{max}. This is due to the fact that two systoles cannot intersect, so applying lemma 2.3 to a closed geodesic of length σ automatically proves the existence of a different minimizing path. We are thus in the case of the following figure:

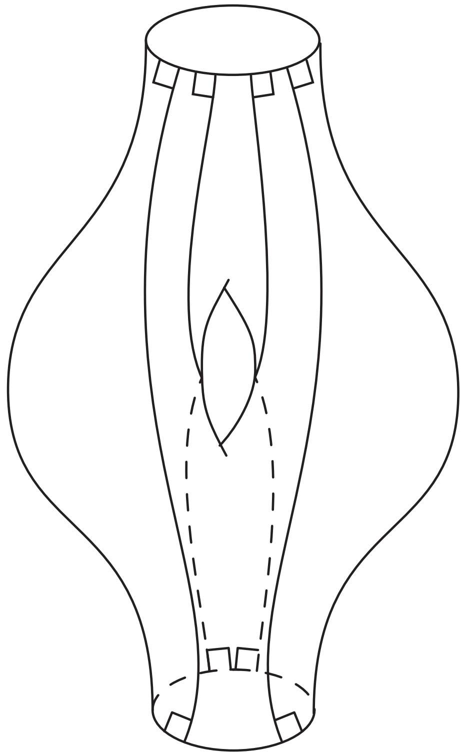


FIGURE 10. Surface with four minimizing paths

Let γ and $\tilde{\gamma}$ be two disjoint simple closed geodesics on S_{max} that each intersect two of the four paths. Lemmas 2.5 and 2.6 prove that $\ell(\gamma) = \ell(\tilde{\gamma})$ and that γ and $\tilde{\gamma}$ are pasted with half twists. There are two possibilities:

1. $\ell(\gamma) = \ell(\tilde{\gamma}) = \sigma$ and the theorem is correct.

2. $\ell(\gamma) = \ell(\tilde{\gamma}) = x > \sigma$. In this case, using the convexity proved in lemma 2.7, it is possible to continuously modify x to increase the distance between α and β. The original surface however was considered maximal so this operation must create a closed geodesic of length inferior to σ, meaning there is a closed geodesic δ of length σ on S_{max}. A systole cannot cross a minimizing path more than once, and thus δ intersects exactly two minimizing paths, and each exactly once. Let $\tilde{\delta}$ be a simple closed geodesic that does not intersect δ and which crosses the two remaining paths exactly once. Once again, applying lemmas 2.5 and 2.6, $\ell(\delta) = \ell(\tilde{\delta})$ and S_{max} is obtained by pasting $(\alpha, \delta, \tilde{\delta})$ to $(\beta, \delta, \tilde{\delta})$ with half twists. The theorem is thus proven. □

Distance between boundary geodesics becomes extremely long when the length of systoles is very small. If systoles are long enough to intersect then their length is necessarily greater than $2\,\text{arcsinh}1$. Having an exact value when systoles are large is not very telling, so here we give an explicit maximal surface for all surfaces with systoles greater or equal to $2\,\text{arcsinh}1$. It is the same surface as before, with the systoles of length $2\,\text{arcsinh}1$.

Theorem 2.9. *Let S be a surface of signature $(1,2)$ with boundary geodesics α and β, and with systole of length $\geq 2\,\text{arcsinh}1$. Then $d_S(\alpha,\beta) \leq d_{S_{max}}(\alpha,\beta)$ where S_{max} is the gluing of two Y-pieces with lengths respectively $(\alpha, 2\,\text{arcsinh}1, 2\text{arcsinh}1)$ and $(\beta, 2\,\text{arcsinh}1, 2\text{arcsinh}1)$ along the equal sides with half-twists.*

Proof. We have, for α and β fixed, $F_{\alpha,\beta,x} \subset F_{\alpha,\beta,x'}$ if $x' < x$. The surface described above has been proved maximal within the set $F_{\alpha,\beta,2\,\text{arcsinh}1}$ and this proves the theorem. □

2.2. **Maximal lengths on surfaces of signature $(0,4)$.** Let $\mathcal{X}$ be an X-piece. Let $X_{\alpha,\beta,\gamma,\delta,\sigma}$ be the set of all X-pieces with boundary lengths α,β,γ and δ and all interior closed geodesics of length superior or equal to σ. Notice that $\mathcal{X} \in X_{\alpha,\beta,\gamma,\delta,\sigma}$ does not necessarily contain a closed geodesic of length σ. The problem solved in this chapter is the following: for given values $(\alpha,\beta,\gamma,\delta,\sigma)$, what $\mathcal{X} \in X_{\alpha,\beta,\gamma,\delta,\sigma}$ has maximal distance between the geodesics α and β? In this section a *maximal surface* will mean a surface with maximum distance between α and β among all elements of $X_{\alpha,\beta,\gamma,\delta,\sigma}$. Finding a solution to this problem is not as technical as the previous section but the methods used are essentially the same.

The topology of an X-piece differs from that of a piece of signature $(1,2)$, essentially in the following properties. Between two given boundary geodesics there are at most two disjoint non-homotopic paths. On an X-piece $\mathcal{X}$, an interior simple closed geodesic is always separating, and separates the X-piece into two Y-pieces. Finding a minimal path on $\mathcal{X}$ between α and β is equivalent to finding the shortest closed geodesic on $\mathcal{X}$ that separates α and β from the other two boundary geodesics. Once again, this comes from the formula for the hyperbolic hexagon (see 4.2). Before answering the question described, it is easier to answer the question on the *shortest* minimal path between α and β among all elements of $X_{\alpha,\beta,\gamma,\delta,\sigma}$. This *minimal surface* is obtained by pasting (α,β,σ) and (γ,δ,σ) along σ as in figure 11.

We can now treat the problem of maximum distance between boundary geodesics.

Theorem 2.10. *Let $\alpha \leq \beta$ and $\gamma \leq \delta$ be positive values. Let $X_{\alpha,\beta,\gamma,\delta,\sigma}$ be the set of all surfaces of signature $(0,4)$ with boundary geodesics of length α,β,γ and δ and with all interior closed geodesics of length superior of equal to σ. Furthermore, let $\sigma \leq 2\,\text{arcsinh}1$. The surface obtained by pasting (α,δ,σ) and (β,γ,σ) along σ with half twists (such as in figure 12) has maximal distance between α and β among all elements of $X_{\alpha,\beta,\gamma,\delta,\sigma}$.*

