FIXED POINT FREE INVOLUTIONS ON RIEMANN SURFACES

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Abstract. Involutions without fixed points on hyperbolic closed Riemann surface are discussed. For an orientable surface X of even genus with an arbitrary Riemannian metric d admitting an involution τ, it is known that $\min_{p\in X} d(p,\tau(p))$ is bounded by a constant which depends on the area of X. The corresponding claim is proved to be false in odd genus, and the optimal constant for hyperbolic Riemann surfaces is calculated in genus 2.

1. Introduction

Involutions play an important role in the study of compact Riemann surfaces. For instance, the study of hyperelliptic surfaces of genus g is the study of surfaces admitting an orientation preserving involution with $2g+2$ fixed points and the study of non-orientable surfaces is essentially the study of the quotient of certain Riemann surfaces by orientation reversing involutions. Furthermore, by the uniformization theorem, surfaces with an orientation reversing involution are conformally equivalent to real algebraic curves. A further motivation can be found in [1] where the so-called filling area conjecture is treated. The conjecture, originally made in [4] for the n-dimensional case, states (in two dimensions) that any surface S with a simple boundary component, endowed with a Riemannian metric with the property that diametrically opposite points on the boundary are of distance $\geq \pi$ on the surface, has area greater or equal to 2π, equality occurring only in the case of the classical hemisphere. The conjecture is equivalent to the following:

Conjecture: Let S be an orientable surface of even genus with a Riemannian metric d that admits an orientation reversing involution τ. Then there is a point $p \in S$ with

$$\frac{d(p,\tau(p))^2}{\text{area}(S)} \leq \frac{\pi}{4}. \tag{1}$$

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In [1], this problem is solved when S is hyperelliptic, thus in particular for genus 2. Furthermore, in the general case, in [5] it is shown that a $p \in S$ can be chosen such that

$$\frac{d(p,\tau(p))^2}{\operatorname{area}(\mathcal{S})} \leq 1 \tag{2}$$

If S is a hyperbolic Riemann surface of even genus g, this shows that there is a $p \in S$ such that

$$d(p,\tau(p)) \leq \sqrt{2\pi(2g-2)}. \tag{3}$$

The idea of the paper is to treat these questions in more detail for hyperbolic Riemann surfaces. The first main result of the article is to show that the conjecture stated above cannot be extended to odd genus, even if one restricts oneself to hyperelliptic surfaces.

Theorem 1.1. *For any odd $g \geq 3$ and positive constant k, there exists a hyperbolic Riemann surface S of genus g admitting an orientation reversing involution τ which satisfies $d(p,\tau(p)) > k$ for all $p \in S$. Furthermore, S can be chosen hyperelliptic.*

The corresponding problem for orientation preserving involutions is also treated, with the same result.

Although the bound of inequality (1) is optimal for hyperelliptic metrics, it does not give a sharp bound for hyperbolic metrics in genus 2. This sharp bound is the object of the second main result of the article.

Theorem 1.2. *Let S be a Riemann surface of genus 2 endowed with a hyperbolic metric and an involution τ. On S, there is a p such that $d(p,\tau(p)) \leq \operatorname{arccosh}\frac{5+\sqrt{17}}{2}$. This upper-bound is sharp and is attained by a unique hyperbolic Riemann surface of genus 2 (up to isometry).*

Compare this optimal bound ($= 2.19...$) with the bound from equation 3 ($= \pi$). The unique surface which attains this upper bound is not the hyperbolic surface found in the conformal class of the Bolza curve. This is somewhat surprising because the hyperbolic representative of the Bolza curve is maximal for systole length, number of systoles (12), and number of automorphisms (48) (see [10]). Also, it has been shown in [6] that the surface that satisfies the case of equality of the optimal systolic inequality for hyperelliptic invariant CAT(0) metrics in genus 2 is in the conformal class of the Bolza curve.

This paper is organized as follows. The purpose of section 2 is to give basic definitions, notations and well known facts concerning the relationship between involutions, simple closed geodesics and hyperbolic metrics. Section 3 contains the proof of theorem 1.1 and section 4 treats the the proof of theorem 1.2.

