Random Matrix Theory (RMT) and Applications in Cognitive Radio Communications

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Outline

- Introduction
- Application Areas
- Basic principle and laws
- Applications in Wireless Communications
- Applications in Cognitive Radio Communications
- Open Problems
- Summary

Introduction

- Important **Multivariate statistics** tool
- Finite size matrices and large dimensional matrices (**limiting results)**
- Allows **easier approximations** and provides **closed-form expressions**
  - Applications in analyzing the statistics of functions having **random matrix arguments**
- **Law of large numbers** and **Central limit theorem**

- **Random matrix**: a matrix whose elements are random drawn from a specific distribution

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \; a_{ij} \sim \mathrm{N}(0,1) \; \forall i,j$$

- Eigenvalue Decomposition (**EVD**)

$$\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^H, \mathrm{diag}(\mathbf{\Lambda}) = [\lambda_1 \dots \lambda_N]$$

- Singular Value Decomposition (**SVD**)

$$\mathbf{B} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H, \mathrm{diag}(\mathbf{\Sigma}) = [\sigma_1 \dots \sigma_N]$$

Introduction

- Two Main Approaches
  - **Analytical method**
    - Utilizes the **asymtotic eigenvalue probability distributions (a.e.p.d.f.)** of large matrices utilizing various laws and transforms such as **Stieltjes transform**
    - Mostly used when the random matrices are **sample covariance matrices**, doubly correlated i.i.d. matrices, information plus noise matrices, isometric matrices, or sum/product of these matrices
  - **Moment-based method**
    - Successive moments of the **asymptotic eigenvalue probability distributions**
    - Not all distributions have **moments of all orders** and for some distributions having all orders, they **may not be uniquely defined by the series of moments**.
    - For **some structured matrices such as Vandermonde**, this method is preferable.

Free probability theory connects analytical methods with the moment-based methods with the help of different transforms!

- R. Couillet and M. Debbah, Random Matrix Methods for Wireless Communications, 1st ed. Cambridge University Press, 2011.

Introduction

- **RMT Tools**
  - **Method of moments**: identify eigenvalue distributions via moments [e.g., Wigner]
  - **Free probability theory**: spectrum of random matrix operations via moments of limiting distributions [e.g., Petz, Biane, Benaych-Georges]
  - **Stieltjes transform method**: spectrum of random matrix operations via Stieltjes transform [e.g., Bai, Silverstein, Pastur]
  - **Replica method**: a tool to study deterministic equivalents [e.g., Tanaka, Moustakas, Riegler]
  - **Gaussian tools on resolvents**: spectrum of Gaussian random matrix operations via Gaussian tricks on the resolvent [e.g., Pastur, Loubaton, Hachem]
  - **Orthogonal polynomials and Fredholm determinants**: study hole probability, e.g., extreme eigenvalue distribution via determinantal equations (e.g., Johnstone, Tracy, Widom, Guionnet)

- R. Couillet and M. Debbah, Random Matrix Methods for Wireless Communications, 1st ed. Cambridge University Press, 2011.

Application Areas

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- **Nuclear Physics**
  - Model the spectra of heavy atoms
  - Spacings between the lines in the spectrum of a heavy atom should resemble the spacings between the eigenvalues of a random matrix

- **Finance**
  - Portfolio theory and return statistics
  - Cases where numerous stocks show price index correlation over short observable time periods

- **Evolutionary biology**
  - The joint presence of multiple genes in the genotype of a given species is analyzed from a few DNA samples

- Wigner, E. (1955). "**Characteristic vectors of bordered matrices with infinite dimensions**". *Ann. Of Math.* **62**(3): 548–564. doi:10.2307/1970079
- J.P. Bouchaud, M. Potters, "**Financial Applications of Random Matrix Theory: a short review**'', http://arxiv.org/abs/0910.1205
- L. Laloux, P. Cizeau, M. Potters, and J. P. Bouchaud, "**Random matrix theory and financial correlations**," International Journal of Theoretical and Applied Finance, vol. 3, no. 3, pp. 391–397, Jul. 2000.
- N. Hansen and A. Ostermeier, "**Adapting arbitrary normal mutation distributions in evolution strategies: the covariance matrix adaptation**," in Evolutionary Computation, Proceedings of IEEE International Conference on. IEEE, 1996, pp. 312–317

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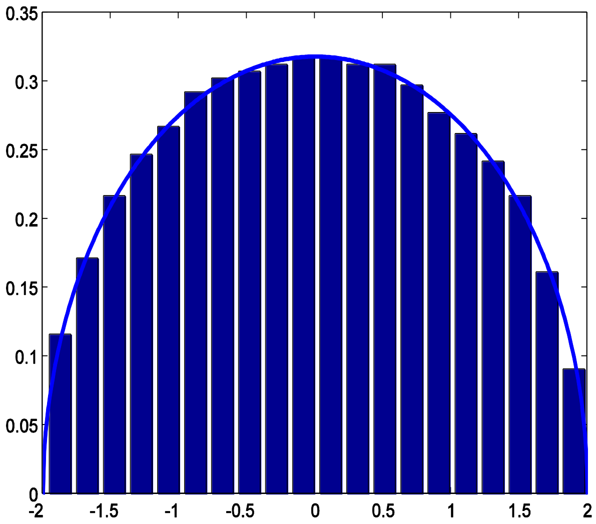
Application Areas

- **Signal Processing and Wireless Communications**
  - Asymptotic capacity analysis
  - Signal detection
  - Parameter estimation
  - Receiver design
  - Modeling transmit/receive correlation
  - Array processing
  - Multidimensional Spectrum Sensing
  - MIMO communications