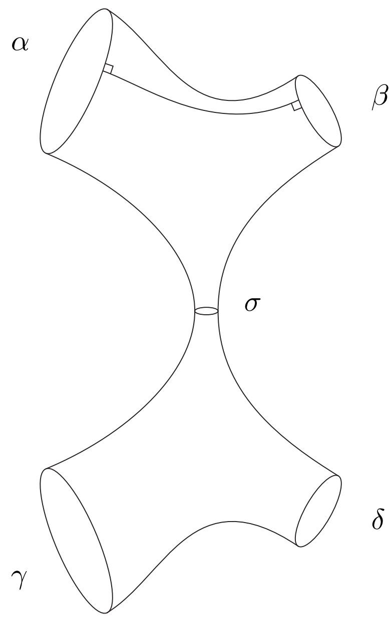


FIGURE 11. The X-piece with minimum distance between α and β

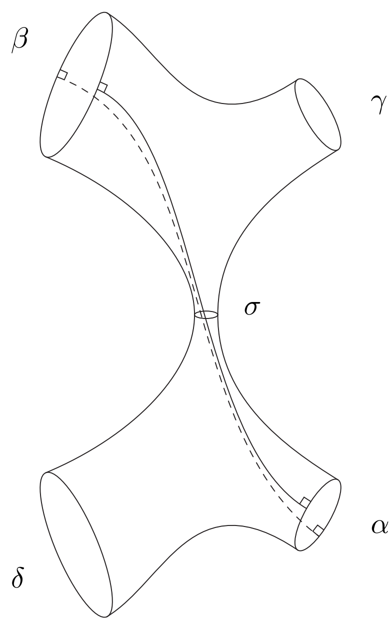


FIGURE 12. The X-piece with maximum distance between boundary geodesics

Proof. Let $\mathcal{X}$ be a maximal surface. Let c be a minimizing path. Let $\tilde{\gamma}$ be a simple closed geodesic with $\mathrm{int}(c, \tilde{\gamma}) = 1$. $\mathcal{X}$ being maximal, the surface obtained by twisting along $\tilde{\gamma}$ to increase the length of c must contain a shorter path between α and β or has a shortest closed geodesic shorter than σ. This means that one of the following two statements holds.

(1) $\mathcal{X}$ has a second minimizing path d that is disjoint from c.
(2) $\mathcal{X}$ contains a closed geodesic γ' of length σ that intersects $\tilde{\gamma}$.

In case 2, γ' separates α and β into two different Y-pieces (otherwise $\mathcal{X}$ is a minimal surface). Twisting along γ' shows that there is necessarily a second minimizing path d that also intersects γ'. We will come back to this case later.

In case 1, there are two minimizing paths on $\mathcal{X}$. Let γ'' be a simple closed geodesic with $\mathrm{int}(c, \gamma'') = \mathrm{int}(d, \gamma'') = 1$. As there are two minimizing paths, as in the case of the piece of signature $(1, 2)$, the twist parameter along γ'' is 0 and α and β are placed on opposite ends of the X-piece, as in figure 12.

Let x be the length of γ''. Let $\mathcal{X}_x$ be the surface obtained with the same pasting conditions as $\mathcal{X}$, only the length of γ'' is left variable. As in the proof of 2.7, the function $f(x) = \cosh(d_{\mathcal{X}_x}(\alpha, \beta))$ can be shown to be strictly convex. (The proof is identical). Thus, x can be chosen so that the distance between α and β is greater than on $\mathcal{X}$. As $\mathcal{X}$ is supposed to be maximal, this means that either $x = \sigma$ or that γ'' crosses a closed geodesic γ''' of length σ. As in case 2, γ''' separates α and β into two different Y-pieces.

In both cases, we have shown that there is a closed geodesic γ_σ of length σ separating $\mathcal{X}$ into two Y-pieces with α on one and β on the other. The pasting conditions along γ'' are predetermined by the fact that there are two minimizing paths between α and β. The only remaining question is on whether $\mathcal{X}$ is obtained by pasting $(\alpha, \gamma, \gamma_\sigma)$ and $(\beta, \delta, \gamma_\sigma)$ (case **(i)**) or by pasting $(\alpha, \delta, \gamma_\sigma)$ and $(\beta, \gamma, \gamma_\sigma)$ (case **(ii)**). To answer the question, we shall calculate the length obtained in both cases, and compare them using the relationships between the different lengths as imposed by hypotheses.

In case **(i)**, the length of the minimizing path can be calculated on the following hyperbolic octogon.

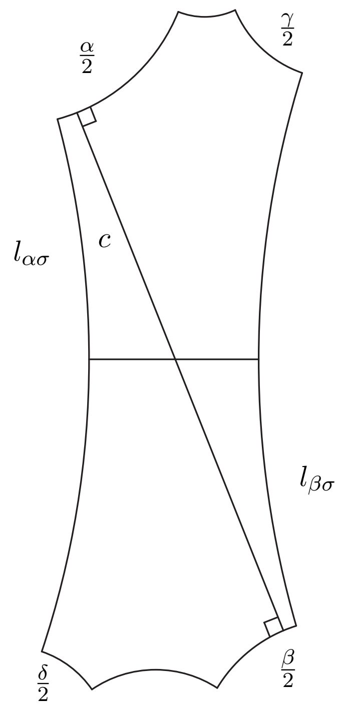


FIGURE 13. Case **(i)**

By using the formulas for hyperbolic hexagons, one obtains the following.

$$\begin{aligned}
\cosh c &= \cosh\frac{\sigma}{2}\sinh l_{\alpha\sigma}\sinh l_{\beta\sigma} + \cosh l_{\alpha\sigma}\cosh l_{\beta\sigma},\\
\cosh l_{\alpha\sigma} &= \frac{\cosh\frac{\gamma}{2} + \cosh\frac{\sigma}{2}\cosh\frac{\alpha}{2}}{\sinh\frac{\sigma}{2}\sinh\frac{\alpha}{2}},\\
\cosh l_{\beta\sigma} &= \frac{\cosh\frac{\delta}{2} + \cosh\frac{\sigma}{2}\cosh\frac{\beta}{2}}{\sinh\frac{\sigma}{2}\sinh\frac{\beta}{2}}.
\end{aligned}$$

In case **(ii)** the length of the minimizing path is calculated on this octogon.