2. Definitions, notations and preliminaries

The article primarily concerns compact surfaces endowed with a hyperbolic metric. The following notations and definitions refer to such surfaces. We shall suppose that the boundary of a surface is a collection of simple closed geodesics. The signature of the surface will be denoted (g,n) (genus and number of boundary components) and verifies $(g,n) \geq (0,3)$ (with lexicographic ordering) and $(g,n) \neq (1,0)$. This condition is imposed by the existence of a hyperbolic metric. A surface of signature $(0,3)$ is commonly called a pair of pants. The area of a hyperbolic surface is given by $\text{area}(S) = -2\pi\chi(S)$ where $\chi(S) = 2-2g-n$ is the Euler characteristic of the surface. Unless specifically stated, a *geodesic* is a simple closed geodesic on S. Distance on S (between subsets, points or curves) is denoted $d(\cdot,\cdot)$.

Curves and geodesics will be considered primitive and non-oriented, and can thus be seen as point sets on S. A geodesic γ (resp. a set of geodesics E) is called *separating* if $S \setminus \gamma$ (resp. $S \setminus E$) is not connected. Let us recall that a non-trivial closed curve c (not necessarily simple) on S is freely homotopic to exactly one closed geodesic γ. The unique geodesic in the free homotopy class of c will be denoted

$$\mathcal{G}(c) = \gamma. \tag{4}$$

If c is simple, then so is γ. We shall denote by $\text{int}(\gamma,\delta)$ the number of transversal intersection points between two geodesics γ and δ. The length of a path or a curve will be denoted $\ell(\cdot)$, although a curve's name and its length may not be distinguished.

We shall readily use the following well known result, commonly called the collar theorem (i.e. [2], [3], [7], [9]).

Theorem 2.1. *Let γ_1 and γ_2 be non-intersecting simple closed geodesics on S. Then the collars*

$$\mathcal{C}(\gamma_i) = \{p \in S \mid d_S(p,\gamma_i) \leq w(\gamma_i)\}$$

of widths

$$w(\gamma_i) = \operatorname{arcsinh}(1/\sinh\frac{\gamma_i}{2})$$

are pairwise disjoint for $i = 1,2$. Furthermore, each $\mathcal{C}(\gamma_i)$ is isometric to the cylinder $[-w(\gamma_i), w(\gamma_i)] \times \mathbb{S}^1$ with the metric $ds^2 = d\rho^2 + \gamma_i^2 \cosh^2\!\rho\, dt^2$.

Notice that the γ_i's divide their collar into two connected spaces which we will call *half-collars*. In the sequel we will make use of the fact that the collars of two disjoint geodesics are also disjoint.

The *systole* σ of a surface is the (or a) shortest non-trivial closed curve on S (sometimes called a systolic loop, although for Riemann surfaces, systole seems to be the standard denomination). A systole is always a simple closed geodesic and cannot intersect another systole more than once.

An *involution* τ is an isometric automorphism of the surface that is of order 2. Involutions can either be orientation preserving or reversing. For Riemann surfaces this is equivalent to whether the involution is holomorphic or antiholomorphic. Simple closed geodesics and orientation reversing involutions are closely related. The following proposition is an extension of what is generally called Harnack's theorem and can be found in [8].

Proposition 2.2. *If a surface S admits τ, an orientation reversing involution, then the fixed point set of τ is a set of n disjoint simple closed geodesics $\mathcal{B} = \{\beta_1, \dots, \beta_n\}$ with $n \leq g + 1$. In the case where the set $\mathcal{B}$ is separating, then $S \setminus \mathcal{B}$ consists of two connected components S_1 and S_2 such that $\partial S_1 = \partial S_2 = \mathcal{B}$ and $S_2 = \tau(S_1)$. If not, then $\mathcal{B}$ can be completed by a set α which consists of one or two simple closed geodesics such that $\mathcal{B} \cup \alpha$ has the properties described above (with the important difference that α does not contain any fixed points of τ). Each of the simple closed geodesics in α is globally fixed by τ.*

3. INVOLUTIONS AND SURFACES OF ODD GENUS

We recall that the main goal of this section is to give a proof of theorem 1.1. Let S be a surface of genus $g \geq 2$, endowed with a hyperbolic metric and a fixed point free involution τ.

