Complex Manifolds

Prof. Dr. Martin SCHLICHENMAIER

lecture notes written by
Alain LEYTEM

Master course, 2^{nd} semester

Contents

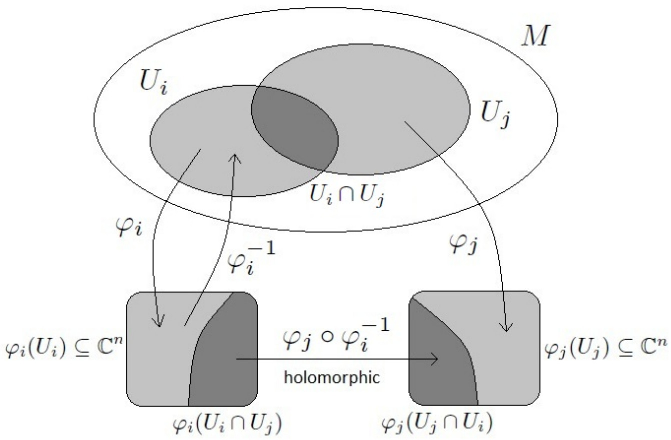
Introduction

Let M be a set with a collection of coordinate charts $(U_i, \varphi_i)_{i\in I}$, i.e.

$$\forall\, i \in I,\ \varphi_i\ :\ U_i \subseteq M \xrightarrow{\sim} \varphi_i(U_i) \subseteq \mathbb{C}^n \cong \mathbb{R}^{2n} \text{ where } \varphi_i(U_i) \text{ is open in } \mathbb{C}^n$$

M is called a complex manifold if these charts satisfy

a) The chart domains U_i cover M : $\bigcup_{i\in I} U_i = M$
b) $\varphi_i(U_i \cap U_j)$ is an open subset of $\mathbb{C}^n$, $\forall\, i, j \in I$
c) The locally defined transition maps $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_j \cap U_i)$ are holomorphic (in n variables).



In this case, M is a manifold of complex dimension n and of real dimension $2n$.

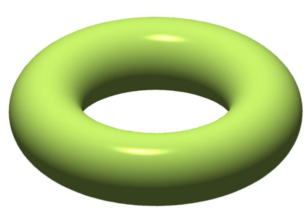
Interpretation :
Complex manifolds locally look like $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

Examples of complex manifolds :
1) $\mathbb{C}^n$: non-compact manifold
2) $\mathbb{P}^n(\mathbb{C}) = \mathbb{CP}^{n-1}$, the complex projective space : compact manifold
3) Complex tori (see figure 1); they can for example be constructed as a quotient of the type

$$T = \mathbb{C}\,/\,L \quad \text{where } L \text{ is a lattice over } \mathbb{Z}^2$$

4) If $\mathbb{K} = \mathbb{C}$, algebraic varieties "without singularities" are non-compact complex manifolds.
5) But not all complex manifolds are of algebraic type.
It turns out that compact and non-compact complex manifolds have a very different behaviour, e.g. if a globally defined function on a compact manifold is globally holomorphic, it must be constant.

Figure 1: the torus $T = \mathbb{C}\,/\,L$ embedded into $\mathbb{R}^3$



Recalls :
− holomorphic functions in 1 variable
− holomorphic functions in n variables for $n > 1$
− differentiable manifolds
− holomorphic differentiable forms

Chapter 1

Holomorphic functions in 1 variable

1.1 Notations and definitions

We know that $\mathbb{C} = \mathbb{R} \oplus i\,\mathbb{R} \equiv \mathbb{R}^2$ as vector spaces.
Let $z \in \mathbb{C} \;\Rightarrow\; \exists!\, x, y \in \mathbb{R}$ such that $z = x + iy \equiv (x, y)$ where $i^2 = -1 \;\Rightarrow\; x = \operatorname{Re} z$, $y = \operatorname{Im} z$, $\bar{z} = x - iy$

$$|z| = \sqrt{x^2 + y^2} \quad\Rightarrow\; |z|^2 = x^2 + y^2 = (x + iy) \cdot (x - iy) = z \cdot \bar{z}$$

Let $U \subseteq \mathbb{C}$ be an open subset of $\mathbb{C}$. A *complex* function is a map

$$f \;:\; U \subseteq \mathbb{C} \to \mathbb{C} \;:\; z \mapsto w = f(z) \quad\Leftrightarrow\quad f \;:\; U \subseteq \mathbb{R}^2 \to \mathbb{R}^2 \;:\; (x, y) \mapsto \big(u(x, y)\,,\, v(x, y)\big)$$

Since $\mathbb{C} \cong \mathbb{R}^2$, these are equivalent descriptions of complex functions : $f(z) \equiv f(x, y) = u(x, y) + i \cdot v(x, y)$.

Thus any complex function has the partial derivatives :

$$\frac{\partial f}{\partial z}\,,\;\frac{\partial f}{\partial \bar{z}} \quad\Leftrightarrow\quad \frac{\partial u}{\partial x}\,,\;\frac{\partial u}{\partial y}\,,\;\frac{\partial v}{\partial x}\,,\;\frac{\partial v}{\partial y}$$

The relation between the variables is given by $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$ and defines a change of variables, so

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial z} = \frac{\partial f}{\partial x} \cdot \frac{1}{2} + \frac{\partial f}{\partial y} \cdot \frac{1}{2i} = \frac{1}{2} \cdot \Big(\frac{\partial f}{\partial x} - i \cdot \frac{\partial f}{\partial y}\Big) \tag{1.1}$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}} = \frac{\partial f}{\partial x} \cdot \frac{1}{2} + \frac{\partial f}{\partial y} \cdot \frac{-1}{2i} = \frac{1}{2} \cdot \Big(\frac{\partial f}{\partial x} + i \cdot \frac{\partial f}{\partial y}\Big) \tag{1.2}$$

because of the chain rule. Notice that these are equations on $\mathbb{R}^2$ since $f : U \to \mathbb{C} \;\Leftrightarrow\; f : U \to \mathbb{R}^2$.

1.1.1 Definition A

Let $U \subseteq \mathbb{C}$ be open, $f : U \to \mathbb{C}$ be a complex function and $w \in U$.
Assume that f is differentiable in the real sense with continuous partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
f is called *holomorphic at* $w \in U \;\Leftrightarrow\; \frac{\partial f}{\partial \bar{z}}(w) = 0$.
f is called *holomorphic in* U if f is holomorphic at all points $w \in U$.

1.1.2 Definition B

Let $U \subseteq \mathbb{C}$ be open. A complex function $f : U \subseteq \mathbb{C} \to \mathbb{C}$ is called *analytic in* U if f has a local series expansion for all $w \in U$, i.e. $\forall\, w \in U$, $\exists\, a_k \in \mathbb{C}$ such that

$$f(z) = \sum_{k=0}^{\infty} a_k \cdot (z - w)^k \tag{1.3}$$

and this series converges in a neighborhood of w, e.g. in an open disc around w : $D(w, \varepsilon) = \{\, z \in \mathbb{C}\,,\; |z - w| < \varepsilon \,\}$.

1.1.3 Remark

As a power series, the convergence will always be absolutely and uniformly, i.e.

$$\sum_{k=0}^{\infty} a_k \cdot (z-w)^k < \infty,\ \forall\, z \in D(w,\varepsilon) \quad \Rightarrow \quad \sum_{k=0}^{\infty} \left| a_k \cdot (z-w)^k \right| < \infty,\ \forall\, z \in D(w,\varepsilon)$$

and $\forall\, w \in U$, the convergence of the series is independent of the chosen point z in $D(w,\varepsilon)$.
Analytic functions are obviously holomorphic since there is no $\bar{z}$ in (1.3), thus Def. B $\Rightarrow$ Def. A.

1.2 The Cauchy-Riemann equations

Let $f = u + i \cdot v$. Using (1.2), the condition of f being holomorphic (Definition A) can be rewritten as

$$\begin{aligned}
\frac{\partial f}{\partial \bar{z}} = 0 \ \Leftrightarrow\ \frac{1}{2} \cdot \Big(\frac{\partial f}{\partial x} + i \cdot \frac{\partial f}{\partial y} \Big) = 0 \ &\Leftrightarrow\ \frac{\partial (u + i \cdot v)}{\partial x} + i \cdot \frac{\partial (u + i \cdot v)}{\partial y} = 0 \\
&\Leftrightarrow\ \frac{\partial u}{\partial x} + i \cdot \frac{\partial v}{\partial x} + i \cdot \frac{\partial u}{\partial y} + i^2 \cdot \frac{\partial v}{\partial y} = 0 \ \Leftrightarrow\ \Big(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \Big) + i \cdot \Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \Big) = 0 \\
&\Leftrightarrow\ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \text{ and } \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad (1.4)
\end{aligned}$$

The identities (1.4) are called the *Cauchy-Riemann differential equations* and are equivalent to Definition A.

1.2.1 Theorem

Let $U \subseteq \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be a complex function such that f is real differentiable, i.e. the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous. Then the following conditions are equivalent :

1) f is holomorphic in U.
2) f is analytic in U.
3) f satisfies the Cauchy-Riemann differential equations in U.

None of these equivalences is true for functions in real variables!
Thus holomorphic functions are (as power series) differentiable up to every order, i.e. they are of class C^∞.

1.2.2 Examples

− holomorphic : polynomial functions $P(z)$, exponential functions $\exp(z)$, trigonometric functions $\sin z$, $\cos z$
rational functions $\frac{P(z)}{Q(z)}$ (without poles, otherwise they are called meromorphic)
− not holomorphic : conjugates $\bar{z}$, modules $|z|^2 = z\bar{z}$

1.3 Cauchy integral formula

1.3.1 Theorem

Let D be an open disc in $\mathbb{C}$ and assume that $f \in C^\infty(\bar{D})$ with f holomorphic in D. Then

$$f(z) = \frac{1}{2\pi i} \cdot \oint_{\partial D} \frac{f(w)}{w - z}\, dw\ , \quad \forall\, z \in D$$

where ∂D is traveled in the trigonometric sense.

1.3.2 Generalization

Let $U \subseteq \mathbb{C}$ be a simply connected open set in $\mathbb{C}$ and $f : U \to \mathbb{C}$ be holomorphic in U. Let γ be a simple closed path in U and $z \in U$ such that $z \notin \operatorname{im} \gamma$. If $n(z,\gamma)$ denotes the winding number of z with respect to γ, then

$$n(z,\gamma) \cdot f(z) = \frac{1}{2\pi i} \cdot \int_\gamma \frac{f(w)}{w - z}\, dw$$

Chapter 2

Holomorphic functions in n variables

2.1 Notations and definitions

We know that $\mathbb{C}^n \cong \left(\mathbb{R}^2\right)^n \cong \mathbb{R}^{2n}$ is a real vector space of dimension $2n$ and an n-dimensional complex vector space. The isomorphism $\mathbb{C}^n \cong \mathbb{R}^{2n}$ is given by

$$(z_1, \dots, z_n) \in \mathbb{C}^n \longmapsto (x_1, y_1, x_2, y_2, \dots, x_n, y_n) \in \mathbb{R}^{2n} \quad \text{with } z_k = x_k + i \cdot y_k,\ \forall\, k \in \{1, \dots, n\}$$

It is not possible to define a scalar product (symmetric, bilinear, positive definite) on $\mathbb{C}^n$. It must be replaced by the *Hermitian product*, defined by

$$\langle\, z, w \,\rangle := \sum_{k=1}^{n} \bar{z}_k \cdot w_k \quad \text{for } z, w \in \mathbb{C}^n$$

This product is linear in the second variable, conjugate-linear in the first argument, positive definite and satisfies $\overline{\langle\, z, w \,\rangle} = \langle\, w, z \,\rangle$. Finally the *module* of $z \in \mathbb{C}^n$ is equal to $\|z\| = \sqrt{\langle\, z, z \,\rangle}$, hence

$$\|z\| = \sqrt{\sum_{k=1}^{n} \bar{z}_k \cdot z_k} = \sqrt{\sum_{k=1}^{n} |z_k|^2} = \sqrt{\sum_{k=1}^{n} \left(x_k^2 + y_k^2\right)} = \sqrt{\sum_{k=1}^{n} x_k^2 + \sum_{k=1}^{n} y_k^2}$$

2.1.1 Definitions

Let $U \subseteq \mathbb{C}^n$ be open, $f : U \to \mathbb{C}$ be a complex function and $w \in U$. f is called *holomorphic at* $w \in U$ if f is continuous at w and if for all $k \in \{1, \dots, n\}$, the function $z_k \mapsto f(z_1, \dots, z_k, \dots, z_n)$ is holomorphic at w_k, i.e. f is holomorphic in each variable separately.
f is called *holomorphic in* U if it is holomorphic at every point $w \in U$.
And a function $F : U \subseteq \mathbb{C}^n \to \mathbb{C}^m$ is holomorphic $\Leftrightarrow$ every component function $F_i : U \to \mathbb{C}$ is holomorphic.

The same argument as in the case $n = 1$ shows that $f : U \to \mathbb{C}$ is holomorphic in U $\Leftrightarrow$ the Cauchy-Riemann equations are satisfied, i.e. if $f = u + i \cdot v$ for $u, v : \mathbb{R}^{2n} \to \mathbb{R}$, then

$$\begin{aligned} f \text{ holomorphic} \quad &\Leftrightarrow \quad \frac{\partial f}{\partial \bar{z}_k} = 0\,,\ \forall\, k \in \{1, \dots, n\} \\ &\Leftrightarrow \quad \frac{\partial u}{\partial x_k} = \frac{\partial v}{\partial y_k} \quad \text{and} \quad \frac{\partial u}{\partial y_k} = -\frac{\partial v}{\partial x_k}\,,\ \forall\, k \in \{1, \dots, n\} \end{aligned}$$

2.2 Power series in n variables

The condition for a complex function in more variables to be analytic in some open set U is more difficult to formulate since we first need to define multi-index power series.
Let $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ and $a_\alpha = a_{\alpha_1, \dots, \alpha_n} \in \mathbb{C}$. The *length* of α is $|\alpha| := \sum_{k=1}^{n} \alpha_i$.

If $z^\alpha := z_1^{\alpha_1} \cdot z_2^{\alpha_2} \cdot \ldots \cdot z_n^{\alpha_n}$, then a *formal power series* in n variables can be written as

$$g(z) = g(z_1, \ldots, z_n) = \sum_{\alpha, |\alpha|=0}^{\infty} a_\alpha \cdot z^\alpha = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} a_\alpha \cdot z^\alpha \tag{2.1}$$

where the sum $\sum\limits_{|\alpha|=k}$ is finite since $\alpha \in \mathbb{N}_0^n$, i.e. only finitely many α can satisfy $|\alpha| = k$.

2.2.1 Example

Let $n = 2$ and consider only $0 \leq k \leq 2$, then

$$\begin{aligned}\sum_{k=0}^{2} \sum_{|\alpha|=k} a_\alpha \cdot z^\alpha &= a_{00} z_1^0 z_2^0 + a_{10} z_1^1 z_2^0 + a_{01} z_1^0 z_2^1 + a_{20} z_1^2 z_2^0 + a_{11} z_1^1 z_2^1 + a_{02} z_1^0 z_2^2 \\ &= a_{00} + a_{10}\, z_1 + a_{01}\, z_2 + a_{20}\, z_1^2 + a_{11}\, z_1\, z_2 + a_{02}\, z_2^2\end{aligned}$$

Saying that the power series is formal just means that we do not worry about convergence problems for the moment. If all a_α are zero except finitely many, we obtain a usual polynomial in n variables. And for polynomials we can plug in values for $z = (z_1, \ldots, z_n)$ since the sum is finite, thus always converges.

2.2.2 Problem

Consider power series of the form $\sum\limits_{|\alpha|=0}^{\infty} a_\alpha \cdot (z - w)^\alpha$.
Recall that convergence of a power series in 1 variable is defined as the convergence of the sequence of partial sums. This definition is however not possible if $n > 1$ since there is natural linear ordering defined on $\mathbb{N}_0^n$; the "partial sums" in (2.1) are not indexed, thus there is no "sequence" which could converge in the usual sense.

2.2.3 Definition : convergence for $n > 1$

Let $z \in \mathbb{C}^n$.
The power series $\sum\limits_{|\alpha|=0}^{\infty} a_\alpha \cdot z^\alpha$ converges to $c \in \mathbb{C} \Leftrightarrow \forall \varepsilon > 0$, there exist a finite index set $I_0 \subset \mathbb{N}_0^n$ such that

$$\text{for all finite set } I \text{ satisfying } I_0 \subseteq I \subset \mathbb{N}_0^n, \text{ we have : } \left| \sum_{\alpha \in I} a_\alpha \cdot z^\alpha - c \right| < \varepsilon$$

In this case we denote $c := \lim \sum a_\alpha\, z^\alpha$ and this limit c is always unique. Moreover this convergence is an absolute convergence, i.e. if $\sum a_\alpha\, z^\alpha \to c$, then $\sum |a_\alpha|\, z^\alpha$ also converges, but not necessarily to $|c|$.

2.3 Analyticity

2.3.1 Definition

Let $U \neq \emptyset$ be open in $\mathbb{C}^n$ and $f : U \to \mathbb{C}$ be a complex function on U.
f is called *complex analytic* in $U \Leftrightarrow \forall\, w \in U$, there exist a neighborhood $U_w \subseteq U$ of w and a power series

$$\sum_{|\alpha|=0}^{\infty} a_\alpha \cdot (z - w)^\alpha \quad \text{where } (z - w)^\alpha = (z_1 - w_1)^{\alpha_1} \cdot (z_2 - w_2)^{\alpha_2} \cdot \ldots \cdot (z_n - w_n)^{\alpha_n}$$

which converges to $f(z)$ for all $z \in U_w$. This means that f can locally be written at any point as a power series which converges in a certain neighborhood of this point.

f is called *real analytic* in $U \Leftrightarrow$ on U, it can locally be expanded as a power series in z and $\bar{z}$.
Obviously : f complex analytic $\Rightarrow$ f real analytic.

2.3.2 Examples

- $f(z) = z^2$: f is a power series as polynomial ; it is complex analytic since it only depends on z
- $g(z) = |z|^2 = z\bar{z}$: g is real analytic, but not complex analytic

2.3.3 Theorem

Let f be a complex function defined on some open set U in $\mathbb{C}^n$.
Then f is holomorphic in U $\Leftrightarrow$ f is complex analytic in U.

2.4 Identity Theorem

Let $A \subseteq \mathbb{C}$ and $a \in \mathbb{C}$. We say that a is an *accumulation point of* A if any open set $U \subseteq \mathbb{C}$ containing a intersects A in some point distinct than a. By taking $U = D(a, \frac{1}{n})$ with $n \to +\infty$, this is equivalent to :

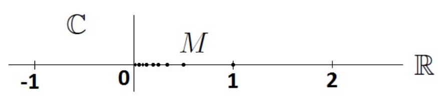
$$a \text{ accumulation point of } A \;\Leftrightarrow\; \exists \text{ sequence } (a_n)_n \text{ with } a_n \in A,\; a_n \neq a,\; \forall\, n, \text{ such that } \lim_{n\to+\infty} a_n = a$$

In particular, the condition $a_n \neq a, \forall\, n$ implies that A must necessarily be infinite.
If $B \subseteq \mathbb{C}$, we say that A *has an accumulation point in* B if $\exists\, z_0 \in B$ such that z_0 in an accumulation point of A.

2.4.1 Theorem ($n = 1$)

Let $U \subseteq \mathbb{C}$ be open, connected and $U \neq \emptyset$ and let $M \subseteq U$ be such that M has an accumulation point in U.
Let $f, g : U \to \mathbb{C}$ be holomorphic in U such that $f_{|_M} = g_{|_M}$. Then $f = g$ on U.

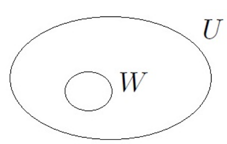
Example :



If 2 globally holomorphic functions $f, g : \mathbb{C} \to \mathbb{C}$ are equal on $M = \{\, \frac{1}{n} \mid n \in \mathbb{N} \,\}$, then they have to coincide everywhere on $\mathbb{C}$ as well.

2.4.2 Theorem ($n > 1$)

Let $U \subseteq \mathbb{C}^n$ be open, connected and $U \neq \emptyset$. Let $W \subseteq U$ be such that $W \neq \emptyset$ and W is open.



If $f, g : U \to \mathbb{C}$ are holomorphic in U such that $f_{|_W} = g_{|_W}$, then $f = g$ on U.

2.4.3 Remark

The condition "$W \subseteq U$ open" is stronger than the condition "$M \subseteq U$ has an accumulation point in U".
Indeed : $\forall\, w \in W$, $\exists\, \varepsilon_w > 0$ such that $w \in D(w, \varepsilon_w) \subset W \subseteq U$ since W is open. Thus one can define a sequence of distinct points in this open disc converging to the center w $\Rightarrow$ w is an accumulation point of W in U.
In fact, the condition about having an accumulation point is not sufficient for $n > 1$:

Consider $\mathbb{C}^2$ with $M = \big\{\, (z_1, z_2) \in \mathbb{C}^2 \mid z_2 = 0 \,\big\}$ $\Rightarrow$ every point in M is an accumulation point of M in $\mathbb{C}^2$.
Let $g, h : \mathbb{C}^2 \to \mathbb{C}$ be holomorphic functions on $\mathbb{C}^2$ such that $g \neq h$ outside of M (this always exists) and define

$$f_1(z_1, z_2) := z_2 \cdot g(z_1, z_2) \qquad , \qquad f_2(z_1, z_2) := z_2 \cdot h(z_1, z_2)$$

Then f_1, f_2 are holomorphic on $\mathbb{C}^2$ with $f_1 = f_2 = 0$ on M, but $f_1 \neq f_2$ on $\mathbb{C}^2$ since $g \neq h$.

2.4.4 Identity Theorem for power series

If $\sum a_\alpha\, z^\alpha$ and $\sum b_\alpha\, z^\alpha$ represent the same holomorphic function, then $a_\alpha = b_\alpha$, $\forall\, \alpha \in \mathbb{N}_0^n$.
Thus : if $\exists\, \alpha_0 \in \mathbb{N}_0^n$ such that $a_{\alpha_0} \neq b_{\alpha_0}$, then the corresponding power series define different holomorphic maps.

2.5 Maximum Principle

2.5.1 Recall

Let $V \subseteq \mathbb{R}^m$ be open. For a function $h : V \to \mathbb{C}$ of real variables $x_1, \ldots, x_m$, the *Laplacian* is defined as

$$\Delta h := \sum_{i=1}^{m} \frac{\partial^2 h}{\partial x_i^2} = \frac{\partial^2 h}{\partial x_1^2} + \frac{\partial^2 h}{\partial x_2^2} + \ldots + \frac{\partial^2 h}{\partial x_m^2}$$

h is called *harmonic* on V if $\Delta h(x) = 0$, $\forall\, x \in V$.

Let now $f : U \to \mathbb{C}$ be a holomorphic function on an open set $U \subseteq \mathbb{C}^n$ and consider the identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$, i.e. we identify $f(z_1, \ldots, z_n) \equiv f(x_1, y_1, \ldots, x_n, y_n)$. Then f, $\operatorname{Re} f$ and $\operatorname{Im} f$ are harmonic on U (as functions of $2n$ real variables).

2.5.2 Theorem

Let $U \neq \emptyset$ be open in $\mathbb{C}^n$ and $f : U \to \mathbb{C}$ be holomorphic in U. Then the absolute value function

$$|f| \;:\; U \to \mathbb{R} \;:\; z \mapsto |f(z)| = \sqrt{f(z) \cdot \overline{f(z)}}$$

is in general not holomorphic or harmonic (unless f is constant), but $|f|$ is continuous and real analytic in U. Moreover $|f|$ does not have a maximum in U.

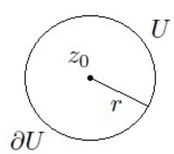
If U is bounded, this means that its maximum "lies on the boundary" of U, given by $\partial U := \overline{U} \cap \overline{(\mathbb{C}^n \setminus U)}$.

2.5.3 Example

Consider the case of an open disc $U = D(z_0, r)$ with center $z_0 \in \mathbb{C}^n$ and radius $r > 0$ as in figure 2.1.

If $|f|$ is continuous on the compact set $\bar{U} = \bar{D}(z_0, r)$, we already know that it must have a maximum in $\bar{U}$. The theorem says that this maximum has to be on the boundary ∂U of $\bar{U}$.

Figure 2.1: the open disc $U = D(z_0, r)$ with boundary ∂U



2.6 Hartog's Lemma

2.6.1 Recalls

$n = 1$: Let $U \subseteq \mathbb{C}$ be open and convex with $p \in U$.

If $f : U \to \mathbb{C}$ is holomorphic in $U \setminus \{p\}$ and continuous in U (in particular at p), then f is holomorphic in U.

If f is holomorphic in $U \setminus \{p\}$ and bounded in a neighborhood of p (without being assumed to be defined or continuous at p), then f extends uniquely to a holomorphic function on U, hence f does not have a pole at p.

2.6.2 Definition

Let $w = (w_1, w_2, \ldots, w_n) \in \mathbb{C}^n$ and $r \in \mathbb{R}$, $r > 0$.

A *polydisc* of radius r centered at w, denoted $\Delta(w, r)$, is equal to the Cartesian product of discs

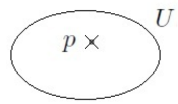
$$\begin{aligned}\Delta(w, r) &= D(w_1, r) \times D(w_2, r) \times \ldots \times D(w_n, r) \\ &= \left\{ (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \;,\; |z_1 - w_1| < r,\, |z_2 - w_2| < r,\, \ldots,\, |z_n - w_n| < r \right\}\end{aligned}$$

It should not be confused with the usual disc $D(w, r) = \left\{ z \in \mathbb{C}^n \;,\; \|z - w\| < r \right\} \subseteq \mathbb{C}^n$.

2.6.3 Hartog's Lemma

Let $n \geq 2$, $U \subseteq \mathbb{C}^n$ be open and $p \in U$. Assume that f is holomorphic in $U \setminus \{p\}$ (see figure 2.2). Then f extends uniquely to a holomorphic function on U (even if f may a priori not be continuous or bounded at p).

Figure 2.2: f is not holomorphic at p



Proof. The extension is necessarily unique because $U \setminus \{p\}$ is open in U. If f has 2 holomorphic extensions, these will coincide on $U \setminus \{p\}$ (as extensions of f), thus by the Identity Theorem 2.4.2 they are also equal on U.

It suffices to prove the statement for $n = 2$ and for $p = (0,0) \in \mathbb{C}^2$ by applying a shift (which is a holomorphic transformation).
Moreover since we work locally around p, it suffices to extend f to a holomorphic function in a small open neighborhood V of p and we may assume that V is a polydisc centered at p of small enough radius. Hence

$$p = (0,0) \in V \;\Rightarrow\; \exists\, r > 0 \text{ such that } V = \Delta\big((0,0), r\big) \subset U$$

f is well-defined and holomorphic on $U \setminus \big\{(0,0)\big\}$, in particular it is holomorphic on the boundary of V. Define

$$F(z_1, z_2) := \frac{1}{2\pi i} \cdot \int_{|w_2|=r} \frac{f(z_1, w_2)}{w_2 - z_2}\, dw_2 \quad , \; \forall\, (z_1, z_2) \in V$$

where z_1 is a parameter in the integral. F is well-defined since $|w_2| = r > 0 \;\Rightarrow\; (z_1, w_2) \neq (0,0)$.
Moreover $(z_1, z_2) \in V \;\Rightarrow\; |z_2| < r$, hence $w_2 - z_2 \neq 0$, so numerator and denominator are well-defined.

Thus $\forall\, w_2 \in \mathbb{C}$ such that $|w_2| = r$, the map

$$(z_1, z_2) \longmapsto g(z_1, z_2, w_2) := \frac{f(z_1, w_2)}{w_2 - z_2}$$

is holomorphic in V. It remains to check that F is holomorphic in V as well. Using the complex version of differentiation of parameter-dependent integrals, we find

$$\begin{aligned} &z_1 \mapsto g(z_1, z_2, w_2) \text{ is holomorphic in } D(0,r),\; \forall\, z_2, w_2 \text{ such that } |z_2| < r,\; |w_2| = r \\ &z_2 \mapsto g(z_1, z_2, w_2) \text{ is holomorphic in } D(0,r),\; \forall\, z_1, w_2 \text{ such that } |z_1| < r,\; |w_2| = r \\ &w_2 \mapsto g(z_1, z_2, w_2) \text{ is continuous on } \partial D(0,r),\; \forall\, z_1, z_2 \text{ such that } |z_1| < r,\; |z_2| < r \end{aligned}$$

Moreover the partial derivatives $\frac{\partial g}{\partial z_1}$, $\frac{\partial g}{\partial z_2}$, $\frac{\partial g}{\partial \bar{z}_1}$ and $\frac{\partial g}{\partial \bar{z}_2}$ are continuous in all variables (whenever defined), hence

$$\frac{\partial F}{\partial \bar{z}_1}(z_1, z_2) = \frac{1}{2\pi i} \cdot \int_{|w_2|=r} \frac{\partial}{\partial \bar{z}_1} \left(\frac{f(z_1, w_2)}{w_2 - z_2} \right) dw_2 = \frac{1}{2\pi i} \cdot \int_{|w_2|=r} \frac{\frac{\partial f}{\partial \bar{z}_1}(z_1, w_2)}{w_2 - z_2}\, dw_2 = 0$$
$$\frac{\partial F}{\partial \bar{z}_2}(z_1, z_2) = \frac{1}{2\pi i} \cdot \int_{|w_2|=r} \frac{\partial}{\partial \bar{z}_2} \left(\frac{f(z_1, w_2)}{w_2 - z_2} \right) dw_2 = \frac{1}{2\pi i} \cdot \int_{|w_2|=r} f(z_1, w_2) \cdot \frac{\partial}{\partial \bar{z}_2} \left(\frac{1}{w_2 - z_2} \right) dw_2 = 0$$

since $z_2 \mapsto \frac{1}{w_2 - z_2}$ is holomorphic $\forall\, w_2 \in \partial D(0,r)$ and f is holomorphic in all variables with $(z_1, w_2) \neq (0,0)$.

