SIMPLE CLOSED GEODESICS OF EQUAL LENGTH ON A TORUS

GREG MCSHANE AND HUGO PARLIER

Abstract. Starting with a classical conjecture of Frobenius on solutions of the Markoff cubic, we are lead, via the work of Harvey Cohn, to explore the multiplicities of lengths of simple geodesics on surfaces. We indicate recent progress on this and related questions stemming from the work of Schmutz-Schaller. As an illustration we compare the cases of multiplicities on euclidean and hyperbolic once punctured tori; in the euclidean case basic number theory gives a complete understanding of the spectrum. We explain an elementary construction using iterated Dehn twists that gives useful information about the lengths of simple geodesics in the hyperbolic case. In particular it shows that the marked simple length spectrum satisfies a rigidity condition: knowing just the order in the marked simple length spectrum is enough to determine the surface up to isometry. These results are special cases of a more general result [10].

1. Introduction

The length spectrum of a hyperbolic surface is defined as the set of lengths of closed geodesics counted with multiplicities, and has been studied extensively in its relationship with the Laplace operator of a surface. A natural subset of the length spectrum is the *simple length spectrum*: the set of lengths of simple closed geodesics counted with multiplicities. This set is more naturally related to Teichmüller space and the mapping class group. In the particular case of the one-holed torus, we shall explore the following question: when, how often and on what type of subsets of Teichmüller space can two distinct simple closed geodesics be of equal length?

The origins of this question can be traced back to Frobenius who conjectured that any solution (a, b, c) of The *Markoff cubic*

$$a^2 + b^2 + c^2 - 3abc = 0 \tag{1.1}$$

admits infinitely many solutions (a, b, c) in positive integers, and such a triple (a, b, c) is called a *Markoff triple*. Frobenius was led to conjecture that a Markoff triple is uniquely determined by $\max\{a, b, c\}$. (The conjecture is generally called the *Markoff uniqueness conjecture*.) Making the change of variable, $(x, y, z) = (3a, 3b, 3c)$ the Markoff cubic becomes

$$x^2 + y^2 + z^2 - xyz = 0. \tag{1.2}$$

Date: December 24, 2007.

By work of Fricke and others, given a once punctured hyperbolic torus $\mathbb{M}$ and α, β, γ a triple of simple closed curves, meeting pairwise in a single point, then

$$(2\cosh\frac{\ell_M(\alpha)}{2}, 2\cosh\frac{\ell_M(\beta)}{2}, 2\cosh\frac{\ell_M(\gamma)}{2}),$$

where $\ell_M(.)$ is the hyperbolic length, is a solution to (1.2). By work of Harvey Cohn [8] and others, a solution over the integers corresponds to the lengths of a triple in the so-called modular torus $\mathbb{M}$. The modular torus is the unique hyperbolic torus with a single cusp as boundary which is conformally equivalent to the flat hexagonal torus. Stated otherwise, it is only once-punctured torus with an isometry group of maximal order (the order is 12). It is called the modular torus because it can be seen as the quotient of $\mathbb{H}$ by a subgroup of index 3 of $PSL(2,\mathbb{Z})$. Frobenius' conjecture on Markoff triples is in fact equivalent to the following conjecture on the modular torus:

Given any two simple closed geodesics of $\mathbb{M}$*, there is an isometry of* $\mathbb{M}$ *which takes one to the other.*

This property of having an isometry between any two simple closed geodesics of equal length on a torus will be called the *Markoff uniqueness property.*

There are a number of partial results which lend weight to the conjecture of Frobenius, notably:

Theorem 1.1 (Baragar [2], Button [6], Schmutz-Schaller [12])**.** *A Markoff number is unique if it is a prime power or 2 times a prime power.*

And:

Theorem 1.2 (Zhang [14])**.** *A Markoff number* c *is unique if one of* $3c+2$*and* $3c+2$ *is a prime power, 4 times a prime power, or 8 times a prime power.*

Zhang's proof is elementary and relies on a clever study of congruences. Unfortunately for a geometer, this leads one to think that the solution of the Markoff uniqueness conjecture is outwith the scope of classical geometry. We think that our study of Schmutz-Schaller's conjecture leads further weight to this point of view.

