

ON-LINE APPENDIX TO SMALL CIRCULANT COMPLEX HADAMARD MATRICES OF BUTSON TYPE

GAURUSH HIRANANDANI AND JEAN-MARC SCHLENKER

APPENDIX A. LISTS OF SMALL CIRCULANT HADAMARD MATRICES OF BUTSON TYPE

This section contains the list, for most values of $n, l \leq 15$ for which no obstruction is known, of all first lines of matrices in $C_l(n)$, up to equivalence. The equivalence considered here is cyclic permutation of lines and columns, and addition of a constant to all lines. For each equivalence class, the first line given here is the smallest possible by lexicographic order. In a few cases (like $n = 4$) we give classification results for l larger than 15.

$n = 4$.

$l = 2$. There is 1 equivalence classes.

- $[0, 0, 0, 1]$

$l = 4$. There are 2 equivalence classes.

- $[0, 0, 0, 2]$
- $[0, 1, 0, 3]$

$l = 6$. There are 4 equivalence classes.

- $[0, 0, 0, 3]$
- $[0, 1, 0, 4]$
- $[0, 1, 3, 1]$
- $[0, 2, 0, 5]$

$l = 8$. There are 5 equivalence classes.

- $[0, 0, 0, 4]$
- $[0, 1, 0, 5]$
- $[0, 1, 4, 1]$
- $[0, 2, 0, 6]$
- $[0, 3, 0, 7]$

$l = 10$. There are 7 equivalence classes.

- $[0, 0, 0, 5]$
- $[0, 1, 0, 6]$
- $[0, 1, 5, 1]$
- $[0, 2, 0, 7]$
- $[0, 2, 5, 2]$
- $[0, 3, 0, 8]$
- $[0, 4, 0, 9]$

2000 *Mathematics Subject Classification.* 05B20.

Key words and phrases. Circulant Hadamard matrix.

$l = 12$. There are 8 equivalence classes.

- $[0, 0, 0, 6]$
- $[0, 1, 0, 7]$
- $[0, 1, 6, 1]$
- $[0, 2, 0, 8]$
- $[0, 2, 6, 2]$
- $[0, 3, 0, 9]$
- $[0, 4, 0, 10]$
- $[0, 5, 0, 11]$

$l = 14$. There are 10 equivalence classes.

- $[0, 0, 0, 7]$
- $[0, 1, 0, 8]$
- $[0, 1, 7, 1]$
- $[0, 2, 0, 9]$
- $[0, 2, 7, 2]$
- $[0, 3, 0, 10]$
- $[0, 3, 7, 3]$
- $[0, 4, 0, 11]$
- $[0, 5, 0, 12]$
- $[0, 6, 0, 13]$

$l = 16$. There are 11 equivalence classes.

- $[0, 0, 0, 8]$
- $[0, 1, 0, 9]$
- $[0, 1, 8, 1]$
- $[0, 2, 0, 10]$
- $[0, 2, 8, 2]$
- $[0, 3, 0, 11]$
- $[0, 3, 8, 3]$
- $[0, 4, 0, 12]$
- $[0, 5, 0, 13]$
- $[0, 6, 0, 14]$
- $[0, 7, 0, 15]$

$l = 18$. There are 13 equivalence classes.

- $[0, 0, 0, 9]$
- $[0, 1, 0, 10]$
- $[0, 1, 9, 1]$
- $[0, 2, 0, 11]$
- $[0, 2, 9, 2]$
- $[0, 3, 0, 12]$
- $[0, 3, 9, 3]$
- $[0, 4, 0, 13]$
- $[0, 4, 9, 4]$
- $[0, 5, 0, 14]$
- $[0, 6, 0, 15]$
- $[0, 7, 0, 16]$
- $[0, 8, 0, 17]$

$n = 6$.

$l = 12$. There are 3 equivalence classes.

- $[0, 1, 0, 9, 4, 9]$
- $[0, 1, 4, 9, 4, 1]$
- $[0, 3, 8, 3, 0, 11]$

$n = 8$.

$l = 4$. There are 4 equivalence classes.

- $[0, 0, 0, 1, 2, 0, 2, 1]$
- $[0, 0, 0, 3, 2, 0, 2, 3]$
- $[0, 0, 1, 0, 0, 2, 1, 2]$
- $[0, 0, 2, 3, 2, 0, 0, 3]$

$l = 8$. There are 9 equivalence classes.

- $[0, 0, 0, 2, 4, 0, 4, 2]$
- $[0, 0, 0, 6, 4, 0, 4, 6]$
- $[0, 0, 2, 0, 0, 4, 2, 4]$
- $[0, 0, 4, 6, 4, 0, 0, 6]$
- $[0, 1, 0, 3, 4, 1, 4, 3]$
- $[0, 1, 0, 7, 4, 1, 4, 7]$
- $[0, 1, 2, 5, 0, 5, 2, 1]$
- $[0, 1, 4, 3, 4, 1, 0, 3]$
- $[0, 1, 4, 7, 4, 1, 0, 7]$

$l = 12$. There are 14 equivalence classes.