Finally F is holomorphic in both variables z_1 and z_2 in the whole polydisc, i.e. we constructed a function which is holomorphic in V. In particular we can plug in

$$F(0,0) = \frac{1}{2\pi i} \cdot \int_{|w_2|=r} \frac{f(0, w_2)}{w_2}\, dw_2 \;\in \mathbb{C}$$

To show that F is an extension of f on V, we need that $F_{|_{V\setminus\{p\}}} = f_{|_{V\setminus\{p\}}}$. Fix some $r_1 \in \,]0, r[$ and define

$$V' := V \cap \left\{ (z_1, z_2) \in \mathbb{C}^2 \ , \ |z_1| > r_1 \right\}$$

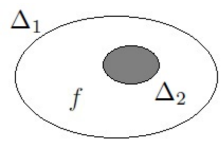
In order to use Cauchy's integral formula, we have to exclude $z_1 = 0$, otherwise the map $w_2 \mapsto f(0, w_2)$ is not holomorphic in $\bar{D}(0, r)$. Hence $\forall\, (z_1, z_2) \in V'$:

$$F(z_1, z_2) = \frac{1}{2\pi i} \cdot \int_{|w_2|=r} \frac{f(z_1, w_2)}{w_2 - z_2}\, dw_2 = f(z_1, z_2) \quad \text{with } (z_1, z_2) \neq (0, 0)$$

So $F = f$ on V' with F and f holomorphic in $V \setminus \{p\}$. But $V' \subset V \setminus \{p\}$ is open, hence by the Identity Theorem $F = f$ on $V \setminus \{p\}$. So F is the required holomorphic extension of f on V, thus on U. □

2.6.4 Remark

As a consequence of Hartog's lemma we obtain that the singularities of holomorphic functions in several variables cannot be isolated points (since it is possible to extend the function in this case).
It can even be shown that such an extension exists if f is holomorphic in some polydisc Δ_1 without knowing its behaviour in a smaller polydisc Δ_2 and $\Delta_1 \setminus \Delta_2$ is still connected.



In this case f can also be uniquely extended to a holomorphic function in the whole big polydisc Δ_1.

This does not hold for arbitrary sets where a function is not defined. Let e.g. $n = 2$ and consider $f(z_1, z_2) = \frac{1}{z_1}$. f is defined and holomorphic on $\mathbb{C} \setminus N$ where $N = \{ (0, z_2) \mid z_2 \in \mathbb{C} \}$. Since N has however empty interior, the set of singularities N does not contain any polydisc and the remark does not apply in this case.

2.6.5 Counter–examples

The condition $n \geq 2$ is necessary.
Consider e.g. the function $z \mapsto \frac{1}{z}$ on $U = \mathbb{C}$ with $p = 0$. It cannot be extended to $\mathbb{C}$ since it is not bounded at 0.

For $n \geq 2$, it is also important that the function is holomorphic, e.g. if $n = 2$, the map

$$z = (z_1, z_2) \longmapsto \frac{1}{\|z\|^2} = \frac{1}{|z_1|^2 + |z_2|^2}$$

is defined on $\mathbb{C}^n \setminus \{0\}$ but, since it is not holomorphic, cannot be extended to a holomorphic function on $\mathbb{C}^n$.

Chapter 3

Real differentiable manifolds

3.1 Introduction

intuitive idea : Real manifolds locally look like $\mathbb{R}^n$.

There are 2 possible approaches in order to define manifolds :

- start with an arbitrary set M, define charts domains and coordinate maps φ_i such that the transition functions satisfy certain properties and construct a topology on M using the atlas of coordinate maps
- start with a given topological space $(M, \mathcal{T})$ and a collection of open sets U_i and homeomorphisms φ_i

We choose the second approach in the sequel.
Let $(M, \mathcal{T})$ be a topological space and $\{U_i\}_{i \in J}$ be a family of non-empty open subsets of M such that

$$M = \bigcup_{i \in J} U_i$$

i.e. the U_i are an open covering of the space M. Let also be given a collection of maps (see figure 3.1)

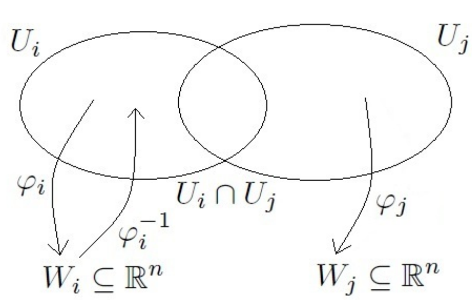
$$\forall\, i \in J,\ \varphi_i \ : \ U_i \to W_i \subseteq \mathbb{R}^n$$

where every φ_i is a topological isomorphism : φ_i and φ_i^{-1} are both bijective and continuous (one also says that the φ_i are *bicontinuous* or *homeomorphisms*). Hence all W_i are open in $\mathbb{R}^n$ and the transition maps

$$\psi_{ji} := \varphi_j \circ \varphi_i^{-1} \ : \ \varphi_i(U_i \cap U_j) \subseteq \mathbb{R}^n \longrightarrow \varphi_j(U_j \cap U_i) \subseteq \mathbb{R}^n$$

are also homeomorphisms (whenever defined).

Figure 3.1: two overlapping coordinate patches



A pair (U_i, φ_i) is called a *coordinate patch* of M and the whole collection $(U_i, \varphi_i)_{i \in J}$ is called an *atlas* of M. Since the transition maps ψ_{ji} are all continuous, one says that M is a *topological manifold*.

3.1.1 Definition

A topological manifold M is called a *(real) differentiable manifold* if the transition maps ψ_{ji} are differentiable, i.e. infinitely often differentiable (of class C^∞), $\forall\,(i,j) \in J \times J$.
Since $\psi_{ji}^{-1} = \psi_{ij}$, this immediately implies that the transition functions are diffeomorphisms.

One can also impose weaker conditions, e.g. the ψ_{ji} should only be C^1, $C^2 \ldots$, or stronger conditions, for instance that the ψ_{ji} have to be real analytic. In this case M would be a *C^k–manifold*, resp. a *real analytic manifold*.
A topological manifold is *of dimension n* if $\varphi_i : U_i \to W_i \subseteq \mathbb{R}^n$, $\forall\, i \in J$.

3.2 Topological properties of manifolds

In order to exclude "exotic" manifold structures, one often requires that :
− M is Hausdorff
− M is connected
− M is paracompact
Unless explicitly mentioned, we always assume that these 3 conditions are satisfied in a differentiable manifold.

3.2.1 Definitions

Let $(M, \mathcal{T})$ be a topological space.
M is called *Hausdorff* if any 2 distinct points in M can be separated, i.e.

$$\forall\, x, y \in M,\ x \neq y\ :\ \exists\, U_x, U_y \text{ open neighborhoods of } x, y \text{ such that } U_x \cap U_y = \emptyset$$

M is called *regular* if any point and a closed set not containing that point can be separated, i.e.

$$\forall\, x \in M,\ \forall\, F \subseteq M \text{ closed such that } x \notin F\ :\ \exists\, U_x, V_F \text{ open such that } x \in U_x,\ F \subseteq V_F,\ U_x \cap V_F = \emptyset$$

M is called *normal* if any 2 disjoint closed sets can be separated by open neighborhoods.

Let $x \in M$. A *basis of neighborhoods* $\mathcal{B}_x$ of x is a family of neighborhoods of x such that for any neighborhood U_x of x, there is a neighborhood $V_x \in \mathcal{B}_x$ such that $x \in V_x \subseteq U_x$. A *basis* $\mathcal{B}$ of M is a collection of open sets such that any open set in M can be written as a union of open sets from $\mathcal{B}$.
M is called *first-countable* if every point in M has a countable basis of neighborhoods. M is called *second-countable* if it has a countable basis.

M is called *connected* $\Leftrightarrow$ $(M = U \cup V$ for some open sets U, V with $U \cap V = \emptyset \ \Rightarrow\ U = \emptyset$ or $V = \emptyset)$, i.e. M is connected if it cannot be written as a disjoint union of 2 non-empty open subsets. An open set $U \subseteq M$ is connected if it is connected with respect to the induced topology.
M is said to be *path-connected* if $\forall\, p, q \in M$, there is a continuous map $\gamma : [0,1] \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$ (see figure 3.2). Such a map γ is called a *path* from p to q.
M is called *locally connected* if every point in M admits a basis of connected neighborhoods and it is *locally path-connected* if every point in M admits a basis of path-connected neighborhoods.

Figure 3.2: γ is a path from p to q

Let now $\mathcal{U} \subseteq \mathcal{T}$ be a family of open subsets of M.
$\mathcal{U}$ is called *pointwise finite* if $\forall\, x \in M$, x only belongs to a finite number of sets in $\mathcal{U}$. $\mathcal{U}$ is called *locally finite* if $\forall\, x \in M$, there is a neighborhood V_x of x such that V_x intersects only a finite number of sets in $\mathcal{U}$.
Assume now that $\mathcal{U}$ is an open covering of M. Another family of subsets $\mathcal{V} \subseteq \mathcal{T}$ is called a *refinement* of $\mathcal{U}$ if it is again an open covering of M and if $\forall\, V \in \mathcal{V}$, $\exists\, U \in \mathcal{U}$ such that $V \subseteq U$.

M is called a *Lindelöf space* if every open covering of M has a countable subcover. M is called *compact* if every open covering of M contains a finite subcover.
M is called *locally compact* if every point in M admits a compact neighborhood. And finally M is called *paracompact* if it is Hausdorff and if every open covering of M has a locally finite refinement.
We also recall that continuous images of compact sets are again compact, i.e.

$$K \subset M \text{ compact}, \ f : M \to N \text{ continuous} \quad \Rightarrow \quad f(K) \subset N \text{ compact}$$

3.2.2 Results

Let $(M, \mathcal{T})$ be a topological space. One can show the following properties, which we are not going to prove :
– M compact $\Rightarrow$ M paracompact (since any finite covering is a locally finite covering)
– M Hausdorff, second-countable and locally compact $\Rightarrow$ M paracompact
– M second-countable $\Rightarrow$ M first-countable and Lindelöf
– M regular and Lindelöf $\Rightarrow$ M paracompact
– M paracompact and Hausdorff $\Rightarrow$ M regular and normal
– M path-connected $\Rightarrow$ M connected
– M connected and locally path-connected $\Rightarrow$ M path-connected
– If M is locally path-connected, then connectedness and path-connectedness are equivalent.
– Any Hausdorff and second-countable manifold is locally compact, locally connected and locally path-connected.

Conclusion :
Any topological manifold which is Hausdorff and second-countable is also paracompact. Moreover connectedness and path-connectedness coincide for differentiable manifolds of our purpose (see section 3.2).
The fact that we consider connected differentiable manifolds M ensures that the dimension of M is well-defined. Otherwise we have to consider each connected component of M separately.

3.2.3 Example

Let M be a manifold and $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering : $M = \bigcup_{i\in I} U_i$.
Again, a refinement of $\mathcal{U}$ is an open covering $\mathcal{V} = \{V_j\}_{j\in J}$ such that $M = \bigcup_{j\in J} V_j$ and

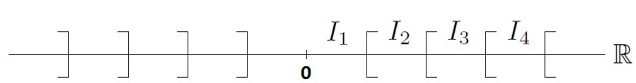
$$\forall\, j \in J, \ \exists\, i \in I \text{ (not necessarily unique) such that } V_j \subseteq U_i$$

Being a Hausdorff and second-countable manifold, we can say that $\mathbb{R}$ is paracompact. As an example, let

$$I_n = \,]-n, n[\quad \Rightarrow \quad \mathbb{R} = \bigcup_{n\in\mathbb{N}} I_n \ \text{ : open covering (see figure 3.3)}$$

But this covering is not locally finite because $\bigcap_{n\in\mathbb{N}} I_n = \{0\}$: $0 \in I_n$, $\forall\, n \in \mathbb{N}$, hence also every neighborhood of 0 intersects infinitely many I_n.

Figure 3.3: the intervals I_n cover $\mathbb{R}$



As long as we have infinitely many I_n necessary to cover $\mathbb{R}$, the cover is not locally finite at any point : every point of $\mathbb{R}$ belongs to infinitely many I_n. But we need infinitely many to cover $\mathbb{R}$. So consider

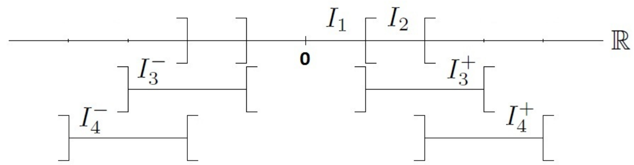
$$I_n^- := \,]-n, -n+2[\, \subseteq I_n \qquad , \qquad I_n^+ := \,]n-2, n[\, \subseteq I_n$$

$\Rightarrow$ $\{\, I_n \mid n \in \mathbb{N}\,\} \cup \{\, I_n^+, I_n^- \mid n \geq 3\,\}$ is again a covering and a refinement of the previous one (even if it contains more sets than before, see definition). And now one can take the subcover (see figure 3.4)

$$\{\, I_1, I_2, I_n^+, I_n^- \mid n \geq 3\,\}$$

It still covers $\mathbb{R}$ and is locally finite at 0 since $0 \in I_1, I_2$ only. And it is also locally finite at any other point of $\mathbb{R}$ since the intervals I_n^+, I_n^- "drift off" to $\pm\infty$.

Figure 3.4: a locally finite covering of $\mathbb{R}$



Note that this is not a proof of the fact that $\mathbb{R}$ is paracompact. In order to show this we need to show that **every** open cover of $\mathbb{R}$ has a locally finite refinement.

3.3 Differentiable functions

3.3.1 Partition of unity

Let M be a topological space and $\mathcal{U} = \{U_i\}_{i\in J}$ be an arbitrary open covering of M. A family of continuous functions $\{\tau_i\}_{i\in J}$ where $\tau_i : M \to [0,1]$, $\forall\, i \in J$ is called a *partition of unity* if

1) $\{\tau_i\}_{i\in J}$ is locally finite, i.e. $\forall\, x \in M$, there is an open neighborhood W of x such that $\tau_{i|_W} = 0$ for all but at most finitely many $i \in J$

2) $\forall\, x \in M : \sum_{i\in J} \tau_i(x) = 1$, which makes sense by 1) since it is a finite sum whenever x is given

A partition of unity is called *subordinated* to $\mathcal{U}$ if $\operatorname{supp} \tau_i = \overline{\{\, x \in M \mid \tau_i(x) \neq 0 \,\}} \subseteq U_i$, $\forall\, i \in J$.

Partitions of unity can be used for "gluing" local objects.
Let again M be a manifold and $\mathcal{U} = \{U_i\}_{i\in J}$ be an open covering of M. Let ψ_i be local objects defined on U_i (e.g. local functions) such that $\psi_i = \psi_j$ on $U_i \cap U_j$. If there exist a partition of unity $\tau_i : M \to [0,1]$ subordinated to $\mathcal{U}$, one can define a global object ψ by setting

$$\psi(x) := \sum_{j\in J} \tau_j(x) \cdot \psi_j(x)$$

Then $\psi_{|_{U_i}} = \psi_i$ since $x \in U_i \;\Rightarrow\; \psi_j(x) = \psi_i(x)$, $\forall\, j$, thus $\psi(x) = \sum_{j\in J} \tau_j(x) \cdot \psi_i(x) = \psi_i(x) \cdot \sum_{j\in J} \tau_j(x) = \psi_i(x) \cdot 1$.

3.3.2 Theorem

Let M be a Haudorff topological space. Then
M is paracompact $\Leftrightarrow$ for any open covering of M there exist a partition of unity subordinated to this covering.

3.3.3 Definition

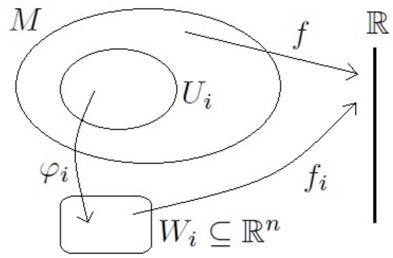
Let M be a differentiable manifold with atlas $(U_i, \varphi_i)_{i\in I}$. A map $f : M \to \mathbb{R}$ is called a *differentiable* function if and only if $f_i := f \circ \varphi_i^{-1} : W_i \subseteq \mathbb{R}^n \to \mathbb{R}$ is a differentiable map, $\forall\, i \in I$ (see figure 3.5).

Strictly speaking, this condition is not really needed. It already suffices the local condition : $\forall\, x \in M$, $\exists\, U \subset M$ open neighborhood of x and $\exists\,(U, \varphi)$ chart of M at x such that $f \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \mathbb{R}$ is differentiable.

3.3.4 Bump functions

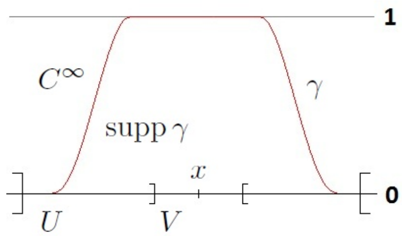
Proposition : Every differentiable manifold has a differentiable partition of unity, called the bump functions. (This does not mean that every partition of unity in a differentiable manifold is differentiable.)

Figure 3.5: composition of the functions f and φ_i^{-1} to give a map $f_i : \mathbb{R}^n \to \mathbb{R}$



Let $x \in M$ and U be an open neighborhood of x. A *bump function* γ is a smooth map $\gamma : M \to [0,1]$ with support contained in U and taking value 1 in a neighborhood V of x. It can be visualized as in figure 3.6.

Figure 3.6: a bump function with support in U

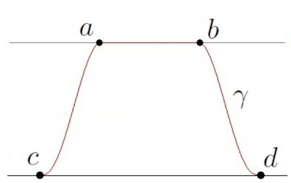


But there does not necessarily exist an analytic partition of unity since these bump functions are constructed by using manipulations and transformations of the map

$$g(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \quad \Rightarrow \; g^{(n)}(0) = 0, \; \forall\, n \in \mathbb{N}$$

Consider small neighborhoods of the points a, b, c, d in figure 3.7 where the bump function γ begins to "go up" and "go down". All the derivatives vanish at these points, so the Taylor series of γ will be constant in any neighborhood of them. Thus γ cannot be analytic since it does not coincide with its Taylor series in a neighborhood of these points.
More generally, bump functions have compact support, but compactly supported functions are never analytic.

Figure 3.7: bump functions are not real analytic



3.4 Orientability of differentiable manifolds

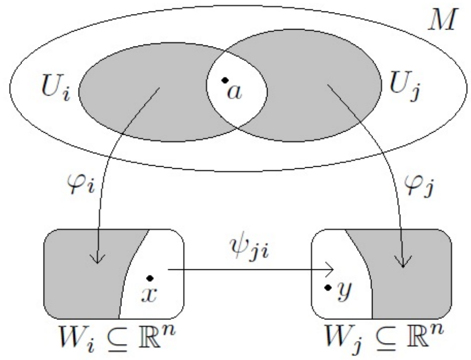
Consider figure 3.8. Let M be a differentiable manifold with atlas $(U_i, \varphi_i)_{i \in I}$. Since all φ_i are bijective, we have

$$\forall\, x \in W_i = \varphi_i(U_i), \; \exists!\, a \in U_i \text{ such that } \varphi_i(a) = x = (x_1, \ldots, x_n)$$

The n-tuple $(x_1, \ldots, x_n)$ represents the *coordinates* of $a \in U_i$ in the considered chart or coordinate system.
These coordinates can however change by using another coordinate chart. If $\exists\, j \neq i$ such that $a \in U_i \cap U_j$, then $\varphi_j(a) = y = (y_1, \ldots, y_n) \in W_j = \varphi_j(U_j) \subseteq \mathbb{R}^n$ as well. The coordinates y and x are then related by the relation

$$y = \psi_{ji}(x) = (\varphi_j \circ \varphi_i^{-1})(x)$$

Figure 3.8: coordinate change in 2 overlapping charts



The ψ_{ji} are thus also called *transition functions* or *coordinate change maps*.
y being a function of x, one can now consider the Jacobian matrix

$$J = J(\psi_{ji}) = \left(\frac{\partial y_k}{\partial x_l}\right)_{k,l} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \dots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \dots & \frac{\partial y_n}{\partial x_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1}(x) & \frac{\partial y_1}{\partial x_2}(x) & \dots & \frac{\partial y_1}{\partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1}(x) & \frac{\partial y_n}{\partial x_2}(x) & \dots & \frac{\partial y_n}{\partial x_n}(x) \end{pmatrix} = J(x)$$

J is a matrix depending on the point $x \in \varphi_i(U_i \cap U_j) \subseteq \mathbb{R}^n$ where it is computed.
Since the transition maps ψ_{ji} are bijective, we have that $\det\big(J(x)\big) \neq 0$, $\forall\, x \in \varphi_i(U_i \cap U_j)$.

3.4.1 Definition

A manifold is called *orientable* $\Leftrightarrow$ it admits an atlas $(U_i, \varphi_i)_{i\in I}$ such that $\det\big(J(\psi_{ji})\big) > 0$, $\forall\, i, j \in I$.
Note that this is an inequality of functions, i.e. for given i, j the relation must hold for any point $x \in \varphi_i(U_i \cap U_j)$.

Fact :
Complex manifolds are always orientable (proof, see section 4.2.3).
Examples of non-orientable manifolds are the Möbius strip and the real projective plane $\mathbb{RP}^2$. Hence by the above fact it is not possible to endow these manifolds with a complex manifold structure.

3.4.2 Formulation with differential forms

Consider the local coordinates $x_1, x_2, \dots, x_n$ with the standard orientation (order) and let $y_1, y_2, \dots, y_n$ be the coordinates after the coordinate change. In order to determine the orientation of the coordinate change, we have to compare the orientation of $dx_1 \wedge dx_2 \wedge \dots \wedge dx_n$ and $dy_1 \wedge dy_2 \wedge \dots \wedge dy_n$. We have :

$$\begin{aligned} &\forall\, k \in \{1, \dots, n\} \ : \ \psi_{ji}^*(dy_k) = \sum_{l=1}^n \frac{\partial(\psi_{ji})_k}{\partial x_l} \cdot dx_l = \sum_{l=1}^n \frac{\partial y_k}{\partial x_l} \cdot dx_l \\ \Rightarrow\ &\psi_{ji}^*(dy_1 \wedge dy_2 \wedge \dots \wedge dy_n) = \det\big(J(\psi_{ji})\big) \cdot dx_1 \wedge dx_2 \wedge \dots \wedge dx_n \qquad (3.1) \end{aligned}$$

where ψ_{ji}^* denotes the pull-back of a differential form under the diffeomorphism ψ_{ji}.
Hence by (3.1) the orientation of both differential forms coincide $\Leftrightarrow$ $\det\big(J(\psi_{ji})\big) > 0$.

3.4.3 Example

Consider the n-dimensional sphere S^n given by

$$S^n = \big\{\, x = (x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \ ,\ \|x\|^2 = 1 \,\big\} \subset \mathbb{R}^{n+1}$$

S^n is compact in R^{n+1} as it is closed and bounded, hence it is paracompact. Moreover $\mathbb{R}^{n+1}$ is Hausdorff and second-countable, thus S^n is also Hausdorff and second-countable since these properties are hereditary.

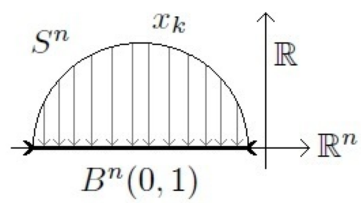
We want to endow S^n with a differentiable manifold structure. For this, consider the hemispheres

$$U_{(k,j)} := \{\, x \in S^n \mid (-1)^j \cdot x_k > 0 \,\}, \quad j = 0,1,\ k = 1,2,\dots,n+1$$

which are open in $\mathbb{R}^{n+1}$ and in S^n. They form an open covering of S^n (the case $n = 1$ is given in figure 3.9), called the *standard covering*. As charts, we take

$$h_{(k,j)} \,:\, U_{(k,j)} \longrightarrow B^n(0,1) \subset \mathbb{R}^n \,:\, (x_1,\dots,x_k,\dots,x_{n+1}) \longmapsto (x_1,\dots,x_{k-1},x_{k+1},\dots,x_{n+1})$$

This just corresponds to dropping the k^{th} position and hence projecting the point on the sphere down into the ball $B^n(0,1) = \{\, y \in \mathbb{R}^n \,,\ \|y\|^2 < 1 \,\}$. Intuitively, this projection corresponds to a flattening of the sphere.



Moreover $h_{(k,j)}$ is well-defined because

$$\sum_{\substack{l=1\\ l\neq k}}^{n+1} x_l^2 \;<\; \sum_{l=1}^{n+1} x_l^2 = 1 \quad\Rightarrow\quad (x_1,\dots,x_{k-1},x_{k+1},\dots,x_{n+1}) \in B^n(0,1)$$

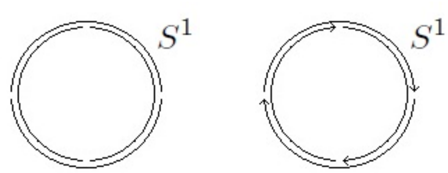
as $x_k > 0$ for $j = 0$ or $x_k < 0$ for $j = 1$. $h_{(k,j)}$ is in addition bijective and its inverse map is equal to

$$h^{-1}_{(k,j)} \,:\, B^n(0,1) \longrightarrow U_{(k,j)} \,:\, (y_1,\dots,y_k,\dots,y_n) \longmapsto \Big(y_1\,,\,\dots\,,\,y_{k-1}\,,\,(-1)^j\cdot\sqrt{1-\sum_{i=1}^n y_i^2}\,,\,y_k\,,\,y_{k+1}\,,\,\dots\,,\,y_n\Big)$$

3.4.4 Exercise

Show that S^n is a differentiable and orientable real manifold.

Figure 3.9: the standard covering of S^1 together with a positive orientation



The domains $U_{(k,j)}$ form a covering of S^n and all images are open. It remains to check that $\psi = h_{(k,j)} \circ h^{-1}_{(l,m)}$ is differentiable.
Observe that the case $k = l$ is trivial since either $j = m \;\Rightarrow\; \psi = \mathrm{id}$ or $j \neq m \;\Rightarrow\; \mathrm{dom}\,\psi = \emptyset$:

$$\begin{aligned}\mathrm{dom}\,\psi = h_{(l,m)}\big(U_{(l,m)} \cap U_{(k,j)}\big) &= h_{(l,m)}\Big(\{\, x \in S^n \mid (-1)^j \cdot x_k > 0 \text{ and } (-1)^m \cdot x_l > 0 \,\}\Big)\\ &= \begin{cases} \{\, x \in \mathbb{R}^n \mid (-1)^j \cdot x_k > 0 \,\} & \text{if } k < l \\ \{\, x \in \mathbb{R}^n \mid (-1)^j \cdot x_{k-1} > 0 \,\} & \text{if } k > l \end{cases}\end{aligned}$$

$$\Rightarrow\quad \psi(x_1,\dots,x_n) = h_{(k,j)}\Big(x_1\,,\,\dots\,,\,x_{l-1}\,,\,(-1)^m\cdot\sqrt{1-\sum_{i=1}^n x_i^2}\,,\,x_l\,,\,x_{l+1}\,,\,\dots\,,\,x_n\Big)$$

$$= \begin{cases} \Big(x_1\,,\,\dots\,,\,x_{k-1}\,,\,x_{k+1}\,,\,\dots\,,\,x_{l-1}\,,\,(-1)^m\cdot\sqrt{1-\sum x_i^2}\,,\,x_l\,,\,x_{l+1}\,,\,\dots\,,\,x_n\Big) \\ \Big(x_1\,,\,\dots\,,\,x_{l-1}\,,\,(-1)^m\cdot\sqrt{1-\sum x_i^2}\,,\,x_l\,,\,x_{l+1}\,,\,\dots\,,x_{k-2}\,,\,x_k\,,\,x_{k+1}\,,\,\dots\;x_n\Big) \end{cases} \tag{3.2}$$

The explicit expressions in (3.2) allow to conclude that ψ, if it is defined, is differentiable and even real analytic (since identity, square and square root are analytic). This atlas does however not (yet) satisfy the orientability condition. But it can nevertheless be used to find an appropriate atlas later on. We set

$$y_u := \frac{\partial}{\partial x_u}\left((-1)^m \cdot \sqrt{1-\sum_{i=1}^n x_i^2}\right) = \frac{(-1)^m \cdot (-2x_u)}{2\sqrt{1-\sum x_i^2}} = \frac{(-1)^{m+1}\cdot x_u}{\sqrt{1-\sum x_i^2}}\ ,\ \forall\, u \in \{1,\dots,n\}$$

Consider the case $k > l$ ($k < l$ is similar). Then the Jacobian matrix $J(\psi)$ is given by

$$\begin{array}{c} 1 \\ \vdots \\ l-1 \\ l \\ l+1 \\ \vdots \\ k-1 \\ k \\ \vdots \\ n \end{array} \begin{pmatrix} 1 &&&&&&&&& \\ & \ddots &&&&&&&& \\ && 1 &&&&&&& \\ y_1 & \cdots & y_{l-1} & y_l & \cdots & y_{k-2} & y_{k-1} & y_k & \cdots & y_n \\ &&& 1 &&&&&& \\ &&&& \ddots &&&&& \\ &&&&& 1 &&&& \\ &&&&&& 0 & 1 && \\ &&&&&&&& \ddots & \\ &&&&&&&&& 1 \end{pmatrix}$$

$$\Rightarrow\ \det\big(J(\psi)\big) = (-1)^{l-1}\cdot(-1)^{n-(k-1)}\cdot \begin{vmatrix} y_1 & y_{l-1} & y_l & y_{k-2} & y_k & y_{k+1} & y_n & y_{k-1} \\ 1 &&&&&&& 0 \\ & 1 &&&&&& 0 \\ && 1 &&&&& 0 \\ &&& 1 &&&& 0 \\ &&&& 1 &&& 0 \\ &&&&& 1 && 0 \\ &&&&&& 1 & 0 \end{vmatrix}$$

and expanding with respect to the last column gives

$$\det\big(J(\psi)\big) = (-1)^{n-k+l}\cdot(-1)^{n+1}\cdot y_{k-1} = (-1)^{2n-k+l+1}\cdot(-1)^{m+1}\cdot\frac{x_{k-1}}{\sqrt{1-\sum x_i^2}}$$

Since $(-1)^j \cdot x_{k-1} > 0$ and $2n$ is even, we obtain that $\operatorname{sign}\Big(\det\big(J(\psi)\big)\Big) = \operatorname{sign}\Big((-1)^{l-k+m+j}\Big)$.