Let us call *simple multiplicity* of a torus the maximum multiplicity which appears in the simple length spectrum. Another rephrasing of the Markoff unicity conjecture is that the modular torus has simple multiplicity equal to 6. Schmutz-Schaller [13] made the following generalization of the Markoff uniqueness conjecture:

All once-punctured tori have simple multiplicity at most 6*.*

Let us now consider the Teichmüller space $\mathcal{T}$ of all hyperbolic tori with either geodesic or cusp boundary. The main result we would like to present is the following:

Theorem 1.3. *The set of hyperbolic tori* $\mathcal{N}eq$ *with all simple closed geodesics of distinct length is Baire dense in* $\mathcal{T}$*. Conversely, the set* $\mathcal{N}eq$ *contains no arcs, and as such is totally disconnected.*

The theorem is in fact true for any Teichmüller space [10]. Here we present only a proof of the converse (which is in fact the interesting part).

As mentioned above, the modular torus is the unique once-puctured torus with an isometry group of order 12, but it is not the only one-*holed* torus (tori with either cusp *or* geodesic boundary). In fact, such tori represent a connected dimension 1 subset of $\mathcal{T}$ which we shall denote $\mathcal{T}^*$. The techniques used to prove the above theorem can be used to show the following:

Theorem 1.4. [10] *The set of one-holed tori with multiplicity at least* 12 *is dense in* $\mathcal{T}^*$.

Thus Schmutz-Schaller's conjecture cannot be generalized to one-holed tori.

Note that theorem 1.3 implies that most tori *do* have the Markoff uniqueness property. However, knowing whether a particular torus has this property is in general a difficult question. An analogy can be made with the case of transcendental real numbers. Although most real numbers are transcendental, given a particular real number, proving that it is transcendental is often a very difficult question, for example we know that $\zeta(3)$ is irrational but we do not know whether it is transcendental.

This note is organized as follows. We begin by showing theorem 1.3 in the case of flat tori. Sections 3 to 5 are dedicated to the proof of theorem 1.3 in the case of hyperbolic tori. In the last section, we discuss bounds on simple multiplicity. First, we show that there are flat tori with unbounded multiplicity. Finally, we end our exposition by presenting certain tori which do have the Markoff uniqueness property, and thus multiplicity bounded by 6.

2. The flat torus

Our general approach is to study the sets of Teichmüller space where two simple closed geodesics are of equal length. In the case of flat tori, these sets are straightforward to characterize.

Recall that Riemann's Uniformization Theorem tells us that every flat or euclidean torus $\mathbb{T}^2$ is obtained as a quotient of its universal cover $\mathbb{C}$ by the group of deck transformations Γ, which is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. The lift to $\mathbb{C}$ of a closed geodesic on $\mathbb{T}^2$ is a straight line $L \subset \mathbb{C}$ invariant by some cyclic subgroup of deck transformations $\langle z \mapsto z + \omega \rangle$. It follows that the length of the geodesic is equal to the translation length of $z \mapsto z + \omega$, that is $|\omega|$. Note that, if $c \in \mathbb{C}$ then $L + c/\langle z \mapsto z + \omega \rangle$ is a closed geodesic, freely homotopic to the original geodesic and of the same length. We identify Γ with the fundamental group of $\mathbb{T}^2$ and note that, contrary to strictly negatively curved spaces, there are infinitely many closed geodesics in each free homotopy class. Thus, in order to make sense of multiplicity in the spectrum, we choose the unique geodesic in the (free) homotopy class which passes through the base point of $\mathbb{T}^2$.

In fact, given a flat torus $\mathbb{T}^2$ there exists τ, $\mathrm{Im}\,(\tau) > 0$ such that $\mathbb{C}/\Gamma$ is conformally equivalent to $\mathbb{T}^2$ where Γ is generated by the translations $z \mapsto z+1, z \mapsto z+\tau$.

Teichmüller space of flat tori can be seen as $\mathbb{H}$ in the following way. Consider a torus obtained by quotienting $\mathbb{C}$ by $\mathbb{T}^2$ generated by two complex translations $z \mapsto z+1$ and $z \mapsto z+\tau$with $\mathrm{Im}\,(\tau) > 0$. Teichmüller space can be seen as the space of deformations of such a torus by letting the parameter τ vary. As we exclude singular tori, we only let τ live in $\mathbb{H}$. Up to homothety, we have described all possible flat tori, and of course a bit more. By the uniformization theorem, we've also described all smooth tori up to *conformal* equivalence. To obtain the *Moduli space* of smooth tori, that is the set of tori up to *conformal equivalence*, one takes our set of flat tori and quotients by homothety. This corresponds to quotienting $\mathbb{H}$ by $PSL(2,\mathbb{Z})$, the mapping class group in this instance. The resulting space is the *modular surface* and has an orbifold structure with three singular points. The modular surface is a rather deep first example of a moduli space and is a very useful source of natural questions one might want to ask for a moduli space in general.