- $[0, 0, 0, 3, 6, 0, 6, 3]$
- $[0, 0, 0, 9, 6, 0, 6, 9]$
- $[0, 0, 3, 0, 0, 6, 3, 6]$
- $[0, 0, 6, 9, 6, 0, 0, 9]$
- $[0, 1, 0, 4, 6, 1, 6, 4]$
- $[0, 1, 0, 10, 6, 1, 6, 10]$
- $[0, 1, 3, 1, 0, 7, 3, 7]$
- $[0, 1, 3, 7, 0, 7, 3, 1]$
- $[0, 1, 6, 4, 6, 1, 0, 4]$
- $[0, 1, 6, 10, 6, 1, 0, 10]$
- $[0, 1, 9, 1, 0, 7, 9, 7]$
- $[0, 1, 9, 7, 0, 7, 9, 1]$
- $[0, 2, 0, 11, 6, 2, 6, 11]$
- $[0, 2, 6, 11, 6, 2, 0, 11]$

$l = 16$. There are 19 equivalence classes.

- $[0, 0, 0, 4, 8, 0, 8, 4]$
- $[0, 0, 0, 12, 8, 0, 8, 12]$
- $[0, 0, 4, 0, 0, 8, 4, 8]$
- $[0, 0, 8, 12, 8, 0, 0, 12]$
- $[0, 1, 0, 5, 8, 1, 8, 5]$
- $[0, 1, 0, 13, 8, 1, 8, 13]$
- $[0, 1, 4, 1, 0, 9, 4, 9]$
- $[0, 1, 4, 9, 0, 9, 4, 1]$
- $[0, 1, 8, 5, 8, 1, 0, 5]$
- $[0, 1, 8, 13, 8, 1, 0, 13]$
- $[0, 1, 12, 1, 0, 9, 12, 9]$
- $[0, 1, 12, 9, 0, 9, 12, 1]$

- $[0, 2, 0, 6, 8, 2, 8, 6]$
- $[0, 2, 0, 14, 8, 2, 8, 14]$
- $[0, 2, 4, 10, 0, 10, 4, 2]$
- $[0, 2, 8, 6, 8, 2, 0, 6]$
- $[0, 2, 8, 14, 8, 2, 0, 14]$
- $[0, 3, 0, 15, 8, 3, 8, 15]$
- $[0, 3, 8, 15, 8, 3, 0, 15]$

$n = 9$.

$l = 3$. There are 6 equivalence classes.

- $[0, 0, 0, 0, 1, 2, 0, 2, 1]$
- $[0, 0, 0, 0, 2, 1, 0, 1, 2]$
- $[0, 0, 0, 1, 0, 2, 2, 0, 1]$
- $[0, 0, 0, 2, 0, 1, 1, 0, 2]$
- $[0, 0, 1, 0, 1, 0, 0, 2, 2]$
- $[0, 0, 1, 1, 0, 0, 2, 0, 2]$

$l = 6$. There are 24 equivalence classes.

- $[0, 0, 0, 0, 2, 4, 0, 4, 2]$
- $[0, 0, 0, 0, 4, 2, 0, 2, 4]$
- $[0, 0, 0, 2, 0, 4, 4, 0, 2]$
- $[0, 0, 0, 4, 0, 2, 2, 0, 4]$
- $[0, 0, 1, 0, 2, 5, 0, 4, 3]$
- $[0, 0, 1, 0, 4, 3, 0, 2, 5]$
- $[0, 0, 1, 2, 0, 5, 4, 0, 3]$
- $[0, 0, 1, 2, 4, 1, 4, 2, 1]$
- $[0, 0, 1, 4, 0, 3, 2, 0, 5]$
- $[0, 0, 1, 4, 2, 1, 2, 4, 1]$
- $[0, 0, 2, 0, 2, 0, 0, 4, 4]$
- $[0, 0, 2, 2, 0, 0, 4, 0, 4]$
- $[0, 0, 3, 0, 2, 1, 0, 4, 5]$
- $[0, 0, 3, 0, 4, 5, 0, 2, 1]$
- $[0, 0, 3, 2, 0, 1, 4, 0, 5]$
- $[0, 0, 3, 2, 4, 3, 4, 2, 3]$
- $[0, 0, 3, 4, 0, 5, 2, 0, 1]$
- $[0, 0, 3, 4, 2, 3, 2, 4, 3]$
- $[0, 0, 5, 0, 2, 3, 0, 4, 1]$
- $[0, 0, 5, 0, 4, 1, 0, 2, 3]$
- $[0, 0, 5, 2, 0, 3, 4, 0, 1]$
- $[0, 0, 5, 2, 4, 5, 4, 2, 5]$
- $[0, 0, 5, 4, 0, 1, 2, 0, 3]$
- $[0, 0, 5, 4, 2, 5, 2, 4, 5]$

$l = 9$. There are 62 equivalence classes.

- $[0, 0, 0, 0, 3, 6, 0, 6, 3]$
- $[0, 0, 0, 0, 6, 3, 0, 3, 6]$
- $[0, 0, 0, 3, 0, 6, 6, 0, 3]$
- $[0, 0, 0, 6, 0, 3, 3, 0, 6]$
- $[0, 0, 1, 0, 3, 7, 0, 6, 4]$
- $[0, 0, 1, 0, 6, 4, 0, 3, 7]$
- $[0, 0, 1, 3, 0, 7, 6, 0, 4]$
- $[0, 0, 1, 3, 6, 1, 6, 3, 1]$
- $[0, 0, 1, 6, 0, 4, 3, 0, 7]$