In general this is not always positive. We admit that the factor $(-1)^{l-k+m+j}$ can be eliminated by adding some powers of (-1) in the definition of $h_{(k,j)}$ such that the orientation is preserved (see figure 3.9). This modification will finally define an orientation-preserving atlas for S^n.

Chapter 4

Complex manifolds

4.1 Definition

Consider figure 4.1. Let $(M,\mathcal{U})$ be a real differentiable manifold with atlas $\mathcal{U} = (U_i, \varphi_i)_{i\in J}$ of (real) dimension $2n$ and assume that M is connected (as topological space) :

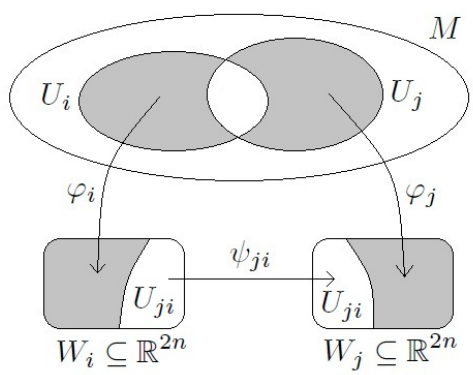
$$\varphi_i : U_i \to W_i \subseteq \mathbb{R}^{2n} \quad , \quad \psi_{ji} : \varphi_i(U_i \cap U_j) \subseteq \mathbb{R}^{2n} \to \varphi_j(U_j \cap U_i) \subseteq \mathbb{R}^{2n}$$

where $W_i \subseteq \mathbb{R}^{2n}$ is open and ψ_{ji} is differentiable whenever defined.
We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ using the standard identification (which is compatible with the orientation of M) :

$$(z_1, z_2, \ldots, z_n) \longleftrightarrow (\operatorname{Re} z_1, \operatorname{Im} z_1, \operatorname{Re} z_2, \operatorname{Im} z_2, \ldots, \operatorname{Re} z_n, \operatorname{Im} z_n) = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$$

Denote $U_{ji} := \varphi_i(U_i \cap U_j)$; then $\psi_{ji} : U_{ji} \subseteq \mathbb{C}^n \to \mathbb{C}^n$ becomes a complex map on the open set $U_{ji} \subseteq \mathbb{C}^n$.

Figure 4.1: a differentiable manifold of real dimension $2n$



Such a manifold $(M,\mathcal{U})$ is called a *complex manifold* $\Leftrightarrow$ the maps ψ_{ji} are biholomorphic $\forall\, i, j \in J$.
Note that it is enough to require that the ψ_{ji} are holomorphic only because $\psi_{ji}^{-1} = \psi_{ij}$:

$$\psi_{ji}, \psi_{ij} \text{ holomorphic} \quad \Rightarrow \quad \psi_{ji}, \psi_{ij} \text{ biholomorphic}$$

As for differentiable manifolds, we always assume in the following that complex manifolds are connected.
A one-dimensional connected compact complex manifold is called a *Riemann surface*.

Remark :
Two different atlases can define the same manifold if they are compatible. Recall that a chart (U, φ) is said to be *compatible* with an atlas $\mathcal{U} = (U_i, \varphi_i)_{i\in I}$ if and only if
− the sets $\varphi_i(U \cap U_i)$ and $\varphi(U \cap U_i)$ are open in $\mathbb{C}^n$, $\forall\, i \in I$
− $\forall\, i \in I$, the functions $\varphi_i \circ \varphi^{-1}$ and $\varphi \circ \varphi_i^{-1}$ are holomorphic whenever defined
Moreover 2 atlases $\mathcal{U}$ and $\mathcal{V}$ are compatible if every chart of $\mathcal{U}$ is compatible with $\mathcal{V}$ and vice-versa.
One can then show that adding charts to a manifold which are compatible with the existing atlas will not change the structure of the manifold. This is why manifolds are in general equipped with an equivalence class of atlases.

4.2 Complex manifolds are orientable

In the sequel, we use the following short-hand notation. For a function

$$f : U \subseteq \mathbb{C}^n \to \mathbb{C}^m : (z_1, \dots, z_n) \longmapsto \big(f_1(z_1, \dots, z_n), f_2(z_1, \dots, z_n), \dots, f_m(z_1, \dots, z_n)\big)$$

$$\frac{\partial f}{\partial z} := \left(\frac{\partial f_i}{\partial z_j}\right)_{i,j} \in \mathrm{Mat}(m \times n, \mathbb{C})$$

4.2.1 Definitions

Fix the indices $i, j \in J$. The holomorphic map $\psi_{ji} : U_{ji} \subseteq \mathbb{C}^n \to \mathbb{C}^n : z \mapsto w = \psi_{ji}(z)$ has the n components

$$(\psi_{ji})_k : U_{ji} \subseteq \mathbb{C}^n \to \mathbb{C}, \ k = 1, \dots, n$$

All components are holomorphic too, thus $\dfrac{\partial w_k}{\partial \bar{z}_l}(z) = \dfrac{\partial (\psi_{ji})_k}{\partial \bar{z}_l}(z) = 0$, $\forall\, k, l \in \{1, \dots, n\}$, $\forall\, z \in U_{ji}$.

For complex functions, one can now define 3 different types of Jacobian matrices :

$$\text{holomorphic Jacobian matrix} : J_{\text{hol}}(z) = \frac{\partial w}{\partial z}(z) = \left(\frac{\partial w_k}{\partial z_l}(z)\right)_{k,l} \in \mathrm{Mat}(n \times n, \mathbb{C})$$

$$\text{complex Jacobian matrix} : J(z) = \begin{pmatrix} \frac{\partial w}{\partial z}(z) & \frac{\partial w}{\partial \bar{z}}(z) \\ \frac{\partial \bar{w}}{\partial z}(z) & \frac{\partial \bar{w}}{\partial \bar{z}}(z) \end{pmatrix} = \begin{pmatrix} \frac{\partial w}{\partial z} & \frac{\partial w}{\partial \bar{z}} \\ \frac{\partial \bar{w}}{\partial z} & \frac{\partial \bar{w}}{\partial \bar{z}} \end{pmatrix}(z) \in \mathrm{Mat}(2n \times 2n, \mathbb{C})$$

$$\text{real Jacobian matrix} : J_{\text{real}}(z) = \begin{pmatrix} \frac{\partial (\mathrm{Re}\, w_1)}{\partial x_1} & \frac{\partial (\mathrm{Re}\, w_1)}{\partial y_1} & \frac{\partial (\mathrm{Re}\, w_1)}{\partial x_2} & \cdots & \frac{\partial (\mathrm{Re}\, w_1)}{\partial y_n} \\ \frac{\partial (\mathrm{Im}\, w_1)}{\partial x_1} & \frac{\partial (\mathrm{Im}\, w_1)}{\partial y_1} & \frac{\partial (\mathrm{Im}\, w_1)}{\partial x_2} & \cdots & \frac{\partial (\mathrm{Im}\, w_1)}{\partial y_n} \\ \frac{\partial (\mathrm{Re}\, w_2)}{\partial x_1} & \frac{\partial (\mathrm{Re}\, w_2)}{\partial y_1} & \frac{\partial (\mathrm{Re}\, w_2)}{\partial x_2} & \cdots & \frac{\partial (\mathrm{Re}\, w_2)}{\partial y_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial (\mathrm{Im}\, w_n)}{\partial x_1} & \frac{\partial (\mathrm{Im}\, w_n)}{\partial y_1} & \frac{\partial (\mathrm{Im}\, w_n)}{\partial x_2} & \cdots & \frac{\partial (\mathrm{Im}\, w_n)}{\partial y_n} \end{pmatrix}(z) \in \mathrm{Mat}(2n \times 2n, \mathbb{R})$$

J can be seen as the complexified version of J_{real}.
M is now called *orientable* $\Leftrightarrow$ there is an atlas of M such that $\det\big(J_{\text{real}}(z)\big) > 0$, $\forall\, z \in U_{ji}$.
Since $w = \psi_{ji}$ is bijective, all 3 determinants are necessarily non-zero. Obviously $J \neq J_{\text{real}}$ (the first one is complex, the second one real only), but there is a relation between their determinants.

4.2.2 Lemma

For any holomorphic function $f : U \subseteq \mathbb{C} \to \mathbb{C}$, we have $\overline{f(z)} = f(\bar{z})$, $\forall\, z \in U$.

Proof. Let $z_0 \in U$. Since f is holomorphic, hence analytic at z_0, we can write f as a power series in z which converges in some neighborhood of z_0 :

$$f(z) = \sum_{n=0}^{\infty} a_n \cdot (z - z_0)^n$$

Hence the result follows from the fact that conjugation is a continuous and linear operation on $\mathbb{C}$. □

4.2.3 Theorem

Any complex manifold is orientable (as a real manifold) and has always even real dimension.

Proof. Even dimension is a consequence of the identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$.
Since the $w_k = (\psi_{ji})_k$ are holomorphic in all variables z_l, we get $\forall\, k, l \in \{1, \dots, n\}$:

$$\frac{\partial w_k}{\partial \bar{z}_l} = 0 \quad \text{and} \quad \frac{\partial \bar{w}_k}{\partial z_l} = \overline{\frac{\partial w_k}{\partial \bar{z}_l}} = 0 \qquad \Rightarrow \quad \frac{\partial w}{\partial \bar{z}} = \frac{\partial \bar{w}}{\partial z} = 0$$

$$\Rightarrow \quad J = \begin{pmatrix} \frac{\partial w}{\partial z} & 0 \\ 0 & \frac{\partial \bar{w}}{\partial \bar{z}} \end{pmatrix} = \begin{pmatrix} \frac{\partial w}{\partial z} & 0 \\ 0 & \overline{\left(\frac{\partial w}{\partial z}\right)} \end{pmatrix} \quad \Rightarrow \ \det(J) = \det\left(\frac{\partial w}{\partial z}\right) \cdot \det\overline{\left(\frac{\partial w}{\partial z}\right)} = \det\left(\frac{\partial w}{\partial z}\right) \cdot \overline{\det\left(\frac{\partial w}{\partial z}\right)}$$

because det is a linear expression. Hence

$$\det(J) = \left|\det\left(\frac{\partial w}{\partial z}\right)\right|^2 = \left|\det(J_{\text{hol}})\right|^2 > 0 \ \text{ since } \det(J_{\text{hol}}) \neq 0$$

It remains to compute $\det\big(J_{\text{real}}(z)\big)$. For this, we want to find a relation between $\det(J)$ and $\det(J_{\text{real}})$, using the relations (1.1) and (1.2) to calculate terms of the type

$$\begin{aligned} \frac{\partial w_k}{\partial z_l} &= \frac{\partial(\operatorname{Re} w_k)}{\partial z_l} + i \cdot \frac{\partial(\operatorname{Im} w_k)}{\partial z_l} = \frac{1}{2} \cdot \left(\frac{\partial(\operatorname{Re} w_k)}{\partial x_l} - i \cdot \frac{\partial(\operatorname{Re} w_k)}{\partial y_l}\right) + \frac{i}{2} \cdot \left(\frac{\partial(\operatorname{Im} w_k)}{\partial x_l} - i \cdot \frac{\partial(\operatorname{Im} w_k)}{\partial y_l}\right) \\ &= \frac{1}{2} \cdot \left(\frac{\partial(\operatorname{Re} w_k)}{\partial x_l} + \frac{\partial(\operatorname{Im} w_k)}{\partial y_l}\right) + \frac{i}{2} \cdot \left(\frac{\partial(\operatorname{Im} w_k)}{\partial x_l} - \frac{\partial(\operatorname{Re} w_k)}{\partial y_l}\right) \end{aligned} \tag{4.1}$$

Note that the terms $\frac{\partial(\operatorname{Re} w_k)}{\partial x_l}$, $\frac{\partial(\operatorname{Re} w_k)}{\partial y_l}$, $\frac{\partial(\operatorname{Im} w_k)}{\partial x_l}$ and $\frac{\partial(\operatorname{Im} w_k)}{\partial y_l}$ form a 2×2–sub-matrix inside J_{real}. Similarly

$$\frac{\partial w_k}{\partial \bar{z}_l} = \frac{1}{2} \cdot \left(\frac{\partial(\operatorname{Re} w_k)}{\partial x_l} - \frac{\partial(\operatorname{Im} w_k)}{\partial y_l}\right) + \frac{i}{2} \cdot \left(\frac{\partial(\operatorname{Im} w_k)}{\partial x_l} + \frac{\partial(\operatorname{Re} w_k)}{\partial y_l}\right) \tag{4.2}$$

$$\frac{\partial \bar{w}_k}{\partial z_l} = \frac{1}{2} \cdot \left(\frac{\partial(\operatorname{Re} w_k)}{\partial x_l} - \frac{\partial(\operatorname{Im} w_k)}{\partial y_l}\right) - \frac{i}{2} \cdot \left(\frac{\partial(\operatorname{Im} w_k)}{\partial x_l} + \frac{\partial(\operatorname{Re} w_k)}{\partial y_l}\right) \tag{4.3}$$

$$\frac{\partial \bar{w}_k}{\partial \bar{z}_l} = \frac{1}{2} \cdot \left(\frac{\partial(\operatorname{Re} w_k)}{\partial x_l} + \frac{\partial(\operatorname{Im} w_k)}{\partial y_l}\right) - \frac{i}{2} \cdot \left(\frac{\partial(\operatorname{Im} w_k)}{\partial x_l} - \frac{\partial(\operatorname{Re} w_k)}{\partial y_l}\right) \tag{4.4}$$

Hence by a rearrangement of J (linear combinations, permutations of lines and columns), one obtains $\det(J_{\text{real}})$ from $\det(J)$. Indeed consider the base case $n = 1$:

$$J = \begin{pmatrix} \frac{\partial w_1}{\partial z_1} & \frac{\partial w_1}{\partial \bar{z}_1} \\ \frac{\partial \bar{w}_1}{\partial z_1} & \frac{\partial \bar{w}_1}{\partial \bar{z}_1} \end{pmatrix} \qquad , \qquad J_{\text{real}} = \begin{pmatrix} \frac{\partial(\operatorname{Re} w_1)}{\partial x_1} & \frac{\partial(\operatorname{Re} w_1)}{\partial y_1} \\ \frac{\partial(\operatorname{Im} w_1)}{\partial x_1} & \frac{\partial(\operatorname{Im} w_1)}{\partial y_1} \end{pmatrix}$$

Replacing the values in J by (4.1), (4.2), (4.3) and (4.4), we find that $\det(J) = \det(J_{\text{real}})$, hence J is obtained by linear combinations of the values in J_{real}. For $n \geq 2$, we then have

$$J = \begin{pmatrix} \frac{\partial w_1}{\partial z_1} & \frac{\partial w_1}{\partial z_2} & \cdots & \frac{\partial w_1}{\partial \bar{z}_1} & \frac{\partial w_1}{\partial \bar{z}_2} & \cdots \\ \frac{\partial w_2}{\partial z_1} & \frac{\partial w_2}{\partial z_2} & \cdots & \frac{\partial w_2}{\partial \bar{z}_1} & \frac{\partial w_2}{\partial \bar{z}_2} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ \frac{\partial \bar{w}_1}{\partial z_1} & \frac{\partial \bar{w}_1}{\partial z_2} & \cdots & \frac{\partial \bar{w}_1}{\partial \bar{z}_1} & \frac{\partial \bar{w}_1}{\partial \bar{z}_2} & \cdots \\ \frac{\partial \bar{w}_2}{\partial z_1} & \frac{\partial \bar{w}_2}{\partial z_2} & \cdots & \frac{\partial \bar{w}_2}{\partial \bar{z}_1} & \frac{\partial \bar{w}_2}{\partial \bar{z}_2} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \end{pmatrix} \begin{matrix} 1 \\ 2 \\ \vdots \\ n+1 \\ n+2 \\ \vdots \end{matrix}$$

In order to obtain the 2×2–sub-matrices as described above, we have to bring the row $n+i$ to position $i+1$ and similarly for the columns. Since we get the <u>same</u> number of changes for rows and columns, the total number of permutations is even, hence the sign of the determinant does not change. And in order to obtain J_{real} we then make the same linear combinations as in the case $n = 1$ in each one of these 2×2–sub-matrices, so finally the whole determinant did not change. Thus $\det(J_{\text{real}}) = \det(J) > 0$, showing that the manifold is orientable. □

4.2.4 Remark

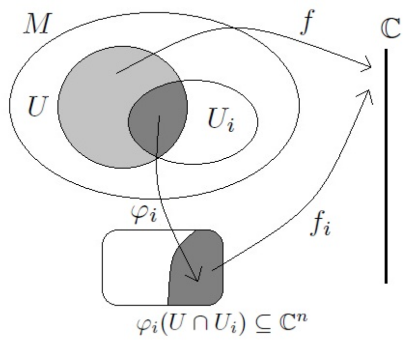
The converse of this theorem is not true : not all orientable manifolds are complex manifolds.
For example S^n is orientable, but S^3 cannot be a complex manifold since $\dim S^3 = 3$ is odd. In fact only S^2 is a complex manifold since it is equal to $S^2 = \mathbb{P}^1(\mathbb{C}) = \mathbb{CP}^1$, the complex projective plane (see section 5.3.8).
More generally one can show that any 2-dimensional orientable compact differentiable manifold admits a complex structure.

4.3 Holomorphic functions on complex manifolds

4.3.1 Definition

Let U be an open subset (not necessarily a chart domain) of a complex manifold $(M,\mathcal{U})$ with atlas $\mathcal{U} = (U_i, \varphi_i)_{i\in J}$.
A complex-valued function $f : U \to \mathbb{C}$ is called a *holomorphic function* on $U \;\Leftrightarrow\; f_i := f \circ \varphi_i^{-1}$ is holomorphic on the open set $\varphi_i(U \cap U_i) \subseteq \mathbb{C}^n$, $\forall\, i \in J$ (see figure 4.2). Note that $U \cap U_i$ can be empty.
Similarly a function $F : U \subseteq M \to \mathbb{C}^m$ is holomorphic on an open subset U of M if all complex-valued coordinate functions are holomorphic on U.

Figure 4.2: composition of the functions f and φ_i^{-1} to give a map $f_i : \mathbb{C}^n \to \mathbb{C}$



More generally : Let $(M,\mathcal{U})$ and $(N,\mathcal{V})$ be 2 complex manifolds of dimension m and n respectively with atlases $\mathcal{U} = (U_i,\varphi_i)_{i\in I}$ and $\mathcal{V} = (V_j,\psi_j)_{j\in J}$ a $f(a) \subseteq \mathbb{C}^m \subseteq \mathbb{C}^n$.
A continuous map $f : M \to N$ is called a *holomorphic map* if and only if (see figure 4.3) the maps

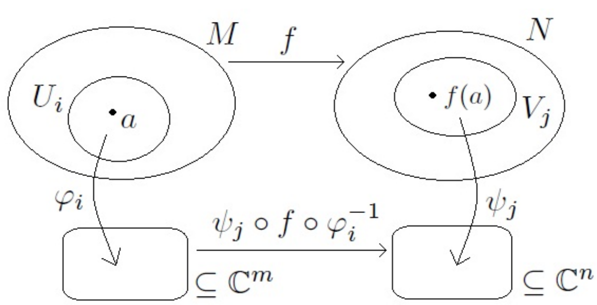
$$\psi_j \circ f \circ \varphi_i^{-1} : \varphi_i\big(f^{-1}(V_j) \cap U_i\big) \subseteq \mathbb{C}^m \to (\psi_j \circ f)\big(f^{-1}(V_j) \cap U_i\big) \subseteq \mathbb{C}^n$$

are holomorphic, $\forall\, i \in I$, $\forall\, j \in J$. Note that $\varphi_i\big(f^{-1}(V_j) \cap U_i\big)$ is open in $\mathbb{C}^m$ because

$$V_j \text{ open, } f \text{ continuous } \Rightarrow f^{-1}(V_j) \text{ open } \quad , \quad U_i \text{ open } \Rightarrow f^{-1}(V_j) \cap U_i \text{ open}$$
$$\varphi_i \text{ bicontinuous } \Rightarrow \varphi_i\big(f^{-1}(V_j) \cap U_i\big) \text{ is open}$$

We have to restrict ourselves to this smaller open set since otherwise some operations may not be well-defined.

Figure 4.3: a holomorphic map $f : M \to N$



Remark :
It is enough to check this condition for a subcover, e.g. in the case where $\mathcal{U}$ or $\mathcal{V}$ contain "superfluous" charts. This is justified by the fact that manifolds are equipped with an equivalence class of atlases.

4.3.2 Proposition

Let $(M,\mathcal{U})$ be a complex manifold and $p \in M$. For any $U \in \mathcal{U}$ with $p \in U$, we can add a coordinate chart (U,φ) which is compatible with the atlas $\mathcal{U}$ such that $\varphi : U \to \mathbb{C}^n$ and $\varphi(p) = 0 \in \mathbb{C}^n$.
One says that (U,φ) is a *centered* coordinate chart at p.

Proof. $p \in M \Rightarrow$ there is a chart (U, ψ) of M such that $p \in U$ (because the chart domains cover M). Define

$$\varphi(x) := \psi(x) - \psi(p) \in \mathbb{C}^n, \ \forall\, x \in U$$

We denote τ_w the translation by $w \in \mathbb{C}^n$, i.e. $\tau_w(z) = z + w$. Since $\tau_w^{-1} = \tau_{-w}$, this map is clearly biholomorphic. So $\varphi = \tau_{-\psi(p)} \circ \psi$, showing that φ is also biholomorphic because ψ and $\tau_{-\psi(p)}$ are. This already ensures that φ is compatible with the existing atlas $\mathcal{U}$. Moreover φ trivially satisfies $\varphi(p) = 0$. □

4.3.3 Consequence

Given a holomorphic function $f : M \to \mathbb{C}$ and a point $p \in M$, one can always suppose that there exists a coordinate chart (U, φ) centered at p. Hence by identifying U with $\varphi(U)$ (which is natural because φ is bijective and biholomorphic), one can identify $p = 0$ and write $f_{|U} = f \circ \varphi^{-1}$ which is holomorphic, thus $f_{|U}$ can be written as a (centered) power series in $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$.

4.4 Complex submanifolds

4.4.1 Proposition

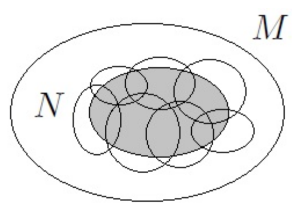
Every connected open set in a complex manifold is again a complex manifold of the same dimension.

Proof. Let $(M, \mathcal{U})$ be a complex manifold with atlas $\mathcal{U} = (U_i, \varphi_i)_{i \in I}$. Let $N \subseteq M$, $N \neq \emptyset$ such that N is open and connected. N can be covered by a family of open subsets in $\mathcal{U}$ (see figure 4.4). Define $\mathcal{V} = (V_i, \psi_i)_{i \in I}$, where

$$V_i := N \cap U_i \quad , \quad \psi_i := \varphi_{i_{|N \cap U_i}}$$

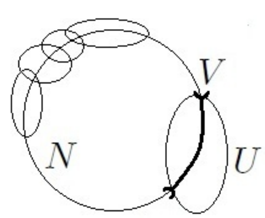
Then $(N, \mathcal{V})$ is also a complex manifold since V_i is open in N and ψ_i is the restriction of the homeomorphism φ_i to an open subset of M, hence ψ_i is still bijective onto $\psi_i(V_i)$. In particular we have that $\dim N = \dim M$. Moreover the transition functions $\psi_j \circ \psi_i^{-1}$ are biholomorphic since they are just restrictions of the $\varphi_j \circ \varphi_i^{-1}$. □

Figure 4.4: an open covering of $N \subset M$



4.4.2 Model of a submanifold

As an example consider the complex manifold $M = \mathbb{C} \cong \mathbb{R}^2$ and the closed subset $N = S^1 = \{ z \in \mathbb{C} \ , \ |z| = 1 \}$. Again N can be covered by open subsets in the atlas of M.



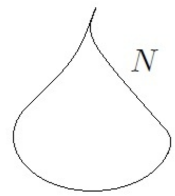
Let $U \subseteq M$ be open such that $U \cap N \neq \emptyset$ and define $V := U \cap N \Rightarrow V$ is open in N. V locally looks like $\mathbb{R}$ and

$$U \overset{\text{loc}}{\cong} \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \overset{\text{loc}}{\cong} V \times \mathbb{R}$$

This is the model of a submanifold : locally N looks like $\mathbb{R}$ in $\mathbb{R}^2 \cong \mathbb{C}$.

Remark :
Arbitrary closed subsets of complex manifolds can in general not define submanifolds. Consider figure 4.5. At the top it is not admissible since it does not look like $\mathbb{R}$ in $\mathbb{R}\times\mathbb{R}$.

Figure 4.5: N is not a complex submanifold of $\mathbb{C}$



But : this is only true if the manifold is complex or differentiable. For topological manifolds, we have that $\Lambda \overset{\text{loc}}{\cong} \mathbb{R}$. Topologically, pointed lines are equivalent to straight lines since no differentiability conditions are required.

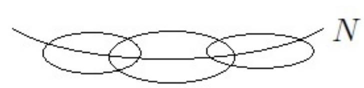
4.4.3 Definition

Let M be a complex manifold of dimension n and $N \subseteq M$ be closed. N is called a *(complex) submanifold* of M $\Leftrightarrow$ $\forall\, y \in N$, there is a coordinate chart (U,φ) of M with $\varphi : U \to W \subseteq \mathbb{C}^n$, W open, $y \in U$ such that

$$\varphi_{|_{U\cap N}} : U \cap N \xrightarrow{\sim} W' = W \cap \left(\mathbb{C}^k \times \{0\}\right)$$

for some $0 \leq k \leq n$ and where $\mathbb{C}^k \hookrightarrow \mathbb{C}^n$ is embedded in the standard way, i.e. $(z_1,\ldots,z_k) \mapsto (z_1,\ldots,z_k,0,\ldots,0)$. In addition k is then equal to the dimension of N.
Note that this chart must not necessarily belong to the chosen atlas $\mathcal{U}$ of M ; it only has to be compatible with $\mathcal{U}$ (since then adding it to the atlas will not change the manifold structure).
In particular, a submanifold should be a manifold itself. Its atlas is given by restrictions of the charts in the atlas of the manifold in which it is contained (see figure 4.6). This makes sense since, as restrictions, the transition functions in the submanifold are then still bijective and biholomorphic. However, by definition, submanifolds do not need to be connected : a submanifold of a complex manifold is thus again a complex manifold if and only if it is connected.

Figure 4.6: an atlas of N is given by restrictions of the atlas of M



Concerning the example of S^1 in 4.4.2, it must therefore be said that S^1 is only a real submanifold of $\mathbb{C}$ since $\dim_{\mathbb{R}} S^1 = 1$ is odd. It is not possible to endow S^1 with a complex manifold structure, thus it cannot be a complex submanifold of $\mathbb{C}$ neither (any connected complex submanifold is a complex manifold).

4.4.4 Equivalent characterization

Using the Implicit Function Theorem and the Constant Rank Theorem, one can show that :

A subset N in a manifold M of dimension n is a complex submanifold of dimension k if and only if it can locally be written as the zero set of locally holomorphic functions for which the Jacobian matrix has maximal rank, i.e. $\forall\, y \in N$, there is an open neighborhood U of y in M (we may choose U sufficiently small such that (U,φ) is a chart at y) and there are holomorphic functions $f_i : U \to \mathbb{C}$, $i = 1,\ldots,n-k$ (as defined in 4.3.1) such that

$$U \cap N = \bigcap_{i=1}^{n-k} f_i^{-1}\big(\{0\}\big) \qquad , \qquad \operatorname{rk}\left(\frac{\partial(f_i\circ\varphi^{-1})}{\partial z_j}(z)\right)_{i,j} = n-k\,,\ \forall\, z \in U\cap N \tag{4.5}$$

This does not mean that submanifolds are affine algebraic varieties since the functions f_i must be holomorphic, which is not an algebraic characterization. Note that there must always exist the same number of functions, i.e. $\forall\, y \in N$ and for any set of functions f_i describing $U \cap N$, exactly $n-k$ of them are necessary and sufficient.

Chapter 5

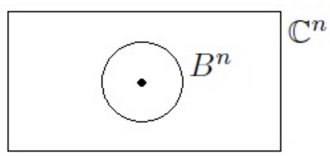
Examples of complex manifolds

5.1 $\mathbb{C}^n$ and open subsets

A global chart is given by $(\mathbb{C}^n, \mathrm{id})$, but this is not the only possibility.
– For any fixed $a \in \mathbb{C}^n$, one can e.g. take $(\mathbb{C}^n, \varphi_a = \mathrm{id} - a)$.
– One can also use the fact that $\mathbb{C}^n$ is a $\mathbb{C}$-vector space and that any basis defines a global coordinate chart :

$$\forall\, z \in \mathbb{C}^n,\ \exists!\, a_i \in \mathbb{C} \text{ such that } z = \sum_{i=1}^{n} a_i\, e_i \quad \Rightarrow \quad \varphi(z) := (a_1, \ldots, a_n) \in \mathbb{C}^n$$

where $\{e_i\}_{i=1,\ldots,n}$ can be any basis of $\mathbb{C}^n$.
Moreover any open connected subset of $\mathbb{C}^n$ is a complex manifold, e.g. the unit ball $B^n := \big\{\, z \in \mathbb{C}^n \ ,\ \|z\| < 1 \,\big\}$.



5.2 Submanifolds of $\mathbb{C}^n$

5.2.1 Linear subspaces

$H \subseteq \mathbb{C}^n$ is called a *linear subspace* of $\mathbb{C}^n$ if it is the solution set of a system of homogeneous linear equations, i.e. if there exist k linear forms $l_i : \mathbb{C}^n \to \mathbb{C}$ such that

$$H = \bigcap_{i=1}^{k} \ker l_i = V(l_1, l_2, \ldots, l_k)$$

In particular, linear subspaces of $\mathbb{C}^n$ define affine linear varieties in $\mathbb{C}^n$ (since linear maps on $\mathbb{C}^n$ are polynomials).