Now simple closed geodesics on a flat torus (up to free homotopy) are naturally associated to rational numbers (union infinity) in the following fashion. Consider the square torus, i.e., when $\tau = i$. Now consider a line in $\mathbb{C}$ of slope σ: clearly the line projects to a simple closed geodesic if and only if $\sigma \in \mathbb{Q}$ (or if the line is vertical, we say the line is of slope $\infty = \frac{1}{0}$). We are interested in *primitive* curves, meaning curves that are not the n-iterate of another curve. Thus up to free homotopy, each simple closed geodesic is described by a unique element of $\mathbb{Q} \cup \infty$.

Consider $\frac{p}{q}, \frac{r}{s} \in \mathbb{Q} \cup \infty$ distinct. The set of $\mathbb{H}$ where their associated simple closed geodesics are equal is the set where τ satisfies

$$|p\tau + q| = |r\tau + s|. \tag{2.1}$$

A straightforward calculation shows this set to be the Poincaré geodesic between endpoints $\frac{q-s}{r-p}$ and $\frac{q+s}{r+p}$.

Conversely, between any given pair of distinct rationals $\frac{a}{b}, \frac{a'}{b'}$, the Poincaré geodesic $[a/b, a'/b']$ between them is the set of $\mathbb{H}$ where two simple closed geodesics are of equal length. To show this, consider the map $z \mapsto -\overline{z}$. It preserves the rationals and fixes $[0,\infty] = \{z \in \mathbb{C} \mid \Re z = 0\}$. For any rational $\frac{m}{n}$ the curves of slope $\frac{m}{n}$ and $-\frac{m}{n}$ have the same length at $\tau \in [0,\infty]$. One maps $[0,\infty]$ onto any geodesic $[a/b, a'/b']$ using $PSL(2,\mathbb{R})$, which is transitive on pairs of rationals and thus finds a pair of curves which are of equal length on $[a/b, a'/b']$.

Using this characterization, we can now show the following.

Theorem 2.1. *The set of flat tori $\mathcal{N}eq$ with all simple closed geodesics of distinct length is Baire dense. Conversely, the set $\mathcal{N}eq$ contains no arcs, and as such is totally disconnected.*

The first statement follows from the fact that rationals are countable and the second from the fact that any two distinct points in $\mathbb{H}$ are separated by a Poincaré geodesic between rationals.

Unfortunately in the case of hyperbolic tori, the set of tori with two simple closed geodesics of equal length is not so easy to characterize.

3. Hyperbolic tori

Let us recall a few definitions and facts from the theory of surfaces; all this is available in a more detailed treatment in either [1], [3] or [5]. Throughout M will denote a surface with constant curvature -1 and we shall insist that M is *complete* with respect to this metric (although we will only be concerned by what happens inside the *Nielsen core* of the surface). This means that M is *locally modeled* on the hyperbolic plane $\mathbb{H}^2$ and there is a natural covering map $\pi : \mathbb{H} \to M$. By $\mathcal{T}$ we mean the Teichmüller space of M, meaning the space of marked complete hyperbolic structures on M. The signature of a surface M will be denoted (g, n) where M is homeomorphic to a surface of genus g with n simple closed boundary curves. In this article, we are interested in hyperbolic tori with one boundary component, which we will consider to be either a cusp or a simple closed geodesic (surfaces of signature $(1, 1)$). (It is worth noting however, that all in fact all of our arguments either apply, or are can easily be made to fit, the case of tori with a cone angle.) In the case of surfaces of signature $(1, 1)$, Teichmüller space is, topologically, $\mathbb{R}^+ \times \mathbb{R}^2$ where the first parameter corresponds to boundary length, and the other two to the lent

Let us recall a few facts about curves on surfaces (see [5] or [7] for details). Firstly, a *simple curve* is a curve which has no self intersections. A curve is said to be *essential* if it bounds neither a disc nor a punctured disc (or an annulus). For each free homotopy class which contains an essential simple loop, there is a unique geodesic representative.