Basic Laws

- 1955, Eugene Wigner
  - **Wigner matrix:** *An N×N Hermitian matrix **W** is a Wigner matrix if its upper-triangular entries are independent zero-mean random variables with identical variance.*
  - **Semi-circle law:** The empirical distribution of **W** converges almost surely to the semicircle law whose density is

$$w(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$$



- **Many Variants**
  - Quarter circle law
  - Full circle law: Uniform distribution on the complex unit disc
  - Haar distribution: Uniform distribution on the complex unit circle
  - Inverse Semi-Circle law
  - Deformed Quarter circle law

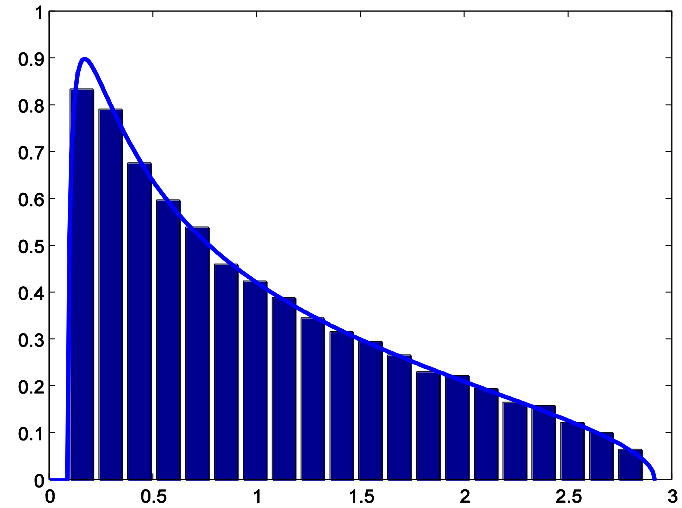
E. Wigner, "Characteristic vectors of bordered matrices with infinite dimensions," The Annals of Mathematics, 1955.

Basic Laws

- **Marchenko Pastur (MP) law** [Vladmir Marchenko and Leonid Pastur, 1967]
  - Consider an *M x N matrix **H** whose entries are independent zero-mean complex (or real) random variables with variance 1/N and fourth moments of order O(1/N²).*
  - As with $M, N \to \infty$ and $\frac{N}{M} \to \beta$, the empirical distribution of eigenvalues of (1/N)$\mathbf{H}^H\mathbf{H}$ converges almost surely to a nonrandom limiting distribution with density

$$f_\beta(\lambda) = (1-\beta)^+ \delta(\lambda) + \frac{\sqrt{(\lambda - a)^+(b-\lambda)^+}}{2\pi\beta\lambda}$$

$$a = (1-\sqrt{\beta})^2 \qquad b = (1+\sqrt{\beta})^2$$



The support is bounded independent of the values of the matrix elements !!

- Useful for getting spectra of so called **Wishart matrices**
  - uncorrelated Rayleigh channels in Wireless Communications

- V. A. Marchenko and L. A. Pastur, ”Distributions of eigenvalues for some sets of random matrices,” Math USSR-Sbornik, vol.1 pp.457-483, 1967.

Free Probability Theory

- **Free probability theory** applies to asymptotically large random matrices
- A form of **independence** for non commutative algebras (multiplication, addition)
- Applicable to expressions that include **sums or products** of **asymptotically free matrices**
  - **Asymptotic moment** as a function of the individual moments
  - **R transform for addition** and **Sigma transform for multiplication**

- **Asymptotic freeness**: Two Hermitian random matrices **A** and **B** are asymptotically free if

for all *l* and for all polymonials $p_i(.)$ and $q_i(.)$ with $1 \leq i \leq l$ such that

$$\phi(p_i(\mathbf{A})) = \phi(q_i(\mathbf{B})) = 0$$

$$\phi(p_1(\mathbf{A})q_1(\mathbf{B})...p_l(\mathbf{A})q_l(\mathbf{B}))) = 0$$

- A. M. Tulino and S. Verdu, "**Random matrix theory and wireless communications**," *Foundations Trends Commun. Inf. Theory*, vol. 1, no. 1, pp. 1–182, 2004.
- D. V. Voiculescu, K. J. Dykema, A. Nica, "**Free random variables**," American Mathematical Society, 1992.

Asymptotically Free Matrices

- Any random matrix and the identity matrix
- Independent **Wigner** matrices
- Independent **Gaussian** matrices
- Independent **Haar** matrices
- Independent **Unitarily Invariant (Wishart)** matrices
- Standard **Wigner** and **deterministic diagonal**
- Standard Gaussian and deterministic diagonal
- Haar matrices and deterministic matrix
- Unitarily invariant and deterministic matrix

- S. Chatzinotas, "**Random Matrix Theory: Basic Tools and Principles**'', Graduate course slides, 2013, available online: http://wwwen.uni.lu/snt/education/graduate_courses/random_matrix_theory_basic_tools_and_principles

Basic Transforms

- **Importance in RMT**
  - Final spectrum of matrix expressions from individual spectra
  - **Non-commutative** property of random matrix multiplication/addition

- **Important Transforms**
  - **Stieltjes Transform**
  - Eta Transform
  - **R transform**
  - **Sigma transform**
  - Shannon transform

- A. M. Tulino and S. Verdu, “**Random matrix theory and wireless communications**,” *Foundations Trends Commun. Inf. Theory*, vol. 1, no. 1, pp. 1–182, 2004.