Equivalently a linear subspace of $\mathbb{C}^n$ can also be given as the solution set of the matrix equation $A \cdot z = 0$:

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{k1} & \ldots & a_{kn} \end{pmatrix},\ a_{ij} \in \mathbb{C} \quad , \quad z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \qquad \Rightarrow \quad H = \big\{\, z \in \mathbb{C}^n \mid A \cdot z = 0 \in \mathbb{C}^k \,\big\}$$

In this case, the linear forms $l_i : \mathbb{C}^n \to \mathbb{C}$ are given by $l_i(z) = a_{i1}\, z_1 + a_{i2}\, z_2 + \ldots + a_{in}\, z_n$. Thus their Jacobian matrix $J(l_1, \ldots, l_k)$ is constant and equal to A. Moreover $\mathrm{rk}\big(J(l_1, \ldots, l_k)\big) = \mathrm{rk}(A) = k \ \Leftrightarrow\ $ the l_r are linearly independent and in this case a linear subspace defines a submanifold of $\mathbb{C}^n$ of codimension k, i.e.

$$\mathrm{rk}\big(J(l_1, \ldots, l_k)\big) = k \ \Leftrightarrow\ \dim H = n - k$$

Indeed this defines a splitting $\mathbb{C}^n = \mathbb{C}^k \times \mathbb{C}^{n-k}$ by the characterization (4.5).

5.2.2 Singular points

Let K be a complex affine variety in $\mathbb{C}^n$ defined by p polynomials $g_1, \dots, g_p$.
A point $z_0 \in K$ is called a *singular point* of K if the rank of the Jacobian matrix of the g_i drops at z_0, i.e.

$$z_0 \in K \text{ is singular} \quad \Leftrightarrow \quad \operatorname{rk}\big(J(g_1, \dots, g_p)(z_0)\big) \text{ is not maximal}$$

A *smooth* variety is an affine variety which does not contain singular points.
Any linear subspace where the defining linear forms are linearly independent is for example a smooth algebraic variety since $J(f_1, \dots, f_k)(z) = A$ is of maximal rank for all $z \in \mathbb{C}^n$.

5.2.3 Exercise

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function on $\mathbb{C}^n$ and consider its vanishing set (which is not a linear subspace!)

$$V(f) = \left\{ z \in \mathbb{C}^n \mid f(z) = 0 \right\} = f^{-1}\big(\{0\}\big)$$

f being holomorphic, thus continuous, $V(f)$ is closed in $\mathbb{C}^n$, but until now it is not yet a submanifold of $\mathbb{C}^n$.

$$\forall\, z \in \mathbb{C}^n \ : \ \operatorname{grad} f(z) = \left(\frac{\partial f}{\partial z_1}(z) \,,\, \frac{\partial f}{\partial z_2}(z) \,,\, \dots \,,\, \frac{\partial f}{\partial z_n}(z) \right)$$

a) Show that $V(f)$ is a (not necessarily connected) submanifold of $\mathbb{C}^n \ \Leftrightarrow\ \operatorname{grad} f(z) \neq 0$, $\forall\, z \in V(f)$.

Since f is globally defined and $V(f)$ is the zero set of the holomorphic function f, we can take $U = \mathbb{C}^n$, thus

$$V(f) \cap \mathbb{C}^n = f^{-1}\big(\{0\}\big)$$

So by (4.5) it is necessary and sufficient to show that the Jacobian matrix associated to f has maximal rank on the set $U \cap V(f) = V(f)$. But there is only one (globally defined) function, so $n - k = 1$ and

$$\operatorname{rk}\big(J(f)(z)\big) = 1 \ \Leftrightarrow\ J(f)(z) \neq 0 \ \Leftrightarrow\ \operatorname{grad} f(z) \neq 0 \,,\ \forall\, z \in V(f)$$

b)

$$B := \{ z \in \mathbb{C}^n \mid \operatorname{grad} f(z) = 0 \} = (\operatorname{grad} f)^{-1}\big(\{0\}\big)$$

B is closed since f is holomorphic, i.e. in particular C^∞. Show that $N := V(f) \setminus B$ is a submanifold of $\mathbb{C}^n \setminus B$.

It is necessary to consider N as a subset of $\mathbb{C}^n \setminus B$ since it is the complement of a closed set in a closed set, hence in general it is not closed in $\mathbb{C}^n$ any more. It is however closed in $\mathbb{C}^n \setminus B$ since

$$N = V(f) \cap \big(\mathbb{C}^n \setminus B\big) \ \text{ and } V(f) \text{ is closed in } \mathbb{C}^n$$

Moreover $\mathbb{C}^n \setminus B$ is open in $\mathbb{C}^n$, hence it is a complex manifold itself.
Proving that N is a submanifold of $\mathbb{C}^n \setminus B$ is now done exactly as in a) since $\operatorname{grad} f(z) \neq 0$, $\forall\, z \in N$.

c) Show that $\dim V(f) = n - 1$ if $V(f)$ is a submanifold of $\mathbb{C}^n$.

This follows from the fact that $V(f)$ is the zero set of a single holomorphic function $\Rightarrow\ n - k = 1$ and

$$\dim V(f) = k = n - (n - k) = n - 1$$

By (4.5) : $\dim N = k \ \Leftrightarrow$ there are $n - k$ (locally) holomorphic functions f_i ; here $n - k = 1 \ \Rightarrow\ k = n - 1$.

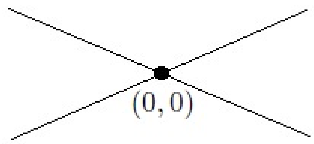
Remark :
If $V(f)$ is a submanifold of $\mathbb{C}^n$, it is called a *hypersurface*. And in the case where f is a polynomial function (in particular it is holomorphic), it is also a hypersurface in the context of affine varieties.

5.2.4 Examples

For $n = 2$, let $g_1(z_1, z_2) = z_1 - z_2^2$, $g_2(z_1, z_2) = z_1^2 - z_2^2$, $g_3(z_1, z_2) = z_1^2 + z_2^2$, $g_4(z_1, z_2) = z_1^2 + z_2^2 - 1$ on $\mathbb{C}^2$.

$$\begin{aligned}
J(g_1)(z_1, z_2) = \operatorname{grad} g_1(z_1, z_2) = \begin{pmatrix} 1 & -2z_2 \end{pmatrix} &\Rightarrow \operatorname{rk}\big(J(g_1)(z_1, z_2)\big) = 1,\ \forall\, (z_1, z_2) \in \mathbb{C}^2 \\
J(g_2)(z_1, z_2) = \operatorname{grad} g_2(z_1, z_2) = \begin{pmatrix} 2z_1 & -2z_2 \end{pmatrix} &\Rightarrow \operatorname{rk}\big(J(g_2)(z_1, z_2)\big) = 1 \ \Leftrightarrow\ (z_1, z_2) \neq (0, 0) \\
J(g_3)(z_1, z_2) = \operatorname{grad} g_3(z_1, z_2) = \begin{pmatrix} 2z_1 & 2z_2 \end{pmatrix} &\Rightarrow \operatorname{rk}\big(J(g_3)(z_1, z_2)\big) = 1 \ \Leftrightarrow\ (z_1, z_2) \neq (0, 0) \\
J(g_4)(z_1, z_2) = \operatorname{grad} g_4(z_1, z_2) = \begin{pmatrix} 2z_1 & 2z_2 \end{pmatrix} &\Rightarrow \operatorname{rk}\big(J(g_4)(z_1, z_2)\big) = 1,\ \forall\, (z_1, z_2) \in V(g_4)
\end{aligned}$$

$V(g_1)$ is a submanifold (hypersurface) of $\mathbb{C}^2$.
$(0, 0) \in V(g_2), V(g_3)$ is a singular point, thus $V(g_2)$ and $V(g_3)$ are not submanifolds of $\mathbb{C}^2$, but of $\mathbb{C}^2 \setminus \{(0, 0)\}$.
Moreover $g_2(z_1, z_2) = (z_1 - z_2)\,(z_1 + z_2)$ and $g_3(z_1, z_2) = (z_1 - i\, z_2)\,(z_1 + i\, z_2)$, hence $V(g_2)$ and $V(g_3)$ consist of 2 complex lines intersecting in the point $(0, 0)$.



$(0, 0)$ is also singular for g_4, but $(0, 0) \notin V(g_4)$, thus $\operatorname{grad} f \neq 0$ on $V(g_4)$, which is hence a submanifold of $\mathbb{C}^n$.

5.2.5 Generalization

Let $f_1, \ldots, f_k$ be holomorphic functions on $\mathbb{C}^n$. Then the common zero set of the f_i is

$$V(f_1, \ldots, f_k) = \big\{\, z \in \mathbb{C}^n \mid f_1(z) = f_2(z) = \ldots = f_k(z) = 0 \,\big\}$$

It is closed and defines a submanifold of $\mathbb{C}^n$ $\Leftrightarrow$ $\operatorname{rk}\big(J(f_1, \ldots, f_k)(z)\big)$ is maximal, $\forall\, z \in V(f_1, \ldots, f_k)$.

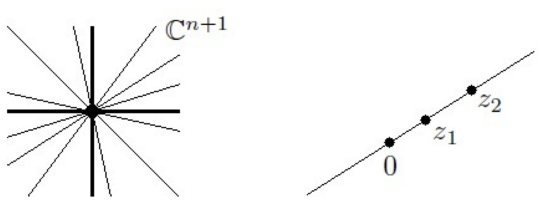
5.3 The projective space

Consider $\mathbb{C}^{n+1}$ and let P be the space of all lines in $\mathbb{C}^{n+1}$ passing through the origin $(0, \ldots, 0)$ (see figure 5.1). Since every line is uniquely determined by 2 points passing through it, we hence obtain that every point in $\mathbb{C}^{n+1} \setminus \{(0, \ldots, 0)\}$ defines such a line. Now we introduce a relation $\sim$ on $\mathbb{C}^{n+1} \setminus \{(0, \ldots, 0)\}$ by

$$z_1 \sim z_2 \ \Leftrightarrow\ \exists\, \lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\} \text{ such that } z_2 = \lambda \cdot z_1$$

i.e. all points lying on a same complex line are identified (figure 5.1). One shows that $\sim$ is an equivalence relation.

Figure 5.1: lines in $\mathbb{C}^{n+1}$ passing through the origin and the relation $\sim$



The *n-dimensional complex projective space* is then given by

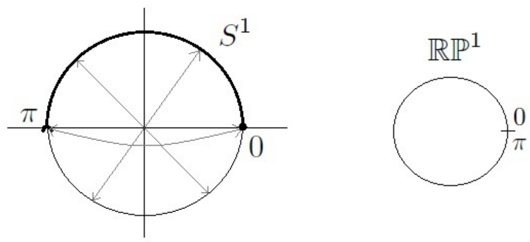
$$\mathbb{P}^n(\mathbb{C}) = \mathbb{CP}^n := \big(\mathbb{C}^{n+1} \setminus \{(0, \ldots, 0)\}\big) \,/\sim\, = \big\{\, [z] \;\big|\; z \in \mathbb{C}^{n+1} \setminus \{(0, \ldots, 0)\} \,\big\}$$

where each equivalence class $[z] = \{\, z' \in \mathbb{C}^{n+1} \setminus \{0\} \mid z' \sim z \,\}$ represents a line in $\mathbb{C}^{n+1}$ passing through the origin. Elements in $\mathbb{P}^n(\mathbb{C})$ are given by such equivalence classes.

Goal :
We want to show that $\mathbb{P}^n(\mathbb{C})$ is a compact complex manifold of dimension n. Compare e.g. the case $\mathbb{RP}^1 \cong S^1$ by identifying antipodal points (see figure 5.2; when arriving at π, the circle closes again). Thus

$$S^1 \text{ compact } \Rightarrow \mathbb{RP}^1 \text{ compact}$$

Figure 5.2: $\mathbb{RP}^1$ is obtained from S^1 by identifying antipodal points



5.3.1 Topology

First we need to define a topology on $\mathbb{P}^n(\mathbb{C})$. Consider the canonical projection map

$$\nu \ : \ \mathbb{C}^{n+1} \setminus \{0\} \twoheadrightarrow \mathbb{P}^n(\mathbb{C}) \ : \ z \mapsto [z]$$

which maps a point to the line it defines. $\mathbb{C}^{n+1} \setminus \{0\}$ is open in $\mathbb{C}^{n+1}$, hence it is a topological space itself. Then we endow $\mathbb{P}^n(\mathbb{C})$ with the usual quotient topology, i.e. the finest topology such that the projection map ν is continuous. This means that $U \subseteq \mathbb{P}^n(\mathbb{C})$ is open $\Leftrightarrow$ $\nu^{-1}(U)$ is open in $\mathbb{C}^{n+1} \setminus \{0\}$. By the properties of preimages, this defines indeed a topology on $\mathbb{P}^n(\mathbb{C})$.

− Anecdote to the quotient topology :
One can of course endow $\mathbb{P}^n(\mathbb{C})$ with the trivial topology since $\nu(\emptyset) = \emptyset$ and $\nu^{-1}\big(\mathbb{P}^n(\mathbb{C})\big) = \mathbb{C}^{n+1} \setminus \{0\}$, thus ν would be continuous. But this topology is too small in order to provide interesting results.
The discrete topology is however not possible since if all points are open in $\mathbb{P}^n(\mathbb{C})$, then $\forall\, [z] \in \mathbb{P}^n(\mathbb{C})$:

$$\nu^{-1}\big(\{[z]\}\big) = \big\{\, z' \in \mathbb{C}^{n+1} \setminus \{0\} \;\big|\; \nu(z') = [z] \,\big\} = \big\{\, z' \in \mathbb{C}^{n+1} \setminus \{0\} \;\big|\; z' \sim z \,\big\} = \big\{\, \lambda \cdot z \mid \lambda \in \mathbb{C}^* \,\big\}$$

This is equal to the line in $\mathbb{C}^{n+1}$ defined by $[z]$ without the origin, which is not open in $\mathbb{C}^{n+1} \setminus \{0\}$, thus ν is not continuous with respect to the discrete topology.

Proposition :
ν is an open map and hence : $U \subseteq \mathbb{P}^n(\mathbb{C})$ is open $\Leftrightarrow$ $\exists\, W \subseteq \mathbb{C}^{n+1} \setminus \{0\}$ open such that $\nu(W) = U$.

Proof. Let $V \subseteq \mathbb{C}^{n+1} \setminus \{0\}$ be open.
We have to show that $\nu(V)$ is open in $\mathbb{P}^n(\mathbb{C})$, i.e. that $\nu^{-1}\big(\nu(V)\big)$ is open in $\mathbb{C}^{n+1} \setminus \{0\}$. But

$$\nu^{-1}\big(\nu(V)\big) = \bigcup_{\lambda \in \mathbb{C}^*} \lambda \cdot V \tag{5.1}$$

$\subseteq$: if $\nu(x) \in \nu(V)$, then $\exists\, v \in V$ s.t. $\nu(x) = \nu(v) \Leftrightarrow [x] = [v] \Leftrightarrow x \sim v \Leftrightarrow \exists\, \lambda \in \mathbb{C}^*$ s.t. $x = \lambda\, v \Rightarrow x \in \lambda \cdot V$
$\supseteq$: if $\exists\, \lambda \in \mathbb{C}^*$ s.t. $x \in \lambda \cdot V$, then $\exists\, v \in V$ s.t. $x = \lambda\, v \Rightarrow [x] = [v] \Leftrightarrow \nu(x) = \nu(v) \in \nu(V) \Rightarrow x \in \nu^{-1}\big(\nu(V)\big)$
Note that $\nu^{-1}\big(\nu(V)\big)$ represents a cone in $\mathbb{C}^{n+1}$ (see figure 5.3).

And $\lambda \cdot V$ is open in $\mathbb{C}^{n+1} \setminus \{0\}$ since the map $\varphi : V \to \lambda \cdot V$, $\varphi(v) = \lambda\, v$ is a homeomorphism if $\lambda \neq 0$, so

$$\lambda \cdot V = \varphi(V) \text{ is open } \Rightarrow \bigcup_{\lambda \in \mathbb{C}^*} \lambda \cdot V = \nu^{-1}\big(\nu(V)\big) \text{ is open in } \mathbb{C}^{n+1} \setminus \{0\} \text{ as a union of open sets}$$

It follows that : $U \subseteq \mathbb{P}^n(\mathbb{C})$ is open $\Leftrightarrow$ $\exists\, W$ open in $\mathbb{C}^{n+1} \setminus \{0\}$ such that $W = \nu^{-1}(U)$, thus $\nu(W) = U$:

$$\nu(W) = \nu\big(\nu^{-1}(U)\big) = \mathrm{id}(U) = U \quad \text{since } \nu \text{ is surjective and has thus a right inverse}$$

And finally $\nu(W)$ is open in $\mathbb{P}^n(\mathbb{C})$ since W is open and ν is an open map. □

5.3.2 Hausdorff

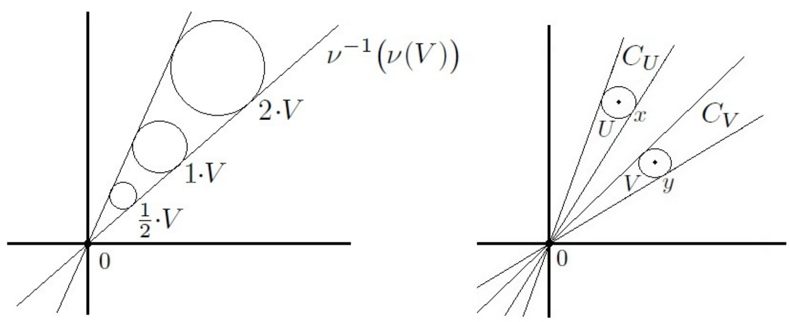
Exercise : Show that $\mathbb{P}^n(\mathbb{C})$ with respect to the quotient topology is a Hausdorff space.

Let $[x],[y]\in\mathbb{P}^n(\mathbb{C})$ such that $[x]\neq[y]\ \Rightarrow\ \exists\, x,y\in\mathbb{C}^{n+1}\setminus\{0\}$ where x and y do not lie on the same line. Let

$$\ell_x := \{\,\lambda\, x \mid \lambda\in\mathbb{C}^*\,\} \quad , \quad \ell_y := \{\,\lambda\, y \mid \lambda\in\mathbb{C}^*\,\}$$

$\ell_x\cap\ell_y=\emptyset$. Now choose U open in $\mathbb{C}^{n+1}\setminus\{0\}$ such that $x\in U$ and $U\cap\ell_y=\emptyset$ (which is possible since x and y are not on the same line) as in figure 5.3. Denote the cone through U by $C_U=\{\,\lambda\, z\mid\lambda\in\mathbb{C}^*,\ z\in U\,\}$. Then choose V open in $\mathbb{C}^{n+1}\setminus\{0\}$ such that $y\in V$ and $V\cap C_U=\emptyset$. We obtain that $C_U\cap C_V=\emptyset$ too. Hence $\nu(U)$ and $\nu(V)$ are open neighborhoods (ν is open) containing $[x]$ and $[y]$ respectively. And $\nu(U)\cap\nu(V)=\emptyset$ because the corresponding cones are disjoint : $\nu^{-1}\big(\nu(U)\big)=C_U$, $\nu^{-1}\big(\nu(V)\big)=C_V$ and $C_U\cap C_V=\emptyset$.

Figure 5.3: preimages under the quotient map ν correspond to cones in $\mathbb{C}^{n+1}$



Note : Not any quotient space of a Hausdorff space is again Hausdorff. Here we really use the fact that we are dealing with $\mathbb{C}^{n+1}$ (which is Hausdorff and regular) and an open projection map.

5.3.3 Compactness

We shall show that $\mathbb{P}^n(\mathbb{C})$ is compact. Let

$$\mathbb{C}^{n+1}\setminus\{0\}\xrightarrow{\ \varphi_1\ } S^{2n+1}\xrightarrow{\ \varphi_2\ }\mathbb{P}^n(\mathbb{C}), \qquad \nu = \varphi_2\circ\varphi_1$$

where $\mathbb{C}^{n+1}\setminus\{0\}\cong\mathbb{R}^{2n+2}\setminus\{0\}$ is non-compact, $\varphi_1(z)=\frac{z}{\|z\|}$ and $\varphi_2(y)=[y],\ \forall\, y\in S^{2n+1}$.
This diagram commutes because

$$(\varphi_2\circ\varphi_1)(z)=\big[\tfrac{z}{\|z\|}\big]=[z]=\nu(z)$$

φ_1 is (obviously) continuous and φ_2 is continuous since similarly as in (5.1), $\forall\, U\subseteq\mathbb{P}^n(\mathbb{C})$ open :

$$\varphi_2^{-1}(U)=\bigcup_{\lambda\in\mathbb{C}^*}\tfrac{\lambda}{|\lambda|}\cdot U=\bigcup_{|\alpha|=1}\alpha\cdot U$$

In addition φ_2 is surjective since for $[z]\in\mathbb{P}^n(\mathbb{C})$, we have $\varphi_2\big(\frac{z}{\|z\|}\big)=[z]$. Hence since S^{2n+1} is compact (closed and bounded in $\mathbb{R}^{2n+2}\setminus\{0\}$), we obtain that $\mathbb{P}^n(\mathbb{C})=\operatorname{im}\varphi_2=\varphi_2(S^{2n+1})$ is also compact (as recalled in 3.2.1).

5.3.4 Atlas

Now we are going to define charts and chart domains for $\mathbb{P}^n(\mathbb{C})$.
Let $\{W_i\}_{i\in J}$ be an open covering of $M=\mathbb{C}^{n+1}\setminus\{0\}$. Then $\{\nu(W_i)\}_{i\in J}$ will be an open covering of $\mathbb{P}^n(\mathbb{C})$ since ν is open and surjective. However it will not be a coordinate covering : since ν is not bijective, the coordinates on $\mathbb{C}^{n+1}\setminus\{0\}$ do not induce coordinates of $\mathbb{P}^n(\mathbb{C})$. "ν^{-1}" does not define a global chart of $\mathbb{P}^n(\mathbb{C})$, so this approach is not helpful for defining a manifold structure.

Definition :
To any $[z] \in \mathbb{P}^n(\mathbb{C})$, we associate the *homogeneous coordinates*

$$[z] = (z_0 : z_1 : z_2 : \ldots : z_n) \in \mathbb{P}^n(\mathbb{C})$$

where $(z_0, z_1, \ldots, z_n)$ are the coordinates of a representative $z \in \mathbb{C}^{n+1} \setminus \{0\}$ of $[z]$, modulo the equivalence relation, i.e. $z' \in [z] \;\Leftrightarrow\; \exists\, \lambda \in \mathbb{C}^*$ such that $z' = \lambda\, z$. Hence

$$(z_0 : z_1 : z_2 : \ldots : z_n) = (\lambda\, z_0 : \lambda\, z_1 : \lambda\, z_2 : \ldots : \lambda\, z_n)\ ,\ \forall\, \lambda \in \mathbb{C}^*$$

Homogeneous coordinates are therefore not unique in general. In particular $(0 : 0 : \ldots : 0) \notin \mathbb{P}^n(\mathbb{C})$ since the origin $(0, 0, \ldots, 0)$ was excluded from $\mathbb{C}^{n+1}$. If $[z] = (z_0 : z_1 : \ldots : z_n)$, then $\exists\, i \in \{0, \ldots, n\}$ such that $z_i \neq 0$.

Now we define for any $k \in \{0, \ldots, n\}$ the subset

$$U_k := \left\{\, [z] = (z_0 : z_1 : \ldots : z_n) \in \mathbb{P}^n(\mathbb{C}) \;\middle|\; z_k \neq 0 \,\right\}$$

This is well-defined since if $[z] = (z_0 : \ldots : z_n) \in \mathbb{P}^n(\mathbb{C})$ such that $z_k \neq 0$ in this homogeneous representation, then $z_k \neq 0$ in all homogeneous representations since we are only allowed to multiply by non-zero constants λ. Thus U_k is a well-defined subset of $\mathbb{P}^n(\mathbb{C})$.
In particular if $[z] \in U_k$, then we can choose $\lambda = \frac{1}{z_k}$ and take as homogeneous coordinates

$$[z] = (z_0 : z_1 : \ldots : z_k : \ldots : z_n) = \left(\frac{z_0}{z_k} : \frac{z_1}{z_k} : \ldots : 1 : \ldots : \frac{z_n}{z_k} \right)$$

Indeed, in $[z]$ (set of points in $\mathbb{C}^{n+1}$) there is a unique representation of $[z]$ with a 1. Hence if $[z] \in U_k$, we can assume without loss of generality that $[z] = (y_0 : y_1 : \ldots : 1 : \ldots : y_n)$ for some $y_i \in \mathbb{C}$. Moreover this representation of $[z]$ is now unique.

Proposition :
U_k is open in $\mathbb{P}^n(\mathbb{C})$, $\forall\, k \in \{0, \ldots, n\}$.

Proof.

$$\nu^{-1}(U_k) = \left\{\, z \in \mathbb{C}^{n+1} \setminus \{0\} \;\middle|\; \nu(z) = [z] \in U_k \,\right\} = \left\{\, z \in \mathbb{C}^{n+1} \setminus \{0\} \;\middle|\; z_k \neq 0 \,\right\}$$

and this is an open subset of $\mathbb{C}^{n+1} \setminus \{0\}$, thus U_k is open. □

Proposition :
$U_k \cong \mathbb{C}^n$, $\forall\, k \in \{0, \ldots, n\}$. Moreover this equivalence is homeomorphic.

Proof. For $[z] \in U_k$, we define the maps

$$\varphi_k \;:\; U_k \longrightarrow \mathbb{C}^n \;:\; [z] \longmapsto \left(\frac{z_0}{z_k}, \frac{z_1}{z_k}, \ldots, \widehat{k}, \ldots, \frac{z_n}{z_k} \right)$$
$$\psi_k \;:\; \mathbb{C}^n \longrightarrow U_k \;:\; (w_1, \ldots, w_n) \longmapsto (w_1 : w_2 : \ldots : w_{k-1} : 1 : w_k : \ldots : w_n)$$

φ_k is well-defined since $\frac{\lambda z_i}{\lambda z_k} = \frac{z_i}{z_k}$, $\forall\, i \neq k$.
This already gives a 1-to-1 correspondence since $\varphi_k \circ \psi_k = \mathrm{id}_{\mathbb{C}^n}$ and $\psi_k \circ \varphi_k = \mathrm{id}_{U_k} \;\Rightarrow\; \psi_k = \varphi_k^{-1}$.
It remains to check that φ_k is bicontinuous, i.e. a homeomorphism.

$-$ φ_k is continuous : let $W \subseteq \mathbb{C}^n$ be open.
We have to show that $\varphi_k^{-1}(W)$ is open in $U_k \;\Leftrightarrow\; \varphi_k^{-1}(W)$ is open in $\mathbb{P}^n(\mathbb{C})$ since U_k is open in $\mathbb{P}^n(\mathbb{C})$ and $\varphi_k^{-1}(W)$ is by definition already contained in U_k. But

$$\nu^{-1}\big(\varphi_k^{-1}(W)\big) = \big(\varphi_k \circ \nu'\big)^{-1}(W) \qquad \text{where } \nu' = \nu_{|\nu^{-1}(U_k)}$$

where one needs to take the restriction of ν, otherwise the composition with φ_k does not make sense.

The map
$$\varphi_k \circ \nu' \;:\; \nu^{-1}(U_k) \subseteq \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{C}^n \;:\; (z_0, z_1, \ldots, z_n) \longmapsto \left(\frac{z_0}{z_k}, \frac{z_1}{z_k}, \ldots, \widehat{k}, \ldots, \frac{z_n}{z_k}\right)$$
is continuous since $z_k \neq 0$ on $\nu^{-1}(U_k)$, thus $\nu^{-1}\big(\varphi_k^{-1}(W)\big)$ is open in $\mathbb{C}^{n+1} \setminus \{0\} \;\Rightarrow\; \varphi_k^{-1}(W)$ is open in U_k.

− φ_k is open : let $U \subseteq U_k$ be open. We have to show that $\varphi_k(U)$ is open in $\mathbb{C}^n$.
$$\varphi_k(U) = \left\{\, y \in \mathbb{C}^n \;\middle|\; y = \varphi_k\big([z]\big) \text{ for some } [z] \in U \,\right\} = \left\{ \left(\frac{z_0}{z_k}, \frac{z_1}{z_k}, \ldots, \widehat{k}, \ldots, \frac{z_n}{z_k}\right) \;\middle|\; [z] \in U \;\Leftrightarrow\; z \in \nu^{-1}(U) \right)$$
By denoting $f_k : \mathbb{C}^{n+1} \to \mathbb{C}^n$, $f_k(z_0, \ldots, z_n) = \big(\frac{z_0}{z_k}, \ldots, \widehat{k}, \ldots, \frac{z_n}{z_k}\big)$, we obtain that
$$\varphi_k(U) = \left\{\, f_k(z_0, \ldots, z_n) \;\middle|\; z \in \nu^{-1}(U) \,\right\} = f_{k|\nu^{-1}(U_k)}\big(\nu^{-1}(U)\big)$$
where $\nu^{-1}(U)$ is open since U is open and ν continuous and f_k is an open map since $(x, y) \mapsto \frac{x}{y}$ is open on $\mathbb{C}^2$, thus the restriction of f_k to an open subset of $\mathbb{C}^{n+1}$ is still open. Finally $\varphi_k(U)$ is open in $\mathbb{C}^n$. □

5.3.5 Connectedness

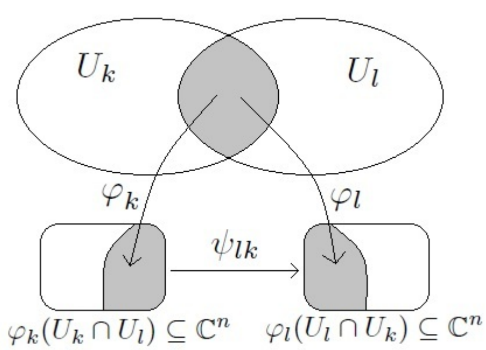
$\mathbb{P}^n(\mathbb{C})$ is connected because ν is surjective : $\mathbb{P}^n(\mathbb{C}) = \operatorname{im}\nu = \nu\big(\mathbb{C}^{n+1} \setminus \{0\}\big)$, where $\mathbb{C}^{n+1} \setminus \{0\}$ is connected and ν is continuous, hence $\operatorname{im}\nu$ is also connected.