There is a natural function, $\ell : \mathcal{T} \times$ essential homotopy classes $\to \mathbb{R}^+$, which takes the pair $M, [\alpha]$ to the length $\ell_M(\alpha)$ of the geodesic in the homotopy class $[\alpha]$ (measured in the Riemannian metric on M). It is an abuse, though common in the literature, to refer merely to *the length of the geodesic* α (rather than, more properly, the length of the geodesic in the appropriate homotopy class). Using length functions one can describe Teichmüller space. In the case of surfaces of signature $(1, 1)$, Teichmüller space is, topologically, $\mathbb{R}^+ \times \mathbb{R}^2$. The first parameter corresponds to boundary length, and the other two correspond to an interior (or essential) simple closed geodesic in the following way. One can think of the first parameter as being the length of the simple closed geodesic (thus formally its lies in $\mathbb{R}^{+,*}$) and the second is a twist parameter, a real valued parameter which tells you how the simple closed geodesic is pasted together to get a torus. (These are the Fenchel-Nielsen parameters.) Note that these parameters are not homogeneous in nature. In the next section, we give a set of homogeneous parameters for the Teichmüller space of one-holed tori.

4. A projectively injective map

In the case of one-holed tori, we will make essential use of the following lemma. Recall that a projectively injective map is a map f such that $f(x) = \lambda f(y)$ for $\lambda \in \mathbb{R}$ implies that $x = y$.

Lemma 4.1. *There are four interior simple closed curves α, β, γ, and δ of a one-holed torus such that the map $\varphi : M \mapsto (\ell_M(\alpha), \ell_M(\beta), \ell_M(\gamma), \ell_M(\delta))$ is projectively injective.*

Proof. Let M be a one-holed torus and let α, β, γ, and δ be the simple closed curves as in figure 1.

We've chosen our curves as follows. We begin by choosing any α and β that intersect once. It is not difficult to see that given α and β, there are exactly two curves (γ and δ) that intersect both α and β exactly once. The curves γ and δ intersect twice. Now the remarkable fact about the geodesic representatives of simple closed curves on a one-holed torus is that they pass through exactly two of the three Weierstrass points of the torus in diametrically opposite points. In the case of the curves α, β, γ, and δ, their intersection points are all necessarily Weierstrass points. Therefore they can be seen in the universal cover as in figure 1.

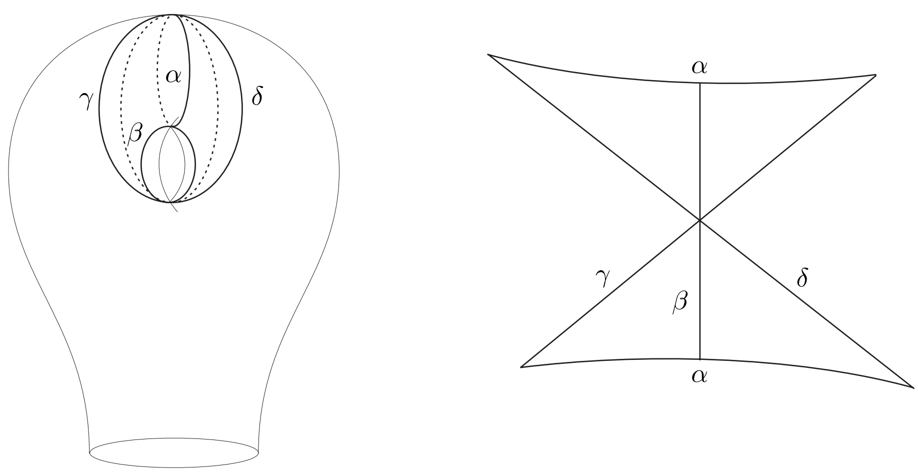

FIGURE 1. The one-holed torus with four interior geodesics and the four curves seen in the universal cover

In fact the lengths of α, β, and γ determine a unique point in the Teichmüller space of one-holed tori. One can show this by recovering the Fenchel-Nielsen parameters from the three lengths (this is done in detail in [4] for instance). However, up to a multiplicative constant, they do not (otherwise the real dimension of the Teichmüller space would be 2 and not 3). For this we need the curve δ. What we need to prove is that if we have two one-holed tori M_1 and M_2 in Teichmüller space with

$$(\ell_{M_1}(\alpha), \ell_{M_1}(\beta), \ell_{M_1}(\gamma), \ell_{M_1}(\delta)) = \lambda(\ell_{M_2}(\alpha), \ell_{M_2}(\beta), \ell_{M_2}(\gamma), \ell_{M_2}(\delta))$$