Proposition 3.1. *If g is even, then τ is orientation reversing.*

Proof. Suppose τ is an orientation preserving involution without fixed points of a surface S of genus g. Then $S/\tau = \tilde{S}$ is a closed surface of Euler characteristic $1 - g$ (because the Euler characteristic is multiplicative under unramified coverings). It follows that the genus $\tilde{g}$ of $\tilde{S}$ is equal to $\frac{g+1}{2}$ which is only possible if g is odd. $\square$

The following propositions concern further relationships between simple closed geodesics and involutions. Suppose that S is a hyperbolic surface of genus $g \geq 2$ and τ is a fixed point free involution.

Proposition 3.2. *Let σ be a systole of S. Then $\sigma \cap \tau(\sigma) = \emptyset$ or $\sigma = \tau(\sigma)$.*

Proof. Suppose that $\sigma \neq \tau(\sigma)$. As $\tau(\sigma)$ is necessarily another systole of S, then the curves σ and $\tau(\sigma)$ cannot intersect more than once (this would imply the existence of a shorter non-trivial closed curve on S). Suppose that $\sigma \cap \tau(\sigma) = p$, where p is a point. Then $\tau(p) = p$ which contradicts the hypotheses. $\square$

As S is compact, the value $\min_{p \in S} d(p, \tau(p))$ exists and is attained for at least one point.

Proposition 3.3. *Let $p \in S$ such that $d(p, \tau(p))$ is minimum. Then p lies on a simple closed geodesic γ of length $2d(p, \tau(p))$ that verifies $\gamma = \tau(\gamma)$.*

Proof. Let p be such a point. Let c_p be a minimal path between p and $\tau(p)$ (thus $d(p, \tau(p)) = \ell(c_p)$). Notice that $c_p \cap \tau(c_p) = \{p, \tau(p)\}$, and the two paths c_p and $\tau(c_p)$ are not freely homotopic among simple paths with endpoints on p and $\tau(p)$. (Given two points on a hyperbolic surface and a homotopy class of simple curves between them, there is a unique geodesic between the two points in the homotopy class.) Thus $c_p \cup \tau(c_p)$ is a non-trivial simple closed curve. Furthermore, it follows that $\gamma = \tau(\mathcal{G}(c_p \cup \tau(c_p)) = \mathcal{G}(c_p \cup \tau(c_p))$ and thus for $q \in \gamma$, $d(q, \tau(q)) \leq d(p, \tau(p))$ with equality occurring only in the case where $\mathcal{G}(c_p \cup \tau(c_p)) = c_p \cup \tau(c_p)$. This implies that p lies on γ and that γ is of length $2d(p, \tau(p))$. ☐

Notice that for such a geodesic γ, all points are diametrically opposite to their images by τ. This is in fact true for any simple geodesic left invariant by τ.

Proposition 3.4. *Let γ be a simple closed geodesic such that $\tau(\gamma) = \gamma$. Then the image $\tau(p)$ of $p \in \gamma$ is the point on γ diametrically opposite from p.*

Proof. In particular, τ is an involution acting on γ, and the only fixed point free isometric involution acting of a circle is the rotation of angle π. ☐

As mentioned in the introduction, a result in [5] implies that for any S of even genus with an involution τ (necessarily orientation reversing by proposition 3.1), then there is a point $p \in S$ such that

$$d(p, \tau(p)) \leq \sqrt{\operatorname{area}(S)}.$$

This is false in odd genus for both orientation preserving and reversing involutions. To give an idea of why this is not the case for odd genus, we shall consider the following subsurface of $\mathbb{R}^3$.