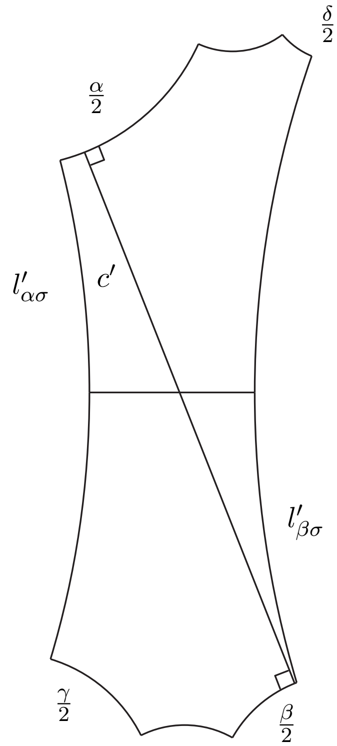


FIGURE 14. Case **(ii)**

The length of the minimizing path c' is given by these expressions.

$$\begin{aligned}\cosh c' &= \cosh\frac{\sigma}{2}\sinh l'_{\alpha\sigma}\sinh l'_{\beta\sigma} + \cosh l'_{\alpha\sigma}\cosh l'_{\beta\sigma},\\ \cosh l'_{\alpha\sigma} &= \frac{\cosh\frac{\delta}{2} + \cosh\frac{\sigma}{2}\cosh\frac{\alpha}{2}}{\sinh\frac{\sigma}{2}\sinh\frac{\alpha}{2}},\\ \cosh l'_{\beta\sigma} &= \frac{\cosh\frac{\gamma}{2} + \cosh\frac{\sigma}{2}\cosh\frac{\beta}{2}}{\sinh\frac{\sigma}{2}\sinh\frac{\beta}{2}}.\end{aligned}$$

By calculation we shall show that $c' \geq c$. First of all

$$\cosh l'_{\alpha\sigma}\cosh l'_{\beta\sigma} - \cosh l_{\alpha\sigma}\cosh l_{\beta\sigma}$$
$$=$$
$$\cosh\frac{\sigma}{2}\frac{\cosh\frac{\gamma}{2}\cosh\frac{\alpha}{2} + \cosh\frac{\beta}{2}\cosh\frac{\delta}{2} - (\cosh\frac{\delta}{2}\cosh\frac{\alpha}{2} + \cosh\frac{\beta}{2}\cosh\frac{\gamma}{2})}{\sinh^2\frac{\sigma}{2}\sinh\frac{\alpha}{2}\sinh\frac{\beta}{2}}.$$

Notice that $\alpha \geq \beta$ and $\gamma \geq \delta$ implies that

$$\cosh\frac{\alpha}{2}(\cosh\frac{\gamma}{2}-\cosh\frac{\delta}{2})\geq\cosh\frac{\beta}{2}(\cosh\frac{\gamma}{2}-\cosh\frac{\delta}{2}).$$

This in turn implies that

$$\cosh\frac{\gamma}{2}\cosh\frac{\alpha}{2}+\cosh\frac{\beta}{2}\cosh\frac{\delta}{2}-(\cosh\frac{\delta}{2}\cosh\frac{\alpha}{2}+\cosh\frac{\beta}{2}\cosh\frac{\gamma}{2})\geq 0,$$

and thus

$$\cosh l'_{\alpha\sigma}\cosh l'_{\beta\sigma}\geq\cosh l_{\alpha\sigma}\cosh l_{\beta\sigma}.$$

Also

$$\sinh l'_{\alpha\sigma}\sinh l'_{\beta\sigma}-\sinh l_{\alpha\sigma}\sinh l_{\beta\sigma}=\frac{\sqrt{N_1N_2}-\sqrt{N_3N_4}}{D}$$

where

$$\begin{aligned}
N_1 &= \cosh^2\frac{\delta}{2}+2\cosh\frac{\delta}{2}\cosh\frac{\alpha}{2}\cosh\frac{\sigma}{2}+\cosh^2\frac{\sigma}{2}+\cosh^2\frac{\alpha}{2}-1,\\
N_2 &= \cosh^2\frac{\gamma}{2}+2\cosh\frac{\gamma}{2}\cosh\frac{\beta}{2}\cosh\frac{\sigma}{2}+\cosh^2\frac{\sigma}{2}+\cosh^2\frac{\beta}{2}-1,\\
N_3 &= \cosh^2\frac{\delta}{2}+2\cosh\frac{\delta}{2}\cosh\frac{\beta}{2}\cosh\frac{\sigma}{2}+\cosh^2\frac{\sigma}{2}+\cosh^2\frac{\beta}{2}-1,\\
N_4 &= \cosh^2\frac{\gamma}{2}+2\cosh\frac{\gamma}{2}\cosh\frac{\alpha}{2}\cosh\frac{\sigma}{2}+\cosh^2\frac{\sigma}{2}+\cosh^2\frac{\alpha}{2}-1,\\
D &= \sinh^2\frac{\sigma}{2}\sinh\frac{\alpha}{2}\sinh\frac{\beta}{2}.
\end{aligned}$$

To prove that this quantity is positive it suffices to prove that

$$N_1N_2-N_3N_4\geq 0.$$

By calculation

$$\begin{gathered}
N_1N_2-N_3N_4\\
=\\
2\cosh\frac{\sigma}{2}(1-\cosh^2\frac{\sigma}{2}+\cosh\frac{\alpha}{2}\cosh\frac{\beta}{2}+\cosh\frac{\gamma}{2}\cosh\frac{\delta}{2})\\
\times\\
(\cosh\frac{\alpha}{2}\cosh\frac{\gamma}{2}+\cosh\frac{\beta}{2}\cosh\frac{\delta}{2}-\cosh\frac{\alpha}{2}\cosh\frac{\delta}{2}-\cosh\frac{\beta}{2}\cosh\frac{\gamma}{2})\\
+\\
\cosh^2\frac{\alpha}{2}\cosh^2\frac{\gamma}{2}+\cosh^2\frac{\beta}{2}\cosh^2\frac{\delta}{2}-\cosh^2\frac{\alpha}{2}\cosh^2\frac{\delta}{2}-\cosh^2\frac{\beta}{2}\cosh^2\frac{\gamma}{2}.
\end{gathered}$$

With our hypotheses $\cosh^2\frac{\sigma}{2}\leq 2$. This combined with

$$\cosh\frac{\alpha}{2}\cosh\frac{\gamma}{2}+\cosh\frac{\beta}{2}\cosh\frac{\delta}{2}-\cosh\frac{\alpha}{2}\cosh\frac{\delta}{2}-\cosh\frac{\beta}{2}\cosh\frac{\gamma}{2}\geq 0$$

proves that $N_1N_2 - N_3N_4 \geq 0$ and thus $c' \geq c$. □

It is interesting to look at why the condition on σ was important in the proof. It was necessary for only two reasons: first of all it was necessary to insure that two interior geodesics of length σ could not cross, and secondly it was necessary in the calculation of the positioning of γ and δ on a maximal surface. If, for other reasons these two problems do not play a role, the proof remains correct. The following corollary is a stronger result in the particular case where the boundary geodesics are of equal length σ.