for some $\lambda \in \mathbb{R}$, then $\lambda = 1$ and then, by what precedes, $M_1 = M_2$. Figure 1 shows four hyperbolic triangles. Consider the two bottom ones. The side lengths of the bottom left triangle are $\frac{\ell(\alpha)}{2}$, $\frac{\ell(\beta)}{2}$, and $\frac{\ell(\gamma)}{2}$. The side lengths of the bottom right triangle are $\frac{\ell(\alpha)}{2}$, $\frac{\ell(\beta)}{2}$, and $\frac{\ell(\delta)}{2}$. The bottom intersection point between α and β forms two angles depending on the surface M, say $\theta_1(M)$ and $\theta_2(M)$ such that $\theta_1 + \theta_2 = \pi$. Suppose without loss of generality that $\lambda \geq 1$. Now if for M_1 the triangle lengths are equal to a, b, c and d, the triangle lengths for M_2 are $\lambda a, \lambda b, \lambda c$ and λd. This implies that $\theta_1(M_1) \leq \theta_1(M_2)$ as well as $\theta_2(M_1) \leq \theta_2(M_2)$, equality occurring only if $\lambda = 1$. As $\theta_1 + \theta_2$ is always equal to π, this concludes the proof. $\square$

We will make essential use of the following corollary to this lemma.

Corollary 4.2. *Let $M_1, M_2 \in \mathcal{T}$ be distinct tori, then there exist two interior simple curves γ_1 and γ_2 such that*

$$\frac{\ell_{M_1}(\gamma_1)}{\ell_{M_1}(\gamma_2)} \neq \frac{\ell_{M_2}(\gamma_1)}{\ell_{M_2}(\gamma_2)}. \tag{4.1}$$

Proof. Suppose inequality 4.1 was in fact an equality for all pairs of simple curves γ_1, γ_2, thus in particular for all pairs in the set of curves α, β, γ and δ. Then the map of lemma 4.1 could not be projectively injective, a contradiction. □

5. Dehn twisting

Here we show the second part of Theorem 1.3 for one-holed tori, namely that a path between two distinct points of $\mathcal{T}$ contains a surface with two simple closed geodesics of equal length.

Let M_1 and M_2 be two distinct points of $\mathcal{T}$. Corollary 4.2 guarantees the existence of two simple closed geodesics who satisfy inequality 4.1 above. For the remainder of this section, these two curves shall be denoted α and β.

Given these curves, we can choose a pair of curves $\tilde{\alpha}$ and $\tilde{\beta}$, such that $\text{int}(\alpha, \tilde{\alpha}) = \text{int}(\beta, \tilde{\beta}) = 1$. Consider the two families of curves $\{\alpha_k\}_{k\in\mathbb{N}}$, $\{\alpha_k\}_{k\in\mathbb{N}}$ obtained by performing k right Dehn twists of α, resp. β, around $\tilde{\alpha}$, resp. $\tilde{\beta}$. These two families satisfy the following lemma.

Lemma 5.1. *For each $k \in \mathbb{N}$, and for any surface M we have*

1. $\text{int}(\alpha, \alpha_k) = 1$, $\text{int}(\beta, \beta_k) = 1$,
2. $k\ell_M(\alpha) - \ell_M(\tilde{\alpha}) < \ell_M(\alpha_k) \leq k\ell_M(\alpha) + \ell_M(\tilde{\alpha})$,
3. $k\ell_M(\beta) - \ell_M(\tilde{\beta}) < \ell_M(\beta_k) \leq k\ell_M(\beta) + \ell_M(\tilde{\beta})$.

Proof. The first statement is obvious and the last two follow by lifting to the universal cover $\mathbb{H}$ and by applying the triangle inequality to the geodesics. □

For a surface M, set $B_i(M) := \{\beta_k : \ell_M(\beta_k) \leq \ell_M(\alpha_i)\}$. Our aim is to calculate the ratio $\ell_M(\alpha)/\ell_M(\beta)$ from the asymptotic formula in lemma 5.1.

Proposition 5.2. *With the notation above:*

$$\frac{\sharp B_i(M)}{i} \longrightarrow \frac{\ell_M(\alpha)}{\ell_M(\beta)}.$$

Proof. As M is fixed, we set $B_i := B_i(M)$ and $\ell := \ell_M$.