- $[0, 0, 1, 6, 3, 1, 3, 6, 1]$
- $[0, 0, 2, 0, 3, 8, 0, 6, 5]$
- $[0, 0, 2, 0, 6, 5, 0, 3, 8]$
- $[0, 0, 2, 3, 0, 8, 6, 0, 5]$
- $[0, 0, 2, 3, 6, 2, 6, 3, 2]$
- $[0, 0, 2, 6, 0, 5, 3, 0, 8]$
- $[0, 0, 2, 6, 3, 2, 3, 6, 2]$
- $[0, 0, 3, 0, 3, 0, 0, 6, 6]$
- $[0, 0, 3, 3, 0, 0, 6, 0, 6]$
- $[0, 0, 4, 0, 3, 1, 0, 6, 7]$
- $[0, 0, 4, 0, 6, 7, 0, 3, 1]$
- $[0, 0, 4, 3, 0, 1, 6, 0, 7]$
- $[0, 0, 4, 3, 6, 4, 6, 3, 4]$
- $[0, 0, 4, 6, 0, 7, 3, 0, 1]$
- $[0, 0, 4, 6, 3, 4, 3, 6, 4]$
- $[0, 0, 5, 0, 3, 2, 0, 6, 8]$
- $[0, 0, 5, 0, 6, 8, 0, 3, 2]$
- $[0, 0, 5, 3, 0, 2, 6, 0, 8]$
- $[0, 0, 5, 3, 6, 5, 6, 3, 5]$
- $[0, 0, 5, 6, 0, 8, 3, 0, 2]$
- $[0, 0, 5, 6, 3, 5, 3, 6, 5]$
- $[0, 0, 7, 0, 3, 4, 0, 6, 1]$
- $[0, 0, 7, 0, 6, 1, 0, 3, 4]$
- $[0, 0, 7, 3, 0, 4, 6, 0, 1]$
- $[0, 0, 7, 3, 6, 7, 6, 3, 7]$
- $[0, 0, 7, 6, 0, 1, 3, 0, 4]$
- $[0, 0, 7, 6, 3, 7, 3, 6, 7]$
- $[0, 0, 8, 0, 3, 5, 0, 6, 2]$
- $[0, 0, 8, 0, 6, 2, 0, 3, 5]$
- $[0, 0, 8, 3, 0, 5, 6, 0, 2]$
- $[0, 0, 8, 3, 6, 8, 6, 3, 8]$
- $[0, 0, 8, 6, 0, 2, 3, 0, 5]$
- $[0, 0, 8, 6, 3, 8, 3, 6, 8]$
- $[0, 1, 2, 0, 4, 8, 0, 7, 5]$
- $[0, 1, 2, 0, 7, 5, 0, 4, 8]$
- $[0, 1, 2, 3, 7, 2, 6, 4, 2]$
- $[0, 1, 2, 6, 1, 5, 3, 1, 8]$
- $[0, 1, 2, 6, 4, 2, 3, 7, 2]$
- $[0, 1, 5, 0, 4, 2, 0, 7, 8]$
- $[0, 1, 5, 3, 7, 5, 6, 4, 5]$
- $[0, 1, 5, 6, 4, 5, 3, 7, 5]$
- $[0, 1, 8, 0, 7, 2, 0, 4, 5]$
- $[0, 1, 8, 3, 7, 8, 6, 4, 8]$
- $[0, 1, 8, 6, 4, 8, 3, 7, 8]$
- $[0, 2, 1, 0, 5, 7, 0, 8, 4]$
- $[0, 2, 1, 0, 8, 4, 0, 5, 7]$
- $[0, 2, 1, 3, 2, 7, 6, 2, 4]$
- $[0, 2, 1, 3, 8, 1, 6, 5, 1]$
- $[0, 2, 1, 6, 5, 1, 3, 8, 1]$
- $[0, 2, 4, 0, 5, 1, 0, 8, 7]$
- $[0, 2, 4, 0, 8, 7, 0, 5, 1]$
- $[0, 2, 7, 0, 5, 4, 0, 8, 1]$

- $[0, 2, 7, 3, 8, 7, 6, 5, 7]$

$l = 12$. There are 108 equivalence classes.

