DESINGULARIZATION OF QUIVER GRASSMANNIANS VIA NAKAJIMA CATEGORIES

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Abstract. In this paper, we show that *generalized Nakajima Categories* provide a framework to construct a desingularization of quiver Grassmannians for self-injective algebras of finite representation type. Furthermore, we show that all standard Frobenius models of orbit categories of the bounded derived category considered in [21] are equivalent to $\operatorname{proj} \mathcal{R}$, the finitely generated projective modules of the regular Nakajima category $\mathcal{R}$.

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Contents

1. Introduction

In this article, we construct desingularizations of quiver Grassmannians for all modules over a self-injective algebra of finite type. A quiver Grassmannian is the variety of subrepresentations with given dimension vector of a fixed quiver representation. Quiver Grassmannianns appeared first in work of Schofield [40] and Crawley-Boevey [9]. Ever since Caldero and Chapoton discovered [6] that the canonical generators of Fomin-Zelevinsky's cluster algebras [15] can be interpreted as generating polynomials of Euler characteristics of quiver Grassmannians, quiver Grassmannians have played an important role in the additive categorification of (quantum) cluster algebras.

The question of finding desingularizations of quiver Grassmannians has attracted a lot of interest, see also [7] and [8]. All results currently available in the literature focus on the desingularization of quiver Grassmannians of Dynkin or tilted Dynkin, see [23] type. We work with a class of algebras of a completely different kind, that is self-injective algebras. Namely we show that generalized quiver varieties, as defined in [38], give a framework to construct desingularization maps for quiver Grassmannians of self-injective algebras of finite type.

In the rest of this Introduction we give a more detailed account of our main results.

1.1. **Nakajima categories and generalized quiver varieties.**

Quiver varieties were first introduced by Nakajima in [27]. Since then they have been of great importance in Nakajima's geometric study of Kac Moody algebras and their representations [27], [28].

In [38], we introduce generalized quiver varieties associated to a *regular Nakajima category* $\mathcal{R}$. Generalized quiver varieties provide a unified approach to classical and cyclic quiver varieties, which arise as special cases of the theory. Furthermore, generalized quiver varieties share many features of classical and cyclic quiver varieties, including smoothness. In [38] we also show that affine quiver varieties associated to Dynkin quivers are equivalent to the moduli spaces of representation of a certain full subcategory $\mathcal{S}$ of $\mathcal{R}$, the *singular Nakajima category.*

1.2. **Frobenius models of orbit categories of the derived category.** Let Q be of Dynkin type and let $\mathcal{D}_Q$ be its bounded derived category of finite-dimensional representations. If F is an equivalence of $\mathcal{D}_Q$ satisfying the assumptions in [21] then the orbit category $\mathcal{D}_Q/F$ is triangulated. A Frobenius model of $\mathcal{D}_Q/F$ is a Frobenius category $\mathcal{E}$ such that its associated stable category $\underline{\mathcal{E}}$, i.e. the quotient of $\mathcal{E}$ by its category of projective-injective, is equivalent to $\mathcal{D}_Q/F$ as a triangulated category.

Let $\mathcal{R}$ be the regular Nakajima category. This category depends on the choice of an admissible configuration C, which is a subset of the vertices of the repetition quiver $\mathbb{Z}Q$ and on an isomorphism F of the mesh category $k(\mathbb{Z}Q)$, which by Happel's Theorem is equivalent to the category of indecomposable objects of $\mathcal{D}_Q$. We show that if $\operatorname{proj}\mathcal{R}$ is Krull-Schmidt, then the category of finitely generated projective $\mathcal{R}$–modules is a Frobenius model for $\mathcal{D}_Q/F$. Conversely, we show that all Frobenius models of $\mathcal{D}_Q/F$ which are standard are indeed equivalent to $\operatorname{proj}\mathcal{R}$. Furthermore, we have that $\mathcal{D}_Q/F \cong \mathcal{R}/\langle\mathcal{S}\rangle$, whose objects are shown in [38] to parametrize the strata of Nakajima quiver varieties and their degeneration order.

The results of this section play a major role in [39], where we establish a connection between the algebraic and geometric realization of the quantum group proposed by Bridgeland [4] and Qin [32] respectively.

1.3. **Desingularization of quiver Grassmanians.** Desingularizations of quiver Grassmanians have been established recently in the case of hereditary algebras of finite representation type in [7] and tilted algebras of Dynkin type [23]. Here we achieve an analogous result for the class of self-injective algebras of finite representation type using generalized quiver varieties. We show that any self-injective algebra of finite representation type can be realized as a singular Nakajima category $\mathcal{S}$:

Theorem 1.4. *(Theorem 4.1) Let A be a standard self-injective algebra of finite representation type. Then there is a singular Nakajima category $\mathcal{S}$ which is Morita equivalent to A. Hence the moduli space of representations of a basic self-injective algebras of finite type is a generalized quiver variety.*

To the singular Nakajima category $\mathcal{S}$, we can associate uniquely up to isomorphism a regular Nakajima category $\mathcal{R}$. We use techniques developed in [23] to construct desingularization maps for quiver Grassmannians of modules of self-injective algebras of finite representation type. The domain of the desingularization maps are smooth quiver Grassmannians of $\mathcal{R}$–modules.

Using Theorem 1.4 and results from [38], we prove that there exist functors

$$K_{LR} : \operatorname{mod} A \to \operatorname{mod}\mathcal{R} \text{ and } \operatorname{res} : \operatorname{mod}\mathcal{R} \to \operatorname{mod} A,$$

that are called respectively *intermediate extension* and *restriction*, and that satisfy

$$\operatorname{res} \circ K_{LR} \cong \mathbf{1}.$$

Given an A-module M, we show that the quiver Grassmannian of submodules with dimension vector d, which is denoted $Gr_d(K_{LR}M)$, is always smooth.