Basic Transforms

- **Stieltjes Transform**
  - uniquely **determines the a.e.p.d.f. and vice versa**
  - Definition: X real valued random variable and z complex argument
  - **Inversion** formula

$$\mathcal{S}_X(z) = \mathbb{E}\left[\frac{1}{X-z}\right] = \int_{-\infty}^{\infty} \frac{1}{\lambda - z}\, dF_X(\lambda)$$

$$\mathrm{f}_X(\lambda) = \lim_{\omega \to 0^+} \frac{1}{\pi} \mathrm{Im}\left[\mathcal{S}_X(\lambda + \mathrm{j}\,\omega)\right]$$

- **R transform**
  - **Definition:** z belongs to the complex plane $\mathrm{R}_X(z) = \mathcal{S}_X^{-1}(-z) - \frac{1}{z}.$
  - **Additive free convolution** property

- **Sigma transform**
  - **Definition:** related to eta transform $\Sigma_X(x) = -\frac{x+1}{x}\eta_X^{-1}(1+x)$ $\quad \eta_X(\gamma) = \mathbb{E}\left[\frac{1}{1+\gamma X}\right]$
  - **Multiplicative free convolution** property

Relations of Transforms

$$\Sigma_{\mathbf{X}}(z) = -\frac{1+z}{z}\eta_{\mathbf{X}}^{-1}(1+z) \qquad \eta_{\mathbf{X}}(z) = \frac{\mathcal{S}_{\mathbf{X}}(-\frac{1}{z})}{z} \qquad \mathcal{R}_{\mathbf{X}}(z) = \mathcal{S}_{\mathbf{X}}^{-1}(-z) - \frac{1}{z}.$$

$$\boxed{\Sigma_{p\hat{\mathbf{R}}_{\hat{\mathbf{H}}}}(z)} \rightarrow \boxed{\mathcal{S}_{\hat{\mathbf{R}}_{\hat{\mathbf{H}}}}(z)} \rightarrow \boxed{\mathcal{R}_{p\hat{\mathbf{R}}_{\hat{\mathbf{H}}}}(z)} \rightarrow \boxed{\mathcal{S}_{\hat{\mathbf{R}}_{\mathbf{Y}}}} \rightarrow \boxed{\text{a.e.p.d.f } f_{\hat{\mathbf{R}}_{\mathbf{Y}}}}$$

$$\Sigma_{p\hat{\mathbf{R}}_{\hat{\mathbf{H}}}}(z) = \Sigma_{\mathbf{\Phi}}(z) \cdot \Sigma_{p\hat{\mathbf{R}}_{\mathbf{H}}}(z) \qquad \lim_{N\to\infty} \hat{\mathbf{R}}_{\mathbf{Y}}(N) \approx p\mathbf{\Phi}^{\frac{1}{2}}\mathbf{H}\mathbf{H}^{H}\mathbf{\Phi}^{\frac{1}{2}} + \hat{\mathbf{R}}_{\mathbf{Z}}$$

$$\mathcal{R}_{\hat{\mathbf{R}}_{\mathbf{Y}}}(z) = \mathcal{R}_{\hat{\mathbf{R}}_{\mathbf{Z}}}(z) + \mathcal{R}_{p\hat{\mathbf{R}}_{\hat{\mathbf{H}}}}(z) \qquad f(x) = \lim_{y\to 0^+} \frac{1}{\pi}\mathrm{Im}\{\mathcal{S}_{\hat{\mathbf{R}}_{\mathbf{Y}}}(x+jy)\}$$

- A. M. Tulino and S. Verdu, “**Random matrix theory and wireless communications**,” *Foundations Trends Commun. Inf. Theory*, vol. 1, no. 1, pp. 1–182, 2004.
- X. Mestre, J. Fonollosa, and A. Pages-Zamora, “**Capacity of MIMO channels: asymptotic evaluation under correlated fading**,” *IEEE J. Sel. Areas Commun.*, vol. 21, no. 5, pp. 829–838, June 2003.

Deterministic Equivalents

- limiting spectral density (lsd) methods such as **Stieltjes, Free Probability Theory** methods are based on the assumption that
  - **aepdf converges asymptotically to a deterministic function.**
- In the cases **where aepdf does not converge**, deterministic equivalents can provide solutions
  - The solution is derived based on **fixed point equations**
- Provides significant advantages over approach on the **system modeling practicalities**
  - **approximation of system performance for every N instead of a single value in lsd based approach**
- Various applications in wireless communications
  - Frequency selective MIMO
  - Information plus noise
  - Doubly correlated MIMO channels

- S. Chatzinotas, “**Random Matrix Theory: Basic Tools and Principles**’’, Graduate course slides, 2013, available online: http://wwwen.uni.lu/snt/education/graduate_courses/random_matrix_theory_basic_tools_and_principles
- Couillet, R.; Debbah, M.; Silverstein, J.W., "**A Deterministic Equivalent for the Analysis of Correlated MIMO Multiple Access Channels**," *Information Theory, IEEE Transactions on* , vol.57, no.6, pp.3493,3514, June 2011

Applications in Wireless Communications

- **High dimensional parameters** in wireless systems
  - multiple users
  - multiple antennas
  - multiple cooperating nodes
- **Matrices with random entries** are the basis for various applications such as MIMO channels, CDMA codes
- **Classical probability** approaches can not handle large dimensional problems
- **Wishart matrices** and **MP law** are of great interest
- **asymptotic eigenvalue probability density function (aepdf)**
  - pdf of the matrix eigenvalues when both dimensions go to infinity with a fixed ratio

- R. Couillet and M. Debbah, **Random Matrix Methods for Wireless Communications**, 1st ed. Cambridge University Press, 2011.