5.3.6 Topological structure

$\mathbb{P}^n(\mathbb{C})$ is a topological manifold. The U_k are open in the projective space and cover $\mathbb{P}^n(\mathbb{C})$ since : if $[z] \in \mathbb{P}^n(\mathbb{C})$, then $[z] \neq (0 : \ldots : 0)$, so $\exists\, l \in \{0, \ldots, n\}$ such that $z_l \neq 0 \;\Rightarrow\; [z] \in U_l$.
$$\Rightarrow\; \{U_k\}_{k=0,\ldots,n} \text{ open covering } : \; \bigcup_{k=0}^{n} U_k = \mathbb{P}^n(\mathbb{C})$$
$\{U_k\}_{k=0,\ldots,n}$ is called the *standard affine covering* of $\mathbb{P}^n(\mathbb{C})$. Hence $\mathbb{P}^n(\mathbb{C})$ already defines a topological manifold of complex dimension n (real dimension $2n$) since we can take the homeomorphism φ_k to be the associated chart to the domain U_k. So $\mathcal{U} = (U_k, \varphi_k)_{k=0,\ldots,n}$ is a continuous atlas of $\mathbb{P}^n(\mathbb{C})$.

5.3.7 Complex structure

Figure 5.4: the transition maps ψ_{lk} are holomorphic



Consider figure 5.4. We have to show that the transition maps
$$\psi_{lk} := \varphi_l \circ \varphi_k^{-1} \;:\; \varphi_k(U_k \cap U_l) \subseteq \mathbb{C}^n \longrightarrow \varphi_l(U_l \cap U_k) \subseteq \mathbb{C}^n$$
are biholomorphic. We assume without restriction that $l < k$, so
$$U_k \cap U_l = \left\{\, [z] \in \mathbb{P}^n(\mathbb{C}) \;\middle|\; z_k \neq 0 \text{ and } z_l \neq 0 \,\right\} \quad \text{and} \quad \varphi_k(U_k \cap U_l) = \left\{\, (w_1, \ldots, w_n) \in \mathbb{C}^n \;\middle|\; w_l \neq 0 \,\right\}$$

Let $(w_1, \dots, w_n) \in \varphi_k(U_k \cap U_l)$, so $w_l \neq 0$ and by applying $\psi_{lk} = \varphi_l \circ \varphi_k^{-1}$ we obtain

$$\begin{aligned}\psi_{lk}(w_1, \dots, w_n) &= \varphi_l\big(\varphi_k^{-1}(w_1, \dots, w_n)\big) = \varphi_l(w_1 : \dots : w_{l-1} : w_l : w_{l+1} : \dots : w_{k-1} : 1 : w_k : \dots : w_n) \\ &= \varphi_l\Big(\frac{w_1}{w_l} : \dots : \frac{w_{l-1}}{w_l} : 1 : \frac{w_{l+1}}{w_l} : \dots : \frac{w_{k-1}}{w_l} : \frac{1}{w_l} : \frac{w_k}{w_l} : \dots : \frac{w_n}{w_l}\Big) \\ &= \Big(\frac{w_1}{w_l}, \dots, \frac{w_{l-1}}{w_l}, \frac{w_{l+1}}{w_l}, \dots, \frac{w_{k-1}}{w_l}, \frac{1}{w_l}, \frac{w_k}{w_l}, \dots, \frac{w_n}{w_l}\Big) \in \mathbb{C}^n\end{aligned}$$

which is well-defined since $w_l \neq 0$. Thus ψ_{lk} is holomorphic from $\mathbb{C}^n \setminus \{\, w \in \mathbb{C}^n \mid w_l = 0 \,\}$ to $\mathbb{C}^n$ as the map $w \mapsto \frac{w_i}{w_l}$ is holomorphic in w, $\forall\, i \neq l$. This now shows that $\mathbb{P}^n(\mathbb{C})$ is a complex manifold of complex dimension n.

5.3.8 Particular case

$\mathbb{CP}^1 := \mathbb{P}^1(\mathbb{C})$ is called the *Riemann sphere* or the *projective line*. Since there are only 2 chart domains in this case, it can be decomposed as :

$$\begin{aligned} U_0 &= \big\{\, (z_0 : z_1) \in \mathbb{P}^1(\mathbb{C}) \;\big|\; z_0 \neq 0 \,\big\} = \big\{\, (1 : w) \;\big|\; w \in \mathbb{C} \,\big\} \cong \mathbb{C} \\ \mathbb{P}^1(\mathbb{C}) \setminus U_0 &= \big\{\, (0 : z_1) \;\big|\; z_1 \in \mathbb{C}^* \,\big\} = \big\{\, (0 : 1) \,\big\} = \{\, \text{pt} \,\}\end{aligned}$$

Since $(0:0) \notin \mathbb{P}^1(\mathbb{C})$, $\mathbb{P}^1(\mathbb{C}) \setminus U_0$ consists of exactly 1 point. This point $(0:1)$ is often also denoted by ∞ and called *the point at infinity*. Hence we obtain the decomposition

$$\mathbb{P}^1(\mathbb{C}) \cong \mathbb{C} \,\dot\cup\, \{\infty\} \cong S^2 \tag{5.2}$$

The first isomorphy is nothing else than the Alexandroff compactification of $\mathbb{C}$ and the second one is obtained by the stereographic projection since topologically $\mathbb{R}^2 \cup \{\infty\} \cong S^2$. Of course (5.2) also holds true with $U_1 \cong \mathbb{C}$ and $\mathbb{P}^1(\mathbb{C}) \setminus U_1 = \{\infty\}$ where $(1:0) = \infty$ in this case.
Using such a compactification argument in order to show that $\mathbb{P}^n(\mathbb{C})$ is compact however only works for $n = 1$. In general one has the following decomposition (which is not a 1-point compactification) :

$$\mathbb{P}^n(\mathbb{C}) \cong \mathbb{C}^n \,\dot\cup\, \mathbb{P}^{n-1}(\mathbb{C}) \cong \mathbb{C}^n \,\dot\cup\, \mathbb{C}^{n-1} \,\dot\cup\, \mathbb{P}^{n-2}(\mathbb{C}) \cong \dots \cong \mathbb{C}^n \,\dot\cup\, \mathbb{C}^{n-1} \,\dot\cup\, \dots \,\dot\cup\, \mathbb{C}^2 \,\dot\cup\, \mathbb{C} \,\dot\cup\, \{\infty\}$$

where $\mathbb{C}^n \cong U_k$, $\mathbb{C}^{n-1} \cong \mathbb{P}^n(\mathbb{C}) \setminus U_k$, etc. This shows in particular that there is no unique way to set ∞ ; one always has to specify if one chooses $\infty = (1:0:\dots:0)$, $\infty = (0:1:\dots:0)$, ..., or $\infty = (0:0:\dots:1)$.

5.4 Submanifolds of $\mathbb{P}^n(\mathbb{C})$

5.4.1 Proposition

Consider the compact complex manifold $\mathbb{P}^n(\mathbb{C})$ and let N be a connected submanifold of $\mathbb{P}^n(\mathbb{C})$.
Then N is also a compact complex manifold.

Proof. Submanifolds are closed by definition and a closed set in a compact set is again compact. Moreover N is a complex manifold because it is connected (see section 4.4.3).

More generally : Let X be compact and $F \subseteq X$ be closed.
We want to show that F is also compact, i.e. if $\{U_i\}_{i \in J}$ is an open covering of F, it admits a finite subcover.
U_i open in F means that $U_i = V_i \cap F$ for some V_i open in X, $\forall\, i \in J$. Since $\{U_i\}_{i \in J}$ covers F, we get

$$F = \bigcup_{i \in J} U_i = \bigcup_{i \in J} (V_i \cap F) = \bigcup_{i \in J} V_i \cap F \quad \Rightarrow \quad F \subseteq \bigcup_{i \in J} V_i$$

Hence $\{V_i\}_{i \in J} \cup \{X \setminus F\}$ is an open covering of X (F is closed). As X is compact, finitely many of them will cover X : $\exists\, I \subset J$ finite such that $\{V_i\}_{i \in I} \cup \{X \setminus F\}$ covers X, so the corresponding $\{U_i\}_{i \in I}$ will cover F. $\square$

5.4.2 Examples

1) zero sets of locally holomorphic functions on $\mathbb{P}^n(\mathbb{C})$
2) zero sets of homogeneous polynomials

We know that $\mathbb{P}^n(\mathbb{C})$ is a compact manifold. In section 6.1, we will show the following properties :
− On a compact complex manifold there are no globally defined non-constant holomorphic functions.
− Let $M \subseteq \mathbb{C}^n$ be a (connected) compact submanifold of $\mathbb{C}^n$. Then M is a point : $\exists\, c \in \mathbb{C}^n$ such that $M = \{c\}$.
Thus it does not make much sense to consider zero sets of globally holomorphic functions on $\mathbb{P}^n(\mathbb{C})$. Note that 2) is not a particular case of 1) since polynomials are not functions on the projective space $\mathbb{P}^n(\mathbb{C})$!

A polynomial on $\mathbb{C}^{n+1}$ with coefficients in $\mathbb{C}$ is called *homogeneous of degree m* if it is of the form

$$f(X_0, X_1, \ldots, X_n) = \sum_{j_0+\ldots+j_n=m} c_{j_0,\ldots,j_n} \cdot X_0^{j_0} \cdot X_1^{j_1} \cdot \ldots \cdot X_n^{j_n} \quad , \; c_{j_0,\ldots,j_n} \in \mathbb{C}$$

Let $g(X_0, X_1) = X_0 X_1$, then is homogeneous of degree 2. But g is not a function on $\mathbb{P}^1(\mathbb{C})$ since evaluation is not well-defined. Indeed let $[z] \in \mathbb{P}^1(\mathbb{C})$ be represented by $z = (z_0, z_1) \in [z] \;\Rightarrow\; g(z) = z_0\, z_1$. But if $z' = (z'_0, z'_1) \in [z]$ with $z'_i = \lambda\, z_i$, then $[z] = [z'] = (z_0 : z_1) = (\lambda z_0 : \lambda z_1)$, but $g(z') = (\lambda z_0)\,(\lambda z_1) = \lambda^2 \cdot z_0\, z_1 \neq g(z)$ in general.

The zero set of a homogeneous polynomial is however well-defined : if f is homogeneous of degree m, then

$$\forall\, \lambda \in \mathbb{C} \;:\; f(\lambda\, z) = \lambda^m \cdot f(z)\,, \text{ hence } \quad f(z) = 0 \;\Leftrightarrow\; f(\lambda\, z) = 0, \; \forall\, \lambda \in \mathbb{C}^*$$

5.4.3 Definition

A *smooth projective variety* of $\mathbb{P}^n(\mathbb{C})$ is a non-singular (see 5.2.2) zero set of finitely many homogeneous polynomials. Hence any smooth projective variety is a (closed) submanifold of $\mathbb{P}^n(\mathbb{C})$. And we even have :

5.4.4 Chow's Theorem

Any (closed) submanifold of $\mathbb{P}^n(\mathbb{C})$ is a smooth projective variety. We will not prove this result.

5.5 Complex tori

Complex tori are examples of compact complex manifolds which are not submanifolds of $\mathbb{P}^n(\mathbb{C})$.
For $n = 1$, a tori T can be visualized as in figure 1, p. 3 (only the surface ; it is "hollow" inside). T is compact as a subset of $\mathbb{R}^3$ and can be embedded into $\mathbb{R}^4 \cong \mathbb{C}^2$. But because of 5.4.2, there are no non-trivial compact complex submanifolds in $\mathbb{C}^2$. So there is no chance to see the complex structure on this picture ; one only sees the topology. As we will see in the following : a way of constructing T starting from $\mathbb{C}$ is to define $T = \mathbb{C} \,/\, L$ where $L = \langle\, 1\,,\, \tau\, \rangle_{\mathbb{Z}}$ is the lattice generated by 1 and $\tau \in \mathbb{C}$, $\operatorname{Im} \tau > 0$ (see figure 5.5).

5.5.1 Definitions

$L \subset \mathbb{C}^n$ is called a *lattice* in $\mathbb{C}^n$ of real dimension $2n$ if it is of the form

$$L = \langle\, \omega_1\,,\, \omega_2\,,\, \ldots\;,\, \omega_{2n}\, \rangle_{\mathbb{Z}} = \left\{\; \sum_{j=1}^{2n} m_j\, \omega_j \;\Big|\; m_j \in \mathbb{Z} \right\}$$

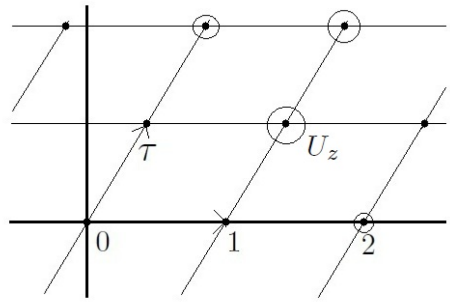
where $\omega_i \in \mathbb{C}^n$ are linearly independent over $\mathbb{R}$.

Remark :
The ω_i cannot be linearly independent over $\mathbb{C}$ since $m > n$ vectors in an n-dimensional vector space are always linearly dependent. But it is possible over $\mathbb{R}$ since $\mathbb{C}^n$ has dimension $2n$ over $\mathbb{R}$. Consider e.g.

- 1 and τ, $\operatorname{Im} \tau > 0$, are linearly dependent over $\mathbb{C}$ since $1 + \frac{-1}{\tau} \cdot \tau = 0$
- over $\mathbb{R}$: let $\alpha, \beta \in \mathbb{R}$ such that $\alpha \cdot 1 + \beta \cdot \tau = 0 \;\Rightarrow\; \beta \cdot \tau \in \mathbb{C} \setminus \mathbb{R}$ since $\operatorname{Im} \tau > 0$, hence $\beta = 0$ and also $\alpha = 0$

L is a *discrete subspace* of $\mathbb{C}^n$, i.e. $\forall\, z \in L$, there is a neighborhood U_z of z in $\mathbb{C}^n$ such that $L \cap U_z = \{z\}$ as it can be seen in figure 5.5.

Figure 5.5: a (discrete) lattice generated by 1 and $\tau \in \mathbb{C}$, $\operatorname{Im} \tau > 0$



Let $z, z' \in \mathbb{C}^n$. We define the relation $z \sim_L z' \;\Leftrightarrow\; z' - z \in L \;\Leftrightarrow\; \exists\, \omega \in L$ such that $z' = z + \omega$.

Exercise :
$\sim_L$ is an equivalence relation on $\mathbb{C}^n$.
− reflexive : $z \sim_L z$ for $\omega = 0 \in L$
− symmetric : $z' = z + \omega \;\Rightarrow\; z = z' + (-\omega)$ since $\omega \in L \;\Rightarrow\; -\omega \in L$
− transitive : $z' = z + \omega$, $z'' = z' + \omega' \;\Rightarrow\; z'' = z + (\omega + \omega')$ with $\omega, \omega' \in L \;\Rightarrow\; \omega + \omega' \in L$

Hence one can define equivalence classes of $\sim_L$ and the quotient of $\mathbb{C}^n$ by $\sim_L$:

$$[z] = \left\{\, z' \in \mathbb{C} \mid z' \sim_L z \,\right\} = \left\{\, z + \omega \mid \omega \in L \,\right\}$$

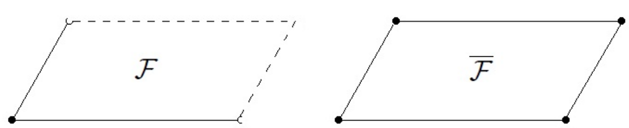
As a set, $\mathbb{C}^n / \sim_L$ can now be identified with $\mathcal{F}$, set consisting of all points from $\mathbb{C}^n$ which are not equivalent to any other element in $\mathcal{F}$. $\mathcal{F}$ is called the *fundamental parallelogram* of L and given by (see figure 5.6)

$$\mathcal{F} = \left\{\, \sum_{j=1}^{2n} \alpha_j\, \omega_j \;\middle|\; 0 \leq \alpha_j < 1 \,\right\}$$

$\mathcal{F}$ is neither open nor closed in $\mathbb{C}^n$, but it gives a 1-to-1 correspondence to the equivalence classes of $\sim_L$. One also considers the (topological) closure of $\mathcal{F}$ (see figure 5.6), which differs from $\mathcal{F}$ just by a set of measure zero :

$$\overline{\mathcal{F}} = \left\{\, \sum_{j=1}^{2n} \alpha_j\, \omega_j \;\middle|\; 0 \leq \alpha_j \leq 1 \,\right\} \;:\; \text{closed in } \mathbb{C}^n$$

Figure 5.6: the fundamental parallelogram $\mathcal{F}$ and its closure



Definition :
The *n-dimensional complex torus* T^n is defined as the quotient of $\mathbb{C}^n$ by the equivalence relation $\sim_L$, i.e.

$$T^n := \mathbb{C}^n / \sim_L = \mathbb{C}^n / L \qquad \Rightarrow \quad T^n \cong \mathcal{F} \text{ as sets}$$

T^n and $\mathcal{F}$ are in bijection only and don't even have the same topological properties. $\mathcal{F}$ is for example simply connected, but the torus is not.

5.5.2 Topology

We again endow T^n with the quotient topology, i.e. with the finest topology such that the natural projection map $\nu : \mathbb{C}^n \to T^n : z \mapsto [z]$ is continuous, so

$$W \subseteq T^n \text{ is open } \;\Leftrightarrow\; \nu^{-1}(W) \text{ is open in } \mathbb{C}^n$$

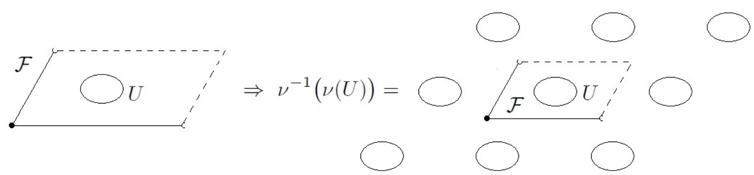
Proposition : ν is an open map.

Proof. Let $U \subseteq \mathbb{C}^n$ be open. ν is open $\Leftrightarrow$ $\nu(U)$ is open in T^n $\Leftrightarrow$ $\nu^{-1}\big(\nu(U)\big)$ is open in $\mathbb{C}^n$. But

$$\Rightarrow \; \nu^{-1}\big(\nu(U)\big) = \bigcup_{\omega \in L} (U + \omega)$$

$\subseteq$: if $\nu(x) \in \nu(U)$, then $[x] = [u]$ for some $u \in U$ $\Rightarrow$ $\exists\, \omega \in L$ such that $x = u + \omega$
$\supseteq$: if $x = u + \omega$ for some $u \in U$, $\omega \in L$, then $\nu(x) = [x] = [u + \omega] = [u] \in \nu(U)$ $\Rightarrow$ $x \in \nu^{-1}\big(\nu(U)\big)$

Figure 5.7: preimages under the projection map ν correspond to translations



And $U + \omega$ is open in $\mathbb{C}^n$ since the map $\varphi : U \to U + \omega$, $\varphi(u) = u + \omega$ is a homeomorphism $\forall\, \omega \in L$, so

$$U + \omega = \varphi(U) \text{ is open } \;\Rightarrow\; \bigcup_{\omega \in L} \big(U + \omega\big) = \nu^{-1}\big(\nu(U)\big) \text{ is open in } \mathbb{C}^n \text{ as a union of open sets}$$

□

5.5.3 Compactness

T^n is compact (hence also paracompact). For this, let $\{U_i\}_{i \in I}$ be an open covering of T^n.
By definition the $\nu^{-1}(U_i)$ are open in $\mathbb{C}^n$ and they form an open covering of $\mathbb{C}^n$ since

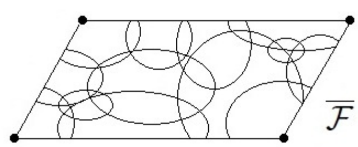
$$\bigcup_{i \in I} \nu^{-1}(U_i) = \nu^{-1}\Big(\bigcup_{i \in I} U_i\Big) = \nu^{-1}(T^n) = \mathbb{C}^n \qquad \Rightarrow \quad \overline{\mathcal{F}} = \bigcup_{i \in I} \big(\nu^{-1}(U_i) \cap \overline{\mathcal{F}}\,\big)$$

$\overline{\mathcal{F}}$ being compact, one can extract a finite subcovering $\{\nu^{-1}(U_j) \cap \overline{\mathcal{F}}\,\}_{j \in J}$ (see figure 5.8), thus $T^n = \bigcup_{j \in J} U_j$:

$$[z] \in T^n \;\Rightarrow\; \exists\, z \in [z],\; \omega \in L \text{ such that } z + \omega \in \mathcal{F} \subset \bigcup_{j \in J} \big(\nu^{-1}(U_j) \cap \overline{\mathcal{F}}\,\big)$$

i.e. $\exists\, j \in J$ such that $z + \omega \in \nu^{-1}(U_j)$ $\Rightarrow$ $[z] = [z + \omega] \in \nu\big(\nu^{-1}(U_j)\big) = U_j$ since ν is surjective.

Figure 5.8: a finite open covering of $\overline{\mathcal{F}}$



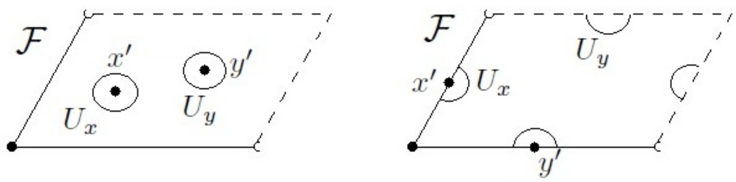
5.5.4 Hausdorff

T^n is Hausdorff. For this, let $[x], [y] \in T^n$ such that $[x] \neq [y]$. $[x]$ and $[y]$ are represented by $x, y \in \mathbb{C}^n$ and $\exists\, \omega, \omega' \in L$ such that $x' = x + \omega$, $y' = y + \omega'$ and $x', y' \in \mathcal{F}$ with $x' \neq y'$ since $[x] \neq [y]$.
Assume first that $x, y \in \mathcal{F}^\circ$ (interior of $\mathcal{F}$). Then there are open neighborhoods U_x and U_y of x' and y' such that $U_x \cap U_y = \emptyset$ since $\mathcal{F}^\circ \subset \mathbb{C}^n$ is Hausdorff (see figure 5.9). ν is open, so $\nu(U_x)$ and $\nu(U_y)$ are open neighborhoods of $[x] = [x']$ and $[y] = [y']$ in T^n. Moreover $\nu(U_x) \cap \nu(U_y) = \emptyset$ since otherwise

$$[z] \in \nu(U_x) \cap \nu(U_y) \;\Rightarrow\; \exists\, w, w' \in L \text{ such that } z + w \in U_x,\; z + w' \in U_y$$

which is a contradiction since there is a unique representative of $[z]$ in $\mathcal{F}$.
The proof is similar if at least one of x' or y' lies in $\mathcal{F} \setminus \mathcal{F}^\circ$. The only difference is that the identification will "disconnect" the neighborhoods in $\mathbb{C}^n$ (see figure 5.9). But $\nu(U_x)$ and $\nu(U_y)$ will not change in this case.

Figure 5.9: open neighborhoods of x' and y' inside of $\mathcal{F}$



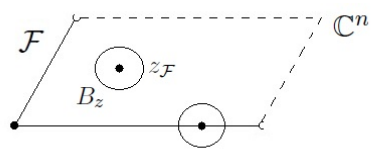
5.5.5 Connectedness

T^n is connected since it is the image of the continuous surjective map $\nu : \mathbb{C}^n \to T^n$ where $\mathbb{C}^n$ is connected, hence $\operatorname{im}\nu = \nu(\mathbb{C}^n) = T^n$ is connected as well.

5.5.6 Complex structure

We have to define charts and chart domains for T^n. In particular we want the map ν to be holomorphic (as defined in 4.3.1) with respect to this atlas.

Let $[z] \in T^n$ and let $z_{\mathcal{F}}$ be the unique representing element of $[z]$ in $\mathcal{F}$. We consider an open ball B_z in $\mathbb{C}^n$ around $z_{\mathcal{F}}$, small enough such that B_z does not meet other points $z' \in [z] : B_z \cap \nu^{-1}\big(\{\nu(z_{\mathcal{F}})\}\big) = \{z_{\mathcal{F}}\}$. Note that B_z is not completely contained in $\mathcal{F}$ if $z_{\mathcal{F}}$ lies on the boundary of $\mathcal{F}$.



Repeating this argument for any $[z] \in T^n$, we obtain a covering of $\overline{\mathcal{F}}$ by these B_z, hence by compactness it suffices to consider only finitely many of them. We denote this finite number of B_z by B_j for $j \in J$ finite. Let

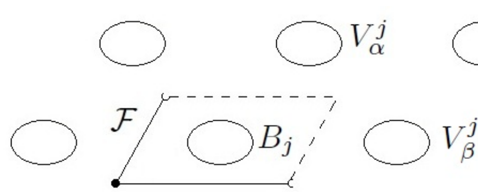
$$\forall\, j \in J \ : \ U_j := \nu(B_j) = \big\{\, [x] \in T^n \;\big|\; x \in B_j \,\big\}$$

U_j is open in T^n since ν is open and $\{U_j\}_{j\in J}$ is an open covering of T^n since

$$\bigcup_{j\in J} U_j = \bigcup_{j\in J} \nu(B_j) = \nu\Big(\bigcup_{j\in J} B_j\Big) = \nu\big(\overline{\mathcal{F}}\,\big) = T^n$$

In addition, $B_j \subsetneq \nu^{-1}(U_j)$. More precisely, we have that $\nu^{-1}(U_j) = \dot{\bigcup}_{\alpha\in L} V_\alpha^j$ (see figure 5.10), where the V_α^j are open in $\mathbb{C}$, $V_0^j = B_j$ and each V_α^j gives a 1-to-1 correspondence with U_j.

Figure 5.10: $\nu^{-1}(U_j)$ consists of several pieces



Now we can define the atlas

$$\begin{aligned} \mathcal{W} = \big(W_{j\alpha}, \varphi_{j\alpha}\big)_{(j,\alpha)\in J\times L} \quad &, \quad W_{j\alpha} = U_j,\ \forall\, \alpha \in L \\ \varphi_{j\alpha} \ : \ U_j \to V_\alpha^j \ : \ [x] \mapsto x_\alpha \quad &, \quad \varphi_{j\alpha}^{-1} \ : \ V_\alpha^j \to U_j \ : \ y \mapsto [y] \end{aligned} \tag{5.3}$$

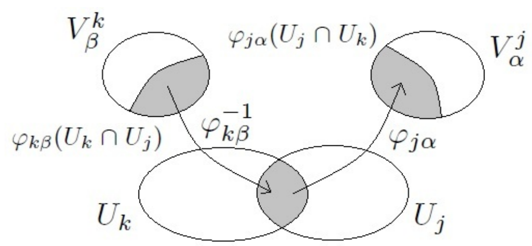
where x_α is the unique representative of $[x]$ in V_α^j, hence $\varphi_{j\alpha}$ is indeed bijective. Moreover $\varphi_{j\alpha}^{-1} = \nu_{|V_\alpha^j}$, so $\varphi_{j\alpha}^{-1}$ is continuous and for $U \subseteq V_\alpha^j$ open, we have $\varphi_{j\alpha}^{-1}(U) = \nu(U)$, which is open, hence $\varphi_{j\alpha}$ is continuous as well.

It remains to check that the maps $\varphi_{j\alpha} \circ \varphi_{k\beta}^{-1} : \varphi_{k\beta}(U_k \cap U_j) \subseteq \mathbb{C}^n \to \varphi_{j\alpha}(U_j \cap U_k) \subseteq \mathbb{C}^n$ are holomorphic. Consider figure 5.11. If $y \in \varphi_{k\beta}(U_k \cap U_j) \subseteq V_\beta^k$, then

$$\big(\varphi_{j\alpha} \circ \varphi_{k\beta}^{-1}\big)(y) = \varphi_{j\alpha}\big(\varphi_{k\beta}^{-1}(y)\big) = \varphi_{j\alpha}\big([y]\big) = y_\alpha \in V_\alpha^j$$

where $y \mapsto y_\alpha$ is just a translation by $\alpha - \beta \in L$, hence it is holomorphic with respect to y.
Finally, $\mathcal{W}$ defines an n-dimensional atlas of T^n, so the torus is a complex manifold of dimension n.

Figure 5.11: the transition maps are holomorphic



With respect to this atlas, one can now show that $\nu : \mathbb{C}^n \to T^n$ is holomorphic (as defined in 4.3.1) because

$$\forall\, (j,\alpha) \in J \times L \ , \ \ \varphi_{j\alpha} \circ \nu \circ \mathrm{id} \ : \ \nu^{-1}(U_j) \subseteq \mathbb{C}^n \longrightarrow V_\alpha^j \subseteq \mathbb{C}^n \ : \ y \longmapsto \varphi_{j\alpha}\big([y]\big) = y_\alpha$$

where $\nu^{-1}(U_j) = \dot{\bigcup}_{\beta \in L} V_\beta^j$ as above. Thus $y \mapsto y_\alpha$ is again a translation, but here the shift depends on y since $\nu^{-1}(U_j)$ is a union of V_β^j (see figure 5.10). This dependence is however holomorphic since $V_\beta^j \cap V_{\beta'}^j = \emptyset$ for $\beta \neq \beta'$, so no discontinuities appear : the shift is constant on any V_β^j. We thus conclude that $\varphi_{j\alpha} \circ \nu$ is holomorphic.

Remark :
By 4.3.2, we can always assume that the atlas $\mathcal{W}$ also contains a centered chart at $[z]$ for any $[z] \in T^n$.

5.5.7 Proposition

Let M_1 and M_2 be 2 complex manifolds. Then $M_1 \times M_2$ carries a natural structure of complex manifolds such that the natural projections $p_i : M_1 \times M_2 \to M_i$ are holomorphic maps and the injections $i_{j,a} : M_j \to M_1 \times M_2$

$$\text{for } j \in \{1,2\},\ a \in M_j \quad : \quad i_{1,a}(m_1) = (m_1, a) \ , \ i_{2,a}(m_2) = (a, m_2)$$

define submanifolds of $M_1 \times M_2$ which are isomorphic to M_j. Furthermore $\dim(M_1 \times M_2) = \dim M_1 + \dim M_2$.