Theorem 1.5. *(Theorem 4.6) Let M be an A-module. Then there are finitely many dimension vectors d such that the map*

$$\pi^{gr} : \sqcup_d \operatorname{Gr}_d(K_{LR}M) \to \operatorname{Gr}_e(M), N \mapsto \operatorname{res} N$$

is proper and surjective with smooth domain. Restricting the domain to the subspace containing only bistable modules $\mathrm{Gr}_d^{bs}(K_{LR}M)$ *yields a desingularization map.*

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2. Generalized Nakajima categories and quiver varieties

2.1. **Notations.** For later use, we introduce the following notations. Let k be an algebraically closed field and $\mathrm{Mod}\, k$ be the category of k-vector spaces. Recall that a *k-category* is a category whose morphism spaces are endowed with a k-vector space structure such that the composition is bilinear. Let $\mathcal{C}$ be a k-category and let $\mathrm{Mod}(\mathcal{C})$ be the category of *right $\mathcal{C}$–modules*, i.e. k-linear functors $\mathcal{C}^{op} \to \mathrm{Mod}(k)$. We denote $\mathrm{mod}(\mathcal{C})$ the category of locally finite-dimensional $\mathcal{C}$-modules, that is all k-linear functors $\mathcal{C}^{op} \to \mathrm{Mod}(k)$ with finite-dimensional image on the objects of $\mathcal{C}$. For each object x of $\mathcal{C}$, we obtain a *free module*

$$x^\wedge = x^\wedge_{\mathcal{C}} = \mathcal{C}(?, x) : \mathcal{C}^{op} \to \mathrm{Mod}\, k$$

and a *cofree module*

$$x^\vee = x^\vee_{\mathcal{C}} = D(\mathcal{C}(x, ?)) : \mathcal{C}^{op} \to \mathrm{Mod}\, k.$$

Here, we write $\mathcal{C}(u, v)$ for the space of morphisms $\mathrm{Hom}_{\mathcal{C}}(u, v)$ and D for the duality over the ground field k. Recall that for each object x of $\mathcal{C}$ and each $\mathcal{C}$–module M, we have canonical isomorphisms

$$\mathrm{Hom}(x^\wedge, M) = M(x) \quad \text{and} \quad \mathrm{Hom}(M, x^\vee) = D(M(x)). \tag{2.1.1}$$

In particular, the module $x^\wedge$ is projective and $x^\vee$ is injective. We will denote $\mathrm{proj}\, \mathcal{C}$ the full subcategory of $\mathcal{C}$–modules with objects the finite direct sums of objects $x^\wedge$ and dually, we denote $\mathrm{inj}\, \mathcal{C}$ the full subcategory of $\mathcal{C}$–modules with objects the finite direct sums of objects $x^\vee$. Furthermore, throughout the paper we denote by $\mathcal{C}_0$ the set of objects of $\mathcal{C}$.

For all triangulated categories $\mathcal{T}$, we shall denote by Σ the shift functor, by τ the Auslander-Reiten translation and by S the Serre functor. We will denote by $\mathcal{D}_Q$ the bounded derived category of finite-dimensional kQ–modules. For a quiver Q, we denote by Q_0 its set of vertices.

2.2. **Generalized Nakajima Categories.** In this section we recall the definition of the singular and regular Nakajima categories $\mathcal{R}$ and $\mathcal{S}$ given in [39]. The regular Nakajima category $\mathcal{R}$ is a mesh category which we can associate to any acyclic finite quiver and $\mathcal{S}$ is a full subcategory of $\mathcal{R}$. We start by introducing the graded Nakajima categories which are defined in [22]. In [22], we denoted the graded Nakajima categories by $\mathcal{R}$ and $\mathcal{S}$, here we will denote them by $\mathcal{R}^{gr}$ and $\mathcal{S}^{gr}$.

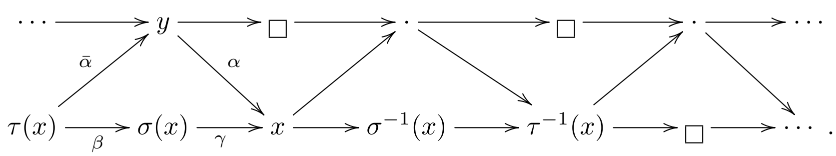
Let Q be a finite acyclic quiver with set of vertices Q_0 and set of arrows Q_1. We denote by $k(\mathbb{Z}Q)$ the *mesh category* of $\mathbb{Z}Q$. By Happel's Proposition 4.6 of [16] and Theorem 5.6 of [17], there is a fully faithful embedding

$$H : k(\mathbb{Z}Q) \hookrightarrow \mathrm{ind}\, \mathcal{D}_Q$$

where $\mathrm{ind}\, \mathcal{D}_Q$ denotes the category of indecomposable complexes in the bounded derived category of $\mathrm{mod}(kQ)$. The functor H is an equivalence if and only if Q is of Dynkin type.

Let us denote by τ the automorphism of $\mathbb{Z}Q$ corresponding to the action of the Auslander-Reiten translation on $\mathcal{D}_Q$. Let C be a *configuration* of $\mathbb{Z}Q$, that is a subset of the set of vertices of the repetition quiver $\mathbb{Z}Q$. We denote by $\mathbb{Z}Q_C$ the quiver obtained from $\mathbb{Z}Q$ by adding for all $x \in C$ a vertex $\sigma(x)$ and arrows $\tau(x) \to \sigma(x)$ and $\sigma(x) \to x$.

For example, if Q is an orientation of the Dynkin diagram A_2 and C all vertices of $\mathbb{Z}Q$, then $\mathbb{Z}Q_C$ is the following quiver:



By a mesh relation R_x in $\mathbb{Z}Q_C$ for any $x \in \mathbb{Z}Q_0$, we mean the sum of all paths from $\tau(x)$ to x. In the above example R_x is $\bar{\alpha}\alpha + \beta\gamma$.

Definition 2.3. *The* regular graded Nakajima category $\mathcal{R}_C^{gr}$ *has objects the vertices of* $\mathbb{Z}Q_C$ *and morphism spaces* $\mathcal{R}_C^{gr}(a,b)$ *given by all* k*-linear combinations of paths from* a *to* b *modulo the ideal generated by the mesh relations* R_x *for all* x *in* $\mathbb{Z}Q_0$*. We will call the objects* $\sigma(x)$ *for* $x \in C$ *frozen objects. We denote by* $\mathcal{S}_C^{gr}$ *the full subcategory of* $\mathcal{R}_C^{gr}$ *formed by the frozen objects.*

Let F be a k-linear isomorphism on $k(\mathbb{Z}Q)$. We make the following assumption on C and F.