By lemma 5.1 we have

$$\sharp B_i \leq \sharp\{k : 2k\ell(\beta) - \ell(\beta_0) \leq 2i\ell(\alpha) + \ell(\alpha_0)\},$$

and

$$\sharp B_i \geq \sharp\{k : 2k\ell(\beta) + \ell(\beta_0) \leq 2i\ell(\alpha) - \ell(\alpha_0)\}.$$

It follows that

$$i\frac{\ell(\alpha)}{\ell(\beta)} - \frac{\ell(\beta_0)}{2\ell(\beta)} - \frac{\ell(\alpha_0)}{2\ell(\beta)} \leq \sharp B_i \leq i\frac{\ell(\alpha)}{\ell(\beta)} + \frac{\ell(\alpha_0)}{2\ell(\beta)} + \frac{\ell(\beta_0)}{2\ell(\beta)}.$$

The statement of the propostion is immediate. □

We can now establish the following.

Corollary 5.3. *There exist two simple closed geodesics* α_k *and* $\beta_{\tilde{k}}$ *such that*

$$\ell_{M_1}(\beta_{\tilde{k}}) > \ell_{M_1}(\alpha_k)$$

and

$$\ell_{M_2}(\beta_{\tilde{k}}) > \ell_{M_2}(\alpha_k).$$

In particular, the marked order in lengths of simple closed geodesics determines a unique surface in $\mathcal{T}$*.*

Proof. Recall that α and β satisfy inequality 4.1. Applying proposition 5.2 to M_1 and M_2, we see that there is an integer k such that $\sharp B_k(M_1) \neq \sharp B_k(M_2)$. In particular, there exists a $\tilde{k}$ such that α_k and $\beta_{\tilde{k}}$ satisfy the desired inequalities. □

Now $M \mapsto \ell_M(\alpha_k) - \ell_M(\beta_{\tilde{k}})$ is a continuous function, so applying the intermediate value theorem to the arc $\mathcal{A}$ between the points M_1 and M_2, yields the existence of a surface $N \in \mathcal{A}$ so that $\ell_Z(\alpha_k) = \ell_Z(\beta_{\tilde{k}})$. This establishes the second part of theorem 1.3.

Remark 5.4. *In fact one can show something stronger than Corollary 5.3, namely that for any given* $M_1 \neq M_2$ *and any integer* N*, there exists a set of simple closed geodesics* α_k*,* $\beta_{\tilde{k}+1}, \dots, \beta_{\tilde{k}+N}$ *such that*

$$\ell_{M_1}(\beta_{\tilde{k}+i}) > \ell_{M_1}(\alpha_k) \text{ and}$$
$$\ell_{M_2}(\beta_{\tilde{k}+i}) > \ell_{M_2}(\alpha_k)$$

for all $i \in \{1, \dots, N\}$*. The proof goes as follows. In the proof of corollary 5.3, we used the fact that* $\sharp B_k(M_1) \neq \sharp B_k(M_2)$*. Suppose by contradiction that* $|\sharp B_k(M_1) - \sharp B_k(M_2)|$ *was bounded by some constant for all* k*. The limit of the ratios from Proposition 5.2 would be then the same for both* M_1 *and* M_2*, a contradiction.*

6. Bounds on multiplicity of simple closed geodesics

Although most surfaces have all simple multiplicities equal to 1, it is an open question as to whether or not hyperbolic surfaces with unbounded simple multiplicity exist. The remark at the end of the last section shows why it might be difficult to prove that simple multiplicity is always bounded. For the full length spectrum, a theorem of Randol [11], based on a construction of Horowitz, shows that multiplicity is *always* unbounded. In the particular case of tori with a single cusp, Schmutz-Schaller [13] conjectured that all simple multiplicities of once-punctured tori are bounded by 6. He also notes that, to the best of his knowledge, one does not know a surface for which we are sure that simple multiplicity is bounded. After having shown why multiplicities can be unbounded in the case of flat tori, we shall give examples of hyperbolic tori for which we are sure that multiplicities are bounded.

6.1. **The multiplicity of the spectrum of a flat torus.**

In this section we give a short account of unboundedness of multiplicities in the length spectrum of a flat or euclidean torus. Our exposition is based on elementary number theory, and we concentrate on only the two "most symmetric" such tori, though a more thorough knowledge of class field theory [9] might allow more cases to be treated.