Corollary 2.11. *Let $\sigma > 0$ be a constant. Let X_σ be the set of all surfaces of signature $(0,4)$ with boundary geodesics α, β, γ and δ of length σ and with all interior closed geodesics of length superior of equal to σ. If X_σ is not empty, then the surface obtained by pasting (α, δ, σ) and (β, γ, σ) along σ with half twists has maximal distance between α and β among all elements of X_σ.*

Proof. What is needed to prove the claim is to insure that the two reasons for the condition $\sigma \leq 2\,\text{arcsinh}1$, necessary in the previous proof, are not necessary in the present case. The first condition no longer applies because if two interior geodesics of length σ intersect, then they necessarily intersect at least twice, and in consequence one of the four boundary geodesics is necessarily of length inferior to σ. The second reason no longer applies because the lengths of γ and δ are equal and the two candidates for maximal surfaces in the previous proof are identical. □

Remark 2.12. *There is a somewhat subtle point in the previous corollary. An X-piece with four equal boundary geodesics obtained by pasting along σ with a half-twist is isometric to the X-piece obtained by pasting along σ with zero twist. There is thus a marking involved in the result. The seam we claim to render maximal is the seam between α and β. If (α, δ, σ) is pasted to (β, γ, σ) along σ with a half-twist, there are shorter seams than the shortest seam between α and β: the ones between α and δ may be shorter, and in any case the seam between α and γ is necessarily shorter.*

3. Extremal surfaces for the length of canonical homology bases

The length of a homology basis $\mathcal{B}(S)$, as defined by M. Seppälä and P. Buser in [4], is $\ell(\mathcal{B}(S)) = \max_{\delta \in \mathcal{B}(S)} \ell(\delta)$. The following question was answered in the affirmative [5]: let S be a given surface of genus g and systole σ, and let $\mathcal{B}(S)$ be the (or a) shortest homology basis on S. Can $\ell(\mathcal{B})$ be bounded by a constant $B_{g,\sigma}$ that depends only on g and σ?

This problem is not unlike the search for Bers' partition constant (a bound on the length of the longest geodesic in a pants decomposition) where the asymptotic bound is not known. In the case of homology bases however, the best bound known (see [6]), namely $B_{g,\sigma} \leq (g-1)(45 + 6\text{arcsinh}\frac{2}{\sigma})$, is asymptotically optimal.

The definition of the length of a homology basis in what follows is going to be modified, and this not only in the interest of confusing the reader. The idea behind this definition is to link the length of homology bases to lengths of separating geodesics. This new definition takes in account the length of both elements in a pair $\{\alpha_i, \beta_i\}$. Furthermore, the new definition has allowed us to prove properties of extremal or maximal surfaces for homology bases. A *maximal surface* for a given genus g and a given systole σ is a surface S_{max} such that a minimal homology basis is of maximal length among all surfaces of same systole and genus.

Notice the following: each pair $\{\alpha_i, \beta_i\}$ is contained in a unique embedded Q-piece $\mathcal{Q}_i$, where the boundary geodesic γ_i is the commutator of α_i and β_i.

Definition 3.1. *The length of a homology basis $\mathcal{B}$ on a surface S is defined as $\ell'(\mathcal{B}) = \max_{i\in\{1,\dots,g\}} \ell(\gamma_i)$.*

There is a certain equivalence between the two definitions of length. For a given length ℓ' we can calculate a sharp bound for ℓ and vice-versa. The relationship between the two lengths functions is discussed in the next section.

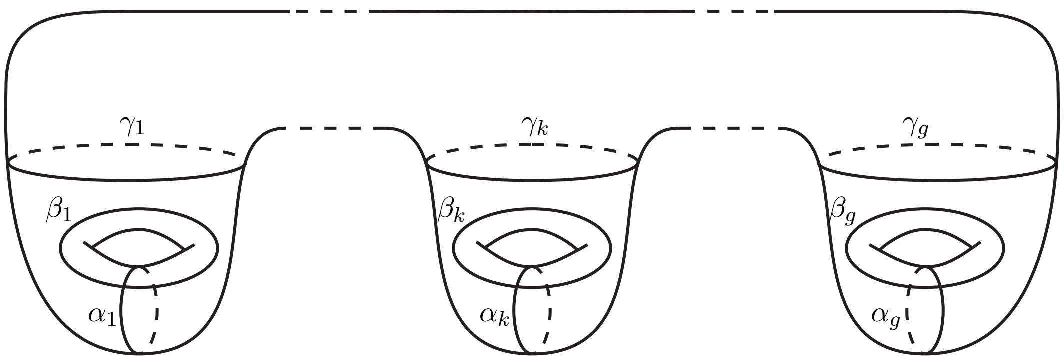


FIGURE 15. A canonical homology basis and the separating geodesics used to calculate ℓ'

3.1. **Lengths of simple closed geodesics on one holed tori.** The two previous definitions of the length of a homology basis are the main reason we shall have to look into length of simple closed geodesics on Q-pieces. The length of the boundary geodesic of a Q-piece naturally bounds lengths of certain simple closed geodesics. This is the object of the following proposition (previously found in [16] expressed differently and without proof).

Proposition 3.2. *Let $\mathcal{Q}$ be a surface of signature* $(1,1)$ *with boundary geodesic γ. Then $\mathcal{Q}$ contains a simple closed geodesic δ satisfying*

$$\cosh\frac{\ell(\delta)}{2} \leq \cosh\frac{\ell(\gamma)}{6} + \frac{1}{2}.$$

This bound is sharp.

Proof. The idea of the proof is to use hyperbolic polygons to obtain an equation from which we can deduce the sharp bound.

Let δ be the shortest closed geodesic on a Q-piece $\mathcal{Q}$ with boundary geodesic γ. Let c be the perpendicular from δ to δ on the Y-piece (δ, δ, γ) obtained by cutting $\mathcal{Q}$ along δ. The length of c is given by the following formula:

$$\cosh c = \frac{\cosh \frac{\gamma}{2} + \cosh^2 \frac{\delta}{2}}{\sinh^2 \frac{\delta}{2}}.$$

Another way of expressing it is using one of the four isometric hyperbolic pentagons that form a symmetric Y-piece as in the following figure.

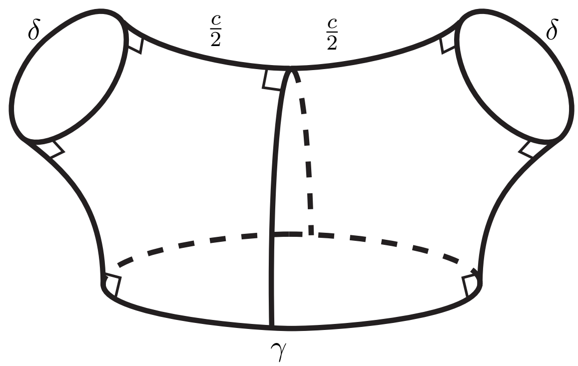


FIGURE 16. A symmetric Y-piece

The pentagon formula implies that

$$\cosh^2 \frac{c}{2} = \frac{\cosh^2 \frac{\gamma}{4} + \cosh^2 \frac{\delta}{2} - 1}{\cosh^2 \frac{\delta}{2} - 1}.$$

The shorter c is, the longer δ is. As we would like an upper bound on δ, we need to find a minimal c, under the constraint that δ is the shortest geodesic in the interior of $\mathcal{Q}$. Any other simple closed geodesic crosses δ. Let δ' be the shortest simple closed geodesic that crosses δ once. For a given δ and γ, this δ' is of maximal length when $\mathcal{Q}$ is a Q-piece obtained by pasting δ with a half twist. The length of this maximal δ' can be calculated in the following quadrilateral.