Applications in Wireless Communications

Matrices of interest

- sample covariance matrices
- i.i.d. matrices multiplied both on the left and right by deterministic matrices (correlated matrices)
- i.i.d. matrices with a variance profile (with independent entries of zero mean but different variances)
- Products and sums of random matrices $\Longrightarrow$ MP law is not directly applicable in many cases

Useful distributions

- Distribution of λ(**H**)
- Distribution of λ($\mathbf{HH}^H$).
- Distribution of λ_{max}(**H**)
- Joint distribution of λ1(**H**), ..., λN(**H**)
- Distribution of the spacings between adjacent eigenvalues $|\lambda_i - \lambda_j|$
- Distribution of the matrix of eigenvectors of $\mathbf{HH}^H$
- Distribution of inverses

Applications in Cognitive Radio Communications

- **Cognitive Radio communications**
  - Possible solution to spectrum scarcity problem
  - Spectral coexistence of licensed (primary) and unlicensed (secondary) systems
  - Spectrum awareness and spectrum exploitation
- **Spectrum awareness and Exploitation**
  - **Spectrum Sensing** (SS)
  - Parameter estimation such as **SNR, sparsity order**
  - Blind multi-source localization
  - Optimal resource allocation for secondary system
- **Multidimensional scenarios** for spectrum awareness
  - Oversampling, multiple antennas, cooperating nodes
- *M X N* **Received Signal:** M receive dimensions, N sample dimensions

$$\mathbf{Y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_M \end{bmatrix} = \begin{bmatrix} y_1(1) & y_1(2) & \dots & y_1(N) \\ y_2(1) & y_2(2) & \dots & y_2(N) \\ \vdots & \vdots & \ddots & \vdots \\ y_M(1) & y_M(2) & \dots & y_M(N) \end{bmatrix}$$

Applications: Multidimensional Spectrum Awareness

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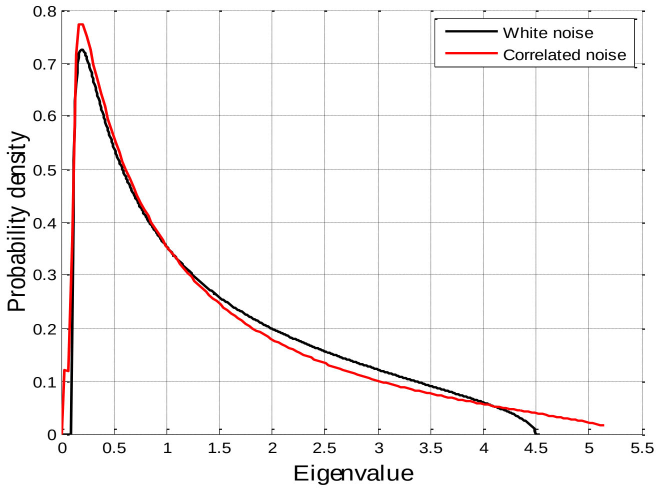
- *M X N* **Received Signal** $\mathbf{Y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_M \end{bmatrix} = \begin{bmatrix} y_1(1) & y_1(2) & \dots & y_1(N) \\ y_2(1) & y_2(2) & \dots & y_2(N) \\ \vdots & \vdots & \ddots & \vdots \\ y_M(1) & y_M(2) & \dots & y_M(N) \end{bmatrix}$ $\beta = N/M$

- Binary hypothesis testing problem
$$\begin{aligned} H_0 : \mathbf{Y} &= \mathbf{Z}, & \text{PU absent} \\ H_1 : \mathbf{Y} &= \sqrt{p}\mathbf{H}\mathbf{S}_d + \mathbf{Z}, & \text{PU present} \end{aligned}$$

- asymptotic eigenvalue probability density function **(aepdf)** of $\mathrm{R}_{\mathbf{Y}}(N)$ in the presence of **noise correlation**

$$\hat{\mathbf{Z}} = \mathbf{\Theta}^{1/2}\mathbf{Z}$$



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Applications: Spectrum Sensing

- **Received Signal's Covariance Matrix**

$$\mathbf{R_Y} = \mathbb{E}[\mathbf{YY}^H] = \mathbb{E}\left[(\sqrt{p}\mathbf{HS})(\sqrt{p}\mathbf{HS})^H\right] + \mathbb{E}[\hat{\mathbf{Z}}\hat{\mathbf{Z}}^H] = p\mathbb{E}[\mathbf{HH}^H] + \mathbf{R}_{\hat{\mathbf{Z}}}$$

- **Sample Covariance Matrix**

$$\hat{\mathbf{R}}_{\mathbf{Y}}(N) = \frac{1}{N}\mathbf{YY}^H \qquad \hat{\mathbf{R}}_{\hat{\mathbf{Z}}}(N) = \frac{1}{N}\hat{\mathbf{Z}}\hat{\mathbf{Z}}^H$$

- **Correlation Modeling**: Exponential correlation model (Spatial correlation)

$$\theta_{ij} = \begin{cases} \rho^{(j-i)}, & i \le j \\ \left(\rho^{(i-j)}\right)^*, & i > j \end{cases} \qquad \hat{\mathbf{Z}} = \mathbf{\Theta}^{1/2}\mathbf{Z}$$

- Under the $\mathbb{H}_0$ **hypothesis** $\hat{\mathbf{R}}_{\mathbf{Y}}(N) = \hat{\mathbf{R}}_{\hat{\mathbf{Z}}}(N) = \mathbf{\Theta}^{1/2}\mathbf{ZZ}^H\mathbf{\Theta}^{1/2}$

- To calculate the **threshold for SS purpose**, we need the **support of a.e.p.d.f.** of $\hat{\mathbf{R}}_{\mathbf{Y}}(N)$

- $1/N\mathbf{ZZ}^H$ is an **uncorrelated Wishart matrix** and follows MP law while $\hat{\mathbf{R}}_{\hat{\mathbf{Z}}}(N) = \frac{1}{N}\hat{\mathbf{Z}}\hat{\mathbf{Z}}^H$ **does not follow MP law**.