Assumption 2.4. *For each vertex* x *of* $\mathbb{Z}Q$*, the sequences*

$$(2.4.1)\quad 0 \to \mathcal{R}_C^{gr}(?,x) \to \bigoplus_{x\to y} \mathcal{R}_C^{gr}(?,y) \quad \text{and} \quad 0 \to \mathcal{R}_C^{gr}(x,?) \to \bigoplus_{y\to x} \mathcal{R}_C^{gr}(y,?)$$

are exact, where the sums range over all arrows of $\mathbb{Z}Q_C$ *whose source (respectively, target) is* x*.*

Furthermore, we have that $F(C) \subset C$ *and* $F^n \neq \mathbf{1}$ *for all* $n \in \mathbb{Z}$*.*

We call all C satisfying the above assumption an *admissible configuration* and (C,F) an *admissible pair.* For example, the set of all vertices of $\mathbb{Z}Q$ is admissible.

Note that F commutes with τ and extends uniquely to an automorphism of $\mathcal{R}_C^{gr}$ by setting $F\sigma(c) := \sigma(F(c))$ for all $c \in C$. Hence F sends frozen objects to frozen objects.

Definition 2.5. *The* generalized Nakajima category $\mathcal{R}$ *associated to an admissible pair* (C,F) *is the orbit category* $\mathcal{R}_C^{gr}/F$*. The* singular Nakajima category $\mathcal{S}$ *is the full subcategory of* $\mathcal{R}$ *with objects the frozen objects. We denote the quotient category* $\mathcal{R}/\langle\mathcal{S}\rangle$ *by* $\mathcal{P}$*.*

The combinatorial properties of F lead to good properties of $\operatorname{proj}\mathcal{P}$.

Lemma 2.6. *Suppose that* Q *is of Dynkin type. Under assumption 2.4,* F *lifts to a triangulated functor of* $\mathcal{D}_Q$ *such that the canonical morphism* $\mathcal{D}_Q \to \operatorname{proj}\mathcal{P}$ *is triangulated. Furthermore* $\mathcal{P}$ *is* Hom*-finite, has only finitely many objects up to isomorphism, and* $\operatorname{proj}\mathcal{P} = \operatorname{inj}\mathcal{P}$*.*

Proof. By Happel's equivalence, we can lift F to a k-linear automorphism on $\mathcal{D}_Q$. By Serre duality and the fact that F commutes with τ, we conclude that

$$\mathcal{D}_Q(x,\Sigma y) \cong D\mathcal{D}_Q(\tau x, y) \cong D\mathcal{D}_Q(\tau F(x), F(y)) \cong \mathcal{D}_Q(Fx, \Sigma Fy)$$

for all $x, y \in \mathcal{D}_Q$. Hence $F(kQ)$ is a tilting object in $\mathcal{D}_Q$ and induces an auto equivalence on $\mathcal{D}_Q$ via $\mathcal{D}_Q \to \mathcal{D}_Q, x \mapsto F(kQ) \otimes^L x$. This functor is isomorphic to F, as both are uniquely determined by the image of the projective indecomposable kQ–modules. Furthermore, it is clearly triangulated. As F acts without torsion, we have that $F^n = \tau^k$ for some $k, n \in \mathbb{Z} - 0$ and as Q is of Dynkin type, we also know that there is an integer s with $\tau^s \cong \Sigma^2$. Hence $F^m = \Sigma^h$ for two non-zero integers m and h. From this equality, it is easy to see that all conditions of [21] are satisfied and as a consequence that $\operatorname{proj} \mathcal{P}$ is naturally a triangulated category. Furthermore, F has only finitely many orbits and therefore $\mathcal{P}$ has only finitely many objects up to isomorphism. As F does not have a fixed point on $\operatorname{ind} \mathcal{D}_Q$, we know that $\mathcal{D}_Q(x, F^i(y))$ vanishes for all but finitely many $i \in \mathbb{Z}$. Hence $\operatorname{proj} \mathcal{P}$ is Hom-finite. As F commutes with the Serre functor S, we find that for all $x \in \operatorname{ind} \mathcal{D}_Q$ the following isomorphism holds

$$\begin{aligned} x^{\vee}_{\mathcal{P}} &= D\mathcal{P}(?, x) \cong D \bigoplus_{i \in \mathbb{Z}} \mathcal{D}_Q(?, F^i x) \\ &\cong \bigoplus_{i \in \mathbb{Z}} D\mathcal{D}_Q(?, F^i x) \cong \bigoplus_{i \in \mathbb{Z}} \mathcal{D}_Q(S^{-1} F^i x, ?) \cong \mathcal{P}(S^{-1}x, ?) = (S^{-1}x)^{\wedge}_{\mathcal{P}}. \end{aligned}$$

√

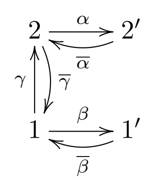
For a locally finite quiver Q' and a fixed point free automorphism a of Q', we denote by Q'/a the quiver with vertices the a–orbits of Q'_0 and the number of arrows $x \to y$ between two fixed representatives x and y of a–orbits is given by the number of arrows from $x \to a^i y$ for all $i \in \mathbb{Z}$ in the quiver Q'. Then Q'/a is locally finite and the canonical map $Q' \to Q'/a$ is a Galois covering.

Proposition 2.7. *The category* $\operatorname{proj} \mathcal{P}$ *admits Auslander-Reiten triangles. Its Auslander-Reiten quiver is given by* $\mathbb{Z}Q/F$ *and* $\operatorname{proj} \mathcal{P}$ *is standard, that is the mesh category* $k(\mathbb{Z}Q)/F \cong k(\mathbb{Z}Q/F)$ *is equivalent to* $\mathcal{P}$.

Proof. As $\mathcal{D}_Q \to \operatorname{proj} \mathcal{P}$ is triangulated, $\operatorname{proj} \mathcal{P}$ is Krull-Schmidt and admits a Serre functor induced from the Serre functor of $\mathcal{D}_Q$. It is easy to see that the images of Auslander-Reiten triangles in $\mathcal{D}_Q$ induce Auslander-Reiten triangles in $\operatorname{proj} \mathcal{P}$. The Auslander-Reiten quiver of $\mathcal{D}_Q$ is given by $\mathbb{Z}Q$ and identifying the F-orbits shows that the Auslander-Reiten quiver of $\operatorname{proj} \mathcal{P}$ is $\mathbb{Z}Q/F$. As the mesh category of $\mathbb{Z}Q/F$ is equivalent to the orbit category of the mesh category of $\mathbb{Z}Q$, it follows from the standardness of $\mathcal{D}_Q$ that $\operatorname{proj} \mathcal{P}$ is also standard. √

In the sequel, we will identify the orbit categories $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{P}$ with their equivalent skeleta categories, in which we identify all objects lying in the same F–orbit. We will denote by $\widetilde{Q}$ the quiver $\mathbb{Z}Q_C/F$.