- $[0, 0, 0, 0, 4, 8, 0, 8, 4]$
- $[0, 0, 0, 0, 8, 4, 0, 4, 8]$
- $[0, 0, 0, 4, 0, 8, 8, 0, 4]$
- $[0, 0, 0, 8, 0, 4, 4, 0, 8]$
- $[0, 0, 1, 0, 4, 9, 0, 8, 5]$
- $[0, 0, 1, 0, 8, 5, 0, 4, 9]$
- $[0, 0, 1, 4, 0, 9, 8, 0, 5]$
- $[0, 0, 1, 4, 8, 1, 8, 4, 1]$
- $[0, 0, 1, 8, 0, 5, 4, 0, 9]$
- $[0, 0, 1, 8, 4, 1, 4, 8, 1]$
- $[0, 0, 2, 0, 4, 10, 0, 8, 6]$
- $[0, 0, 2, 0, 8, 6, 0, 4, 10]$
- $[0, 0, 2, 4, 0, 10, 8, 0, 6]$
- $[0, 0, 2, 4, 8, 2, 8, 4, 2]$
- $[0, 0, 2, 8, 0, 6, 4, 0, 10]$
- $[0, 0, 2, 8, 4, 2, 4, 8, 2]$
- $[0, 0, 3, 0, 4, 11, 0, 8, 7]$
- $[0, 0, 3, 0, 8, 7, 0, 4, 11]$
- $[0, 0, 3, 4, 0, 11, 8, 0, 7]$
- $[0, 0, 3, 4, 8, 3, 8, 4, 3]$
- $[0, 0, 3, 8, 0, 7, 4, 0, 11]$
- $[0, 0, 3, 8, 4, 3, 4, 8, 3]$
- $[0, 0, 4, 0, 4, 0, 0, 8, 8]$
- $[0, 0, 4, 4, 0, 0, 8, 0, 8]$
- $[0, 0, 5, 0, 4, 1, 0, 8, 9]$
- $[0, 0, 5, 0, 8, 9, 0, 4, 1]$
- $[0, 0, 5, 4, 0, 1, 8, 0, 9]$
- $[0, 0, 5, 4, 8, 5, 8, 4, 5]$
- $[0, 0, 5, 8, 0, 9, 4, 0, 1]$
- $[0, 0, 5, 8, 4, 5, 4, 8, 5]$
- $[0, 0, 6, 0, 4, 2, 0, 8, 10]$
- $[0, 0, 6, 0, 8, 10, 0, 4, 2]$
- $[0, 0, 6, 4, 0, 2, 8, 0, 10]$
- $[0, 0, 6, 4, 8, 6, 8, 4, 6]$
- $[0, 0, 6, 8, 0, 10, 4, 0, 2]$
- $[0, 0, 6, 8, 4, 6, 4, 8, 6]$
- $[0, 0, 7, 0, 4, 3, 0, 8, 11]$
- $[0, 0, 7, 0, 8, 11, 0, 4, 3]$
- $[0, 0, 7, 4, 0, 3, 8, 0, 11]$
- $[0, 0, 7, 4, 8, 7, 8, 4, 7]$
- $[0, 0, 7, 8, 0, 11, 4, 0, 3]$
- $[0, 0, 7, 8, 4, 7, 4, 8, 7]$
- $[0, 0, 9, 0, 4, 5, 0, 8, 1]$
- $[0, 0, 9, 0, 8, 1, 0, 4, 5]$
- $[0, 0, 9, 4, 0, 5, 8, 0, 1]$
- $[0, 0, 9, 4, 8, 9, 8, 4, 9]$
- $[0, 0, 9, 8, 0, 1, 4, 0, 5]$
- $[0, 0, 9, 8, 4, 9, 4, 8, 9]$
- $[0, 0, 10, 0, 4, 6, 0, 8, 2]$
- $[0, 0, 10, 0, 8, 2, 0, 4, 6]$

- [0, 0, 10, 4, 0, 6, 8, 0, 2]
- [0, 0, 10, 4, 8, 10, 8, 4, 10]
- [0, 0, 10, 8, 0, 2, 4, 0, 6]
- [0, 0, 10, 8, 4, 10, 4, 8, 10]
- [0, 0, 11, 0, 4, 7, 0, 8, 3]
- [0, 0, 11, 0, 8, 3, 0, 4, 7]
- [0, 0, 11, 4, 0, 7, 8, 0, 3]
- [0, 0, 11, 4, 8, 11, 8, 4, 11]
- [0, 0, 11, 8, 0, 3, 4, 0, 7]
- [0, 0, 11, 8, 4, 11, 4, 8, 11]
- [0, 1, 2, 0, 5, 10, 0, 9, 6]
- [0, 1, 2, 0, 9, 6, 0, 5, 10]
- [0, 1, 2, 4, 1, 10, 8, 1, 6]
- [0, 1, 2, 4, 9, 2, 8, 5, 2]
- [0, 1, 2, 8, 1, 6, 4, 1, 10]
- [0, 1, 2, 8, 5, 2, 4, 9, 2]
- [0, 1, 3, 0, 5, 11, 0, 9, 7]
- [0, 1, 3, 0, 9, 7, 0, 5, 11]
- [0, 1, 3, 4, 1, 11, 8, 1, 7]
- [0, 1, 3, 4, 9, 3, 8, 5, 3]
- [0, 1, 3, 8, 1, 7, 4, 1, 11]
- [0, 1, 3, 8, 5, 3, 4, 9, 3]
- [0, 1, 6, 0, 5, 2, 0, 9, 10]
- [0, 1, 6, 4, 1, 2, 8, 1, 10]
- [0, 1, 6, 4, 9, 6, 8, 5, 6]
- [0, 1, 6, 8, 1, 10, 4, 1, 2]
- [0, 1, 6, 8, 5, 6, 4, 9, 6]
- [0, 1, 7, 0, 5, 3, 0, 9, 11]
- [0, 1, 7, 4, 1, 3, 8, 1, 11]
- [0, 1, 7, 4, 9, 7, 8, 5, 7]
- [0, 1, 7, 8, 5, 7, 4, 9, 7]
- [0, 1, 10, 0, 9, 2, 0, 5, 6]
- [0, 1, 10, 4, 1, 6, 8, 1, 2]
- [0, 1, 10, 4, 9, 10, 8, 5, 10]
- [0, 1, 10, 8, 5, 10, 4, 9, 10]
- [0, 1, 11, 4, 9, 11, 8, 5, 11]
- [0, 1, 11, 8, 5, 11, 4, 9, 11]
- [0, 2, 1, 0, 6, 9, 0, 10, 5]
- [0, 2, 1, 0, 10, 5, 0, 6, 9]
- [0, 2, 1, 4, 2, 9, 8, 2, 5]
- [0, 2, 1, 4, 10, 1, 8, 6, 1]
- [0, 2, 1, 8, 2, 5, 4, 2, 9]
- [0, 2, 1, 8, 6, 1, 4, 10, 1]
- [0, 2, 5, 0, 6, 1, 0, 10, 9]
- [0, 2, 5, 0, 10, 9, 0, 6, 1]
- [0, 2, 5, 4, 2, 1, 8, 2, 9]
- [0, 2, 5, 4, 10, 5, 8, 6, 5]
- [0, 2, 5, 8, 2, 9, 4, 2, 1]
- [0, 2, 5, 8, 6, 5, 4, 10, 5]
- [0, 2, 9, 0, 6, 5, 0, 10, 1]
- [0, 2, 9, 0, 10, 1, 0, 6, 5]
- [0, 2, 9, 4, 2, 5, 8, 2, 1]