Applications: Spectrum Sensing (Contd.)

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- **Tilted Semicircular Law**: Let Θ be a positive definite matrix which is normalized as $(1/M)\mathrm{trace}\{\boldsymbol{\Theta}\} = 1$ and whose asymptotic spectrum has the p.d.f.

$$f_{\Theta}(\lambda) = \frac{1}{2\pi\mu\lambda^2}\sqrt{\left(\frac{\lambda}{\sigma_1}-1\right)\left(1-\frac{\lambda}{\sigma_2}\right)} \qquad \text{with} \qquad \sigma_1 \le \lambda \le \sigma_2 \qquad \mu = \frac{(\sqrt{\sigma_2}-\sqrt{\sigma_1})^2}{4\sigma_1\sigma_2}$$

If **F** is an $M \times N$ standard complex Gaussian matrix which follows the MP law, then as $M, N \to \infty$ with $\frac{N}{M} \to \beta$ the asymptotic distribution of W= $\Theta^{1/2}\mathbf{F}\mathbf{F}^H\boldsymbol{\Theta}^{1/2}$ has the following p.d.f.

$$f_{\mathbf{W}}(\lambda) = (1-\beta)^+\delta(\lambda) + \frac{\sqrt{(\lambda-\tilde{a})^+(\tilde{b}-\lambda)^+}}{2\pi\lambda(1+\lambda\mu)}$$

$$\tilde{a} = 1+\beta+2\mu\beta-2\sqrt{\beta}\sqrt{(1+\mu)(1+\mu\beta)}$$

$$\tilde{b} = 1+\beta+2\mu\beta+2\sqrt{\beta}\sqrt{(1+\mu)(1+\mu\beta)}$$

- Mestre, X.; Fonollosa, J.R.; Pages-Zamora, A., "**Capacity of MIMO channels: asymptotic evaluation under correlated fading**," *Selected Areas in Communications, IEEE Journal on* , vol.21, no.5, pp. 829-838, June 2003.

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Applications: Spectrum Sensing (Contd.)

- **Application of Tilted circular law in our problem**

$$\hat{\mathbf{R}}_{\mathbf{Y}}(N) = \hat{\mathbf{R}}_{\mathbf{Z}}(N) = \mathbf{\Theta}^{1/2}\mathbf{Z}\mathbf{Z}^H\mathbf{\Theta}^{1/2}$$

$$\Sigma_{\mathbf{X}}(z) = -\frac{1+z}{z}\eta_{\mathbf{X}}^{-1}(1+z) \qquad \eta_{\mathbf{X}}(z) = \frac{\mathcal{S}_{\mathbf{X}}(-\frac{1}{z})}{z} \qquad f(x) = \lim_{y\to 0^+}\frac{1}{\pi}\mathrm{Im}\{\mathcal{S}_{\hat{\mathbf{R}}_{\mathbf{Y}}}(x+jy)\}$$

$$\boxed{\Sigma_{\hat{\mathbf{R}}_{\mathbf{Y}}}} \Rightarrow \boxed{\eta_{\hat{\mathbf{R}}_{\mathbf{Y}}}} \Rightarrow \boxed{\mathcal{S}_{\hat{\mathbf{R}}_{\mathbf{Y}}}} \Rightarrow \boxed{\text{a.e.p.d.f } f_{\hat{\mathbf{R}}_{\mathbf{Y}}}}$$

$$\Sigma_{\mathbf{Z}\mathbf{Z}^H} = \frac{1}{\beta+z} \qquad \Sigma_{\Theta}(z) = 1-\mu z \qquad \Sigma_{\hat{\mathbf{R}}_{\mathbf{Y}}}(z) = \Sigma_{\mathbf{\Theta}}(z)\frac{1}{z+\beta}$$

$$f_{\hat{\mathbf{R}}_{\mathbf{Y}}}(\lambda) = (1-\beta)^+\delta(\lambda) + \frac{\sqrt{(\lambda-\tilde{a})^+(\tilde{b}-\lambda)^+}}{2\pi\lambda(1+\lambda\mu)}$$

$$\text{decision} = \begin{cases} \mathbb{H}_0, & \text{if SCN} \le \frac{\tilde{b}}{\tilde{a}} \\ \mathbb{H}_1, & \text{otherwise} \end{cases}$$

$$\text{SCN} = \frac{1+\rho}{1-\rho} \qquad \mu = \frac{\rho^2}{1-\rho^2}$$

- S. K. Sharma, S. Chatzinotas, and B. Ottersten, "**Eigenvalue Based Sensing and SNR Estimation for Cognitive Radio in Presence of Noise Correlation**", *IEEE Trans. Vehicular Technology*, vol. 42, no. 8, pp. 3671-3684, Oct. 2013.

Applications: SNR Estimation (Contd.)