Example 2.8. *In the case that* $Q = A_2$, $F = \tau$ *and that* C *is the set of all vertices of* $\mathbb{Z}Q$, *our assumptions are satisfied and* $\mathcal{R}$ *is equivalent to the path category of* $\widetilde{Q}$



modulo the mesh relations, which are given by

$$\alpha\overline{\alpha} + \overline{\gamma}\gamma = 0 \text{ and } \beta\overline{\beta} + \gamma\overline{\gamma} = 0.$$

We see that $\mathcal{P}$ is equivalent to the preprojective algebra associated to A_2.

2.9. **Kan extensions and Stability.** In [23], we introduced the notion of intermediate extensions and stability. We briefly recall these definitions, which we have adapted to the setup of the present paper. We call an $\mathcal{R}$–module M *stable* if $\mathrm{Hom}_{\mathcal{R}}(S, M) = 0$ vanishes for all modules S supported only in non-frozen vertices. Equivalently, M does not contain any non zero submodule supported only on non frozen vertices. We call M *costable* if we have $\mathrm{Hom}_{\mathcal{R}}(M, S) = 0$ for each module S supported only in non-frozen vertices. Equivalently, M does not contain any non zero quotient supported only on non frozen vertices. A module is *bistable*, if it is both stable and costable.

As the restriction functor

$$\mathrm{res} : \mathrm{Mod}\,\mathcal{R} \to \mathrm{Mod}\,\mathcal{S}$$

is a localization functor in the sense of [11], it admits a right and a left adjoint which we denote K_R and K_L respectively: the left and right Kan extension cf. [25].

We obtain the following recollement of abelian categories:

$$\mathrm{Mod}\,\mathcal{P} \;\underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}}\; \mathrm{Mod}\,\mathcal{R} \;\underset{\overset{\longleftarrow}{K_R}}{\overset{\overset{K_L}{\longleftarrow}}{\xrightarrow{\mathrm{res}}}}\; \mathrm{Mod}\,\mathcal{S}.$$

We define the intermediate extension

$$K_{LR} : \mathrm{Mod}\,\mathcal{S} \to \mathrm{Mod}\,\mathcal{R}$$

as the image of the canonical map $K_R \to K_L$. We refer to [23] for general properties. Recall

Proposition 2.10. *An $\mathcal{R}$-module M is bistable if and only if $M \cong K_{LR}\,\mathrm{res}\,M$.*

3. Frobenius models and Nakajima categories

Throughout this section we assume that Q is an orientation of a Dynkin diagram. Recall that $\mathrm{proj}\,\mathcal{P}$ is a triangulated category. In this section, by a *Frobenius category*, we mean a k-linear, Krull–Schmidt category $\mathcal{E}$ endowed with the structure of an exact category for which it is Frobenius. Then the stable category $\underline{\mathcal{E}}$, which is the quotient of $\mathcal{E}$ by all morphisms factoring through projective-injective objects of $\mathcal{E}$, is naturally a triangulated category. A *Frobenius model for* $\mathrm{proj}\,\mathcal{P}$ is a Frobenius category $\mathcal{E}$ together with an equivalence of triangulated categories: $\mathrm{proj}\,\mathcal{P} \xrightarrow{\sim} \underline{\mathcal{E}}$.

3.1. **Gorenstein projective modules of $\mathcal{S}$.** Recall that, for a k-category $\mathcal{C}$, a $\mathcal{C}$–module M is *Gorenstein projective* [10] if there is an acyclic complex

$$P : \ldots \to P_1 \to P_0 \to P_{-1} \to \ldots$$

of finitely generated projective modules such that M is isomorphic to the cokernel of $P_1 \to P_0$, and such that the complex $\mathrm{Hom}(P, P')$ is acyclic for each finitely generated projective $\mathcal{C}$–module P'. We denote the category of Gorenstein projective modules by $\mathrm{gpr}(\mathcal{C})$. By Proposition 5.1 of [2] it follows easily that $\mathrm{gpr}(\mathcal{C})$ is a Frobenius exact category and that the subcategory of projective–injective objects is the subcategory of finitely generated projective $\mathcal{C}$–modules.

We have shown in Theorem 5.18 of [22] that the stable category of the Frobenius category $\operatorname{gpr}(\mathcal{S}_C^{gr})$ is equivalent to the bounded derived category

$$\mathcal{D}_Q \cong \underline{\operatorname{gpr}}(\mathcal{S}_C^{gr}).$$

In this section, we denote by F_* both the exact automorphism on $\operatorname{Mod}\mathcal{R}_C^{gr}$ and its restriction to $\operatorname{Mod}\mathcal{S}_C^{gr}$ induced by the functor F.

As the pushforward functors

$$p_* : \operatorname{gpr}(\mathcal{S}_C^{gr}) \to \operatorname{gpr}(\mathcal{S}) \text{ and } p_* : \operatorname{proj}\mathcal{R}_C^{gr} \to \operatorname{proj}\mathcal{R}$$

are invariant under F_*, we obtain functors

$$\operatorname{gpr}(\mathcal{S}_C^{gr})/F_* \to \operatorname{gpr}(\mathcal{S})$$

and

$$\operatorname{proj}(\mathcal{R}_C^{gr})/F_* \to \operatorname{proj}\mathcal{R}.$$

sending the finitely generated projective $\mathcal{S}_C^{gr}$–modules to finitely generated projective $\mathcal{S}$–modules. These functors satisfy the following properties.