Proof. $m \in M_1 \times M_2 \ \Leftrightarrow \ m = (m_1, m_2)$ for some $m_i \in M_i$
Let $\mathcal{U} = (U_i, \varphi_i)_{i \in I}$ be an atlas of M_1 and $\mathcal{V} = (V_j, \psi_j)_{j \in J}$ be an atlas of M_2. Then we can define an atlas of $M_1 \times M_2$ by setting $\mathcal{U} \times \mathcal{V} := (U_i \times V_j, \varphi_i \times \psi_j)_{(i,j) \in I \times J}$ where $\{U_i \times V_j\}$ is an open covering of $M_1 \times M_2$ and

$$(\varphi_i \times \psi_j)(m) = (\varphi_i \times \psi_j)(m_1, m_2) = \big(\varphi_i(m_1), \psi_j(m_2)\big) \in \mathbb{C}^{\dim M_1} \times \mathbb{C}^{\dim M_2}$$

With respect to this atlas, the projections are holomorphic since in any local coordinate chart, we have

$$\Big(\varphi_k \circ p_1 \circ (\varphi_i \times \psi_j)^{-1}\Big)(x) = \varphi_k\Big(p_1\big(\varphi_i^{-1}(x_1), \psi_j^{-1}(x_2)\big)\Big) = \varphi_k\big(\varphi_i^{-1}(x_1)\big) = \big(\varphi_k \circ \varphi_i^{-1}\big)(x_1)$$
$$\Big(\psi_k \circ p_2 \circ (\varphi_i \times \psi_j)^{-1}\Big)(x) = \psi_k\Big(p_2\big(\varphi_i^{-1}(x_1), \psi_j^{-1}(x_2)\big)\Big) = \psi_k\big(\psi_j^{-1}(x_2)\big) = \big(\psi_k \circ \psi_j^{-1}\big)(x_2)$$

and this is holomorphic since the transition maps are holomorphic. Concerning the injections, we have

$$i_{1,a}(M_1) = M_1 \times \{a\} = p_2^{-1}\big(\{a\}\big) \qquad \text{and} \qquad i_{2,a}(M_2) = \{a\} \times M_2 = p_1^{-1}\big(\{a\}\big)$$

hence $i_{1,a}(M_1)$ and $i_{2,a}(M_2)$ are closed with respect to the product topology. In order to show that they indeed define submanifolds of $M_1 \times M_2$, we use the characterization in (4.5). Denote $n_1 = \dim M_1$, $n_2 = \dim M_2$ and

$$m \in M_1 \times M_2 \ \Rightarrow \ (\varphi_i \times \psi_j)(m) = z = (z_1, z_2) = (z_1^{(1)}, z_1^{(2)}, \ldots, z_1^{(n_1)}, z_2^{(1)}, z_2^{(2)}, \ldots, z_2^{(n_2)})$$

$m = (m_1, m_2) \in M_1 \times M_2 \ \Rightarrow\ \exists\, V_j \in \mathcal{V}$ such that $(m_1, m_2) \in M_1 \times V_j$. For $l \in \{1, \dots, n_2\}$, let

$$f_l \ : \ M_1 \times V_j \longrightarrow \mathbb{C} \ : \ (m_1, m_2) \longmapsto \big(\psi_j(m_2) - \psi_j(a)\big)^{(l)} \ : \ l^{\text{th}} \text{ coordinate}$$

So $f_l(m_1, m_2) = 0 \ \Leftrightarrow\ \big(\psi_j(m_2)\big)^{(l)} = \big(\psi_j(a)\big)^{(l)}$ and $f_l(m_1, m_2) = 0, \forall\, l \ \Leftrightarrow\ \psi_j(m_2) = \psi_j(a) \ \Leftrightarrow\ m_2 = a$, hence

$$\Big(M_1 \times V_j\Big) \cap \Big(M_1 \times \{a\}\Big) = M_1 \times \{a\} = \bigcap_{l=1}^{n_2} f_l^{-1}\big(\{0\}\big)$$

and this is exactly (4.5) with $U = M_1 \times V_j$, $n \to n_1 + n_2$ and $k = n_1$. And the f_l are holomorphic since locally

$$\big(f_l \circ (\varphi_i \times \psi_j)^{-1}\big)(z_1, z_2) = f_l\big(\varphi_i^{-1}(z_1), \psi_j(z_2)^{-1}\big) = \Big(\psi_j\big(\psi_j(z_2)^{-1}\big) - \psi_j(a)\Big)^{(l)} = z_2^{(l)} - \big(\psi_j(a)\big)^{(l)} \tag{5.4}$$

It remains to check that the rank of the Jacobian matrix is maximal. But by (5.4), this is simply given by

$$J(f_1, \dots, f_{n_2}) = \begin{pmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

since f_l is independent of z_1 and just a projection with respect to z_2. Finally $M_1 \times \{a\} \cong M_1$ is a submanifold of $M_1 \times M_2$ of dimension n_1. □

5.5.8 Additive structure

Next we want to define an additive structure on T^n.
As a vector space $(\mathbb{C}^n, +)$ (addition of vectors) is an abelian group. The lattice $L \subset \mathbb{C}^n$ is an additive subgroup of $\mathbb{C}^n$, thus also called the free abelian group generated by $2n$ elements. L being abelian, hence normal, the quotient $T^n = \mathbb{C}^n \,/\, L$ carries a natural group structure given by $[z] + [y] = [z + y]$ and $[z]^{-1} = [-z]$.

Exercise :
The group operations $+ : T^n \times T^n \to T^n$ and ${}^{-1} : T^n \to T^n$ are holomorphic.

1) To check this, we have to compose $+$ with the charts of T^n given in (5.3). If we work locally, then

$$\begin{aligned}\Big(\varphi_{j\alpha} \circ + \circ (\varphi_{k\beta} \times \varphi_{l\gamma})^{-1}\Big)(x_1, x_2) &= \varphi_{j\alpha}\Big(\varphi_{k\beta}^{-1}(x_1) + \varphi_{l\gamma}^{-1}(x_2)\Big) = \varphi_{j\alpha}\Big([x_1] + [x_2]\Big) \\ &= \varphi_{j\alpha}\Big([x_1 + x_2]\Big) = (x_1 + x_2)_\alpha\end{aligned}$$

i.e. locally we only add the components of $x = (x_1, x_2)$, followed by a constant shift. This is holomorphic.

2) Similarly we obtain for ${}^{-1}$:

$$\Big(\varphi_{j\alpha} \circ {}^{-1} \circ \varphi_{k\beta}^{-1}\Big)(x) = \varphi_{j\alpha}\Big(\big(\varphi_{k\beta}^{-1}(x)\big)^{-1}\Big) = \varphi_{j\alpha}\Big([x]^{-1}\Big) = \varphi_{j\alpha}\Big([-x]\Big) = (-x)_\alpha$$

Locally we just take the inverse of the coordinate and shift it again by a constant, so this is holomorphic too.
Hence T^n admits a holomorphic group structure.

5.6 Complex Lie groups

5.6.1 Definition

Let $(G, \cdot)$ be a complex manifold with a group law $\cdot : G \times G \to G$. $(G, \cdot)$ is called a *complex Lie group* if
1) the multiplication $\cdot : G \times G \to G$ is a holomorphic map with respect to the product structure
2) the inverse operation ${}^{-1} : G \to G : g \mapsto g^{-1}$ is also holomorphic.

$(T^n, +)$ is thus a complex Lie group by the previous exercise.

5.6.2 Example

Consider the space of matrices $M = \mathrm{Mat}(n \times n, \mathbb{C}) \cong \mathbb{C}^{n \cdot n}$.
This is complex manifold with respect to the global coordinate chart $\varphi :\ A = (a_{ij})_{ij} \mapsto (a_{11}, a_{12}, \dots, a_{nn})$. But M is not yet a group since there is no inverse : A^{-1} does not exist for any $A \in M$. So let $\mathrm{GL}(n, \mathbb{C}) \subsetneq M$:

$$B\ A \in \mathrm{GL}(n, \mathbb{C}) \;\Leftrightarrow\; A \text{ is invertible } \;\Leftrightarrow\; \det(A) \neq 0$$

Denote $Y := \{\, A \in M \mid \det(A) = 0 \,\} = \det^{-1}\big(\{0\}\big)$. det is continuous and holomorphic since it is a polynomial function in the entries a_{ij} of A, so Y is closed in $\mathbb{C}^{n \cdot n}$ and $\mathrm{GL}(n, \mathbb{C}) = M \setminus Y \neq \emptyset$. $\mathrm{GL}(n, \mathbb{C})$ is thus an open subset of M. Moreover $\mathrm{GL}(n, \mathbb{C})$ is connected since it is path-connected. Indeed :

$$Y = M \cap Y = \det^{-1}\big(\{0\}\big) = \bigcap_{i=1}^{n^2-k} f_i\big(\{0\}\big)$$

which is (4.5) with $U = M$, $n \to n^2$, $k = n^2 - 1$ and $f_1 = \det$. Moreover if $A = (a_{ij})_{ij} \in Y$, then

$$J(\det)(A) = \left(\frac{\partial (f_1 \circ \varphi^{-1})}{\partial a_{11}}\big((a_{ij})_{ij}\big) \quad \frac{\partial (f_1 \circ \varphi^{-1})}{\partial a_{12}}\big((a_{ij})_{ij}\big) \quad \dots \quad \frac{\partial (f_1 \circ \varphi^{-1})}{\partial a_{nn}}\big((a_{ij})_{ij}\big) \right) \neq 0 \tag{5.5}$$

since the expression $\det(A)$ cannot be independent of all its coordinates a_{ij}. This shows that Y is a 1-complex codimensional submanifold of M.

Remark :
det is a homogeneous polynomial of degree n since $\det(\lambda \cdot A) = \lambda^n \cdot \det(A)$. Hence Euler's relation holds :

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \cdot \frac{\partial \det(A)}{\partial a_{ij}} = n \cdot \det(A)$$

Now consider figure 5.12 : $\operatorname{codim}_{\mathbb{C}} Y = 1$, which means that $\operatorname{codim}_{\mathbb{R}} Y = 2$, so in the real picture one can always connect any $A, B \in \mathrm{GL}(n, \mathbb{C})$ by a continuous path which does not intersect Y. Finally $\mathrm{GL}(n, \mathbb{C})$ is open and connected in M, hence it is a complex manifold itself.
Note that $\mathrm{GL}(n, \mathbb{C})$ cannot be a submanifold of M since it is not closed. Moreover we recall that $\mathrm{GL}(n, \mathbb{C})$ carries the structure of an affine variety of $\mathbb{C}^{n \cdot n}$.

Figure 5.12: $\mathrm{GL}(n, \mathbb{C})$ is path-connected, but $\mathrm{GL}(n, \mathbb{R})$ is not

The above argument is not valid in a real manifold X. If Y is of codimension 1 in X, then $X \setminus Y$ is in general disconnected. Consider again figure 5.12 : if for example $A, B \in \mathrm{GL}(n, \mathbb{R})$ with $\det(A) < 0$ and $\det(B) > 0$, then any path relating A and B must pass through a matrix with determinant zero since there is no continuous way to pass from negative to positive numbers in $\mathbb{R}$ without passing through 0. In $\mathbb{C}$, this is however possible by "going around" 0 using complex-valued determinants.

Figure 5.13: passing from negative to positive numbers without hitting 0 is not possible in $\mathbb{R}$

5.6.3 Proposition

$\big(\mathrm{GL}(n;\mathbb{C}), \cdot\big)$ with respect to matrix multiplication is a complex Lie group.

Proof. First of all $\mathrm{GL}(n,\mathbb{C})$ is closed with respect to $\cdot$ and $^{-1}$ since for $A, B \in \mathrm{GL}(n;\mathbb{C})$, we have :

$$\det(A \cdot B) = \det(A) \cdot \det(B) \neq 0 \qquad , \qquad \det\left(A^{-1}\right) = \frac{1}{\det(A)} \neq 0$$

1) For showing that $\cdot$ is holomorphic, we prove that $\varphi \circ \cdot \circ (\varphi^{-1} \times \varphi^{-1})$ is holomorphic on $\mathbb{C}^{n \cdot n} \times \mathbb{C}^{n \cdot n}$. Let $A, B \in \mathrm{GL}(n,\mathbb{C})$ with $A = (a_{ij})$, $B = (b_{kl})$ and $C = A \cdot B$, so $C = (c_{rs})$ where

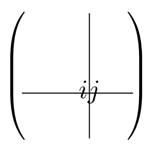
$$c_{rs} = \sum_{k=1}^{n} a_{rk}\, b_{ks} \qquad \Rightarrow \quad \big(\varphi \circ \cdot \circ (\varphi^{-1} \times \varphi^{-1})\big)(a_{11}, \ldots, a_{nn}, b_{11}, \ldots, b_{nn}) = (c_{11}, \ldots, c_{nn})$$

c_{rs} is just a polynomial expression in the entries a_{ij} and b_{kl}, thus $\varphi \circ \cdot \circ (\varphi^{-1} \times \varphi^{-1})$ and hence $\cdot$ are holomorphic.

2) For the inversion of A, recall that $A^{-1} = \frac{1}{\det(A)} \cdot {}^t(A^{\mathrm{ad}})$ where A^{ad} is the algebraic adjoint of A given by

$$(A^{\mathrm{ad}})_{ij} = (-1)^{i+j} \cdot \det(A^{ij})$$

where A^{ij} is the $(n-1) \times (n-1)$–submatrix of A obtained by erasing line i and column j :

$A =$ 

For example in the case $n = 2$, this gives $\det(A) = ad - bc \neq 0$ and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \Rightarrow \quad A^{\mathrm{ad}} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \qquad \Rightarrow \quad A^{-1} = \frac{1}{ad - bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Hence in a local coordinate, we obtain in general

$$\left(A^{-1}\right)_{ij} = \frac{1}{\det(A)} \cdot \left({}^t(A^{\mathrm{ad}})\right)_{ij} = \frac{1}{\det(A)} \cdot (A^{\mathrm{ad}})_{ji} = \frac{1}{\det(A)} \cdot (-1)^{i+j} \cdot \det(A^{ji})$$

det is a (non-vanishing) polynomial in a_{ij} and the entries in ${}^t(A^{\mathrm{ad}})$ are also polynomials, hence $^{-1}$ is holomorphic. So $\mathrm{GL}(n,\mathbb{C})$ is a complex Lie group of dimension n^2. □

Remark :
In the real case, $\cdot$ and $^{-1}$ are therefore real analytic operations $\Rightarrow$ $\mathrm{GL}(n,\mathbb{R})$ is a real Lie group.
Similarly one can show that $\mathrm{U}(n,\mathbb{R})$ and $\mathrm{SU}(n,\mathbb{R})$ are also real Lie groups. However $\mathrm{U}(n,\mathbb{C})$ and $\mathrm{SU}(n,\mathbb{C})$ are not complex Lie groups since the definitions of these involve the conjugate-transpose matrix A^*, whose entries are not holomorphic functions in the variables $a_{11}, a_{12}, \ldots, a_{nn}$. A more general theorem even states that :

Any compact connected complex Lie group is holomorphically isomorphic to a torus T^n. (no proof)

5.6.4 Exercise

Show that $\mathrm{SL}(n,\mathbb{C}) = \{\, A \in \mathrm{GL}(n,\mathbb{C}) \mid \det(A) = 1 \,\}$ is a closed submanifold of $\mathrm{GL}(n,\mathbb{C})$ of dimension $n^2 - 1$.

$\mathrm{SL}(n,\mathbb{C}) = \det^{-1}\big(\{1\}\big)$, hence $\mathrm{SL}(n,\mathbb{C})$ is closed in $\mathrm{GL}(n,\mathbb{C})$. If $f : \mathrm{GL}(n,\mathbb{C}) \to \mathbb{C} : A \mapsto \det(A) - 1$, then

$$\mathrm{SL}(n,\mathbb{C}) = f^{-1}\big(\{0\}\big)$$

which is (4.5) with $U = \mathrm{GL}(n,\mathbb{C})$, $n \to n^2$ and $k = n^2 - 1$. Moreover $\mathrm{rk}\,\big(J(f)\big) = 1$ since f differs from det only by a constant, hence $J(f)(A)$ is the same expression as in (5.5).

Remark :
Since f is a polynomial expression, $\mathrm{SL}(n,\mathbb{C})$ is even an affine variety of $\mathbb{C}^{n \cdot n}$.

5.7 Grassmannians

This is in fact a generalization of the projective space $\mathbb{P}^n(\mathbb{C})$:
Instead of considering the lines in $\mathbb{C}^n$ passing through the origin, one can also look at the set of linear subspaces of $\mathbb{C}^n$ of dimension $k \leq n$ (for the projective space, $k = 1$ since lines are 1-dimensional).
By a similar construction, one then obtains the so-called *Grassmannians*, denoted by $\mathrm{Gr}(n, k)$. It consists of the k-planes in $\mathbb{C}^n$ and will again define a compact complex manifold. In particular, $\mathrm{Gr}(n+1, 1) = \mathbb{P}^n(\mathbb{C})$.

Chapter 6

Sheaf of holomorphic functions

6.1 Global holomorphic functions on a compact complex manifold

6.1.1 Theorem

Let M be a (connected) compact complex manifold and $f : M \to \mathbb{C}$ be a global holomorphic function on M. Then f is constant, i.e. there are no non-constant globally holomorphic functions on a compact complex manifold.

Proof. f is holomorphic, hence continuous. So $|f|$ is also continuous on M (compact). Therefore $|f|$ takes its maximum value on M : $\exists\, x_0 \in M$ such that $|f(x_0)| \geq |f(x)|$, $\forall\, x \in M$. Let

$$S := \left\{ x \in M \;\middle|\; f(x) = f(x_0) \right\} = f^{-1}\big(\{f(x_0)\}\big)$$

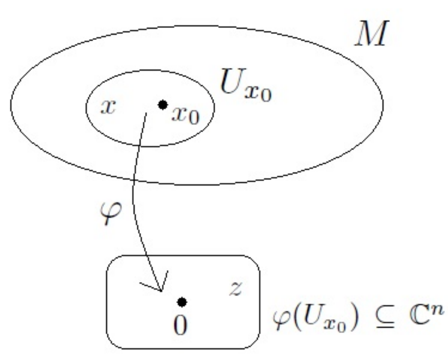
Then S is a closed subset of M since f is continuous and the points are closed in $\mathbb{C}^n$. $S \neq \emptyset$ because $x_0 \in S$. We want to show that $S = M$.
Consider figure 6.1. Let $(U_{x_0}, \varphi = (z_1, \ldots, z_n))$ be a local complex coordinate chart around and centered at x_0 (which exists by 4.3.2), i.e. $\varphi(x_0) = 0$ and we denote $z := \varphi(x)$, $\forall\, x \in U_{x_0}$. Let also $F := f \circ \varphi^{-1}$ and consider

$$g_z \;:\; \mathbb{C} \longrightarrow \mathbb{C} \;:\; \lambda \longmapsto g_z(\lambda) = F(\lambda\, z)$$

where $z \in \varphi(U_{x_0}) \subseteq \mathbb{C}^n$ is a parameter. By construction g_z is holomorphic in the variable λ and it satisfies $g_z(1) = F(z) = (f \circ \varphi^{-1})\big(\varphi(x)\big) = f(x)$. Moreover $|g_z|$ takes its maximum value at $\lambda = 0$ because

$$|g_z(0)| = |F(0)| = |(f \circ \varphi^{-1})(0)| = |f\big(\varphi^{-1}(0)\big)| = |f(x_0)|$$

Figure 6.1: centered coordinate chart at x_0



Hence $|g_z(\lambda)| \leq |g_z(0)|$, $\forall\, \lambda \in \mathbb{C}$. By the maximum principle of holomorphic functions in 1 variable (λ) :
The absolute value of a non-constant holomorphic functions cannot take its maximum value in the interior of its domain of definition (see section 2.5.2). But $|g_z|$ takes its maximum at $0 \in \mathbb{C} = \mathbb{C}^\circ$, so g_z must be constant. In particular $g_z(0) = g_z(1)$, $\forall\, z \in \varphi(U_{x_0})$. Now we let z vary, i.e. if $\forall\, z, z' \in \varphi(U_{x_0})$, we have

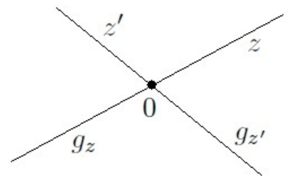
$$F(z') = g_{z'}(1) = g_{z'}(0) = f(x_0) = g_z(0) = g_z(1) = F(z)$$

g_z and $g_{z'}$ are both constant and coincide at 0 (see figure 6.2), hence they are equal and

$$F(z') = F(z),\ \forall\, z, z' \in \varphi(U_{x_0}) \quad \Leftrightarrow \quad f(x') = f(x),\ \forall\, x, x' \in U_{x_0}$$

i.e. $f(x) = f(x_0)$, $\forall\, x \in U_{x_0}$. f is constant in the coordinate neighborhood around x_0.

Figure 6.2: g_z and $g_{z'}$ coincide everywhere

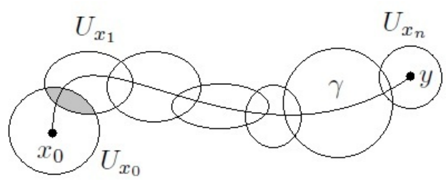


Now let $y \in M$ be arbitrary. Since the manifold M is connected, hence path-connected by 3.2.2, there is a continuous path $\gamma : [0,1] \to M$ such that $\gamma(0) = x_0$ and $\gamma(1) = y$. Consider a covering of open sets in M of this path (see figure 6.3), for example

$$\operatorname{im}\gamma \subset \bigcup_{x \in \operatorname{im}\gamma} U_x$$

where U_x denotes a small open neighborhood of x in M. But $\operatorname{im}\gamma = \gamma\big([0,1]\big)$, so $\operatorname{im}\gamma$ is compact and we can extract a finite subcovering open neighborhoods U_{x_i}, $i = 0, \dots, n$.

Figure 6.3: $\operatorname{im}\gamma$ can be covered by finitely many U_{x_i}



We hence know that f is constant on U_{x_0}. $U_{x_0} \cap U_{x_1}$ is non-empty and open in $U_{x_0} \cup U_{x_1}$, hence by the Identity Theorem (via using local coordinates), f is constant on $U_{x_0} \cup U_{x_1}$ since it is holomorphic. Iterating this process (finitely many U_{x_i}) finally gives that f is constant on $U_{x_0} \cup \ldots \cup U_{x_n}$ with $y \in U_{x_n}$, i.e. $f(y) = f(x_0)$.
Since this can be done for any $y \in M$, we obtain that f is constant everywhere. □

6.1.2 Theorem

1) There are no connected compact submanifolds in $\mathbb{C}^n$ but the points.
2) Any (not necessarily connected) compact submanifold of $\mathbb{C}^n$ only consists of finitely many points.
Hence there is no non-trivial compact complex substructure in $\mathbb{C}^n$.

Proof. 1) Let $M \subset \mathbb{C}^n$ be a connected compact submanifold and $i : M \hookrightarrow \mathbb{C}^n$ the canonical injection. Then

$$M \xhookrightarrow{\ i\ } \mathbb{C}^n \xrightarrow{\ p_i\ } \mathbb{C}$$

where i is holomorphic (since it the restriction of the identity) and $p_i : \mathbb{C}^n \to \mathbb{C}$ are the coordinate functions, which are also holomorphic. Thus the $f_i := p_i \circ i$ are holomorphic from M to $\mathbb{C}$, hence all f_i are constant by the previous theorem since M is connected. This means that the coordinate evaluation on M always gives the same value : $\forall\, z \in M,\ f_i(z) = p_i(z) = z_i = c_i$ for some $c_i \in \mathbb{C}$, i.e. all points in M have the same coordinate. This implies that M can only consist of a single point : $M = \{c\} = \{(c_1, \dots, c_n)\}$.

2) Now let M be an arbitrary compact submanifold of $\mathbb{C}^n$. Since M is locally connected (see section 3.2.2), the connected components C_i of M are open and closed and M writes as $M = \bigcup_i C_i$. By compactness, there is a finite subfamily of C_i covering M. Since the components do not intersect, this implies that M can only have finitely many connected components, which are in addition compact since they are closed. It follows from 1) that every of these finitely many components is just given by a point, hence M consists of finitely many points. □

6.1.3 Conclusion

As a consequence we conclude that Whitney's Embedding Theorem does not apply to complex manifolds since non-trivial compact complex manifolds of dimension k (which exist) cannot be embedded into $\mathbb{C}^{2k+1}$. Moreover we cannot deduce the structure of a compact complex (sub)manifold from its algebra of holomorphic functions (since there are no interesting ones). For this we have to introduce the notion of a sheaf.

6.2 Sheaves : definitions

Let M be a complex manifold and $U \subseteq M$ be open, so U is again a (not necessarily connected) complex manifold. We denote

$$\mathcal{O}(U) := \{\, f : U \to \mathbb{C} \text{ is a holomorphic function on } U \,\}$$

This definition makes sense since holomorphic functions on U do not need to be restrictions of globally holomorphic functions. By convention $\mathcal{O}(\emptyset) = \{0\}$, the zero function.

6.2.1 Proposition

$\mathcal{O}(U)$ is a $\mathbb{C}$–algebra for all $U \subseteq M$ open.

Proof. $0 \in \mathcal{O}(U)$
$f, g \in \mathcal{O}(U) \;\Rightarrow\; f + g \in \mathcal{O}(U)$ and $f \in \mathcal{O}(U)$, $\alpha \in \mathbb{C} \;\Rightarrow\; \alpha \cdot f \in \mathcal{O}(U)$, so $(\mathcal{O}(U), +)$ is a vector space over $\mathbb{C}$
moreover $f, g \in \mathcal{O}(U) \;\Rightarrow\; f \cdot g \in \mathcal{O}(U)$ and $\cdot$ is compatible with $+$, thus $(\mathcal{O}(U), \cdot)$ is a (commutative) ring
Hence is an algebra over $\mathbb{C}$. □

$\mathcal{O}(M)$ denotes the $\mathbb{C}$–algebra of globally holomorphic functions.
Let $U, V \subseteq M$ be open such that $V \subseteq U$. If f is holomorphic on U, then $f_{|V}$ is holomorphic on V because holomorphy is a local condition, i.e. $f \in \mathcal{O}(U) \;\Rightarrow\; f_{|V} \in \mathcal{O}(V)$.
Restriction of holomorphic functions can be seen as a map $\rho_V^U : \mathcal{O}(U) \to \mathcal{O}(V) : f \mapsto f_{|V}$ where $\mathcal{O}(U)$ and $\mathcal{O}(V)$ are both $\mathbb{C}$–algebras. This restriction is compatible with the $\mathbb{C}$–algebra structures, e.g.

$$(f+g)_{|V} = f_{|V} + g_{|V} \quad , \quad (\alpha \cdot f)_{|V} = \alpha \cdot f_{|V} \quad , \quad (f \cdot g)_{|V} = f_{|V} \cdot g_{|V} \tag{6.1}$$

6.2.2 Definition

A *sheaf* $\mathcal{F}$ of abelian groups / vector spaces / rings / algebras is an assignment $U \mapsto \mathcal{F}(U)$ where U is an open set in M and $\mathcal{F}(U)$ is an abelian group / vector space / ring / algebra, such that $\forall\, V \subseteq U$ open, there are maps

$$\rho_V^U \;:\; \mathcal{F}(U) \to \mathcal{F}(V) \;:\; f \mapsto \rho_V^U(f) = f_{|V}$$

called the *restrictions morphisms*, which are a family of homomorphisms of abelian groups / linear maps between vector spaces / ring homomorphisms / algebra homomorphisms such that

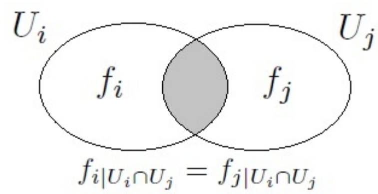
1) $\rho_U^U = \mathrm{id}_{\mathcal{F}(U)}$, $\forall\, U \subseteq M$ open
2) for any open sets $V \subseteq U \subseteq W$, we have $\rho_V^W = \rho_V^U \circ \rho_U^W$

For any open set $U \subseteq M$ and any open covering $\{U_i\}_{i \in J}$ of U, the following conditions hold :

3) if $f, g \in \mathcal{F}(U)$ such that $\rho_{U_i}^U(f) = \rho_{U_i}^U(g) \;\Leftrightarrow\; f_{|U_i} = g_{|U_i}$, $\forall\, i \in J$, then $f = g$
4) if $f_i \in \mathcal{F}(U_i)$ such that $f_{i|U_i \cap U_j} = f_{j|U_i \cap U_j}$, $\forall\, i, j \in J$, then $\exists\, f \in \mathcal{F}(U)$ such that $f_{|U_i} = f_i$, $\forall\, i \in J$

Remarks :
– By 3), the f in 4) is always unique. Consider figure 6.4 for an interpretation of 4).
– Note that f and g are not necessarily functions and that ρ_V^U is not necessarily the restriction of functions.
– However, when considering functions with the usual restriction, then 1), 2) and 3) are always satisfied.
– If $\mathcal{F}$ only satisfies the conditions 1) and 2), then $\mathcal{F}$ is called a *presheaf*.

Figure 6.4: f_i and f_j coincide on $U_i \cap U_j$



6.2.3 Theorem

The holomorphic functions on a complex manifold M define a sheaf of $\mathbb{C}$–algebras.

Proof. because holomorphy is a local condition and the restriction is compatible with the $\mathbb{C}$–algebra structure :

$$\rho^U_V(f+g) = \rho^U_V(f) + \rho^U_V(g) \qquad , \qquad \rho^U_V(f\cdot g) = \rho^U_V(f)\cdot\rho^U_V(g)$$ □

The sheaf of $\mathbb{C}$-valued holomorphic functions on a complex manifold M is denoted by $\mathcal{O}_M$. Similarly one can show that the differentiable and the analytic functions on M also define sheaves, denoted by C^∞_M and C^ω_M.