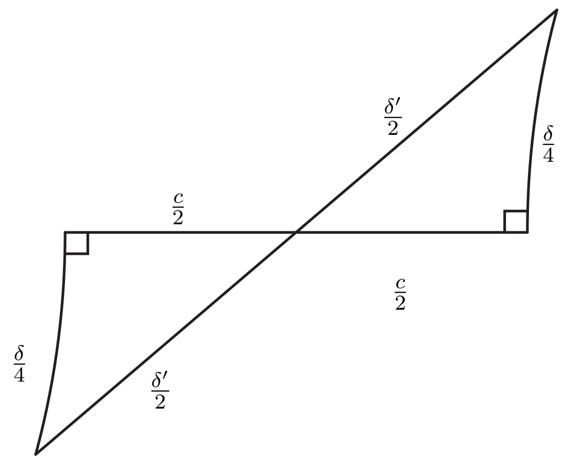


FIGURE 17. Quadrilateral for a maximal δ

From one of the two right-angled triangles that compose the quadrilateral we have

$$\cosh\frac{\delta'}{2} = \cosh\frac{c}{2}\cosh\frac{\delta}{4}.$$

Using the fact the $\delta \leq \delta'$ we can deduce

$$\begin{aligned}\cosh^2\frac{\delta}{2} &\leq \cosh^2\frac{c}{2}\cosh^2\frac{\delta}{4}\\ &= \frac{\cosh^2\frac{\gamma}{4}+\cosh^2\frac{\delta}{2}-1}{\cosh^2\frac{\delta}{2}-1}\cosh^2\frac{\delta}{4}\\ &= \frac{\cosh^2\frac{\gamma}{4}+\cosh^2\frac{\delta}{2}-1}{2(\cosh\frac{\delta}{2}-1)}.\end{aligned}$$

From this we obtain the following condition:

$$2\cosh^3\frac{\delta}{2} - 3\cosh^2\frac{\delta}{2} + 1 - \cosh^2\frac{\gamma}{4} \leq 0.$$

With $x = \cosh\frac{\delta}{2}$ and $C = \cosh^2\frac{\gamma}{4} > 1$ we can study the following degree 3 polynomial

$$f(x) = 2x^3 - 3x^2 + 1 - C$$

and find out when it is negative for $x > 1$. The function f verifies $f(1) = -C < 0$ and $f'(x) > 0$ for $x > 1$. The sharp condition we are looking for is given by the unique solution x_3 to $f(x) = 0$ with $x > 0$. Using for instance Cardano's method we have

$$x_3' = \frac{1}{2}(-1 + 2C + 2\sqrt{-C + C^2})^{\frac{1}{3}} + \frac{1}{2}\frac{1}{(-1 + 2C + 2\sqrt{-C + C^2})^{\frac{1}{3}}} + \frac{1}{2}.$$

Now we replace x and C by their original values. Using hyperbolic trigonometry we can show

$$\cosh\frac{\delta}{2} \leq \frac{1}{2}((\cosh\frac{\ell(\gamma)}{2} + \sinh\frac{\ell(\gamma)}{2})^{\frac{1}{3}} + (\cosh\frac{\ell(\gamma)}{2} + \sinh\frac{\ell(\gamma)}{2})^{-\frac{1}{3}} + 1)$$

which in turn can be simplified to

$$\cosh\frac{\ell(\delta)}{2} \leq \cosh\frac{\ell(\gamma)}{6} + \frac{1}{2}.$$

The bound is sharp as the bound value can be used to construct a Q-piece using the bound value as the length of an interior geodesic and by pasting it along half twists. Notice that in this case there are three distinct closed geodesics with the length of the bound. □

Now suppose that we have a given value for the shortest closed geodesic α in the interior of $\mathcal{Q}$ with boundary geodesic γ. Suppose that β is the shortest simple closed geodesic that crosses α once. A sharp bound on β is given in the following proposition.

Proposition 3.3. *Let $\mathcal{Q}$ be a surface of signature* $(1, 1)$ *with γ as a boundary geodesic. Let α be the shortest simple closed geodesic on $\mathcal{Q}$. Let β be the shortest simple closed geodesic that crosses α. Then*

$$\cosh\frac{\beta}{2} \leq \sqrt{\frac{1}{2}\frac{\cosh^2\frac{\gamma}{4} + \cosh^2\frac{\alpha}{2} - 1}{\cosh\frac{\alpha}{2} - 1}}.$$

This bound is sharp.

Furthermore the above bound on β is optimal among all Q-pieces with same boundary length and shortest closed geodesic α' such that $\alpha' \geq \alpha$.

Proof. This has been essentially proved during the proof of the previous proposition. For a given α, the value of the perpendicular c (between α and α) is given by

$$\cosh^2\frac{c}{2} = \frac{\cosh^2\frac{\gamma}{4} + \cosh^2\frac{\alpha}{2} - 1}{\cosh^2\frac{\alpha}{2} - 1}.$$

As is the previous proposition, the largest possible β is given by the following quadrilateral.

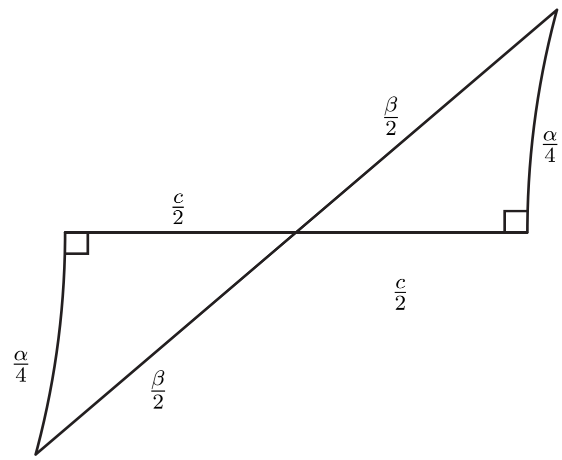


FIGURE 18. Quadrilateral with maximal β

This gives

$$\cosh^2\frac{\beta}{2} \leq \frac{1}{2}\frac{\cosh^2\frac{\gamma}{4}+\cosh^2\frac{\alpha}{2}-1}{\cosh\frac{\alpha}{2}-1}.$$

What has now been proven is that the above bound is optimal for all Q-pieces with same boundary length and who contain a closed geodesic of length exactly α. We must now show that the bound strictly decreases in α. For this it suffices to study the following function.