Consider a flat torus $\mathbb{T}^2$. As explained in Section 2, there exists τ, $\operatorname{Im}\tau > 0$ such that $\mathbb{C}/\Gamma$ is conformally equivalent to $\mathbb{T}^2$ where Γ is generated by the translations $z \mapsto z+1, z \mapsto z+\tau$. For certain values of τ one can compute the multiplicities of the numbers which appear in the length spectrum by studying the ring of integers $\mathbb{Z}[\tau]$ of a quadratic field $\mathbb{Q}(\tau)$. Throughout we will assume that $\mathbb{Z}[\tau]$ is a unique factorization domain domain, that is every $\omega \in \mathbb{Z}[\tau]$ factors as $uq_1q_2\dots q_n$ where u is a unit and q_i are irreducible elements of $\mathbb{Z}[\tau]$ and this factorization is unique up to permutation of q_i and multipliction by the units of $\mathbb{Z}[\tau]$. Whenever $\mathbb{Z}[\tau]$ is a Euclidean domain e.g. $\tau = i, \sqrt{-2}, \sqrt{-3}, \sqrt{-7}, \sqrt{-11}$ then it is a unique factorization domain, the former condition being easier to verify [9].

We restrict our attention to τ such that τ is a *quadratic irrational* that is it satisfies a quadratic with integer coefficients

$$\tau^2 + B\tau + C = 0,$$

since, for such τ, the ring $\mathbb{Z}[\tau]$ embeds in $\mathbb{C}$ as a lattice and there is an isomorphism of abelian groups

$$\mathbb{Z}[\tau] \to \Gamma,\; x+\tau y \mapsto (z \mapsto z + x + \tau y),$$

where, as above, Γ denotes the group of deck transformations of $\mathbb{T}^2$. We are interested primarily in $\tau = i, \frac{-1+\sqrt{-3})}{2}$ as the resulting torus, $\mathbb{C}/\Gamma$, is respectively the square torus and the regular hexagonal (or modular) torus.

By convention, the *norm* of $\omega \in \mathbb{Z}[\tau]$ is defined to be $\omega\bar{\omega}$; this is an integer and it is evidently the square of the translation length of $z \mapsto z+\omega$. For example when $\tau = i$ the norm of $x+\tau y \in \mathbb{Z}[\tau]$ is just x^2+y^2 and when $\tau = \frac{-1+\sqrt{-3}}{2}$ the norm is x^2+xy+y^2. Now a prime $p \neq 2$ can be written as a sum of squares x^2+y^2, $x, y \in \mathbb{N}$ if and only if p is congruent to 1 modulo 4. and it can be written as x^2+xy+y^2, $x, y \in \mathbb{N}$ if and only if it is congruent to 1 modulo 3. It is a celebrated theorem of Dirichlet that there are infinitely many primes in any arithemetic progression and so there are infinitely many congruent to 1 modulo 4 and to 1 modulo 3. Such a prime p admits a factorization $p = (x+\tau y)(x+\bar{\tau}y)$ where $x+\tau y, x+\bar{\tau}y$ are irreducible elements of the ring of $\mathbb{Z}[\tau]$. By Dirichlet's theorem we may choose n such distinct primes $p_k \in \mathbb{N}$, $1 \le k \le n$, let $a_k \in \mathbb{Z}[i]$, $p_k = a_k\bar{a_k}$ and let N denote their product. Now N factorizes over $\mathbb{Z}[\tau]$ and

$$N = (a_1\bar{a}_1)(a_2\bar{a}_2)\dots(a_n\bar{a}_n).$$

Consider the set $R_N \subset \mathbb{Z}[\tau]$ of the form $c_1c_2\dots c_n$ where $c_k \in \{a_k, \bar{a}_k\}$. Note that the norm of each element of R_N is N. It is easy to check, using the fact that $\mathbb{Z}[\tau]$ is a unique factorization domain, that R_N contains exactly 2^{n-1} distinct elements. Note further that if $c_1c_2\dots c_n \in R_N$ and $c_1c_2\dots c_n = x+iy$ then x, y are coprime integers, for otherwise there is a prime p that divides x, y hence $x+\tau y$, now as the c_i are irreducible p factors as

$$p = uc_{i_1}\dots c_{i_l}$$

for some unit $u \in \mathbb{Z}[\tau]$. Considering the norms of both sides of the above one has

$$p^2 = p_{i_1}\dots p_{i_l}$$

which is contradicts the hypothesis that the p_i were distinct. The set of deck transformation $z \mapsto z+\omega, \omega \in R_N$ yields a set of pairwise non-homotopic, primitive, simple closed geodesics of length $\sqrt{N}$ on the torus.