- Analysis under H_1 hypothesis (both **noise and channel uncorrelated case**)

$$\lim_{N\to\infty} \hat{\mathbf{R}}_{\mathbf{Y}}(N) \approx p\mathbf{H}\mathbf{H}^H + \hat{\mathbf{R}}_{\mathbf{Z}}$$

- Both terms follow MP law
- **Free additive convolution property using R transform**

$$\mathcal{R}_{p\hat{\mathbf{R}}_{\mathbf{H}}}(z) = p\mathcal{R}_{\hat{\mathbf{R}}_{\mathbf{H}}}(pz) = \frac{p\beta}{1-pz}. \qquad \mathcal{R}_{\hat{\mathbf{R}}_{\mathbf{Z}}}(z) = \frac{\beta}{1-z}$$

$$\mathcal{R}_{\hat{\mathbf{R}}_{\mathbf{Y}}}(z) = \frac{p\beta}{1-pz} + \frac{\beta}{1-z}$$

- Cubic polymonial for the **Stieltjes Transform**

$$(pz)\mathcal{S}^3_{\hat{\mathbf{R}}_{\mathbf{Y}}}(z) + (p(-2\beta+z+1)+z)\mathcal{S}^2_{\hat{\mathbf{R}}_{\mathbf{Y}}}(z) + ((1-\beta)(1+p)+z)\mathcal{S}_{\hat{\mathbf{R}}_{\mathbf{Y}}}(z) + 1$$

- **Inversion formula for calculating a.e.p.d.f.** $f^{\infty}_{\mathbf{Y}} = \lim_{y\to 0^+} \frac{1}{\pi}\mathrm{Im}\{\mathcal{S}_{\hat{\mathbf{R}}_{\mathbf{Y}}}(x+jy)\}$

- S. K. Sharma, S. Chatzinotas, and B. Ottersten, “**SNR Estimation for Multi-dimensional Cognitive Receiver under Correlated Channel/Noise**", *IEEE Trans. Wireless Commun.*, vol. 12, no. 12, pp. 6392-6405, Dec. 2013.

Applications: SNR Estimation (Contd.)

- Analysis under H_1 hypothesis (**correlated noise and uncorrelated channel**)

$$\lim_{N\to\infty} \hat{\mathbf{R}}_{\mathbf{Y}}(N) \approx p\mathbf{H}\mathbf{H}^H + \hat{\mathbf{R}}_{\hat{\mathbf{Z}}}$$

- The first term follows MP law and for the **2nd term, tilted semicircular law**
- R transform for the 2nd term

$$\mathcal{R}_{\hat{\mathbf{R}}_{\hat{\mathbf{Z}}}}(z) = -\frac{1}{2}\frac{(z-1+\sqrt{(z^2-2z+1-4\mu\beta z)})}{\mu z}$$

- Combined R transform

$$\mathcal{R}_{\hat{\mathbf{R}}_{\mathbf{Y}}}(z) = \frac{p\beta}{(1-pz)} - \frac{1}{2}\frac{(-1+z+\sqrt{(1-2z+z^2-4\mu\beta z)})}{z\mu}$$

- Polynomial for Stieltjes transform

$$(zp^2(1+\mu z))\mathcal{S}^4_{\hat{\mathbf{R}}_{\mathbf{Y}}}(z) + (2z\mu p(z-p\beta) + p^2(1+2z\mu$$
$$+z-2\beta) + 2zp)\mathcal{S}^3_{\hat{\mathbf{R}}_{\mathbf{Y}}}(z) + (p^2(\mu(1-\beta)^2+1-\beta) + 2p$$
$$(1+z+\mu z(2-\beta)) + z - 3p\beta + z^2\mu)\mathcal{S}^2_{\hat{\mathbf{R}}_{\mathbf{Y}}}(z) + (2p(1+$$
$$\mu(1-\beta)) + z(1+2\mu) - \beta(1+p)+1)\mathcal{S}_{\hat{\mathbf{R}}_{\mathbf{Y}}}(z) + 1 + \mu$$

Applications: Compressive Estimation

- **Multiple Measurement Vector** (MMV) model

$$\mathbf{Y} = \mathbf{AUBX} + \mathbf{Z} = \mathbf{AUS} + \mathbf{Z}$$

where

- **AU** is an $N \times N$ **sensing matrix** with P[$\mathbf{A}_{i,i}$ = 1] = ρ = 1 − P[$\mathbf{A}_{i,i}$ = 0].
- **S** = **BX** is an $N \times N$ sparse signal matrix with uniform sparsity across all the columns with P[$\mathbf{B}_{i,i}$ = 1] = σ = 1 − P[$\mathbf{B}_{i,i}$ = 0].

- **Assumptions**
  - **A**, **U**, **B, X** and **Z** are mutually independent.
  - **U** and **X** to be $N \times N$ random matrices having i.i.d. entries with zero mean and variance 1/N.
  - The sensing matrix **AU** is assumed to be known by the receiver.

- Shree K. Sharma, S. Chatzinotas, and B. Ottersten, "**Compressive Sparsity Order Estimation for Wideband Cognitive Radio Receiver**", *IEEE Trans. Signal Process.*, vol. 62, no. 19, Oct. 2014.

Applications: Compressive Estimation (Contd.)