$$f(\alpha) = \frac{\cosh^2\frac{\gamma}{4}+\cosh^2\frac{\alpha}{2}-1}{\cosh\frac{\alpha}{2}-1}$$

Replacing $\cosh\frac{\alpha}{2}$ with $x > 1$ gives a new function

$$g(x) = \frac{\cosh^2\frac{\gamma}{4}+x^2-1}{x-1}.$$

We will show that g has a negative derivative for $x > 1$ and under the conditions imposed by proposition 3.2. We obtain

$$g'(x) = \frac{(x-1)^2-\cosh^2\frac{\gamma}{4}}{(x-1)^2}.$$

The denominator is positive, and the numerator is, using the bound on α from proposition 3.2, at most equal to

$$n(\gamma) = \cosh^2\frac{\gamma}{6}+\frac{1}{4}-\cosh\frac{\gamma}{6}-\cosh^2\frac{\gamma}{4}.$$

Notice that $n(0) = -\frac{3}{4}$ and that

$$n'(\gamma) = \frac{\sinh\frac{\gamma}{3}}{3} - \frac{\sinh\frac{\gamma}{2}}{2} - \frac{\sinh\frac{\gamma}{6}}{6} < 0.$$

Thus g has a negative derivative for $\alpha > 0$. The result follows. □

Corollary 3.4. *Let $\mathcal{B}$ be a canonical homology basis for S a surface with length of the systole at least σ. Then*

$$\cosh\frac{\ell(\mathcal{B})}{2} \leq \sqrt{\frac{1}{2}\frac{\cosh^2\frac{\ell'(\mathcal{B})}{4} + \cosh^2\frac{\sigma}{2} - 1}{\cosh\frac{\sigma}{2} - 1}}.$$

Proof. Let γ be the maximal geodesic according to ℓ' for $\mathcal{B}$ and let $\mathcal{Q}_\gamma$ be the Q-piece separated by γ. Then on $\mathcal{Q}_\gamma$ the shortest closed geodesic is at least of length σ. The previous proposition now proves the result. □

We would also like to have a bound on $\ell'(\mathcal{B})$ for a given value of $\ell(\mathcal{B})$.

Proposition 3.5. *Let $\mathcal{B}$ be a homology basis on S. Then*

$$\cosh\frac{\ell'(\mathcal{B})}{2} \leq 2\sinh^4\frac{\ell(\mathcal{B})}{2} - 1.$$

This bound is sharp.

Proof. Let $\alpha, \beta \in \mathcal{B}$ be a pair such that $\ell(\mathcal{B}) = \beta$. This of course implies that $\alpha \leq \beta$. Let $\mathcal{Q}$ be the Q-piece containing α and β. Cut $\mathcal{Q}$ along β. Let c be the perpendicular from β to β on the resulting Y-piece. The length of γ, the boundary geodesic of $\mathcal{Q}$ is given by

$$\cosh\frac{\gamma}{2} = \sinh^2\frac{\beta}{2}\cosh c - \cosh^2\frac{\beta}{2}.$$

It is easy to see that $c \leq \alpha$, and thus $c \leq \beta$. A surface such that $c = \beta$ will then give the largest possible γ. From this:

$$\cosh\frac{\ell'(\mathcal{B})}{2} \leq \sinh^2\frac{\beta}{2}\cosh\beta - \cosh^2\frac{\beta}{2}$$

and the result is obtained using hyperbolic trigonometry. □

Notice that an extremal surface for ℓ' will be extremal for ℓ but the contrary may not be true. As far as the problem that we are concerned with goes, this means that a bound on ℓ' is stronger than a bound on ℓ.

3.2. **Homology bases and short separating curves in genus 2.** Let S be a genus 2 surface and let σ be its shortest closed geodesic. If σ is a separating geodesic then there is a homology basis $\mathcal{B}$ on S such that $\ell'(\mathcal{B}) = \sigma$. For all surfaces of genus 2 with systole of length greater or equal to σ, such a surface S has a homology basis of minimal length for ℓ'. Also, $\ell(\mathcal{B})$ is short in virtue of corollary 3.4. Let σ be non-separating and let F be the surface of signature $(1, 2)$ obtained by cutting S along σ. The length of the shortest seams on F, combined with the value of σ, allows us to calculate the explicit length of the shortest separating geodesic on F, which is also the length of a separating geodesic on S. This gives us an upper bound on the length of a shortest separating geodesic on S (thus an upper bound on a minimal $\mathcal{B}$ for ℓ'). It suffices to construct a surface which attains this upper bound to show that the value is sharp. The following theorem relies heavily on the results of section 2.

Theorem 3.6. *Let S be a surface of genus 2 with a systole σ. Then S has a separating geodesic γ of length*

$$\ell(\gamma) \leq 2\operatorname{arccosh}\frac{2\cosh^3\frac{\sigma}{2}+3\cosh^2\frac{\sigma}{2}-\cosh\frac{\sigma}{2}}{\cosh\frac{\sigma}{2}-1}.$$

This bound is sharp.

Proof. If $\sigma \leq 2\operatorname{arcsinh}1$ then cut S along σ to obtain a surface F of signature $(1,2)$ with boundary geodesics of length σ. The length of the shortest separating geodesic γ that does not intersect σ can be determined with the length of the shortest seam c on F. In virtue of Theorem 2.8, we know that c can be bounded by

$$\cosh c \leq \cosh\frac{\sigma}{2}\frac{3\cosh\frac{\sigma}{2}-1}{(\cosh\frac{\sigma}{2}-1)^2}.$$

The length of γ can now be bounded using the maximum value of c.

$$\begin{aligned}\cosh\frac{\gamma}{2} &= \sinh^2\frac{\sigma}{2}\cosh c-\cosh^2\frac{\sigma}{2}\\ &\leq \frac{2\cosh^3\frac{\sigma}{2}+3\cosh^2\frac{\sigma}{2}-\cosh\frac{\sigma}{2}}{\cosh\frac{\sigma}{2}-1}.\end{aligned}$$

Notice that the bound is sharp, and is obtained when F is a maximal surface (for the length between boundary geodesics on a surface of signature $(1,2)$). The maximal surface S is thus obtained by pasting two identical Y-pieces (σ,σ,σ) along boundary geodesics with half twists.

If $\sigma > 2\operatorname{arcsinh}1$ then either the surface described above is maximal for γ, or we are in the situation where two systoles of length σ intersect. Suppose we are in the latter case, meaning that the maximal surface for a given σ is not the one described. Here we will prove that such a surface has at least one separating geodesic of length inferior or equal to the bound given by the surface described.