If M is a compact complex manifold, we know by 6.1 that $\mathcal{O}_M(M) = \mathbb{C}$. This is not true for $C^\infty(M)$. Moreover $\mathcal{O}_M(U)$ can be non-trivial for some (non-compact) open subset $U \subset M$ because if U is small enough, it can be identified with some open subset of $\mathbb{C}^n$, which has a lot of holomorphic functions.

6.2.4 Counter–example

Consider the complex plane $M = \mathbb{C}$ and $\mathcal{B}(U)$, the $\mathbb{C}$–algebra of bounded holomorphic functions on $U \subseteq M$ open together with the restriction of functions ρ^U_V. Thus 1), 2), 3) are satisfied. But 4) is violated because boundedness is not a local property. Indeed consider the case $U = \mathbb{C}$ and U is covered by $\{U_i\}_{i\in\mathbb{N}}$ where

$$U_i = \left\{\, z\in\mathbb{C}\ ,\ |z| < i \,\right\}$$

are open, i.e. $\mathbb{C} = \bigcup_{i\in\mathbb{N}} U_i$. Let $f_i \in \mathcal{B}(U_i)$ be given by $f_i(z) = z$. So any f_i is bounded on U_i and the f_i glue at the intersections : $f_{i|U_i\cap U_j} = f_{j|U_i\cap U_j}$, $\forall\, i,j \in \mathbb{N}$. But there is no global holomorphic function which extends the f_i since $\mathcal{B}(\mathbb{C}) = \mathbb{C}$: by Liouville, every bounded globally holomorphic (entire) function on $\mathbb{C}$ must be constant. Hence no one of the f_i can be extended to $\mathbb{C}$ since it is non-constant in any neighborhood of every point in U_i.

other argument :
If there is an extension f of the f_i, it is necessarily of the form $f(z) = z$ (this is the only candidate). But this f is not bounded on $\mathbb{C}$: $\nexists\, K \in \mathbb{R}$ such that $|f(z)| = |z| \le K$, $\forall\, z \in \mathbb{C}$. Finally

$$\nexists\, f \in \mathcal{B}(\mathbb{C}) \ \text{ such that } \ f_{|U_i} = f_i,\ \forall\, i \in \mathbb{N}$$

6.2.5 Exercise

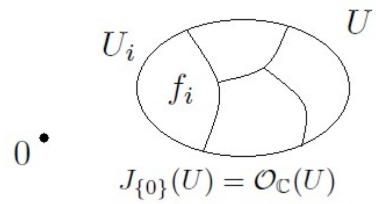
Let $M = \mathbb{C}$ and consider $\mathcal{F} : U \mapsto \mathcal{F}(U) = J_{\{0\}}(U)$ where $J_{\{0\}}(U) = \mathcal{O}(U)$ if $0 \notin U$ and for $0 \in U$, we set

$$J_{\{0\}}(U) = \left\{\, f \in \mathcal{O}_\mathbb{C}(U) \mid f(0) = 0 \,\right\} \subsetneq \mathcal{O}_\mathbb{C}(U)$$

(strict inclusion because of the non-zero constant functions) together with the usual restriction of functions ρ^U_V. Show that $\mathcal{F}$ is a sheaf on M. It is called the *vanishing sheaf of the point* $0 \in \mathbb{C}$. More generally, one can also consider the vanishing sheaf of arbitrary subsets of a more general manifold M (see section 6.5.2).
The conditions 1), 2) and 3) are satisfied. So let $U \subseteq \mathbb{C}$ be open with an open covering $U = \bigcup_{i\in J} U_i$ and let $f_i \in J_{\{0\}}(U_i)$ such that $f_{i|U_i\cap U_j} = f_{j|U_i\cap U_j}$, $\forall\, i,j \in J$.
If $0 \notin U$, then $\exists\, f \in J_{\{0\}}(U) = \mathcal{O}_\mathbb{C}(U)$ such that $f_{|U_i} = f_i$, $\forall\, i \in J$ since $\mathcal{O}_\mathbb{C}$ is a sheaf (see figure 6.5).
Hence we can assume that $0 \in U$. $\forall\, i \in J$, $f_i \in J_{\{0\}}(U_i) \subset \mathcal{O}_\mathbb{C}(U_i) \ \Rightarrow\ \exists\, f \in \mathcal{O}_\mathbb{C}(U)$ such that $f_{|U_i} = f_i$, $\forall\, i \in J$. We have to check that this f satisfies $f(0) = 0$. $0 \in U \ \Rightarrow\ \exists\, i_0 \in J$ such that $0 \in U_{i_0}$, hence

$$f(0) = f_{|U_{i_0}}(0) = f_{i_0}(0) = 0 \ \text{ since } f_{i_0} \in J_{\{0\}}(U_{i_0}) \quad\Rightarrow\quad f \in J_{\{0\}}(U)$$

Figure 6.5: if $0 \notin U$, then $J_{\{0\}}$ coincide with the sheaf of holomorphic functions on U



6.2.6 Definition

Since $\mathcal{O}_M$ is a sheaf of $\mathbb{C}$–algebras, $\mathcal{O}_M(U)$ is in particular a commutative ring for any $U \subseteq M$ open. Hence one can consider modules over these rings.
A sheaf $\mathcal{F}$ with restrictions ρ_V^U is called a *sheaf of $\mathcal{O}_M$–modules* if $\mathcal{F}(U)$ is a module over $\mathcal{O}_M(U)$ for any $U \subseteq M$ open and the module structure is compatible with the restrictions, i.e.

$$\forall\, U \subseteq M \text{ open}\,,\ \forall\, f \in \mathcal{F}(U)\,,\ \forall\, h \in \mathcal{O}_M(U)\ :\ \rho_V^U(h *_U f) = \rho_V^U(h) *_V \rho_V^U(f) = h_{|V} *_V \rho_V^U(f) \qquad (6.2)$$

where $h *_U f \in \mathcal{F}(U)$ and $h_{|V} *_V \rho_V^U(f) \in \mathcal{F}(V)$.

6.2.7 Examples

1) If $\mathcal{F}$ is a sheaf of $\mathcal{O}_M$–modules and M is compact, then $\mathcal{F}(M)$ is a module over $\mathcal{O}_M(M) = \mathbb{C}$, i.e. $\mathcal{F}(M)$ is a vector space over $\mathbb{C}$.

2) $J_{\{0\}}$ is a sheaf of $\mathcal{O}_{\mathbb{C}}$–modules with the definition $h * f := h \cdot f$ for $h \in \mathcal{O}_{\mathbb{C}}(U)$, $f \in J_{\{0\}}(U)$, $U \subseteq \mathbb{C}$ open. This is well-defined because $(h \cdot f)(0) = 0$ too, hence $h \cdot f \in J_{\{0\}}(U)$. Moreover this module structure is compatible with the restrictions as in (6.1).

3) $\mathcal{O}_M$ is a sheaf of $\mathcal{O}_M$–modules since any commutative ring $\mathcal{O}_M(U)$ can be considered as a module over itself.

6.3 Morphisms of sheaves

Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of the same type, i.e. both are sheaves of abelian groups / vector spaces / rings or algebras. Denote the restrictions of $\mathcal{F}$ by $\rho^{\mathcal{F}}_{VU}$ and those of $\mathcal{G}$ by $\rho^{\mathcal{G}}_{VU}$. A *morphism of sheaves* $\psi : \mathcal{F} \to \mathcal{G}$ is a family $\{\psi_U\}_{U \subseteq M}$ of maps $\psi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ indexed by the open sets in M such that if $V \subseteq U$ are open :

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\psi_U} & \mathcal{G}(U) \\ \downarrow{\scriptstyle \rho^{\mathcal{F}}_{VU}} & & \downarrow{\scriptstyle \rho^{\mathcal{G}}_{VU}} \\ \mathcal{F}(V) & \xrightarrow{\psi_V} & \mathcal{G}(V) \end{array} \qquad\qquad \rho^{\mathcal{G}}_{VU} \circ \psi_U = \psi_V \circ \rho^{\mathcal{F}}_{VU}$$

and any ψ_U is a homomorphism of abelian groups / linear map between vector spaces / ring homomorphism or algebra homomorphism.
If $\mathcal{F}$ and $\mathcal{G}$ are sheaves of $\mathcal{O}_M$–modules, it is in addition required that $\forall\, V \subseteq U$ open :

$$\begin{array}{ccccccc} \mathcal{O}_M(U) \times \mathcal{F}(U) & \xrightarrow{*_U} & \mathcal{F}(U) & \xrightarrow{\psi_U} & \mathcal{G}(U) & \xleftarrow{*'_U} & \mathcal{O}(U)_M \times \mathcal{G}(U) \\ & & \downarrow{\scriptstyle \rho^{\mathcal{F}}_{VU}} & & \downarrow{\scriptstyle \rho^{\mathcal{G}}_{VU}} & & \\ \mathcal{O}_M(V) \times \mathcal{F}(V) & \xrightarrow{*_V} & \mathcal{F}(V) & \xrightarrow{\psi_V} & \mathcal{G}(V) & \xleftarrow{*'_V} & \mathcal{O}_M(V) \times \mathcal{G}(V) \end{array} \qquad \begin{array}{c} \mathcal{O}_M(U) \\ \downarrow{\scriptstyle \rho^{\mathcal{O}}_{VU}} \\ \mathcal{O}_M(V) \end{array} \qquad (6.3)$$

where any ψ_U is a module homomorphism, i.e. $\psi_U(h *_U f) = h *'_U \psi_U(f)$, $\forall\, h \in \mathcal{O}_M(U)$, $\forall\, f \in \mathcal{F}(U)$.

6.4 Restriction of sheaves

6.4.1 Definition

Let $\mathcal{F}$ be a sheaf over a complex manifold M and $U \subseteq M$ be open. The *restriction* of $\mathcal{F}$ to U, denoted by $\mathcal{F}_{|U}$, is defined by $\mathcal{F}_{|U}(U') := \mathcal{F}(U')$, $\forall\, U' \subseteq U$ open (hence U' is also open in M).
The restriction morphisms of $\mathcal{F}_{|U}$ are coming from the restrictions of $\mathcal{F}$. Hence $\mathcal{F}_{|U}$ is a sheaf on U since the conditions 1), 2), 3) and 4) hold for any open set in M, thus also for U and open subsets of U.

Example :
If M is a complex manifold, $U \subseteq M$ open, non-empty and connected, then $(\mathcal{O}_M)_{|U} = \mathcal{O}_U$.

6.4.2 Definitions

A sheaf $\mathcal{F}$ of $\mathcal{O}_M$–modules is called a *free sheaf of rank* $k \in \mathbb{N}_0$ if $\mathcal{F} \cong \mathcal{O}_M^k = \mathcal{O}_M \oplus \ldots \oplus \mathcal{O}_M$ as sheaves of $\mathcal{O}_M$–modules. $\mathcal{F} \cong \mathcal{O}_M^k$ means that $\mathcal{F}$ and $\mathcal{O}_M^k$ are isomorphic in the category of sheaves of $\mathcal{O}_M$–modules. As a consequence, we obtain that for all $U \subseteq M$ open :

$$\mathcal{F}(U) \cong \mathcal{O}_M(U)^k = \mathcal{O}_M(U) \times \ldots \times \mathcal{O}_M(U) \tag{6.4}$$

In general (6.4) is however not sufficient to say that $\mathcal{F} \cong \mathcal{O}_M \oplus \ldots \oplus \mathcal{O}_M$ since the compatibility conditions with the restrictions still need to be satisfied. In fact : $\mathcal{F} \cong \mathcal{O}_M^k \;\Leftrightarrow\;$ (6.4) and (6.3).
It is however sufficient if the $\mathcal{O}_M(U)$–module $\mathcal{F}(U)$ consists of functions and ρ_V^U is the restriction of functions since (6.3) is always satisfied for functions with usual restriction.

A sheaf $\mathcal{F}$ of $\mathcal{O}_M$–modules is called *locally free of finite rank k*
$\Leftrightarrow\;\; \forall\, x \in M$, $\exists\, k \in \mathbb{N}_0$ and there is an open neighborhood U of x in M such that $\mathcal{F}_{|U} \cong \mathcal{O}_U^k \cong \big((\mathcal{O}_M)_{|U}\big)^k$.

This k is necessarily the same for any $x \in M$:
Let $x, y \in M$ with $x \in U$, $y \in V$, $U, V \subseteq M$ open such that $\mathcal{F}(U) \cong \mathcal{O}^k(U)$ and $\mathcal{F}(V) \cong \mathcal{O}^l(V)$. M being (path-)connected, let $\gamma : [0,1] \to M$ be a continuous path from x to y. For any $z \in \operatorname{im}\gamma$, consider an open neighborhood U_z of z such that $\mathcal{F}(U_z) \cong \mathcal{O}^{k_z}(U_z)$. $\operatorname{im}\gamma$ is compact, hence there are finitely many z_i such that

$$\operatorname{im}\gamma \subset \bigcup_{z \in \operatorname{im}\gamma} U_z \quad\Rightarrow\quad \operatorname{im}\gamma \subset \bigcup_{i=1}^{n} U_i$$

with $U_1 = U$, $U_n = V$ and $\mathcal{F}(U_i) \cong \mathcal{O}^{k_i}(U_i)$, $\forall\, i \in \{1, \ldots, n\}$, $k = k_1$, $l = k_n$. In particular, we have for example

$$\mathcal{F}(U) \cong \mathcal{O}^k(U) \quad , \quad \mathcal{F}(U \cap U_2) \cong \mathcal{O}^k(U \cap U_2) \quad , \quad \mathcal{F}(U_2) \cong \mathcal{O}^{k_2}(U_2) \quad , \quad \mathcal{F}(U_2 \cap U) \cong \mathcal{O}^{k_2}(U_2 \cap U)$$

hence $\mathcal{O}^k(U \cap U_2) \cong \mathcal{O}^{k_2}(U_2 \cap U)$, which implies that $k = k_2$ since the rank of a module is uniquely given. By induction, we obtain that $k = k_2 = k_3 = \ldots = k_{n-1} = k_n = l$ since there are only finitely many indices.

6.4.3 Examples

1) Let $M = \mathbb{C}$. Then $J_{\{0\}}$ is a sheaf of $\mathcal{O}_{\mathbb{C}}$–modules as shown in 6.2.7. Moreover $J_{\{0\}}$ is a free sheaf of rank 1 : $J_{\{0\}}(\mathbb{C}) = \mathcal{O}_{\mathbb{C}}(\mathbb{C}) \cdot p$ where $p : \mathbb{C} \to \mathbb{C}$, $p(z) = z \;\Rightarrow\; p(0) = 0$. This means that $\forall\, \varphi \in J_{\{0\}}(\mathbb{C})$, $\exists\, \psi \in \mathcal{O}_{\mathbb{C}}(\mathbb{C})$ such that $\varphi = \psi \cdot p$: φ has no constant term. φ is holomorphic on $\mathbb{C}$, hence saying that $\varphi \in J_{\{0\}}(\mathbb{C})$ means that the power series expansion of φ around 0 satisfies $a_0 = 0$:

$$\varphi(z) = \sum_{k=0}^{\infty} a_k \cdot z^k = 0 + z \cdot \sum_{k=1}^{\infty} a_k \cdot z^k \quad\Rightarrow\quad \varphi(0) = 0$$

In order to satisfy the compatibility conditions of the restrictions, we thus need that $J_{\{0\}}(U) = \mathcal{O}_{\mathbb{C}}(U) \cdot p_{|U}$ for all $U \subseteq M$ open. But this is true, hence (6.4) is satisfied with $k = 1$ and we get $J_{\{0\}} \cong \mathcal{O}_{\mathbb{C}}$.

2) Let $M = \mathbb{P}^1(\mathbb{C})$ with $a = (0:1) \neq \infty$, $U \subseteq \mathbb{P}^1(\mathbb{C})$ open and define

$$J_a(U) := \big\{\, f \in \mathcal{O}_M(U) \;\big|\; \text{if } a \in U, \text{ then } f(a) = 0 \,\big\}$$

J_a is a sheaf on $\mathbb{P}^1(\mathbb{C})$ (same proof as for $J_{\{0\}}$), called the *vanishing sheaf of a*. It is locally free of rank 1 because $\mathbb{C} \cong \{ (\alpha : 1) \mid \alpha \in \mathbb{C} \}$ and $a = (0:1)$, so $J_a(U) \cong J_{\{0\}}(U)$ for all $U \subsetneq \mathbb{P}^1(\mathbb{C})$ open.

J_a is however not a free sheaf because there are no non-zero holomorphic functions on $\mathbb{P}^1(\mathbb{C})$ that vanish at a given point :

$$\mathcal{O}_M\big(\mathbb{P}^1(\mathbb{C})\big) = \mathbb{C} \quad , \quad J_a\big(\mathbb{P}^1(\mathbb{C})\big) = \{0\} \quad \Rightarrow \quad J_a(M) \ncong \mathcal{O}_M(M)$$

Thus (6.4) is not satisfied, meaning that J_a cannot be free of rank 1.

other argument :
Assume that $J_a(M) \cong \mathcal{O}_M(M)$. Since $\mathcal{O}_M(M)$ has rank 1 (basis given by $f = 1$), $J_a(M)$ must therefore also be generated by 1 element over $\mathcal{O}_M(M)$, i.e. $\exists\, g \in \mathcal{O}_M(M)$ such that $J_a(M) = \mathcal{O}_M(M) \cdot g$. Since $J_a(M) = \{0\}$, we have $g = 0$. In order to satisfy the compatibility conditions of the restrictions, we thus need that

$$J_a(U) = \mathcal{O}_M(M) \cdot g_{|U} = \mathcal{O}_M(M) \cdot 0_{|U} = \{0\}$$

which is not true since $J_a(U) \neq \{0\}$ for a small open set $U \subset \mathbb{P}^1(\mathbb{C})$ (because U locally looks like $\mathbb{C}$).

6.5 Ideal sheaves

6.5.1 Definition

Let M be a complex manifold and $\mathcal{O}_M$ be the sheaf of $\mathbb{C}$-valued holomorphic functions on M.
A sheaf $\mathcal{J}$ on M is called an *ideal sheaf* if $\mathcal{J}$ is a subsheaf of $\mathcal{O}_M$, i.e. $\mathcal{J}(U) \subseteq \mathcal{O}_M(U)$, $\forall\, U \subseteq M$ open, denoted by $\mathcal{J} \subseteq \mathcal{O}_M$, such that $\mathcal{J}(U)$ is an ideal in the commutative ring $\mathcal{O}_M(U)$ for any U.
This implies in particular that $\mathcal{J}$ is also a sheaf of $\mathcal{O}_M$–modules since every ideal is a module over the considered ring by setting :

$$\mathcal{O}_M(U) \times \mathcal{J}(U) \longrightarrow \mathcal{J}(U) \;:\; (h, f) \longmapsto h * f := h \cdot f \in \mathcal{J}(U)$$

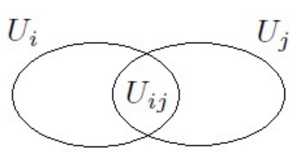
6.5.2 Example

Let $A \subseteq M$ be an arbitrary subset of the complex manifold M. For $U \subseteq M$ open, we define

$$\mathcal{J}_A(U) := \big\{\, f \in \mathcal{O}_M(U) \;\big|\; f(x) = 0,\ \forall\, x \in A \cap U \,\big\}$$

$\mathcal{J}_A(U)$ is an ideal in $\mathcal{O}_M(U)$: for all $h \in \mathcal{O}_M(U)$, $h \cdot f$ still vanishes at any $x \in A \cap U$ if $f \in \mathcal{J}_A(U)$.
$\mathcal{J}_A$ defines an ideal sheaf and is called the *vanishing sheaf of A over $\mathcal{O}_M$*.

Proof. The conditions 1), 2) and 3) are satisfied since we consider functions and usual restrictions.
Let $U \subseteq M$ be open and $U = \bigcup_{i \in I} U_i$ be an open covering. Denote $U_{ij} := U_i \cap U_j$.



For all $i \in I$, let $f_i \in \mathcal{J}_A(U_i)$ such that $f_{i|U_{ij}} = f_{j|U_{ij}}$, $\forall\, i, j \in I$. We have to show that $\exists\, f \in \mathcal{J}_A(U)$ such that $f_{|U_i} = f_i$, $\forall\, i \in I$. By 3), the only candidate for functions with the canonical restriction is

$$f(x) := f_i(x) \;\text{ if } x \in U_i \text{ for some } i \in I$$

a) f is well-defined since if $x \in U_i \cap U_j$, then $f_i(x) = f_{i|U_{ij}}(x) = f_{j|U_{ij}}(x) = f_j(x) = f(x)$.

b) f is holomorphic on U since holomorphy is a local condition, i.e. it is sufficient to check it on a small open neighborhood of any point $x \in U$. But this is true since $\forall\, x \in U$, $\exists\, i_0 \in I$ such that $x \in U_{i_0}$ and $f_{|U_{i_0}} = f_{i_0}$ is holomorphic on the open set U_{i_0} since $f_{i_0} \in \mathcal{O}_M(U_{i_0})$ by assumption.

c) check that $f(x) = 0$, $\forall\, x \in A \cap U$:

$$A \cap U = A \cap \textstyle\bigcup_i U_i = \bigcup_i (A \cap U_i)$$

Hence if $x \in A \cap U$, then $\exists\, i \in I$ (not necessarily unique) such that $x \in A \cap U_i$, implying that

$$f(x) = f_i(x) = 0 \quad \text{since } f_i \in \mathcal{J}_A(U_i)$$

□

6.5.3 Proposition

If $A \subseteq M$ is open, $A \neq \emptyset$ and $U \subseteq M$ is open and connected such that $A \cap U \neq \emptyset$, then $\mathcal{J}_A(U) = \{0\}$.

Proof. follows from the Identity Theorem : If $f \in \mathcal{O}_M(U)$ where U is connected and contains the non-empty open set $A \cap U$ on which f vanishes, then f vanishes on the whole open set U. □

Remark :
This result does not have much applications in practise since one often requires that A has to be closed, e.g.

$$M = \mathbb{C}^2 \ , \ A = \{(0,0)\} \quad \Rightarrow \ \mathcal{J}_A(U) = \left\{ \, f \in \mathcal{O}_M(U) \ \middle| \ f(0,0) = 0 \ \text{ if } (0,0) \in U \, \right\} = J_{\{(0,0)\}}(U)$$

6.6 Germs and stalks

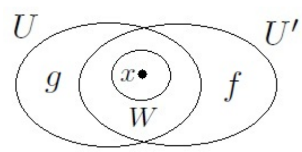
Let M be a complex manifold and $x \in M$ be fixed.
The idea of germs and stalks is to consider holomorphic functions in a (non-determined) neighborhood of x.

6.6.1 Definition

Let U and U' be open in M such that $x \in U \cap U'$ and let $g \in \mathcal{O}_M(U)$, $f \in \mathcal{O}_M(U')$. We define

$$(g, U) \sim_x (f, U') \ \Leftrightarrow \ \exists\, W \subseteq U \cap U' \text{ open}, \ x \in W \text{ such that } g_{|W} = f_{|W}$$

i.e. two functions with their corresponding domains are equivalent with respect to x if there exists a smaller open neighborhood of x on which both functions concide.



$\sim_x$ is an equivalence relation (reflexivity and symmetry are clear).
− transitivity : if $g_{|W} = f_{|W}$ and $f_{|W'} = h_{|W'}$, then $g_{|W\cap W'} = h_{|W\cap W'}$ where $W \cap W'$ is open and non-empty

We denote the equivalence class simply by $[f] = f_x := \left\{ (g, U) \ \middle| \ (f, U') \sim_x (g, U) \right\}$; the domain of f does not need to be specified. Such an equivalence class is called a *germ* of holomorphic functions. And the set of all germs of holomorphic functions (the set of all equivalence classes) is called the *stalk at x* and denoted by

$$\mathcal{O}_{M,x} = \left\{ \, f_x \mid f \text{ holomorphic in some open neighborhood of } x \, \right\}$$

We point out that germs do not have a domain of definition, but the representatives of a given germ have one.

This construction can also be done for arbitrary sheaves :
Let $\mathcal{F}$ be a sheaf on M and $U, V \subseteq M$ be open. Fix $x \in U \cap V$ and let $f \in \mathcal{F}(U)$, $g \in \mathcal{G}(V)$. Then

$$f \sim_x g \ \Leftrightarrow \ \exists\, W \subseteq U \cap V \text{ open}, \ x \in W \text{ such that } \rho^U_W(f) = \rho^V_W(g)$$

The germs are again given by the f_x and the stalk at x (the set of all germs) is denoted by $\mathcal{F}_x$.

6.6.2 Proposition

$\mathcal{F}_x$ has the same algebraic properties as the "objects of $\mathcal{F}$", i.e. as the $\mathcal{F}(U)$ where $U \subseteq M$ is open.

Proof. Consider for example the case where $\mathcal{F}(U)$ is an abelian group. We have to show that $\mathcal{F}_x$ can also be endowed with an abelian group structure. Let $U, V \subseteq M$ open, $x \in U \cap V$, $f \in \mathcal{F}(U)$, $g \in \mathcal{F}(V)$ and set

$$[f] + [g] := [f + g] := \left[f_{|U\cap V} + g_{|U\cap V} \right] \tag{6.5}$$

where $f_{|U\cap V}, g_{|U\cap V} \in \mathcal{F}(U \cap V)$ since the sum of f and g can only be defined in a smaller neighborhood.

This definition is also independent of the representing elements of f and g. Let $f' \sim_x f$ and $g' \sim_x g$ with $f_{|W} = f'_{|W}$ and $g_{|W'} = g'_{|W'}$. Since x belongs to all of these open set, we have that $A := U \cap V \cap W \cap W' \neq \emptyset$:

$$f_{|A} = f'_{|A} \quad \text{and} \quad g_{|A} = g'_{|A}$$
$$\Rightarrow \; f'_{|A} + g'_{|A} = f_{|A} + g_{|A} \quad \Rightarrow \; f' + g' \sim_x f + g$$

Finally definition (6.5) implies that $[f]+[g]$ has exactly the same properties as $f+g$, so $\mathcal{F}_x$ is an abelian group. □

Remark :
A more sophisticated way to prove this result is to use the concept of a filtrant inductive limit. Indeed :

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U)$$

which intuitively means that we take $\mathcal{F}(U)$ and let $U \to \{x\}$. Inductive limits preserve the properties of $\mathcal{F}(U)$.

6.6.3 Corollary

$\mathcal{O}_{M,x}$ is a $\mathbb{C}$–algebra for all $x \in M$. In particular it is a commutative ring.

Proof. follows from proposition 6.6.2 since $\mathcal{O}_M(U)$ is a $\mathbb{C}$–algebra for all $U \subseteq M$ open (as showed in 6.2.1) □

6.6.4 Proposition

$\forall\, x \in M$, $\mathcal{O}_{M,x}$ is a local ring, i.e. a ring which contains only one maximal ideal.

Proof. Recall that a ring R is a local ring $\Leftrightarrow$ $R \setminus R^\times$ is an ideal. We define

$$\mathfrak{M}_x := \left\{ [f] \in \mathcal{O}_{M,x} \;\middle|\; f(x) = 0 \right\}$$

$\mathfrak{M}_x$ is well-defined since if $g \sim_x f$ and $f(x) = 0$, then $g(x) = 0$ too since x belongs to any open neighborhood around x : all representatives of $[f]$ have the same value at x.
Moreover $\mathfrak{M}_x$ is an ideal in $\mathcal{O}_{M,x}$ with the definitions $[f_1] + [f_2] = [f_1 + f_2]$, $\forall\, [f_1], [f_2] \in \mathfrak{M}_x$ as above and $[g] \cdot [f] = [g \cdot f]$, where $[g] \in \mathcal{O}_{M,x}$ and the product $g \cdot f$ is done in the neighborhood $U \cap V$ if the representing elements are $f \in \mathcal{O}_M(U)$, $g \in \mathcal{O}_M(V)$ because $f(x) = 0 \;\Rightarrow\; (g \cdot f)(x) = 0$ too.

It remains to show that the elements in $\mathcal{O}_{M,x} \setminus \mathfrak{M}_x$ are invertible. Let $[g] \notin \mathfrak{M}_x \;\Rightarrow\; g(x) \neq 0$, $\forall\, g \in [g]$.
Let (g, U) be a representing element of $[g]$. Since g is holomorphic, hence continuous on U, $\exists\, U' \subseteq U$ open such that $x \in U'$ and $g(y) \neq 0$, $\forall\, y \in U'$. Thus $\frac{1}{g}$ is well-defined and holomorphic on U'. Moreover

$$[g]^{-1} = \left[\tfrac{1}{g}\right] \quad \text{since} \quad [g] \cdot \left[\tfrac{1}{g}\right] = [1]$$

Thus $\left(\frac{1}{g}, U'\right)$ satisfies all the conditions and $[g]$ is invertible in $\mathcal{O}_{M,x}$. This shows that $\mathcal{O}_{M,x} \setminus \mathfrak{M}_x \subseteq \mathcal{O}^\times_{M,x}$.

And $\mathcal{O}^\times_{M,x} \subseteq \mathcal{O}_{M,x} \setminus \mathfrak{M}_x$ because $\mathfrak{M}_x \neq \mathcal{O}_{M,x}$. If an element in $\mathfrak{M}_x$ would be invertible, then $[1] \in \mathfrak{M}_x$, hence $\mathfrak{M}_x = \mathcal{O}_{M,x}$ because it is an ideal. This contradiction finishes the proof. □

Remark :
This does not hold for arbitrary $\mathcal{F}_x$ (since not any $\mathcal{F}_x$ must be a ring).

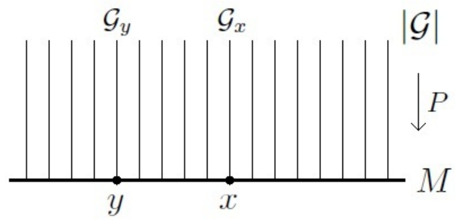
6.7 Remarks on sheafification

Assume that $\mathcal{G}$ is only a presheaf on M. Then $\mathcal{G}_x$ also exists for all $x \in M$ y $\mathcal{G}_y$. Set $|\mathcal{G}| := \bigsqcup_{x \in M} \mathcal{G}_x$ and consider

$$\forall\, x \in M \;,\; p_x \;:\; \mathcal{G}_x \to \{x\} \;:\; [g] \mapsto x$$

$|\mathcal{G}|$ is defined as the disjoint union of all the $\mathcal{G}_x$ and called the *total space of presheaves*. Hence all the maps p_x induce a map $P : |\mathcal{G}| \to M : [g] \mapsto x$ where x is the unique x such that $[g] \in \mathcal{G}_x$ (see figure 6.6).
We endow $|\mathcal{G}|$ with the initial topology with respect to P, i.e. the smallest topology that makes P continuous. This topology hence consists of all the preimages under P of open sets in M.