Lemma 3.2. *The functor* $\operatorname{proj}\mathcal{R}_C^{gr}/F_* \to \operatorname{proj}\mathcal{R}$ *is an equivalence and* $\operatorname{gpr}(\mathcal{S}_C^{gr})/F_* \hookrightarrow \operatorname{gpr}(\mathcal{S})$ *is fully faithful and exact.*

Proof. Let $\mathcal{C}^{gr}$ be either $\mathcal{S}_C^{gr}$ or $\mathcal{R}_C^{gr}$, and let $\mathcal{C}$ be respectively either $\mathcal{S}$ or $\mathcal{R}$. Let $x^\wedge$ and $y^\wedge$ be two projective $\mathcal{C}^{gr}$–modules associated with $x, y \in \mathcal{C}_0^{gr}$. Then

$$\begin{aligned} \operatorname{Mod}\mathcal{C}^{gr}/F_*(x^\wedge, y^\wedge) &= \bigoplus_{i\in\mathbb{Z}} \operatorname{Mod}\mathcal{C}^{gr}(x^\wedge, F^i(y)^\wedge) \\ &= \bigoplus_{i\in\mathbb{Z}} \mathcal{C}^{gr}(x, F^i(y)) \cong \mathcal{C}(x,y) \cong \operatorname{Mod}\mathcal{C}(x^\wedge, y^\wedge), \end{aligned}$$

where we identify the vertices x and y with their images under p. Hence the first functor is fully faithful. It is an equivalence as every indecomposable projective $\mathcal{R}$–module lifts to a projective indecomposable $\mathcal{R}_C^{gr}$–module. This proves the first part.

To prove the second part, we use that all Gorenstein projective module are finitely presented. Hence the claim follows from the fully faithfulness on the finitely generated projective modules, which has been shown above. √

We show next that $\operatorname{proj}\mathcal{R}$ is a Frobenius model for $\operatorname{proj}\mathcal{P}$.

Theorem 3.3. *The category* $\operatorname{gpr}(\mathcal{S}_C^{gr})/F_*$ *is equivalent to* $\operatorname{proj}\mathcal{R}$ *and the stable category of* $\operatorname{proj}\mathcal{R}$ *is equivalent to* $\operatorname{proj}\mathcal{P}$.

Proof. In Theorem 5.23 of [22] we have shown that the map $\operatorname{proj}\mathcal{R}_C^{gr} \to \operatorname{gpr}(\mathcal{S}_C^{gr}), x^\wedge \mapsto \operatorname{res} x^\wedge$ induces an isomorphism of exact categories. This implies that $\operatorname{gpr}(\mathcal{S}_C^{gr})/F_* \cong \operatorname{proj}\mathcal{R}_C^{gr}/F_*$. Furthermore, we have shown in the Lemma 3.2 that $\operatorname{proj}\mathcal{R}_C^{gr}/F_*$ is equivalent to $\operatorname{proj}\mathcal{R}$. Hence the category $\operatorname{gpr}(\mathcal{S}_C^{gr})/F_*$ is equivalent to $\operatorname{proj}\mathcal{R}$. Let $x, y \in \mathcal{R}_0 - \mathcal{S}_0$, then

$$\underline{\operatorname{proj}}\ \mathcal{R}(x^\wedge, y^\wedge) \cong \mathcal{R}/\langle\mathcal{S}\rangle(x,y) \cong \operatorname{proj}\mathcal{P}(x^\wedge, y^\wedge).$$

Hence the stable category $\underline{\operatorname{proj}}\ \mathcal{R}$ is equivalent to $\operatorname{proj}\mathcal{P}$. This finishes the proof. √

Combining the results of this section, yields the following.

Corollary 3.4. *There is a fully faithful embedding*

$$\operatorname{proj}\mathcal{P} \to \underline{\operatorname{gpr}}(\mathcal{S}), x^\wedge \mapsto \operatorname{res} x^\wedge$$

for all $x \in \mathcal{R}_0 - \mathcal{S}_0$.

Proof. By the previous results, we have that $\operatorname{res} : \operatorname{proj}\mathcal{R} \to \operatorname{gpr}(\mathcal{S}), x^\wedge \mapsto \operatorname{res} x^\wedge$ yields a fully faithful functor between two Frobenius categories, such that the indecomposable projective-injective objects in $\operatorname{proj}\mathcal{R}$ are mapped to the projective-injective objects in $\operatorname{gpr}(\mathcal{S})$ given by $x^\wedge$ for all $x \in \mathcal{S}_0$. Hence this functor lifts to a functor on the stable categories. As $\underline{\operatorname{proj}}(\mathcal{R}) \cong \operatorname{proj}\mathcal{P}$, we obtain the above statement. $\surd$

Theorem 3.5. *Suppose that* $\mathcal{S}$ *is* Hom*-finite. Then* $\operatorname{gpr}(\mathcal{S}_C^{gr})/F_* \to \operatorname{gpr}(\mathcal{S})$ *is an equivalence.*

Proof. Using the same argument as in Theorem 5.18 of [22], we can show that $\operatorname{proj}\mathcal{P} \to \underline{\operatorname{gpr}}(\mathcal{S})$ is essentially surjective, hence an equivalence by corollary 3.4. Note also that by Theorem 3.3 the stable category of $\operatorname{gpr}(\mathcal{S}_C^{gr})/F_*$ is equivalent to $\operatorname{proj}\mathcal{P}$. As the embedding $\operatorname{gpr}(\mathcal{S}_C^{gr})/F_* \to \operatorname{gpr}(\mathcal{S})$ is one to one on the projective-injective objects, we obtain that it is essentially surjective. $\surd$

3.6. **Frobenius models of self-injective algebras.** In this section we classify all Frobenius models of $\operatorname{proj}\mathcal{P}$ that are standard (cf. section 2.3, page 63 of [37]) in the following sense:

- (P1) For each indecomposable non projective object X of $\mathcal{E}$, there is an almost split sequence starting and an almost split sequence ending at X.
- (P2) The category of indecomposables is equivalent to the mesh category of its Auslander–Reiten quiver.
- (P3) Every projective-injective indecomposable object appears in exactly one mesh of the Auslander-Reiten quiver.

Recall from Proposition 2.7 that $\mathcal{P}$ is equivalent to the orbit category $k(\mathbb{Z}Q)/F$ where we view F as an automorphism on the mesh category $k(\mathbb{Z}Q)$. Similarly, $\mathcal{R}$ is defined by the choice of an automorphism F of $k(\mathbb{Z}Q)$ and an admissible configuration $C \subset (\mathbb{Z}Q)_0$.

Theorem 3.7. *There is a bijection between the admissible* F*-stable configurations* $C \subset \mathbb{Z}Q_0$ *such that* $\mathcal{R}$ *is Krull-Schmidt, and the equivalence classes of Frobenius models* $\mathcal{E}$ *of* $\operatorname{proj}\mathcal{P}$ *which are standard in the above sense. The bijection maps* C *to the Nakajima category* $\operatorname{proj}\mathcal{R}$ *where* $\mathcal{R}$ *is defined by the datum* (C, F).