- Assuming that the source signal is independent from the noise

$$\mathbf{R_Y} = \mathbb{E}[\mathbf{Y}\mathbf{Y}^{\dagger}] = \mathbb{E}\left[(\mathbf{AUBX})(\mathbf{AUBX})^{\dagger}\right] + \mathbb{E}[\mathbf{Z}\mathbf{Z}^{\dagger}]$$

- Since all the matrices **A,U,B,X** and **Z** are square $f_{\mathbf{R_Y}}(\lambda) = f_{\bar{\mathbf{R}}_\mathbf{Y}}(\lambda)$

$$\bar{\mathbf{R}}_\mathbf{Y} = \mathbf{R}\mathbf{R}_1 + \mathbf{R_Z} \qquad \mathbf{R} = \mathbb{E}\left[\mathbf{U}^{\dagger}\mathbf{A}^{\dagger}\mathbf{A}\mathbf{U}\right] \qquad \mathbf{R}_1 = \mathbb{E}\left[\mathbf{B}\mathbf{X}\mathbf{X}^{\dagger}\mathbf{B}^{\dagger}\right]$$

- Sample covariance matrices $\hat{\mathbf{R}}_\mathbf{Y}(N) = \frac{1}{N}\mathbf{Y}\mathbf{Y}^{H}$ $\hat{\mathbf{R}}_\mathbf{Z}(N) = \frac{1}{N}\mathbf{Z}\mathbf{Z}^{\dagger}$

- **3 different cases**
  - Constant power case $\mathrm{Y} = \mathrm{AU}\sqrt{p}\mathrm{BX} + \mathrm{Z}$
  - Varying power case $\mathrm{Y} = \mathrm{AUP}^{1/2}\mathrm{BX} + \mathrm{Z}$
  - Correlated case $\mathrm{Y} = \mathrm{AU}\Theta^{1/2}\mathrm{BX} + \mathrm{Z}$

Applications: Compressive Estimation (Contd.)

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- Assuming that signal and noise are uncorrelated

$$\lim_{N\to\infty} \hat{\mathbf{R}}_{\mathbf{Y}}(N) \approx p\hat{\mathbf{R}}\hat{\mathbf{R}}_1 + \hat{\mathbf{R}}_{\mathbf{Z}}$$

- **Problem**: ***To find the aepdf of the observed signal's sample covariance matrix, whose maximum eigenvalue is related to the sparsity order of the wideband PU signal to be estimated***
- The η transform of $\hat{\mathbf{R}}$

$$1 = \frac{1-\eta_{\hat{\mathbf{R}}}(z)}{1-\eta_{\mathbf{F}}(z\eta_{\hat{\mathbf{R}}}(z))} \qquad \mathbf{F} = \mathbf{A}^{\dagger}\mathbf{A}$$

$$z\eta^2_{\hat{\mathbf{R}}}(z) - ((1-\rho)z-1)\eta_{\hat{\mathbf{R}}}(z) - 1 = 0$$

$$z\eta^2_{\hat{\mathbf{R}}_1}(z) - ((1-\sigma)z-1)\eta_{\hat{\mathbf{R}}_1}(z) - 1 = 0$$

⇒ correspond to the η transform of the **HH†** with *N × ρN and N × σN* random matrix **H** with i.i.d. elements having zero mean and variance 1/N

⇒ follow **Marchenko Pastur (MP) Law**

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- A. Tulino, G. Caire, S. Verdu, and S. Shamai, "**Support recovery with sparsely sampled free random matrices**," IEEE Trans. Info. Th., vol. 59, no. 7, pp. 4243-4271, July 2013

Open Problems

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- Distributions of **test statistics very important** in signal detection, parameter estimation problems
- In the presence of noise correlation $\hat{\mathbf{R}}_{\mathbf{Y}}(N) = \hat{\mathbf{R}}_{\hat{\mathbf{Z}}}(N) = \mathbf{\Theta}^{1/2}\mathbf{Z}\mathbf{Z}^H\mathbf{\Theta}^{1/2}$
- Distributions of eigenvalue based decision statistics
  - Maximum eigenvalue $\lambda_{max}(\hat{\mathbf{R}}_{\mathbf{Y}})$ Maximum to Minimum eigenvalue $\frac{\lambda_{max}}{\lambda_{min}}$
  - Scaled largest eigenvalue $\frac{\lambda_{max}}{\frac{1}{M}tr(\hat{\mathbf{R}}_{\mathbf{Y}})}$ Average eigenvalue $\frac{1}{N}\sum_{i=1}^{N}\lambda_i$
  - Spherical test $T_{ST} = \frac{(\det(\mathbf{R}_{\mathbf{Y}}(N)))^{1/M}}{\frac{1}{M}\mathrm{tr}(\mathbf{R}_{\mathbf{Y}}(N))} = \frac{\left(\prod_{i=1}^{M}\lambda_i\right)^{1/M}}{\frac{1}{M}\sum_{i=1}^{M}\lambda_i}$
  - John's detector $T_J = \frac{\sqrt{\sum_{i=1}^{M}\lambda_i^2}}{\sum_{i=1}^{M}\lambda_i}$
- Distributions of some statistics for finite case are available for $\mathbf{Z}\mathbf{Z}^H$
- For correlated case, only the approximated distribution of the maximum eigenvalue available in the form of **Tracy Widom** distribution
- **Any known distributions for different eigenvalue metrics** of the **following form** ?

$$\hat{\mathbf{R}}_{\mathbf{Y}}(N) = \hat{\mathbf{R}}_{\hat{\mathbf{Z}}}(N) = \mathbf{\Theta}^{1/2}\mathbf{Z}\mathbf{Z}^H\mathbf{\Theta}^{1/2}$$

- Z. Bao, G. Pan, and W. Zhou "Universality for the largest eigenvalue of a class of sample covariance matrices," online, July 2013, http://arxiv.org/abs/1304.5690v4.