- $[0, 2, 9, 4, 10, 9, 8, 6, 9]$
- $[0, 2, 9, 8, 2, 1, 4, 2, 5]$
- $[0, 2, 9, 8, 6, 9, 4, 10, 9]$
- $[0, 3, 2, 0, 11, 6, 0, 7, 10]$
- $[0, 3, 6, 0, 7, 2, 0, 11, 10]$
- $[0, 3, 10, 4, 11, 10, 8, 7, 10]$

$l = 15$. There are 172 equivalence classes.

- $[0, 0, 0, 0, 5, 10, 0, 10, 5]$
- $[0, 0, 0, 0, 10, 5, 0, 5, 10]$
- $[0, 0, 0, 5, 0, 10, 10, 0, 5]$
- $[0, 0, 0, 10, 0, 5, 5, 0, 10]$
- $[0, 0, 1, 0, 5, 11, 0, 10, 6]$
- $[0, 0, 1, 0, 10, 6, 0, 5, 11]$
- $[0, 0, 1, 5, 0, 11, 10, 0, 6]$
- $[0, 0, 1, 5, 10, 1, 10, 5, 1]$
- $[0, 0, 1, 10, 0, 6, 5, 0, 11]$
- $[0, 0, 1, 10, 5, 1, 5, 10, 1]$
- $[0, 0, 2, 0, 5, 12, 0, 10, 7]$
- $[0, 0, 2, 0, 10, 7, 0, 5, 12]$
- $[0, 0, 2, 5, 0, 12, 10, 0, 7]$
- $[0, 0, 2, 5, 10, 2, 10, 5, 2]$
- $[0, 0, 2, 10, 0, 7, 5, 0, 12]$
- $[0, 0, 2, 10, 5, 2, 5, 10, 2]$
- $[0, 0, 3, 0, 5, 13, 0, 10, 8]$
- $[0, 0, 3, 0, 10, 8, 0, 5, 13]$
- $[0, 0, 3, 5, 0, 13, 10, 0, 8]$
- $[0, 0, 3, 5, 10, 3, 10, 5, 3]$
- $[0, 0, 3, 10, 0, 8, 5, 0, 13]$
- $[0, 0, 3, 10, 5, 3, 5, 10, 3]$
- $[0, 0, 4, 0, 5, 14, 0, 10, 9]$
- $[0, 0, 4, 0, 10, 9, 0, 5, 14]$
- $[0, 0, 4, 5, 0, 14, 10, 0, 9]$
- $[0, 0, 4, 5, 10, 4, 10, 5, 4]$
- $[0, 0, 4, 10, 0, 9, 5, 0, 14]$
- $[0, 0, 4, 10, 5, 4, 5, 10, 4]$
- $[0, 0, 5, 0, 5, 0, 0, 10, 10]$
- $[0, 0, 5, 5, 0, 0, 10, 0, 10]$
- $[0, 0, 6, 0, 5, 1, 0, 10, 11]$
- $[0, 0, 6, 0, 10, 11, 0, 5, 1]$
- $[0, 0, 6, 5, 0, 1, 10, 0, 11]$
- $[0, 0, 6, 5, 10, 6, 10, 5, 6]$
- $[0, 0, 6, 10, 0, 11, 5, 0, 1]$
- $[0, 0, 6, 10, 5, 6, 5, 10, 6]$
- $[0, 0, 7, 0, 5, 2, 0, 10, 12]$
- $[0, 0, 7, 0, 10, 12, 0, 5, 2]$
- $[0, 0, 7, 5, 0, 2, 10, 0, 12]$
- $[0, 0, 7, 5, 10, 7, 10, 5, 7]$
- $[0, 0, 7, 10, 0, 12, 5, 0, 2]$
- $[0, 0, 7, 10, 5, 7, 5, 10, 7]$
- $[0, 0, 8, 0, 5, 3, 0, 10, 13]$
- $[0, 0, 8, 0, 10, 13, 0, 5, 3]$
- $[0, 0, 8, 5, 0, 3, 10, 0, 13]$