Proof. Let $\mathcal{R}$ be defined by the admissible pair (C, F). Then $\operatorname{proj}\mathcal{R}$ is a Krull-Schmidt category whose Auslander-Reiten quiver is $\widetilde{Q} \cong \mathbb{Z}Q_C/F$. Now, by Theorem 3.3, the category $\operatorname{proj}\mathcal{R}$ is exact with projective-injective objects $\sigma(x)^\wedge$ for all $x \in C$ and we have a natural equivalence of triangulated categories between the stable category of $\operatorname{proj}\mathcal{R}$ and $\operatorname{proj}\mathcal{P}$. Hence $\operatorname{proj}\mathcal{R}$ is a Frobenius model of $\operatorname{proj}\mathcal{P}$.

Conversely, suppose $\mathcal{E}$ is a Frobenius model of $\operatorname{proj}\mathcal{P}$ satisfying (P1)–(P3). Then the stable Auslander-Reiten quiver of $\mathcal{E}$ is $\mathbb{Z}Q/F$. Let $C_{\mathcal{E}} \subset (\mathbb{Z}Q/F)_0$ be the vertices which correspond to objects $c \in \mathcal{E}$, such that a projective-injective object appears as middle term of the Auslander-Reiten sequence starting in c. Then $C_{\mathcal{E}}$ lifts to an F-invariant subset $C \subset \mathbb{Z}Q_0$. Clearly, this configuration gives rise to a category $\mathcal{R}$ which is equivalent to $\mathcal{E}$ by (P1)–(P3). It remains to show that C is admissible. Applying $\mathrm{Hom}(-,-)$ to the almost split sequences of $\mathcal{E}$ starting in x yields the exact sequences

$$0 \to \mathcal{R}(?,x) \to \bigoplus_{x \to y} \mathcal{R}(?,y)$$

and

$$0 \to \mathcal{R}(x,?) \to \bigoplus_{y \to x} \mathcal{R}(y,?)$$

where the arrows run between representatives of F–orbits in $\widetilde{Q}$. By the definition of $\mathcal{R}$ as an orbit category, this yields exact sequences in $\mathcal{R}_C^{gr}$ as in 2.4. Hence C is an admissible configuration. $\surd$

Remark 3.8. *If $\mathcal{R}$ is not Krull-Schmidt for an admissible pair (C,F), then one can consider its orbifold completion $\widehat{\mathcal{R}}$ as defined in* [26]*, which is Krull-Schmidt as the endomorhism ring of indecomposable objects are local. Then* $\mathrm{proj}\,\widehat{\mathcal{R}}$ *is a Frobenius model of $\widehat{\mathcal{D}_Q/F}$ which is equivalent to* $\mathrm{proj}\,\mathcal{P}$*, as $\mathcal{P}$ is* Hom*-finite. Furthermore, the mesh category of its Auslander-Reiten quiver is equivalent to $\mathcal{R}$: the irreducible maps are induced by the arrows in the quiver of $\mathcal{R}$ and the indecomposable objects are in bijection with the vertices.*

It follows by [22] that $\mathrm{proj}\,\mathcal{R}_C^{gr} \cong \mathrm{gpr}(\mathcal{S}_C^{gr})$ is a Frobenius model of $\mathcal{D}_Q$ and every standard Hom-finite Frobenius model arises in this way. Hence we know that all standard Hom-finite Frobenius models of $\mathrm{proj}\,\mathcal{P}$ are orbit categories of standard Hom-finite Frobenius models of $\mathcal{D}_Q$.

4. Desingularization of quiver Grassmannians

We can apply our results to desingularize quiver Grassmanianns of modules of self-injective algebras of finite representation type. We refer to the survey [41] for an overview on self-injective algebras. We denote by $\widehat{B}$ the repetitive algebra associated with a finite-dimensional algebra B, see [18] and [16]. Furthermore, we denote by $\Gamma(A)$ the mesh category associated to the Auslander-Reiten quiver of an algebra A.

Theorem 4.1. *Let A be a standard finite-dimensional self-injective algebra of finite representation type. Then there is a Nakajima category $\mathcal{R}$ such that $\Gamma(A)$ is isomorphic to $\mathcal{R}$ and A is Morita equivalent to $\mathcal{S}$.*

Proof. By [35], every self-injective algebra of finite representation type over a field of characteristic $\neq 2$ is necessarily standard. Furthermore, if A is self-injective, standard and of finite representation type, then by [34] and [42] there is an algebra B which is tilted of Dynkin type Q such that $A \cong \widehat{B}/F$, where F is an autoequivalence on $\widehat{B}$. Also $\widehat{B}$ is standard and its stable Auslander-Reiten quiver is $\mathbb{Z}Q$. By Proposition 4.3 of [23] the mesh category $\Gamma(\widehat{B})$ is equivalent to the Nakajima category $\mathcal{R}_C^{gr}$ associated with the quiver Q, and the configuration C such that the vertices $\sigma(x)$ with $x \in C$ correspond to the positions of the projective-injective $\widehat{B}$–modules in the Auslander-Reiten quiver.

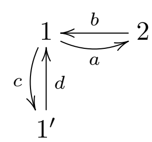
Hence F acts on $k(\mathbb{Z}Q)$ and satisfies the assumptions in 2.4 by [34], [42]. By [13] (see also [41] Section 3.2) the mesh category $\Gamma(A)$ is given by the quotient $\Gamma(\widehat{B})/F$ and the canonical pushforward functor induces a Galois cover $\Gamma(\widehat{B}) \to \Gamma(A)$. Hence

$\Gamma(A) \cong \mathcal{R}$ where $\mathcal{R}$ is determined by Q, F and C as above. As $\widehat{B}$ is standard, it is Morita equivalent to $\mathcal{S}_C^{gr}$, the subcategory of $\mathcal{R}_C^{gr}$ generated by the projective-injective objects $\sigma(x)$ with $x \in C$. Hence A is Morita equivalent to $\mathcal{S}$. $\surd$

We define A_n to be the quiver $1 \to 2 \to \cdots \to n$.