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Open Problems (Contd.)

- **RMT in compressive sensing** application

$$\mathbf{Y} = \mathbf{AUBX} + \mathbf{Z} = \mathbf{AUS} + \mathbf{Z} \qquad \hat{\mathbf{R}} = \frac{1}{N}\left[\mathbf{U}^H\mathbf{A}^H\mathbf{AU}\right] \quad \hat{\mathbf{R}}_1 = \frac{1}{N}\left[\mathbf{BXX}^\dagger\mathbf{B}^\dagger\right]$$

- Do the basic laws of RMT hold in compressive settings ?
  - It has been known that MP law holds true for the terms $\hat{\mathbf{R}}$ and $\hat{\mathbf{R}}_1$
- **Stieltjest transform** method complicated due to involvement of higher order polymonials for correlated scenarios.
  - Any other suitable approach for the following case (**to find aepdf**) ?

$$\hat{\mathbf{R}}_{\mathbf{Y}}(N) \approx p\hat{\mathbf{R}}\mathbf{\Theta}^{1/2}\hat{\mathbf{R}}_1\mathbf{\Theta}^{1/2} + \mathbf{\Phi}^{1/2}\mathbf{Z}\mathbf{Z}^H\mathbf{\Phi}^{1/2}$$

- **Hypothesis testing in compressive settings (C:** a K × K diagonal matrix with the diagonal representing the sparsity level and **B**: a K ×N PU transmitted signal matrix)

$$\begin{aligned} \mathbb{H}_0 : \mathbf{Y} &= \mathbf{\Phi}\mathbf{Z}, \\ \mathbb{H}_1 : \mathbf{Y} &= \mathbf{\Phi}(\mathbf{S} + \mathbf{Z}) \end{aligned} \qquad \mathbf{S} = \mathbf{CB}$$

  - Is there any suitable **eigenvalue based metric** which can **effectively distinguish the above two hypotheses** ?

- Shree K. Sharma, S. Chatzinotas, and B. Ottersten, "**Compressive Sparsity Order Estimation for Wideband Cognitive Radio Receiver**", *IEEE Trans. Signal Process.*, vol. 62, no. 19, Oct. 2014.

Open Issues (Contd.)

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- **Spectrum sensing using multiple decision statistics having arbitrary distributions**
- Bivariate PDF: $f(T_1, T_2)$ Bivariate CDF: $F(T_1, T_2)$ Thresholds λ_1 λ_2
- Expression for **probability of false alarm**

$$P_f = 1 - F(\lambda_1, \lambda_2) = 1 - \int_{-\infty}^{\lambda_1} \int_{-\infty}^{\lambda_2} f(T_1, T_2) dT_2 dT_1 \quad (\lambda_1, \lambda_2) = F^{-1}(1 - P_f)$$

  - There **exist multiple pairs of thresholds** which satisfy the **same target P_f**.

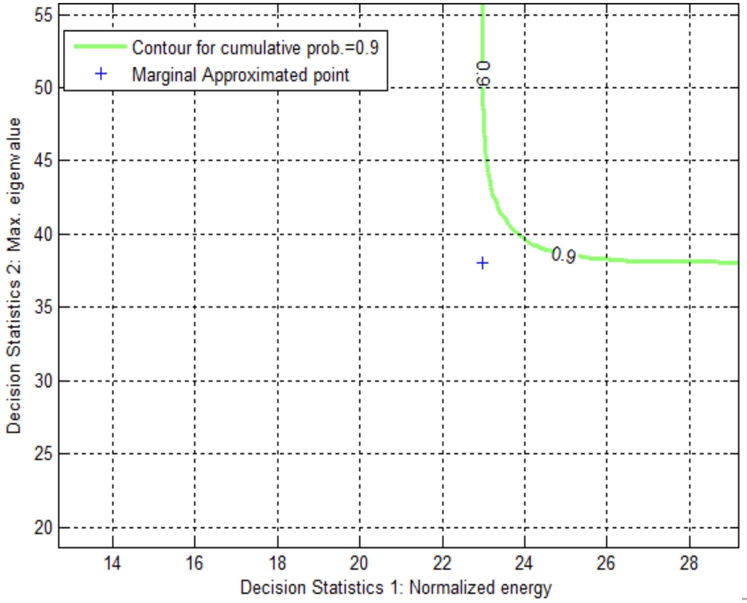
- **Problems**
  - What is the best **approximation method** to find the approximated **threshold pair** ?
  - **Extension** for more than 2 variables.

- **Marginal Approximation**

$$F_{T_1}(\lambda_1) = \lim_{\lambda_2 \to \infty} \int_{-\infty}^{\lambda_1} \int_{-\infty}^{\lambda_2} f(T_1, T_2) dT_2 dT_1$$

$$F_{T_2}(\lambda_2) = \lim_{\lambda_1 \to \infty} \int_{-\infty}^{\lambda_1} \int_{-\infty}^{\lambda_2} f(T_1, T_2) dT_2 dT_1$$

- Euclidean approximation
- Any other **good approximation or method** for thresholds computation?



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Summary

- Asymptotics plays a key role in **removing the randomness of large dimensional matrices** and also provides closed approximations for finite cases
- Long-term system performance can be characterized deterministically
- Different RMT tools exist depending on the nature of the problem
- Several applications in wireless communications
- Closed form solutions for the metric of interest
- Applications in **Cognitive radio communications**
  - Spectrum sensing
  - SNR estimation
  - Sparsity order estimation

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Thank you for your attention!

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