- $[0, 0, 8, 5, 10, 8, 10, 5, 8]$
- $[0, 0, 8, 10, 0, 13, 5, 0, 3]$
- $[0, 0, 8, 10, 5, 8, 5, 10, 8]$
- $[0, 0, 9, 0, 5, 4, 0, 10, 14]$
- $[0, 0, 9, 0, 10, 14, 0, 5, 4]$
- $[0, 0, 9, 5, 0, 4, 10, 0, 14]$
- $[0, 0, 9, 5, 10, 9, 10, 5, 9]$
- $[0, 0, 9, 10, 0, 14, 5, 0, 4]$
- $[0, 0, 9, 10, 5, 9, 5, 10, 9]$
- $[0, 0, 11, 0, 5, 6, 0, 10, 1]$
- $[0, 0, 11, 0, 10, 1, 0, 5, 6]$
- $[0, 0, 11, 5, 0, 6, 10, 0, 1]$
- $[0, 0, 11, 5, 10, 11, 10, 5, 11]$
- $[0, 0, 11, 10, 0, 1, 5, 0, 6]$
- $[0, 0, 11, 10, 5, 11, 5, 10, 11]$
- $[0, 0, 12, 0, 5, 7, 0, 10, 2]$
- $[0, 0, 12, 0, 10, 2, 0, 5, 7]$
- $[0, 0, 12, 5, 0, 7, 10, 0, 2]$
- $[0, 0, 12, 5, 10, 12, 10, 5, 12]$
- $[0, 0, 12, 10, 0, 2, 5, 0, 7]$
- $[0, 0, 12, 10, 5, 12, 5, 10, 12]$
- $[0, 0, 13, 0, 5, 8, 0, 10, 3]$
- $[0, 0, 13, 0, 10, 3, 0, 5, 8]$
- $[0, 0, 13, 5, 0, 8, 10, 0, 3]$
- $[0, 0, 13, 5, 10, 13, 10, 5, 13]$
- $[0, 0, 13, 10, 0, 3, 5, 0, 8]$
- $[0, 0, 13, 10, 5, 13, 5, 10, 13]$
- $[0, 0, 14, 0, 5, 9, 0, 10, 4]$
- $[0, 0, 14, 0, 10, 4, 0, 5, 9]$
- $[0, 0, 14, 5, 0, 9, 10, 0, 4]$
- $[0, 0, 14, 5, 10, 14, 10, 5, 14]$
- $[0, 0, 14, 10, 0, 4, 5, 0, 9]$
- $[0, 0, 14, 10, 5, 14, 5, 10, 14]$
- $[0, 1, 2, 0, 6, 12, 0, 11, 7]$
- $[0, 1, 2, 0, 11, 7, 0, 6, 12]$
- $[0, 1, 2, 5, 1, 12, 10, 1, 7]$
- $[0, 1, 2, 5, 11, 2, 10, 6, 2]$
- $[0, 1, 2, 10, 1, 7, 5, 1, 12]$
- $[0, 1, 2, 10, 6, 2, 5, 11, 2]$
- $[0, 1, 3, 0, 6, 13, 0, 11, 8]$
- $[0, 1, 3, 0, 11, 8, 0, 6, 13]$
- $[0, 1, 3, 5, 1, 13, 10, 1, 8]$
- $[0, 1, 3, 5, 11, 3, 10, 6, 3]$
- $[0, 1, 3, 10, 1, 8, 5, 1, 13]$
- $[0, 1, 3, 10, 6, 3, 5, 11, 3]$
- $[0, 1, 4, 0, 6, 14, 0, 11, 9]$
- $[0, 1, 4, 0, 11, 9, 0, 6, 14]$
- $[0, 1, 4, 5, 1, 14, 10, 1, 9]$
- $[0, 1, 4, 5, 11, 4, 10, 6, 4]$
- $[0, 1, 4, 10, 1, 9, 5, 1, 14]$
- $[0, 1, 4, 10, 6, 4, 5, 11, 4]$
- $[0, 1, 7, 0, 6, 2, 0, 11, 12]$

- $[0, 1, 7, 5, 1, 2, 10, 1, 12]$
- $[0, 1, 7, 5, 11, 7, 10, 6, 7]$
- $[0, 1, 7, 10, 1, 12, 5, 1, 2]$
- $[0, 1, 7, 10, 6, 7, 5, 11, 7]$
- $[0, 1, 8, 0, 6, 3, 0, 11, 13]$
- $[0, 1, 8, 0, 11, 13, 0, 6, 3]$
- $[0, 1, 8, 5, 1, 3, 10, 1, 13]$
- $[0, 1, 8, 5, 11, 8, 10, 6, 8]$
- $[0, 1, 8, 10, 1, 13, 5, 1, 3]$
- $[0, 1, 8, 10, 6, 8, 5, 11, 8]$
- $[0, 1, 9, 0, 6, 4, 0, 11, 14]$
- $[0, 1, 9, 5, 1, 4, 10, 1, 14]$
- $[0, 1, 9, 5, 11, 9, 10, 6, 9]$
- $[0, 1, 9, 10, 6, 9, 5, 11, 9]$
- $[0, 1, 12, 0, 11, 2, 0, 6, 7]$
- $[0, 1, 12, 5, 1, 7, 10, 1, 2]$
- $[0, 1, 12, 5, 11, 12, 10, 6, 12]$
- $[0, 1, 12, 10, 6, 12, 5, 11, 12]$
- $[0, 1, 13, 0, 6, 8, 0, 11, 3]$
- $[0, 1, 13, 0, 11, 3, 0, 6, 8]$
- $[0, 1, 13, 5, 1, 8, 10, 1, 3]$
- $[0, 1, 13, 5, 11, 13, 10, 6, 13]$
- $[0, 1, 13, 10, 1, 3, 5, 1, 8]$
- $[0, 1, 13, 10, 6, 13, 5, 11, 13]$
- $[0, 1, 14, 5, 11, 14, 10, 6, 14]$
- $[0, 1, 14, 10, 6, 14, 5, 11, 14]$
- $[0, 2, 1, 0, 7, 11, 0, 12, 6]$
- $[0, 2, 1, 0, 12, 6, 0, 7, 11]$
- $[0, 2, 1, 5, 2, 11, 10, 2, 6]$
- $[0, 2, 1, 5, 12, 1, 10, 7, 1]$
- $[0, 2, 1, 10, 2, 6, 5, 2, 11]$
- $[0, 2, 1, 10, 7, 1, 5, 12, 1]$
- $[0, 2, 4, 0, 12, 9, 0, 7, 14]$
- $[0, 2, 4, 10, 2, 9, 5, 2, 14]$
- $[0, 2, 6, 0, 7, 1, 0, 12, 11]$
- $[0, 2, 6, 0, 12, 11, 0, 7, 1]$
- $[0, 2, 6, 5, 2, 1, 10, 2, 11]$
- $[0, 2, 6, 5, 12, 6, 10, 7, 6]$
- $[0, 2, 6, 10, 2, 11, 5, 2, 1]$
- $[0, 2, 6, 10, 7, 6, 5, 12, 6]$
- $[0, 2, 9, 0, 7, 4, 0, 12, 14]$
- $[0, 2, 11, 0, 7, 6, 0, 12, 1]$
- $[0, 2, 11, 0, 12, 1, 0, 7, 6]$
- $[0, 2, 11, 5, 2, 6, 10, 2, 1]$
- $[0, 2, 11, 5, 12, 11, 10, 7, 11]$
- $[0, 2, 11, 10, 2, 1, 5, 2, 6]$
- $[0, 2, 11, 10, 7, 11, 5, 12, 11]$
- $[0, 2, 14, 10, 7, 14, 5, 12, 14]$
- $[0, 3, 1, 0, 8, 11, 0, 13, 6]$
- $[0, 3, 1, 0, 13, 6, 0, 8, 11]$
- $[0, 3, 1, 5, 3, 11, 10, 3, 6]$
- $[0, 3, 1, 5, 13, 1, 10, 8, 1]$