Example 4.2. *Let A be given as the path algebra of*

modulo x^2. Then A is equivalent to the Nakajima category $\mathcal{S}$ given by the data $Q = A_1$, $C = \mathbb{Z}A_1$ and $F = \tau$. That is $\mathcal{R}$ is



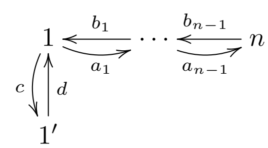
with relations $cd = b_1a_1$ and $ba = 0$.

This holds in greater generality.

Proposition 4.3. *Let A be the path algebra of*

modulo x^{n+1} for $n \geq 1$.Then A is equivalent to $\mathcal{S}$, where $Q = A_n$, C is the τ-orbit of a point in $\mathbb{Z}A_n$ with only one incoming arrow and $F = \tau$.

Proof. Let $\mathcal{R}$ be given by the datum Q, C and τ as in the proposition. Then $\mathcal{R}$ is given by the quiver



satisfying $a_ib_i = b_{i-1}a_{i-1}$ for $2 \leq i \leq n-1$ and $cd = ab$ and $b_{n-1}a_{n-1} = 0$. By [38], the quiver of $\mathcal{S}$ is given as the Ext^1-quiver of P_1 in $\mathrm{proj}\,\mathcal{P}$, where $\mathcal{P}$ is the preprojective algebra of A_n. This is one-dimensional and hence there is one loop. Furthermore, the number of minimal relations between the paths is given by the dimension of $\mathrm{Ext}^2(P_1, P_1)$ which is also one-dimensional. Now it is easy to see that $\mathcal{S}$ has one object and the morphisms are spanned by powers of the loop dc. Furthermore, one sees that the minimal relation is $(dc)^{n+1} = 0$. $\surd$

Let A, $\mathcal{R}$ and $\mathcal{S}$ be as in Theorem 4.1. As shown in Theorem 4.1, the algebra A is Morita equivalent to $\mathcal{S}$ which is the full subcategory of $\mathcal{R}$. Furthermore, as A is standard, we have equivalences

$$\mathrm{mod}\,A \cong \Gamma(A) \cong \mathcal{R}.$$

Note also that $\mathrm{proj}\,\mathcal{P} \cong \mathrm{proj}\,\mathcal{R}/\mathcal{S} \cong \underline{\mathrm{mod}}A$. Hence the intermediate extension $K_{LR} : \mathrm{mod}\,\mathcal{S} \to \mathrm{mod}\,\mathcal{R}$ can be seen as a functor

$$K_{LR} : \mathrm{mod}\,A \to \mathrm{mod}\,\mathrm{mod}\,A.$$

Theorem 4.4. *Let A be a self-injective algebra of finite representation type and $M \in \operatorname{mod} A$. Then the projective variety $\operatorname{Gr}_d(K_{LR}M)$ is smooth. Furthermore there are finitely many dimension vectors $d_1, \ldots, d_n$ such that the restriction induces a map*

$$\bigsqcup \operatorname{Gr}_{d_i}(K_{LR}M) \to \operatorname{Gr}_e(M), \ L \mapsto \operatorname{res} L$$

which is proper and surjective.

Proof. By Lemma 2.7 of [23], the image under the intermediate extension K_{LR} of every $M \in \operatorname{mod} A$ is rigid and has projective dimension one. As $\Gamma(A)$ is an Auslander algebra, it has global dimension at most two and using Proposition 7.1 of [7] we have that $\operatorname{Gr}_d(K_{LR}M)$ is smooth and equi-dimensional. Now the restriction induces a map $\operatorname{Gr}_d(K_{LR}M) \to \operatorname{Gr}_e(M)$ for any dimension vector d of $\Gamma(A)$, where e is the restriction of d to objects $\sigma(x)$ for $x \in C$. This map is proper as its domain is projective. Furthermore as K_{LR} preserves monomorphisms, every submodule $N \subset M$ gives rise to a submodule $K_{LR}N \subset K_{LR}M$. Hence there are finitely many dimension vectors $d_1 \ldots d_r$ such that

$$\bigsqcup_i \operatorname{Gr}_{d_i}(K_{LR}M) \to \operatorname{Gr}_e(M)$$

is surjective. √

We define $\operatorname{Gr}_d^{bs}(K_{LR}M)$ to be the closure of the open subset

$$\{L \in \operatorname{Gr}_d(K_{LR}M) | L \text{ is bistable}\}.$$

It follows that $\operatorname{Gr}_d^{bs}(K_{LR}M)$ is smooth. Following [7], let us denote by $C(N)$ the irreducible subvariety of $\operatorname{Gr}_e(M)$ containing all submodules isomorphic to an A–module N. As A is of finite representation type, all irreducible components of $\operatorname{Gr}_e(M)$ are of the form $C(N)$. Let $C(N_i)$ for $i = 1, \ldots, n$ denote the irreducible components of $\operatorname{Gr}_e(M)$ for some representatives $N_i \in \operatorname{Gr}_e(M)$ and let d_i be the dimension vector of $K_{LR}(N_i)$. We denote $\mathcal{V}(M) := \{d_1, \ldots, d_n\}$ the set of dimension vectors.

Lemma 4.5. *The restriction $\pi^{gr} : \operatorname{Gr}_{d_i}^{bs}(K_{LR}M) \to \operatorname{Gr}_e(M)$ maps birationally onto all components $C(N_i)$ with $\underline{\dim}\, K_{LR}(N_i) = d_i$.*

Proof. Clearly, to every $N \in \operatorname{Gr}_e(M)$ which is isomorphic to some N_i as above, we obtain an element $K_{LR}N \in \operatorname{Gr}_{d_i}^{bs}(K_{LR}M)$ such that $\pi^{gr}(K_{LR}N)$ maps to N. Furthermore all bistable modules form an open subset of $\operatorname{Gr}_{d_i}^{bs}(K_{LR}M)$ and are of the form $K_{LR}L$ for some $L \in \operatorname{mod} A$. Hence the open subset of bistable modules in $\operatorname{Gr}_{d_i}^{bs}(K_{LR}M)$ is mapped bijectively to the open subsets of modules isomorphic to some N_i as in the Lemma. Therefore π^{gr} is surjective and birational. √

Recall that a *desingularization map* between algebraic varieties is a proper, surjective and birational map with smooth domain. We conclude with the following result.

Theorem 4.6. *The map*

$$\pi^{gr} : \sqcup_{d \in \mathcal{V}(M)} \operatorname{Gr}_d^{bs}(K_{LR}M) \to \operatorname{Gr}_e(M)$$

is a desingularization map.