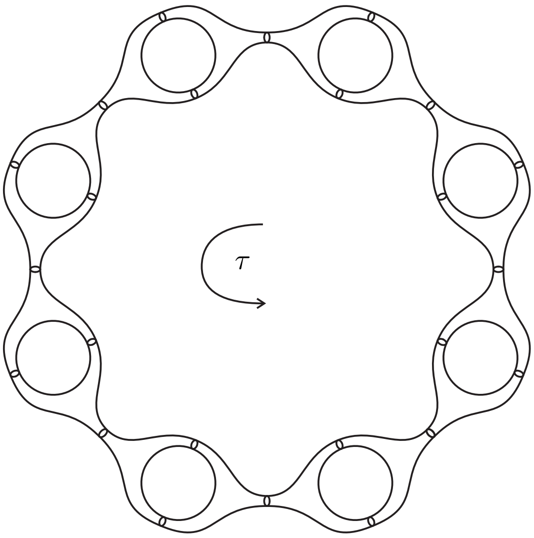

Figure 1. An example in $\mathbb{R}^3$

Let us suppose that the surface is as symmetric as it looks. In particular, it admits an obvious orientation preserving isometric involution τ given by the rotation of angle π of the surface. Now for any fixed area, the surface can be chosen "thin" enough (i.e. the simple closed curves on depicted on the figure are short) so that points are sent arbitrarily far away by τ. Notice that the odd genus is essential in this construction. To obtain orientation reversing involutions with the same property, one could compose τ with the reflection along a plane perpendicular to the axis of τ such that the surface remains invariant (roughly speaking the plane containing the figure).

Theorem 3.5. *Let $g \geq 3$ be an odd integer and k a positive constant. There exists a hyperbolic Riemann surface S of genus g admitting an orientation preserving involution τ which verifies $d(p,\tau(p)) > k$ for all $p \in S$. The same result holds for orientation reversing involutions. Furthermore, in both cases, S can be chosen hyperelliptic.*

Proof. We shall begin by showing the general idea for constructing surfaces with orientation preserving involutions, then surfaces with orientation reversing involutions which satisfy the conditions of the theorem. Finally, we shall give an explicit example of a hyperelliptic surface which has both an orientation preserving involution, and an orientation reversing involution which verifies the conditions of the theorem.

Consider a surface $\tilde{S}$ of signature $(\tilde{g},2)$ with boundary geodesics α and β of equal length x. Let p_α and q_α be two diametrically opposite points on α, and p_β and q_β be two diametrically opposite points on β. Consider two identical copies of $\tilde{S}$, say $\tilde{S}_1$ and $\tilde{S}_2$. For $k \in \{1,2\}$, denote the copies of α, β, p_α, q_α, p_β and q_β on $\tilde{S}_k$ by α_k, β_k, $p_{\alpha k}$, $q_{\alpha k}$, $p_{\beta k}$ and $q_{\beta k}$. Paste $\tilde{S}_1$ and $\tilde{S}_2$ along their boundary geodesics such that α_1 is pasted to β_2, and α_2 is pasted to β_1. Further ensure that the pasting is such that the points labeled p are pasted together (for instance $p_{\alpha 1}$ is pasted to $p_{\beta 2}$).

The resulting surface S is an orientable surface of genus $2g+1$ admitting an orientation preserving involution τ_o without fixed points which acts as follows: a point originally on $\tilde{S}_1$ is sent to it's corresponding point on $\tilde{S}_2$ and vice-versa. The simple closed geodesics of S, previously the boundary geodesics of $\tilde{S}_1$ and $\tilde{S}_2$, are of length x and are reversed by τ_o.

The image $\tau_o(p)$ of a point $p \in S$ is at least "half a collar away" from p, and by the collar theorem the following inequality is thus satified:

$$d(p,\tau_o(p)) > \operatorname{arcsinh}(1/\cosh(\frac{x}{2})).$$

The half-collar length tends to infinity as x tends to 0, thus for any $k > 0$, it suffices to chose x such that $k < \operatorname{arcsinh}(1/\cosh(\frac{x}{2})$, and the result follows.

Now let us treat the case of orientation reversing involutions. Consider a $\tilde{S}$ as above. Instead of pasting two identical copies of $\tilde{S}$, consider $\tilde{S}$ and a symmetric copy of $\tilde{S}$ (a mirror image), say $\tilde{S}_-$. Denote by α_- and β_- the images of α and β on $\tilde{S}_-$ as in the following figure.