As done previously, if one cuts open the maximal surface along a systole, one obtains a surface F of signature $(1,2)$ with two boundary geodesics σ_1 and σ_2 of length σ. If there is interior geodesic of length σ on F, then cutting along this geodesic we obtain a surface X of signature $(0,4)$ with boundary geodesics $\sigma_1,\dots,\sigma_4$ all of length σ. The length of a separating geodesic on the initial surface of genus 2 is given by the length of a geodesic path between σ_1 and σ_2. According to corollary 2.11, the longest possible path is given by the surface described maximal. Thus if the maximal surface is not of the type described, then there are no interior geodesics of length σ on F.

In that case, we can apply the methods of the previous chapter. We now are searching for a maximal surface of signature $(1,2)$ for distance between boundary geodesics with boundary geodesics of length σ. By applying lemmas 2.3 and 2.4, either there is an interior systole, in which case we refer to the above discussion, or there are four minimizing paths between the boundary geodesics. Thus the surface is obtained by pasting two Y-pieces with half twists (lemmas 2.5 and 2.6). By the convexity proved in lemma 2.7, the maximal surface F has at least one interior systole.

What we have proved is that our bound works, and that it is optimal for $\sigma \leq 2\,\mathrm{arcsinh} 1$. In genus 2 it is well known that $\cosh\frac{\sigma}{2} \leq 1+\sqrt{2}$ (see [9]). For values of σ between $2\mathrm{arcsinh}1 = 2\mathrm{arccosh}\sqrt{2}$ and $2\mathrm{arccosh}(1+\sqrt{2})$ a surface of genus 2 may indeed be constructed by pasting (σ,σ,σ) to an identical Y-piece with half twists. There are no shorter closed geodesics that appear on S, and this proves that the bound is optimal for all surfaces of genus 2. □

Let γ be a minimal separating geodesic on a maximal surface. It is interesting to see how this bound evolves in function of the length of the systole σ of a surface. Let $f = \cosh\frac{\gamma}{2}$. Then for $s = \cosh\frac{\sigma}{2}$ we have just seen that

$$f(s) = \frac{2s^3+3s^2-s}{s-1}$$

with $1 < s \leq 1+\sqrt{2}$. This function is continuous, decreases until it reaches a strict minimum for s between $]\sqrt{2}, 1+\sqrt{2}[$ and then increases until $s = 1+\sqrt{2}$. The s in which the function reaches its minimum is given by the following equation, obtained by calculating the derivative of f:

$$f'(s) = \frac{4s^3-3s^2-6s+1}{(s-1)^2} = \frac{s+1}{s-1}(4s^2-7s+1).$$

The value $s \in]1, 1+\sqrt{2}[$ for which $f(s)$ is minimum is $s = \frac{7}{8}+\frac{1}{8}\sqrt{33}$ which in turn gives

$$\sigma = 2\mathrm{arccosh}(\frac{7}{8}+\frac{1}{8}\sqrt{33}).$$

In genus 2, finding a shortest separating geodesic and finding a shortest homology basis are the same problem, thus the following corollary is immediate.

Corollary 3.7. *Let S be a surface of genus 2 with systole σ. Then there is a canonical homology basis $\mathcal{B}$ on S with*

$$\ell'(\mathcal{B}) \leq 2\,\mathrm{arccosh}\frac{2\cosh^3\frac{\sigma}{2}+3\cosh^2\frac{\sigma}{2}-\cosh\frac{\sigma}{2}}{\cosh\frac{\sigma}{2}-1}.$$

This bound is sharp. □

In []

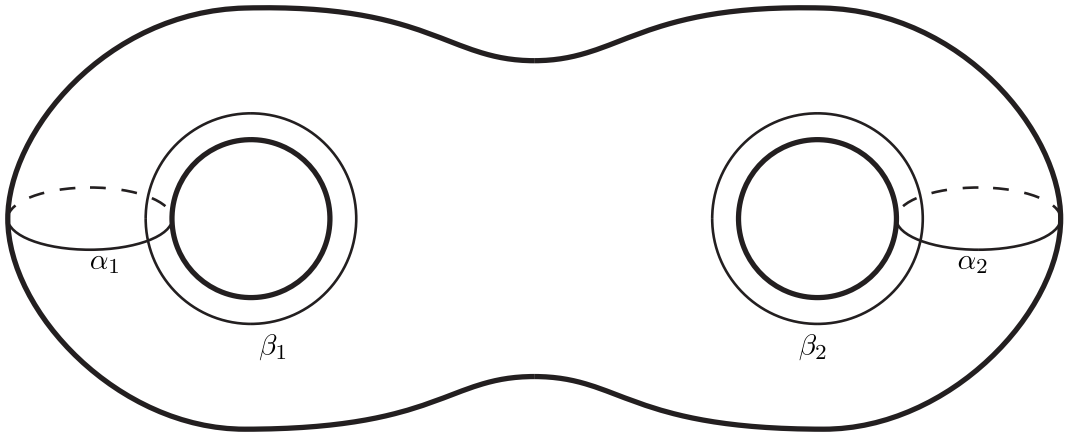


FIGURE 19. A canonical homology basis in genus 2

3.3. **Properties in genus g.** Find a maximal surface for a given genus and given systole length seems to be a very difficult problem. There is no immediate reason to believe that it is easier than the well known problem of finding surfaces that have the largest size systole in a given genus. Recall that with the exception of genus 2, the surface extremal for the systole is not known. The combinatorial difficulties that arise seem similar in nature to the corresponding problem for canonical homology bases.

Using the main theorem of [15], and the convexity of geodesic length functions along earthquake paths, the following property of maximal surfaces can be proved.

Proposition 3.8. *Let S be a maximal surface of genus g and of systole σ for homology bases. Let γ be a simple closed geodesic. Then γ intersects two distinct geodesics which are either systoles, or maximal separating geodesics for homology bases.*

Proof. Cut S along γ to obtain a surface S' of signature $(g-1,2)$. The main theorem in [15] ensure us that by equally enlarging the lengths of the boundary geodesics (in this case the two copies of γ) we can strictly increase the length of *all* interior geodesics of S'. By pasting along the images of γ under this operation one obtains a new surface $\tilde{S}$ of genus g. The geodesics on S that would not be increased are necessarily those that transversally intersect γ. This operation can be performed such that the effect on geodesic length is continuous. Because S is maximal, γ must transversally intersect a significant geodesic, in other words either a systole or a maximal separating geodesic. By twisting the significant geodesic along γ one can increase its length. It is now possible to re-perform the operation described above. This proves that γ must intersect two distinct significant geodesics, and the property is proved. □

The collar theorem [10] ensures that geodesics intersecting small systoles are long, and in section 2 it was proved that maximal boundary to boundary distance on surfaces of signature $(1,2)$ and $(0,4)$ are attained when paths are obliged to pass through systoles. These facts, combined with the properties of maximal surfaces and with the following propositions, are the basis of the conjecture.