6.2. **Hyperbolic tori.** Given a hyperbolic structure on a surface M, not necessarily of finite volume, then the holonomy of the metric gives a representation of the fundamental group into the group of isometries of the hyperbolic plane $PSL(2,\mathbb{R})$. In fact, since there is no 2-torsion, one can lift this representation $\hat{\rho} : \pi_1 \to SL(2,\mathbb{R})$ and for any element $\gamma \in \pi_1$

$$2\cosh(\ell_M(\gamma)) = \operatorname{tr} \hat{\rho}(\gamma).$$

In the case of the once punctured torus one obtains a representation of the free group on two generators $\langle \alpha, \beta \rangle$ into $SL(2,\mathbb{R})$.

Theorem 6.1 (Fricke, Horowitz, Keen)**.** *Let A, B be matrices in $SL(2,\mathbb{C})$. If W is a word in A, B then there is a polynomial $P_W \in \mathbb{Z}[x,y,z]$ such that*

$$\operatorname{tr} W = P_W(\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB).$$

A celebrated construction of Horowitz, see [5] and [11] for details, yields pairs of words W, W' such that W, W', W^{-1}, W'^{-1} are pairwise inconjugate but $P_W = P_{W'}$. However, the Horowitz construction cannot be applied to W representing a simple closed curve (see [10]):

FACT: If W represents a simple closed curve then $P_W = P_{W'}$ then W' is conjugate to W or W^{-1}. Thus, we remark that if W, W' represent simple closed curves such that W, W', W^{-1}, W'^{-1} are pairwise inconjugate then $P_W - P_{W'}$ is a non-zero element of $\mathbb{Z}[x,y,z]$.

An immediate corollary of this remark is that, given $\lambda > 2$ a transcendental real number and A, B, AB such that $\operatorname{tr} A = \operatorname{tr} B = \operatorname{tr} AB = \lambda$, then the quotient $\mathbb{H}/\langle A, B \rangle$ is a hyperbolic one-holed torus which satisfies Markoff uniqueness.

References

[1] William Abikoff. *The real analytic theory of Teichmüller space*, volume 820 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[2] Arthur Baragar. On the unicity conjecture for Markoff numbers. *Canad. Math. Bull.*, 39(1):3–9, 1996.

[3] Alan F. Beardon. *The geometry of discrete groups*, volume 91 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. Corrected reprint of the 1983 original.

[4] P. Buser and K.-D. Semmler. The geometry and spectrum of the one-holed torus. *Comment. Math. Helv.*, 63(2):259–274, 1988.

[5] Peter Buser. *Geometry and spectra of compact Riemann surfaces*, volume 106 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1992.

[6] J. O. Button. The uniqueness of the prime Markoff numbers. *J. London Math. Soc. (2)*, 58(1):9–17, 1998.

[7] Andrew J. Casson and Steven A. Bleiler. *Automorphisms of surfaces after Nielsen and Thurston*, volume 9 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1988.

[8] Harvey Cohn. Representation of Markoff's binary quadratic forms by geodesics on a perforated torus. *Acta Arith.*, 18:125–136, 1971.

[9] Harvey Cohn. *A classical invitation to algebraic numbers and class fields*. Springer-Verlag, New York, 1978. With two appendices by Olga Taussky: "Artin's 1932 Göttingen lectures on

class field theory" and "Connections between algebraic number theory and integral matrices", Universitext.

[10] Greg McShane and Hugo Parlier. Multiplicities of simple closed geodesics and hypersurfaces in Teichmüller space. *Preprint*, 2007.

[11] Burton Randol. The length spectrum of a Riemann surface is always of unbounded multiplicity. *Proc. Amer. Math. Soc.*, 78(3):455–456, 1980.

[12] Paul Schmutz. Systoles of arithmetic surfaces and the Markoff spectrum. *Math. Ann.*, 305(1):191–203, 1996.

[13] Paul Schmutz Schaller. Geometry of Riemann surfaces based on closed geodesics. *Bull. Amer. Math. Soc. (N.S.)*, 35(3):193–214, 1998.

[14] Ying Zhang. Congruence and uniqueness of certain Markoff numbers. *Acta Arith.*, 128(3):295–301, 2007.

Laboratoire Emile Picard, Universit Paul Sabatier, Toulouse, France

Institut de Géométrie, Algèbre et Topologie, Ecole Polytechnique Fédérale de Lausanne, Lausanne, Switzerland

E-mail address: greg.mcshane@gmail.com

E-mail address: hugo.parlier@epfl.ch