- $[0, 3, 1, 10, 3, 6, 5, 3, 11]$
- $[0, 3, 1, 10, 8, 1, 5, 13, 1]$
- $[0, 3, 2, 0, 13, 7, 0, 8, 12]$
- $[0, 3, 2, 5, 13, 2, 10, 8, 2]$
- $[0, 3, 2, 10, 3, 7, 5, 3, 12]$
- $[0, 3, 2, 10, 8, 2, 5, 13, 2]$
- $[0, 3, 6, 0, 8, 1, 0, 13, 11]$
- $[0, 3, 6, 0, 13, 11, 0, 8, 1]$
- $[0, 3, 6, 10, 3, 11, 5, 3, 1]$
- $[0, 3, 6, 10, 8, 6, 5, 13, 6]$
- $[0, 3, 7, 0, 8, 2, 0, 13, 12]$
- $[0, 3, 7, 0, 13, 12, 0, 8, 2]$
- $[0, 3, 7, 5, 3, 2, 10, 3, 12]$
- $[0, 3, 7, 10, 3, 12, 5, 3, 2]$
- $[0, 3, 11, 0, 8, 6, 0, 13, 1]$
- $[0, 3, 11, 5, 13, 11, 10, 8, 11]$
- $[0, 3, 11, 10, 3, 1, 5, 3, 6]$
- $[0, 3, 11, 10, 8, 11, 5, 13, 11]$
- $[0, 3, 12, 5, 13, 12, 10, 8, 12]$
- $[0, 3, 12, 10, 8, 12, 5, 13, 12]$
- $[0, 4, 3, 0, 14, 8, 0, 9, 13]$
- $[0, 4, 8, 0, 9, 3, 0, 14, 13]$
- $[0, 4, 13, 5, 14, 13, 10, 9, 13]$

$n = 12$.

$l = 6$. There are 37 equivalence classes.

- $[0, 0, 0, 1, 2, 0, 0, 3, 0, 4, 2, 3]$
- $[0, 0, 0, 1, 2, 4, 0, 3, 0, 4, 2, 1]$
- $[0, 0, 0, 1, 4, 0, 0, 3, 0, 4, 4, 3]$
- $[0, 0, 0, 1, 4, 4, 0, 3, 0, 4, 4, 1]$
- $[0, 0, 0, 3, 2, 2, 0, 3, 0, 0, 2, 5]$
- $[0, 0, 0, 3, 2, 4, 0, 3, 0, 0, 2, 1]$
- $[0, 0, 0, 3, 4, 2, 0, 3, 0, 0, 4, 5]$
- $[0, 0, 0, 3, 4, 4, 0, 3, 0, 0, 4, 1]$
- $[0, 0, 0, 5, 2, 0, 0, 3, 0, 2, 2, 3]$
- $[0, 0, 0, 5, 2, 2, 0, 3, 0, 2, 2, 5]$
- $[0, 0, 0, 5, 4, 0, 0, 3, 0, 2, 4, 3]$
- $[0, 0, 0, 5, 4, 2, 0, 3, 0, 2, 4, 5]$
- $[0, 0, 1, 0, 0, 2, 3, 0, 4, 0, 3, 2]$
- $[0, 0, 1, 0, 0, 4, 3, 0, 4, 0, 3, 4]$
- $[0, 0, 1, 0, 4, 2, 3, 0, 4, 0, 1, 2]$
- $[0, 0, 1, 0, 4, 4, 3, 0, 4, 0, 1, 4]$
- $[0, 0, 1, 2, 0, 2, 3, 0, 4, 2, 3, 2]$
- $[0, 0, 1, 2, 4, 2, 3, 0, 4, 2, 1, 2]$
- $[0, 0, 1, 4, 0, 4, 3, 0, 4, 4, 3, 4]$
- $[0, 0, 2, 1, 0, 4, 0, 3, 2, 4, 0, 1]$
- $[0, 0, 2, 1, 2, 0, 0, 3, 2, 4, 2, 3]$
- $[0, 0, 2, 1, 2, 4, 0, 3, 2, 4, 2, 1]$
- $[0, 0, 2, 3, 0, 2, 0, 3, 2, 0, 0, 5]$
- $[0, 0, 2, 3, 2, 2, 0, 3, 2, 0, 2, 5]$
- $[0, 0, 2, 3, 2, 4, 0, 3, 2, 0, 2, 1]$
- $[0, 0, 2, 5, 0, 2, 0, 3, 2, 2, 0, 5]$