Proposition 3.9. *Let γ be a simple closed geodesic. Let $\gamma_1,\dots,\gamma_n$ be disjoint simple closed geodesics such that* $\mathrm{int}(\gamma,\gamma_i)=1$*. Then $n\leq 2g-2$. Furthermore, there exists $\gamma_{n+1},\dots,\gamma_{2g-2}$ such that the set $\gamma_1,\dots,\gamma_{2g-2}$ have the same property.*

Proof. Complete $\gamma_1,\dots,\gamma_n$ to obtain a partition. If γ enters a Y-piece $\mathcal{Y}$ of the partition then it must leave $\mathcal{Y}$ by a different boundary geodesic. It is thus easy to see that three geodesics γ_i,γ_j and γ_k cannot form a Y-piece of the partition, otherwise one of them is intersected at least twice. This bounds the maximum number of γ_i's at $2g-2$. It is easy to see that the bound is sharp. The completion of the set of γ_is is easy to see as well. □

Notice that all the above geodesics are non-separating, otherwise the intersection numbers could never be 1.

There is an equivalent for separating simple closed geodesics.

Proposition 3.10. *Let γ be separating simple closed geodesic. Let $\gamma_1,\dots,\gamma_n$ be disjoint simple closed geodesics such that* $\mathrm{int}(\gamma,\gamma_i)=2$*. Then $n\leq 2g-2$. Furthermore, there exists $\gamma_{n+1},\dots,\gamma_{2g-2}$ such that the set $\gamma_1,\dots,\gamma_{2g-2}$ have the same property.*

Proof. Identical to the proof above. Notice that the γ_is are all non-separating. □

A simple non-separating closed geodesic, cut open along pieces of signature $(1,2)$, crosses thus a maximum of $2g-2$ of these pieces. Section 2 informs us that obliging minimal length paths to pass through systoles seems to be what lengthens them most. For systoles of length $\sigma\leq 2\,\mathrm{arcsinh}1$, the above proposition proves that, a simple closed geodesic intersects at most $2g-2$ systoles. This leads us to the following. Let us call *neck lace surface* the surface constructed by taking $2g-2$ copies of the Y-piece (σ,σ,σ) as in figure 20 and pasting along all systoles with half twists.

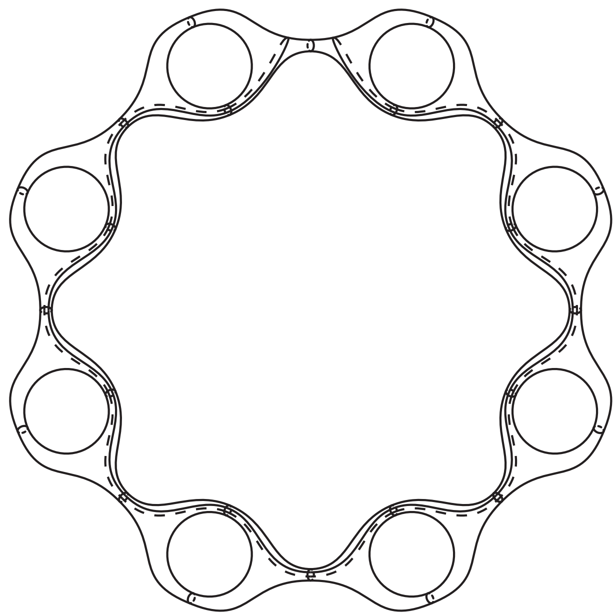


FIGURE 20. The necklace surface and a maximal geodesic for ℓ'

Conjecture *The necklace surface is maximal for canonical homology bases.*

This example is not entirely new. In [5] and [6] the same surface is described without the half twists. It is used as an example to prove that there is a surface S such that $\ell(\mathcal{B}(S)) \geq (g-1)(2+2\,\mathrm{arcsinh}\frac{2}{\sigma})$, which proves that the upper bound in [6] is asymptotically optimal.

Appendix 4. Trigonometric formulae

For convenience, we include the following list of well known propositions concerning right-angled hyperbolic polygons. Their proofs can be found in [3] or [8]. Unless specifically mentioned, all polygons are considered right-angled.

Proposition 4.1. *Let P be a pentagon with adjacent sides a and b. Let c be the only remaining side non-adjacent to either a or b. Then*

$$\sinh a \sinh b = \cosh c.$$

Proposition 4.2. *Let H be a hexagon with a, b and c be non-adjacent sides. Let α be the remaining edge adjacent to b and c, β the remaining edge adjacent to a and c and γ the remaining edge adjacent to a and b. Then*

$$\cosh c = \sinh a \sinh b \cosh \gamma - \cosh a \cosh b.$$

Proposition 4.3. *Let H be a non-convex hexagon with a, b and c be non-adjacent sides. Let α be the remaining edge adjacent to b and c, β the remaining edge adjacent* *to a and c and γ the remaining edge adjacent to a and b. Let H be such that γ and c intersect. Then*

$$\cosh c = \sinh a \sinh b \cosh \gamma + \cosh a \cosh b.$$

We shall also need a few formulas for polygons that are not entirely right-angled. First of all, a formula for a triangle with one right angle.

Proposition 4.4. *Let T be a triangle with sides a, b, and c and a right angle between a and b. Then*

$$\cosh c = \cosh a \cosh b.$$

A *trirectangle* is a quadrilateral with three right angles.

Proposition 4.5. *Let R be a trirectangle with interior angle φ being the only non right angle situated between sides α and β. Let a and b be the remaining sides with a adjacent to β and b adjacent to α. Then the following formulas hold:*

$$\begin{aligned} \cos\varphi &= \sinh a \sinh b, \\ \sinh\alpha &= \sinh a \cosh\beta. \end{aligned}$$

The following proposition deals with quadrilaterals with only two right angles.

Proposition 4.6. *Let R be a convex quadrilateral with two right interior angles. Let γ be the side of R between the two right angles. Let c be the side opposite γ and a and b the remaining sides. Then*

$$\cosh c = \cosh a \cosh b \cosh \gamma - \sinh a \sinh b.$$

And in the non-convex case the following proposition holds.

Proposition 4.7. *Let R be a non-convex quadrilateral with two right interior angles. Let γ be the side of R between the two right angles and c be the side that intersects γ. Let a and b be the remaining sides. Then*

$$\cosh c = \cosh a \cosh b \cosh \gamma + \sinh a \sinh b.$$

Finally:

Proposition 4.8. *Let P be a pentagon with four right angles. Let φ be the (only non-right) interior angle between two sides, a and b. Let α be the other side adjacent to b and β the other side adjacent to a. Let c be the remaining edge. Then the following formulas hold:*

$$\begin{aligned} \cosh c &= -\cosh a \cosh b \cos\varphi + \sinh a \sinh b, \\ \frac{\cosh a}{\cosh\alpha} &= \frac{\cosh b}{\cosh\beta} = \frac{\cosh c}{\cosh\varphi}. \end{aligned}$$

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