Paste the boundary geodesics together (α to α_-, β to β_-) while, as before, respecting the choice of ps and qs. The resulting surface S is of genus $2g+1$ and the

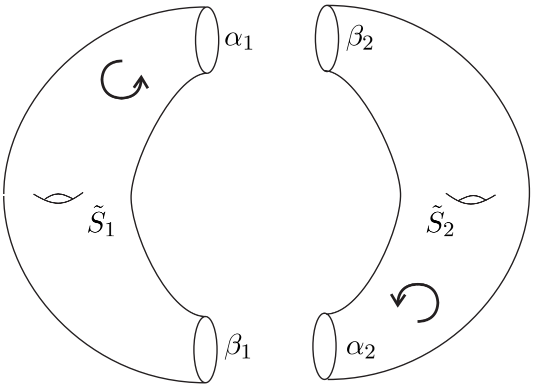

Figure 2. Example in genus 3 with orientation preserving involution

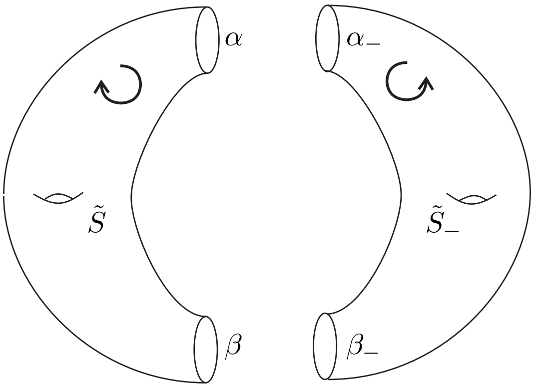

Figure 3. Example in genus 3 with orientation reversing involution

orientation reversing involution τ_r is the isometry taking a point of $\tilde{S}$ to its corresponding point on $\tilde{S}_-$ and vice-versa. As in the first examples, for any k, x, the length of both α and β, can be chosen such that $d(p, \tau_r(p)) > k$ for all $p \in S$.

We shall now give an explicit example of a hyperelliptic surface with both an orientation preserving involution and an orientation reversing involution which verifies the conditions of the theorem.

Consider a right-angled $2g+4$-gon in the hyperbolic plane, say P, with edges labeled in cyclic ordering $\{a_1, b_1, \ldots, a_{g+2}, b_{g+2}\}$. The figures are all done when $g = 3$.

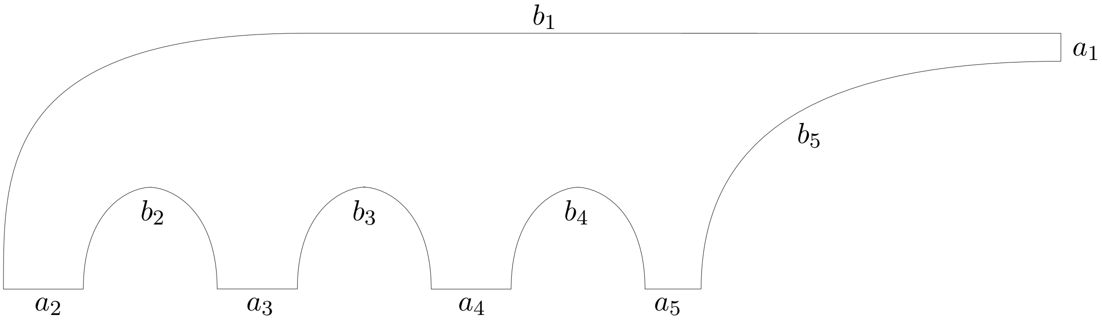

FIGURE 4. The polygon P

Take two isometric copies of P, say P_1 and Q_1, and two symmetric images of P, say P_2 and Q_2 and paste them along the sides labeled a_i as in the following figure to obtain a surface S^+ of signature $(0, 2g+2)$. The bold curves represent the boundary geodesics of S^+.