- $[0, 0, 3, 4, 2, 4, 3, 0, 0, 4, 5, 4]$
- $[0, 0, 4, 3, 0, 2, 0, 3, 4, 0, 0, 5]$
- $[0, 0, 4, 3, 4, 2, 0, 3, 4, 0, 4, 5]$
- $[0, 0, 4, 5, 0, 2, 0, 3, 4, 2, 0, 5]$
- $[0, 0, 4, 5, 4, 2, 0, 3, 4, 2, 4, 5]$
- $[0, 0, 5, 0, 2, 4, 3, 0, 2, 0, 5, 4]$
- $[0, 0, 5, 4, 0, 4, 3, 0, 2, 4, 3, 4]$
- $[0, 0, 5, 4, 2, 4, 3, 0, 2, 4, 5, 4]$
- $[0, 1, 0, 2, 4, 5, 0, 4, 0, 5, 4, 2]$
- $[0, 1, 2, 4, 0, 5, 0, 4, 2, 1, 0, 2]$
- $[0, 1, 3, 5, 4, 5, 3, 1, 0, 5, 1, 5]$

APPENDIX B. VALUES OF l, n FOR WHICH THE GENERALIZED TURYN OBSTRUCTION APPLIES

We list here the values of l and n between 2 and 15 for which the generalized Turyn obstruction, Proposition 5.2 in the paper, applies.

- $l = 2, n = 2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15$.
- $l = 3, n = 2, 5, 6, 8, 10, 11, 14, 15$.
- $l = 4, n = 3, 6, 7, 11, 12, 14, 15$.
- $l = 5, n = 2, 3, 4, 6, 7, 8, 9, 10, 12, 13, 14, 15$.
- $l = 6, n = 2, 5, 6, 8, 10, 11, 14, 15$.
- $l = 7, n = 2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 15$.
- $l = 8, n = 7, 14$.
- $l = 9, n = 2, 5, 6, 8, 10, 11, 14, 15$.
- $l = 10, n = 2, 3, 6, 7, 8, 10, 12, 13, 14, 15$.
- $l = 11, n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 13, 15$.
- $l = 12, n = 11$.
- $l = 13, n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 14$.
- $l = 14, n = 2, 3, 5, 6, 10, 12, 13, 15$.
- $l = 15, n = 2, 6$.

This list is obtained by using the following short *python* program. When called, it computes and lists the values of l, n between 2 and 15 for which the obstruction applies, and writes the output to a file.

```
# For given values of l and n, checks whether the generalized
# Turyn obstruction applies.
# program writes results for all values of l and n between 2 and 15

import math
pi=math.pi
cos=math.cos
sin=math.sin

def Complement(b,l,n):
    '''Returns the list of vectors with integer entries that can complement b to
    length l, with sum of entries equal to n and with all entries at most equal to
    the first.'''
    if len(b)==l-1:
        if n-sum(b)<=b[0]:
            return [b+[n-sum(b)]]
        else:
            return []
    else:
        L=[]
```

```

    for bb in range(min(b[0],n-sum(b))+1):
        L.extend(Complement(b+[bb],l,n))
    return L

def Turyn(l,n):
    '''returns True if the Turyn obstruction applies'''
    R=True # will change to False if there is a vector a solution of the equation
    for a0 in range(1+(n-1)/l, n):
        A=Complement([a0],l,n)
        for a in A:
            RR=sum([sum([a[i]*a[(i+k)%l] for i in range(l)])*cos(2*pi*k/l) for k in range(l)])-n
            II=sum([sum([a[i]*a[(i+k)%l] for i in range(l)])*sin(2*pi*k/l) for k in range(l)])
            if abs(RR)<0.0001 and abs(II)<0.0001:
                R=False
                break
    return R

outfile=open('Turyn_n_2-15.txt','w')
for n in range(2,16):
    for l in range(2,16):
        rr=Turyn(l,n)
        print 'l='+str(l)+', n='+str(n)+', '+str(rr)
        if rr:
            outfile.write('l='+str(l)+', n='+str(n)+'\n')
            outfile.flush()
    print
outfile.close()

GH: SENIOR UNDERGRADUATE, DEPARTMENT OF MATHEMATICS AND STATISTICS, IIT KANPUR, KANPUR, UTTAR
PRADESH, INDIA - 208016. gaurushh@iitk.ac.in

J.-M.S.: UNIVERSITY OF LUXEMBOURG, CAMPUS KIRCHBERG, MATHEMATICS RESEARCH UNIT, BLG, 6, RUE
RICHARD COUDENHOVE-KALERGI, L-1359 LUXEMBOURG, LUXEMBOURG
E-mail address: jean-marc.schlenker@math.uni.lu

```