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References

[1] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński, *Elements of the representation theory of associative algebras. Vol. 1*, London Mathematical Society Student Texts, vol. 65, Cambridge University Press, Cambridge, 2006, Techniques of representation theory.

[2] Maurice Auslander and Idun Reiten, *Applications of contravariantly finite subcategories*, Adv. Math. **86** (1991), no. 1, 111–152.

[3] Otto Bretscher, Christian Läser, and Christine Riedtmann, *Self-injective and simply connected algebras*, Manuscripta Math. **36** (1981/82), no. 3, 253–307.

[4] Tom Bridgeland, *Quantum groups via Hall algebras of complexes* Annals of Maths. **177** (2013), 739–759.

[5] Aslak Buan, Robert Marsh, Idun Reiten, Markus Reinecke and Gordana Todorov, *Clusters and seeds in acyclic cluster algebras*, Proc. Amer. Math. Soc. 135 (2007) 3049–3060.

[6] Philippe Caldero and Frédéric Chapoton, *Cluster algebras as Hall algebras of quiver representations*, Comment. Math. Helv. **81** (2006), no. 3, 595–616.

[7] Giovanni Cerulli Irelli, Evgeny Feigin, and Markus Reineke, *Desingularization of quiver Grassmannians for Dynkin quivers*, Adv. Math. 245 (2013) 182–207.

[8] ———, *Homological approach to the Hernandez–Leclerc construction and quiver varieties*, Representation Theory 18, 1–14 (2014).

[9] William Crawley-Boevey, *Geometry of the moment map for representations of quivers*, Compositio Math. **126** (2001), no. 3, 257–293.

[10] Edgar E. Enochs and Overtoun M. G. Jenda, *Gorenstein injective and projective modules*, Math. Z. **220** (1995), no. 4, 611–633. MR 1363858 (97c:16011)

[11] Peter Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323–448.

[12] ———, *Auslander-Reiten sequences and representation-finite algebras*, Representation theory, I (Proc. Workshop, Carleton Univ., Ottawa, Ont., 1979), Springer, Berlin, 1980, pp. 1–71.

[13] ———, *The universal cover of a representation-finite algebra*, Representations of Algebras, Lecture Notes in Math., vol. 903, Springer Verlag, 1981, pp. 68–105.

[14] Karin Erdmann and Andrzej Skowronski, *The stable Calabi-Yau dimension of tame symmetric algebras*, J. Math. Soc. Japan Volume 58, Number 1 (2006), 1-297.

[15] Sergey Fomin and Andrei Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. **15** (2002), no. 2, 497–529
(electronic).

[16] Dieter Happel, *On the derived category of a finite-dimensional algebra*, Comment. Math. Helv. **62** (1987), no. 3, 339–389.

[17] ———, *Triangulated categories in the representation theory of finite-dimensional algebras*, Cambridge University Press, Cambridge, 1988.

[18] David Hughes and Josef Waschbüsch, *Trivial extensions of tilted algebras*, Proc. London Math. Soc. (3) **46** (1983), no. 2, 347–364. MR 693045 (84m:16023)

[19] Bernhard Keller, *Chain complexes and stable categories*, Manuscripta Math. **67** (1990), no. 4, 379–417.

[20] Bernhard Keller, *Derived categories and universal problems*, Comm. in Algebra **19** (1991), 699–747.

[21] ———, *On triangulated orbit categories*, Documenta Math. 10 (2005), 551-581.

[22] Bernhard Keller and Sarah Scherotzke, *Graded quiver varieties and derived categories*, J. reine angew. Math.,DOI 10.1515 / crelle-2013-0124.

[23] ———, *Desingularizations of quiver Grassmannians via graded quiver varieties*, Advances in Mathematics 256 (2014) 318-347.

[24] Bernard Leclerc and Pierre-Guy Plamondon, *Nakajima varieties and repetitive algebras*, Publications of RIMS, Volume 49, Issue 3, 2013, pp. 531–561.

[25] Saunders Mac Lane, *Categories for the working mathematician*, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998.

[26] Alfredo Nájera Chávez, *A 2-Calabi-Yau realization of universal cluster algebras of finite type*, in preparation.

[27] Hiraku Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. 76 (1994) 365–416.
[28] ———, *Quiver Varieties and Kac-Moody Algebras*, Duke Math., 91, (1998), 515–560.
[29] ———, *Quiver varieties and finite-dimensional representations of quantum affine algebras*, J. Amer. Math. Soc. **14** (2001), no. 1, 145–238 (electronic).
[30] ———, *Quiver varieties and cluster algebras*, Kyoto Journal of Mathematics **51** (2011), no. 1, 71–126.
[31] Fan Qin, *Algèbres amassées quantiques acycliques*, Ph. D. Thesis, Université Paris Diderot – Paris 7, May 2012.
[32] ———, *Quantum groups via cyclic quiver varieties I*, http://arxiv.org/abs/1312.1101.
[33] Christine Riedtmann, *Algebren, Darstellungsköcher, Überlagerungen und zurück*, Comment. Math. Helv. **55** (1980), no. 2, 199–224.
[34] ———, *Representation-finite self-injective algebras of class A_n*, Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), Lecture Notes in Math., vol. 832, Springer, Berlin, 1980, pp. 449–520.
[35] ———, *Representation-finite self-injective algebras of class D_n*, Compositio Math. 49 (1983), 231–282.
[36] Idun Reiten, *Dynkin diagrams and the representation theory of algebras*, Notices of the AMS 44 (1997), 546–556.
[37] Claus Michael Ringel, *Tame algebras and integral quadratic forms*, Lecture Notes in Mathematics, vol. 1099, Springer Verlag, 1984.
[38] Sarah Scherotzke, *Generalized quiver varieties and self-injective algebras* arXiv:1405.4729.
[39] Sarah Scherotzke and Nicolo Sibilla, *Quiver varieties and Hall algebras*, Proc. London Math. Soc. (2016) 112 (6), 1002-1018.
[40] Aidan Schofield, *General representations of quivers*, Proc. London Math. Soc. (3) 65 (1992), no. 1, 46–64.
[41] Andrzej Skowronski, *Selfinjective algebras; finite and tame type*, Contemporary Mathematics vol 406(2006).
[42] J. Waschbusch, *On selfinjective algebras of finite representation type*, Monographs of Institute of Mathematics, vol. 14, UNAM Mexico, 1983.