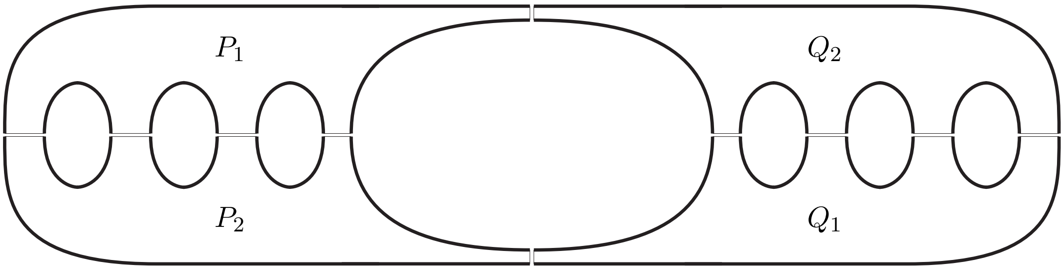

FIGURE 5. The surface S^+

By taking a symmetric copy S^- of the surface thus obtained, and denote by P_1^- etc. the various images of the polygons of S^+. By pasting the two surfaces S^+ and S^- along the $2g+2$ boundary geodesics, as in the following figure, one obtains a surface S of genus $2g+1$. We require the pasting to be exact - each edge of a polygon is pasted exactly to its corresponding edge with end points of each edge coinciding.

The surface thus obtained necessarily admits a number of involutions. One of these is the hyperelliptic involution τ_h, with fixed points the end points of all the images of $a_3, \dots, a_{g+1}$. (To be precise, τ_h exchanges P_1 and P_2^-, P_2 and P_1^-, Q_1 and Q_2^-, Q_2 and Q_1^-.)

The remaining involutions we are interested in are the ones without fixed points. The orientation preserving involution τ_o defined by exchanging P_1 with Q_1, P_2 with Q_2, P_1^- with Q_1^-, and P_2^- with Q_1^-, does not have any fixed points. The orientation reversing involution τ_r defined by exchanging P_1 with Q_1^-, P_2 with Q_2^-, P_1^- with

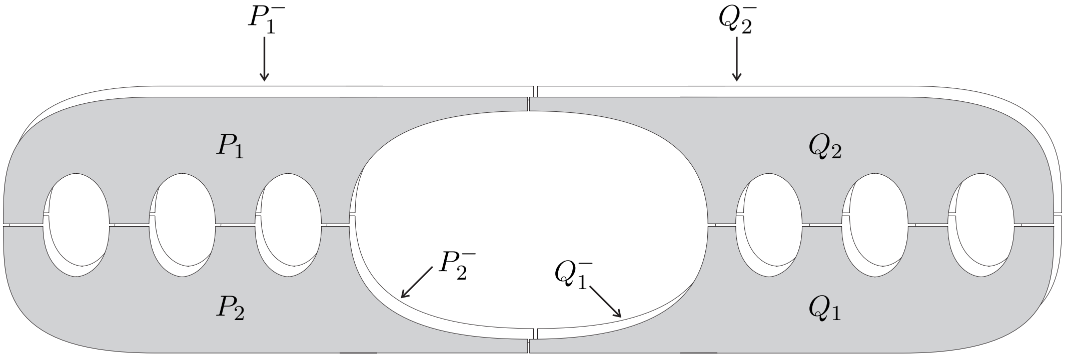

FIGURE 6. The surface S

Q_1, and P_2^- with Q_1, does not have any fixed points either. The eight images of the edge a_1 on S form two simple closed geodesics (after pasting) of length $2a_1$. As in the more general case, described above, for any k, we can chose a sufficiently short a_1 such that we have $d(p, \tau_o(p)) > k$ as well as $d(p, \tau_r(p)) > k$ for all $p \in S$. ☐

4. THE CASE OF GENUS 2

For all hyperbolic surfaces of genus 2, inequality 3 of the introduction implies that the minimum distance between a point and the image by an involution is bounded. The object of this section is a detailed study of fixed point free involutions on surfaces of genus 2 leading to the sharp bound on this distance.

Lemma 4.1. *Let τ be an involution without fixed points on a surface of genus 2. Then the following statements are true:*

(1) *τ reverses orientation.*

(2) *S contains a separating simple closed geodesic β such that $\beta = \tau(\beta)$. The two parts of S separated by β are interchanged by τ.*

(3) *For any $p \in \beta$, p and $\tau(p)$ are diametrically opposite.*

Proof. Parts (1) and (3) have been treated earlier in the general case and part (2) is a direct consequence of proposition 2.2 (Harnack's theorem). ☐

The geodesic β from the previous lemma divides S into two surfaces of signature $(1,1)$. The following lemma recalls some essential facts about these surfaces. These facts are either well known, or their proofs can be found in [10] (theorem 4.2, p. 578). For such surfaces, and a choice of interior simple closed geodesic α, we denote h_α the unique simple geodesic path which goes from boundary to boundary and intersects boundary at two right angles and does not cross α. We will refer to the geodesic path h_α as the height associated to α (see figure 7).

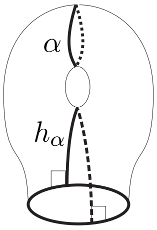

FIGURE 7. The curve h_α associated to α

Lemma 4.2. *Let Q be a surface of signature $(1,1)$ with boundary geodesic β. Then the following statements are true:*

1. *Q is hyperelliptic and its hyperelliptic involution has three fixed points in the interior of Q (the Weierstrass points of Q).*

2. *Let γ be an interior simple closed geodesic of Q and denote its associated height h_γ. Then γ passes through exactly two of the three Weierstrass points and the remaining Weierstrass point is the midpoint of h_γ. Furthermore, the length of γ is directly proportional to the length of h_γ.*

3. *Among all surfaces of boundary length $\ell(\beta)$, the unique surface (up to isometry) with maximum length systole is the surface with three distinct systoles.*
4. *If Q has three systoles, then their associated heights do not intersect and are evenly spaced along β.*

We shall now proceed to the main result of this section.

Theorem 4.3. *Let S be a Riemann surface of genus 2 endowed with a hyperbolic metric and a fixed point free involution τ. On S, there is a p such that $d(p,\tau(p)) \leq \operatorname{arccosh}(\frac{5+\sqrt{17}}{2})$. This upper-bound is sharp and is attained by a unique hyperbolic Riemann surface of genus 2 (up to isometry).*

Proof. For S of genus 2, let β be the separating geodesic described in the previous lemma 4.1. Let Q be one of the two surfaces of signature $(1,1)$ separated by β. Proposition 3.3 implies that a p which minimizes $d(p,\tau(p)$ is found on a simple closed geodesic left invariant by τ. As all simple closed geodesics in the interior of both Q and $\tau(Q)$ are distinct from their images, p must be found on a simple geodesic that either crosses β or is β.

Let h be a height on Q. Then $h \cup \tau(h) = \gamma_h$ is a simple closed geodesic with the property that for $p \in \gamma_h$, $\tau(p)$ is diametrically opposite to p (lemma 4.1). Furthermore, if σ is a systole on Q, then h_σ is the shortest height on Q. Notice that all simple closed geodesics that cross β are longer that $2h_\sigma$, thus $d(p,\tau(p)) \geq h_\sigma$, equality occurring when $2h_\sigma \leq \beta$. For any β, the maximum value that h_σ can attain is attained in the situation described in lemma 4.2. Furthermore, the maximum value of h_σ is strictly proportional to the length of β. The maximum value for $d(p, \tau(p))$ is thus obtained when both $\beta/2$ and maximum h_σ are equal. These conditions define a unique surface of signature $(1,1)$, whose lengths satisfy (i.e. [10])

$$4\cosh^3(\frac{\sigma}{2}) + 6\cosh^2(\frac{\sigma}{2}) + 1 = \cosh(\frac{\beta}{2}). \tag{5}$$

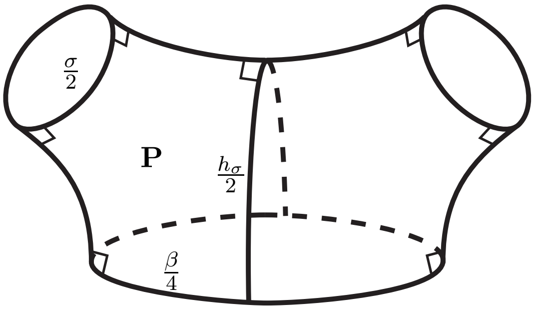

Figure 8. Maximal Q cut along a systole

The figure shows this surface after cutting along a systole σ. The hyperbolic right angled pentagon **P** has equal length adjacent edges of lengths $\frac{h_\sigma}{2}$, and $\frac{\beta}{4}$, and opposite edge of length $\frac{\sigma}{2}$. Using the hyperbolic trigonometry formula for such pentagons, one obtains

$$\sinh^2(\frac{\beta}{4}) = \cosh(\frac{\sigma}{2})$$

and thus

$$\cosh(\frac{\beta}{2}) - 1 = 2\cosh(\frac{\sigma}{2}). \tag{6}$$

Using equations 5 and 6, the value for a systole σ verifies

$$2\cosh^2(\frac{\sigma}{2}) - 3\cosh(\frac{\sigma}{2}) - 1 = 0. \tag{7}$$

From this we can deduce the value

$$h_\sigma = \frac{\beta}{2} = \operatorname{arccosh}(\frac{5+\sqrt{17}}{2}).$$

In order to obtain the maximal surface of genus 2, it suffices to take two copies of the maximum surface of signature $(1,1)$ and to paste them such that minimum length heights touch. As the heights are evenly spaced along β, whichever way this is done one will obtain the same surface. This surface, say $S_{\max}$, is clearly unique up to isometry. The following figure explicitly illustrates how to obtain $S_{\max}$ by pasting 8 copies of **P**. On **P**, consider the midpoint of the edge labeled $\frac{\sigma}{2}$ on figure 8. The points labeled p_1, p_2, q_1 and q_2 are the 8 copies of this point and the points p_3 and q_3 are as labeled on the figure. The pasting of the boundary geodesics is as indicated in the figure (p_1 pasted to p_1 etc.). The Weierstrass points of $S_{\max}$ are exactly the points p_1, p_2, p_3 and q_1, q_2, q_3.

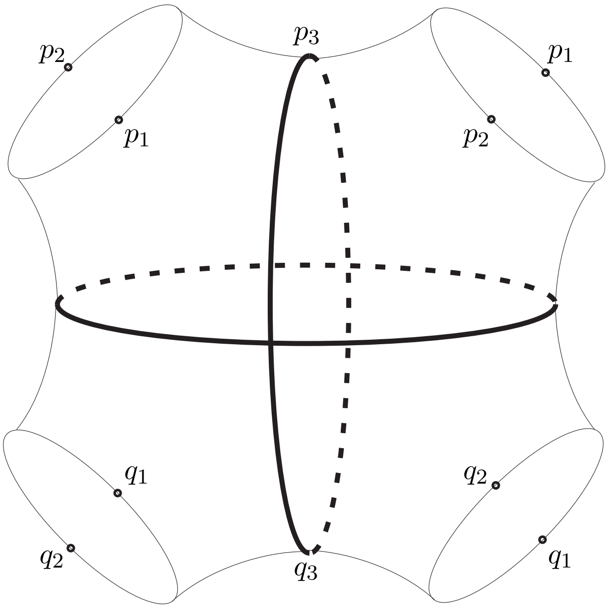

FIGURE 9. $S_{\max}$

Notice that their are exactly 6 systoles on this surface, and the length of a systole of the surface is the same as a systole on the maximal $(1,1)$ surface. The length of a systole, by equation 7 is

$$\sigma = 2\operatorname{arccosh}\left(\frac{3+\sqrt{17}}{4}\right).$$

□

The maximal surface $S_{\max}$ we have constructed above has a remarkable property: it is *not* the Bolza curve. The Bolza curve, with its unique hyperbolic metric, is constructed in the same fashion, namely it is obtained by pasting two maximal surfaces of signature $(1,1)$. It is the unique maximal genus 2 surface for systole length. Compared to $S_{\max}$, its separating geodesic β is longer and its systole length is $2\operatorname{arccosh}(1+\sqrt{2})$.

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