Figure 6.6: for any $[g] \in |\mathcal{G}|$ there is a unique $x \in M$ such that $[g] \in \mathcal{G}_x$



For $U \subseteq M$ open, we then define

$$\hat{\mathcal{G}}(U) := \left\{ s : U \to P^{-1}(U) \subset |\mathcal{G}| \text{ such that } s \text{ is continuous and } P \circ s = \mathrm{id}_U \right\}$$

Such an element $s \in \hat{\mathcal{G}}(U)$ is called a *continuous section* on U.
One can show that $\hat{\mathcal{G}}$ is now a sheaf; it is the *associated sheaf* of $\mathcal{G}$. And if $\mathcal{G}$ was already a sheaf, then $\hat{\mathcal{G}} = \mathcal{G}$.

Chapter 7

Meromorphic functions

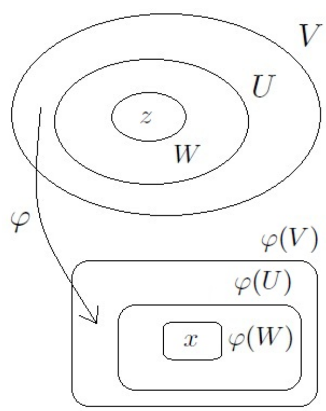
7.1 Construction of meromorphic functions

7.1.1 Proposition

Let M be a complex manifold and $U \subseteq M$ be open and connected.
Then $\mathcal{O}_M(U)$ is an integral domain, i.e. if $f, g \in \mathcal{O}_M(U)$ are such that $f \cdot g = 0$ on U, then $f = 0$ or $g = 0$.

Proof. Let $f \cdot g = 0$ and assume that $f \neq 0$. We have to show that $g = 0$ on U, i.e. $g(z) = 0, \forall z \in U$.

We first consider the case where U is completely contained in some chart domain V. Let (φ, V) be the associated coordinate chart. $f \neq 0 \Rightarrow \exists z_0 \in U$ such that $f(z_0) \neq 0$. f continuous $\Rightarrow \exists W \subseteq U$ open such that $z_0 \in W$ and $f(z) \neq 0, \forall z \in W$. By bijectivity of φ, W is in 1-to-1 correspondence with $\varphi(W)$, i.e. any $z \in W \subseteq U \subseteq V$ can uniquely be written as $z = \varphi^{-1}(x)$ for some $x \in \varphi(W) \subseteq \varphi(U) \subseteq \varphi(V)$.



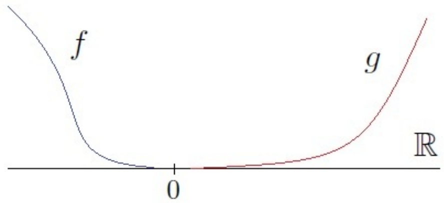
$$\forall z \in W \ : 0 = (f \cdot g)(z) = f(z) \cdot g(z) = f(z) \cdot (g \circ \varphi^{-1})(x) \quad \text{with } f(z) \neq 0$$
$$\Rightarrow \ (g \circ \varphi^{-1})(x) = 0 \, , \ \forall x \in \varphi(W) \subseteq \varphi(U)$$

$g \circ \varphi^{-1}$ is holomorphic on $\varphi(U) \subseteq \mathbb{C}^n$, which is open and connected (as image of U, which is connected), and vanishes on $\varphi(W) \subseteq \varphi(U)$, which is open and non-empty since $\varphi(z_0) \in \varphi(W)$. Thus $g \circ \varphi^{-1} = 0$ on $\varphi(U)$ by the Identity Theorem and since φ is bijective this exactly means that $g = 0$ on U.

Now let $U \subseteq M$ be covered by chart domains : $U = \bigcup_{i \in I} V_i$. $f \neq 0 \Rightarrow \exists z_0 \in U$ and $\exists i_0 \in I$ such that $z_0 \in V_{i_0}$ with $f(z_0) \neq 0$. Moreover $\exists W \subseteq V_{i_0} \subseteq U$ open such that $z_0 \in W$ and $f(z) \neq 0, \forall z \in W$. By the same argument as above, we conclude that $g = 0$ on V_{i_0}. Let $y \in U$ be arbitrary. We have to show that $g(y) = 0$ too.
Since U is connected (hence path-connected), there is a continuous path $\gamma : [0,1] \to M$ such that $\gamma(0) = z_0$ and $\gamma(1) = y$. $\operatorname{im} \gamma$ is compact and covered by the V_i, hence it can be covered by finitely many of them, say $V_{i_0}, V_1, \ldots, V_n$ with $y \in V_n$. We know that $g = 0$ on V_{i_0}, hence it is also zero on $V_{i_0} \cap V_1$, which is non-empty and open in V_1, hence (again via local coordinates), $g = 0$ on V_1. We repeat this argument until V_n, so finally $g = 0$ on V_n and $g(y) = 0$. Hence $g = 0$ on U since $y \in U$ was arbitrary. ☐

Remark :
This is not true for C^∞_M or C^ω_M if M is a real differentiable manifold. Consider e.g. the following example :

Figure 7.1: C^∞_M is not an integral domain



$f(x) = 0, \forall\, x > 0$ and $g(x) = 0, \forall\, x < 0 \;\Rightarrow\; f \cdot g = 0$ on $\mathbb{R}$, but $f \neq 0$ and $g \neq 0$

7.1.2 Definition 1

Let $U \subseteq M$ be open and connected. Since $\mathcal{O}_M(U)$ is an integral domain, it does not contain zero divisors, hence

$$\overline{\mathcal{M}}(U) := \text{Quot}\left(\mathcal{O}_M(U)\right) = S^{-1}\mathcal{O}_M(U)$$

exists and is a field, where $S = \mathcal{O}_M(U) \setminus \{0\}$ is a multiplicative set. Its elements are called *meromorphic "functions"* on U. This definition however implies a certain number of problems.

1) An element $\bar{h} \in \overline{\mathcal{M}}(U)$ is given by an equivalence class $\bar{h} = \frac{f}{g}$ where $f, g \in \mathcal{O}_M(U)$, but it is not a function.

$$\text{Recall} \;:\; \frac{f}{g} \sim \frac{f'}{g'} \;\Leftrightarrow\; f \cdot g' - f' \cdot g = 0$$

If $\bar{h}$ is represented by f and g, we may define

$$h \;:\; U \to \mathbb{C}\,,\; h(z) := \frac{f(z)}{g(z)} \quad \text{(division)}$$

First of all, this is only a function on $U \setminus V(g)$ where $V(g) = \{\, z \in U \mid g(z) = 0\,\}$ is the zero set of g.
Problem : This is not independent of the representatives f, g since maybe $V(g) \subsetneq V(g')$ if $\frac{f}{g} \sim \frac{f'}{g'}$.
At the moment we are not yet able to solve this problem. For this we need the notion of analytic sets (this will be explained in section 7.4).

2) For $n > 1$, putting $h(z) := \infty$ if $z \in V(g)$ does not work, see example 7.1.3.

3) $\overline{\mathcal{M}}$ is in general not a sheaf. Consider an open subset U which is not connected, i.e. $U = U_1 \cup U_2$, $U_1 \cap U_2 = \emptyset$ for $U_1, U_2 \subseteq M$ open. Then $\mathcal{O}_M(U)$ contains zero divisors, e.g.

$$f_1(z) = \begin{cases} 1 & \text{if } z \in U_1 \\ 0 & \text{if } z \in U_2 \end{cases} \quad , \quad f_2(z) = \begin{cases} 0 & \text{if } z \in U_1 \\ 1 & \text{if } z \in U_2 \end{cases}$$

$f_1 \in \mathcal{O}_M(U)$, $f_2 \in \mathcal{O}_M(U)$, $f_1 \neq 0$, $f_2 \neq 0$, but $f_1 \cdot f_2 = 0$.
But Quot(R) is not defined if R is not an integral domain (since $R \setminus \{0\}$ is not multiplicative), hence the mapping

$$U \longmapsto \overline{\mathcal{M}}(U)$$

is not even well-defined and $\overline{\mathcal{M}}$ cannot be a sheaf.

In order to solve this problem, one can do the following (which we do not develop in detail) :
Since the manifold M is locally connected, it admits a basis consisting of connected open sets, on which the mapping $U \mapsto \overline{\mathcal{M}}(U)$ makes sense. This can be extended to all open sets and $\overline{\mathcal{M}}$ will define a presheaf on M. And finally we may take as $\mathcal{M}$ the sheaf associated to $\overline{\mathcal{M}}$.

7.1.3 Example

<u>$n=1$</u> : in local coordinates, 2 holomorphic functions f and g on M can locally be written as

$$f(z) = (z-z_0)^k \cdot \hat{f}(z) \quad , \quad g(z) = (z-z_0)^l \cdot \hat{g}(z)$$

where $z_0 \in \mathbb{C}$, $\hat{f}(z_0) \neq 0$, $\hat{g}(z_0) \neq 0$ and $k,l \in \mathbb{Z}$ can be zero. Then one defines $h(z) := (z-z_0)^{k-l} \cdot \dfrac{\hat{f}(z)}{\hat{g}(z)}$.

− if $k > l$, h is a holomorphic function with a zero at z_0
− if $k = l$, h is a holomorphic function with with a non-zero value at z_0
− if $k < l$, then h is not holomorphic at z_0, but we can put $h(z_0) := \infty$

In this last case, we say that z_0 is a *pole of order* $k-l$ of h. Moreover one can show that the definition of h is independent of the chosen coordinates.

<u>$n>1$</u> : poles are in general not well-defined for $n > 1$
Consider $M = \mathbb{C}^2$ and $f(z) = f(z_1, z_2) = \frac{z_1}{z_2} \;\Rightarrow\; V(z_2) = \{ (z_1, 0) \mid z_1 \in \mathbb{C} \}$. Let $x = (z_1, 0) \in V(f)$; we approach x from different directions (see figure 7.2). Let α and β be zero-sequences and set

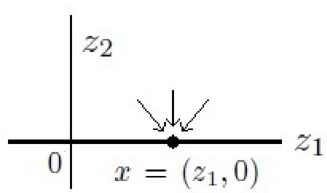
$$x_n := \big(z_1 + \alpha(n), 0 + \beta(n)\big) \quad \Rightarrow \quad f(x_n) = \frac{z_1 + \alpha(n)}{\beta(n)} = \frac{z_1}{\beta(n)} + \frac{\alpha(n)}{\beta(n)}$$

If $z_1 \neq 0$, then $\left|\frac{z_1}{\beta(n)}\right| \to \infty$, hence $|f(x_n)| \to \infty$, so the points $(z_1, 0)$, $z_1 \neq 0$ could be considered to be poles. However for $z_1 = 0$ and $x = (0,0)$, we get $f(x_n) = \frac{\alpha(n)}{\beta(n)}$. By definition $\lim\limits_{z \to 0} f(z)$ exists if we obtain the same limit for any zero-sequence. But this is not satisfied here, e.g. for $c \in \mathbb{C}$, let

$$\alpha(n) = \frac{c}{n} \;,\; \beta(n) = \frac{1}{n} \quad \Rightarrow \; f(x_n) = c \longrightarrow c \;\text{ as } n \to +\infty$$
$$\alpha(n) = \frac{1}{n^2} \;,\; \beta(n) = \frac{1}{n} \quad \Rightarrow \; f(x_n) = \frac{1}{n} \longrightarrow 0 \;\text{ as } n \to +\infty$$
$$\alpha(n) = \frac{1}{n} \;,\; \beta(n) = \frac{1}{n^2} \quad \Rightarrow \; f(x_n) = n \longrightarrow \infty \;\text{ as } n \to +\infty$$

We conclude that by approaching $(0,0)$ from different directions, one can obtain any complex value, including 0 and ∞. Hence putting $f(0,0) := \infty$ is not a good choice.

Figure 7.2: approaching the point x from different directions



7.1.4 Conclusion

If $\bar{h} \in \overline{\mathcal{M}}(U)$, then $\bar{h} = \frac{f}{g}$ and one can define $h(x) := \frac{f(x)}{g(x)}$ as a function as long as

$$x \notin V(g) = \big\{ y \in U \;\big|\; g(y) = 0 \big\}$$

$V(g)$ is closed since g is holomorphic, hence continuous. We assume that g has no essential singularities.
A meromorphic function is only a (holomorphic) functions outside of this zero set. Moreover if $\frac{f}{g} \sim \frac{f'}{g'}$ where g' has other zeros than g, we have to look for the smallest set of points we have to take out.

7.2 Definition using UFDs

Recall :
Let $m, n \in \mathbb{N}$ and $d = \gcd(n,m)$.
n and m are *relatively prime* $\Leftrightarrow$ $d = 1$ $\Leftrightarrow$ n and m do not have common prime factors.
$\mathbb{Z}$ is a unique factorization domain ("unique" means up to order of the factors and multiplication by units).

7.2.1 Theorem

If M is a complex manifold and $x \in M$, then the local rings $\mathcal{O}_{M,x}$ are UFDs. No proof will be given.

Hence $f_x, g_x \in \mathcal{O}_{M,x}$ are relatively prime $\Leftrightarrow$ $\exists\, U, V, W \subseteq M$ open, $f \in \mathcal{O}_M(U)$, $g \in \mathcal{O}_M(V)$ such that $W \subseteq U \cap V$ and $f_{|W}$, $g_{|W}$ are relatively prime (i.e. they do not have common factors).
Given $g \in \mathcal{O}_M(U)$, we can decompose g as $g = p \cdot g_1 \cdot \ldots \cdot g_l$ where all factors are holomorphic, p is a unit and all g_i are indecomposable (up to units). The units are the holomorphic functions f without zeros in U (since the inverse $\frac{1}{f}$ is then still holomorphic in U).

Example : for $c \in \mathbb{C} \setminus \{0\}$, $z_1\, z_2^3 \cdot c = c \cdot z_1 \cdot z_2 \cdot z_2 \cdot z_2$ and c is a unit.

7.2.2 Definition

A *Weierstrass polynomial* of degree m is a function $W : \mathbb{C}^n \to \mathbb{C}$ of the form

$$W(z_1, \ldots, z_n) = z_n^m + \sum_{j=0}^{m-1} a_j(z_1, \ldots, z_{n-1}) \cdot z_n^j$$

where the a_j are holomorphic functions in a neighborhood of $(0, \ldots, 0) \in \mathbb{C}^{n-1}$ and $a_j(0, \ldots, 0) = 0$. Hence a Weierstrass polynomial is a monic polynomial in the variable z_n whose coefficients are holomorphic functions in the remaining variables and vanish at the origin.

7.2.3 Weierstrass preparation theorem

Let $p \in \mathbb{C}^n$ and $f : U \to \mathbb{C}$ be a holomorphic function on an open neighborhood $U \subseteq \mathbb{C}^n$ of $p = (p_1, \ldots, p_n)$. Let k be the order of p_n as a zero of $f(p_1, \ldots, p_{n-1}, \cdot\,)$ (where $k = 0$ is possible; $k \geq 1$ means that $f(p) = 0$).
Then locally around p (e.g. in some small polydisc) f writes uniquely as $f(z) = h(z) \cdot W(z - p)$ where h is a holomorphic function in a neighborhood of p with $h(p) \neq 0$ and W is a Weierstrass polynomial of degree k.

Example : if $k = 0$, then one can choose $h = f$ and $W \equiv 1$.

7.2.4 Definition 2

Let $U \subseteq \mathbb{C}^n$ be an arbitrary open subset. We define

$$\mathcal{M}(U) := \big\{ \text{objects that are locally given as quotients of 2 holomorphic functions} \big\}$$

i.e. $\forall\, f \in \mathcal{M}(U)$, there exists an open covering $\{U_i\}_{i \in J}$ of U such that $f_{|U_i} = \frac{g_i}{h_i}$ where $g_i, h_i \in \mathcal{O}_M(U_i)$ are relatively prime and $g_i \cdot h_j = g_j \cdot h_i$ on $U_i \cap U_j$, i.e.

$$\left(\frac{g_i}{h_i}\right)_{|U_i \cap U_j} = \left(\frac{g_j}{h_j}\right)_{|U_i \cap U_j} , \ \forall\, i, j \in J \tag{7.1}$$

This is an alternative definition of meromorphic functions on U. Note that it suffices to define $\mathcal{M}(U)$ for open sets in $\mathbb{C}^n$ instead of open sets in a general complex manifold M since the definition of $\mathcal{M}$ only depends of the local structure of U and M locally looks like $\mathbb{C}^n$.
$\mathcal{M}$ with restriction of functions defines a sheaf since all conditions are local, so 3) and 4) are immediately satisfied.

In general, $\mathcal{M}(U) \neq \overline{\mathcal{M}}(U)$ for $U \subseteq M$ open and connected. Let e.g. M be a compact complex manifold. Then

$$\mathcal{O}_M(M) = \mathbb{C} \quad \Rightarrow \quad \overline{\mathcal{M}}(M) = \mathrm{Quot}\big(\mathcal{O}_M(M)\big) = \mathrm{Quot}(\mathbb{C}) = \mathbb{C}$$

but $\overline{\mathcal{M}}(M) \subsetneq \mathcal{M}(M)$ since $\mathcal{M}(M)$ is given by quotients of locally holomorphic functions (no global condition).

Proposition : (no proof)
If $U \subseteq M$ is contractible (in particular connected), then $\mathcal{M}(U) = \overline{\mathcal{M}}(U)$.

7.2.5 Remark

Elements in $\mathcal{M}(U)$ can also be interpreted as functions outside of a certain set A. Let $f \in \mathcal{M}(U)$ with an open covering $U = \bigcup_i U_i \;\Rightarrow\; f_{|U_i} = \frac{g_i}{h_i}$ and f is a well-defined (holomorphic) function on $U_i \setminus V(h_i)$. $\forall\, x \in U_i \cap U_j$:

$$\big(f_{|U_i}\big)_{|U_j}(x) = \Big(\frac{g_i}{h_i}\Big)_{|U_{ij}}(x) = \Big(\frac{g_j}{h_j}\Big)_{|U_{ij}}(x) = \big(f_{|U_j}\big)_{|U_i}(x) \;\Leftrightarrow\; g_i \cdot h_j = g_j \cdot h_i \text{ on } U_i \cap U_j$$

because of (7.1). Now let $A := \bigcup_i V(h_i)$. Using local compactness of U and the fact that g_i and h_i are relatively prime (all common factors are already taken out, which is possible in a UFD, so h_i may only change by a unit), one can show that $A \neq U$, hence f is a holomorphic function on $U \setminus A \neq \emptyset$.

7.3 Definition using exceptional sets

In this approach, we are looking for the smallest possible set A which must be removed in order to define meromorphic functions on a complex manifold M that are holomorphic on $M \setminus A$.

7.3.1 Definition 3

A *meromorphic function* on a complex manifold M is a pair (A, f) where $A \subseteq M$, f is a holomorphic function on $M \setminus A$ and A is minimal, i.e. $\forall\, x_0 \in A$, there exist an open neighborhood U of x_0 and holomorphic functions g, h on U such that

a) $A \cap U = V(h) = \{\, x \in U \mid h(x) = 0 \,\}$

b) the germs g_{x_0} and h_{x_0} are relatively prime (as defined in 7.2.1), i.e. have no common factors except the units

c) $f(x) = \frac{g(x)}{h(x)}$, $\forall\, x \in U \setminus A$

A is called the *exceptional set* of f. In particular, f is holomorphic on $M \;\Leftrightarrow\; A = \emptyset$.

7.3.2 Remarks

These conditions already imply that A cannot be open (in particular $A \neq M$), otherwise $V(h) = A \cap U$ is open, non-empty since it contains x_0 and $h = 0$ on $A \cap U \subseteq M$, hence $h = 0$ on M by the Identity Theorem (via local coordinates) since h is holomorphic on the non-empty open set $A \cap U \subseteq U$.

A is minimal means that $A \cap U$ is exactly equal to the vanishing set of h, i.e. exactly all the zeros of h have to be removed so that f is well-defined. This is the case because g_{x_0} and h_{x_0} are relatively prime, i.e. g cannot compensate a vanishing factor of h in the denominator.

The functions g and h may differ from neighborhood to neighborhood, but they always need to define the same function on non-empty intersections : if $f = \frac{g_i}{h_i}$ on $U_i \setminus A$ and if $f = \frac{g_j}{h_j}$ on $U_j \setminus A$, then we need

$$g_i(x) \cdot h_j(x) = g_j(x) \cdot h_i(x)\,,\ \forall\, x \in U_i \cap U_j \quad\Rightarrow\; f = \frac{g_i}{h_i} = \frac{g_j}{h_j} \ \text{ on } \ (U_i \cap U_j) \setminus A$$

7.4 Analytic sets

7.4.1 Lemma

For $n = 1$, the zero set $V(f)$ of a non-constant holomorphic function f on an open set U is a *discrete set*, i.e. $\forall\, z \in V(f)$, there is an open neighborhood V of z such that $V \cap V(f) = \{z\}$.

Proof. Assume that $V(f) \subseteq U$ is not discrete. Then $\exists\, z_0 \in V(f)$ such that any open neighborhood of z_0 intersects $V(f)$ in some point which is distinct from z_0, i.e. z_0 is an accumulation point of $V(f)$ (see section 2.4). But since $f = 0$ on $V(f)$ and f is holomorphic on U, the Identity Theorem for $n = 1$ implies that $f = 0$ on U, which contradicts the fact that f is non-constant. Hence $V(f)$ must be a discrete set. □

Remark :

This does not hold for $n \geq 2$. Consider e.g. $f(z_1, z_2) = z_1 \;\Rightarrow\; V(f) = \{\, (0, z_2) \mid z_2 \in \mathbb{C} \,\}$ is not discrete.

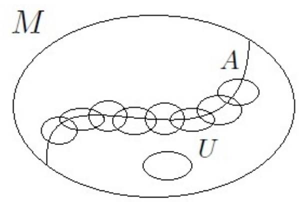
7.4.2 Definition

Let M be a complex manifold. $A \subseteq M$ is called an *analytic set* if $\forall\, z \in M$, there exist an open neighborhood U of z and $\exists\, f_1, \ldots, f_k \in \mathcal{O}_M(U)$ such that (see figure 7.3)

$$A \cap U = \{\, x \in U \mid f_1(x) = f_2(x) = \ldots = f_k(x) = 0 \,\} = \bigcap_{j=1}^{k} f_j^{-1}\big(\{0\}\big) \tag{7.2}$$

Hence analytic sets locally look like the zero set of finitely many holomorphic functions. These functions are not uniquely determined by A. Moreover k may depend on z since the functions f_i are in general not the same for any $z \in M$. Note that (7.2) is trivially satisfied if $A \cap U = \emptyset$ by taking $f_1 \equiv 1 \in \mathcal{O}_M(U)$. If $V \subseteq M$ is open and connected, we can also talk about analytic sets in V.

Figure 7.3: $A \subseteq M$ is an analytic set



Equivalently, A is an analytic set if and only if there is an open covering $\{U_i\}_{i\in J}$ of M such that $\forall\, i \in J$, $\exists\, f_1^i, \ldots, f_{k_i}^i \in \mathcal{O}_M(U_i)$ satisfying $A \cap U_i = \{\, x \in U_i \mid f_1^i(x) = \ldots = f_{k_i}^i(x) = 0 \,\}$.
Note that the k_i are not uniquely given since there may be many possibilities to describe $A \cap U_i$ as a zero set. But one can always extract a minimal number of functions which are necessary and sufficient to describe $A \cap U_i$. In particular if A is connected, then it is possible to choose $k_i = k_j$, $\forall\, i, j \in J$.

7.4.3 Examples

1) Zero sets of holomorphic functions are by definition analytic sets (take $U = M$).

2) Algebraic varieties in $\mathbb{C}^n$ are analytic sets since they are given by the zero set of finitely many polynomials, which are globally holomorphic functions on $\mathbb{C}^n$.

3) The exceptional set A of a meromorphic function (A, f) is always an analytic set by condition a). In particular if $A \neq \emptyset$, then any exceptional set is a 1-codimensional analytic subset of M (since there is just 1 function).

4) For an analytic set, $A = M$ is possible by taking the zero function everywhere : $A \cap M = \{\, x \in M \mid 0(x) = 0 \,\}$.

5) There are analytic sets which are not of codimension 1. Consider e.g. $M = \mathbb{C}^2$ and $A = \{(0,0)\}$. In order to show that A is an analytic set, we have to find an open covering $\{U_i\}_{i\in I}$ of M and holomorphic functions $f_1^i, \ldots, f_{k_i}^i \in \mathcal{O}_M(U_i)$ such that $A \cap U_i = V(f_1^i, \ldots, f_{k_i}^i) \cap U_i$.
Take $i = 1$, $U_1 = \mathbb{C}^2$, $f_1(z) = z_1$ and $f_2(z) = z_2$ for $z = (z_1, z_2) \in \mathbb{C}^2$. Hence $f_1, f_2 \in \mathcal{O}_M(\mathbb{C}^2)$ and

$$V(f_1, f_2) \cap U_1 = \big\{\, z \in \mathbb{C}^2 \;\big|\; f_1(z) = f_2(z) = 0 \,\big\} = \big\{\, (z_1, z_2) \in \mathbb{C}^2 \;\big|\; z_1 = z_2 = 0 \,\big\} = \{(0,0)\} = A = A \cap U_1$$

Thus A is an analytic set, but it is not of codimension 1 in $\mathbb{C}^2$.

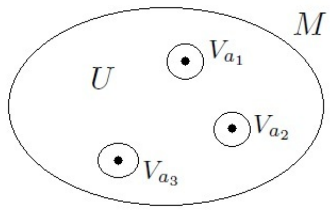
6) Points, a finite number of points and, more generally, any discrete set is an analytic set.

Let M be a complex manifold and $A \subseteq M$ be discrete, i.e. $\forall\, a \in A$, there is an open neighborhood V_a of a such that $V_a \cap A = \{a\}$. By choosing the V_a small enough, we may assume that (V_a, φ_a) is a coordinate chart around a. Hence one can take as open covering of M the collection $\mathcal{V} = \{U, V_a \mid a \in A\}$ where $U \subseteq M$ is open and chosen such that $U \cup \bigcup_{a\in A} V_a = M$ and $A \cap U = \emptyset$. This is possible since M is Hausdorff, e.g.

$$U = \bigcup_{x\in F} U_x \qquad \text{where } \; F = M \setminus \bigcup_{a\in A} V_a$$

and U_x is a small open neighborhood of $x \in F$ not containing any points from A (see figure 7.4).

Figure 7.4: U and the V_{a_i} form an open covering of M



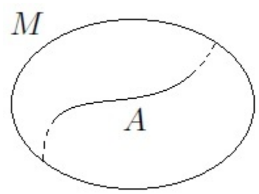
Since $A \cap U = \emptyset$, we can take the constant function 1 on U. And on V_a we may consider the holomorphic function

$$f^a(x) = \varphi_a(x) - \varphi_a(a) \quad \Rightarrow \; f^a(x) = 0 \;\Leftrightarrow\; \varphi_a(x) = \varphi_a(a) \;\Leftrightarrow\; x = a$$

since φ_a is bijective. Hence $f^a \in \mathcal{O}_M(V_a)$ and $A \cap V_a = \{a\} = \{\, x \in V_a \mid f^a(x) = 0 \,\} = V(f^a) \cap V_a$, $\forall\, a \in A$.

7) But not all analytic sets are discrete, as e.g. zero sets of holomorphic functions in several variables. Now we show that such a zero set cannot "end" somewhere.

Figure 7.5: $A = V(f)$ cannot "end" at a certain point



Let $U \subseteq M$ open, $f \in \mathcal{O}_M(U)$ and assume that $A = V(f)$ is "bounded in U". Consider an "endpoint" $a \in A$ and a chart (V, φ) around it (see figure 7.6). In the local situation $\varphi(V) \subseteq \mathbb{C}^n$, we obtain that $\varphi(A \cap V)$ is closed in $\varphi(V)$ since φ is a homeomorphism and zero sets of holomorphic functions are closed. Moreover

$$\varphi(A \cap V) = \varphi(A) \cap \varphi(V) \quad \Rightarrow \quad \text{closed in } \varphi(V)$$

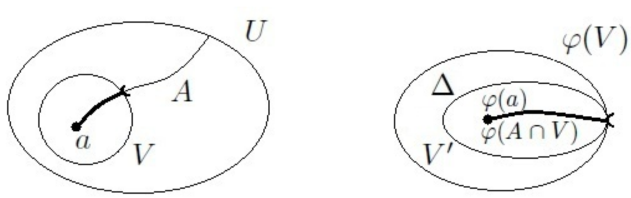
$$\varphi(A \cap V) = \varphi\Big(\big\{\, x \in V \;\big|\; f(x) = 0 \,\big\}\Big) = \big\{\, \varphi(x) \in \varphi(V) \;\big|\; f(x) = 0 \,\big\} = \big\{\, z \in \mathbb{C}^n \;\big|\; (f \circ \varphi^{-1})(z) = 0 \,\big\} = V(f \circ \varphi^{-1})$$

Hence we can choose a polydisc Δ in $\mathbb{C}^n$ such that $\varphi(A \cap V) \subsetneq \Delta \subsetneq \varphi(V)$ and denote $V' := \varphi(V) \setminus \Delta$. Let

$$g(z) := \frac{1}{(f \circ \varphi^{-1})(z)} \,, \; \forall\, z \in V'$$

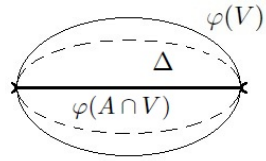
g is well-defined and holomorphic in V' since $V(f \circ \varphi^{-1}) \subset \Delta$. A consequence of Hartog's Lemma (section 2.6.4) then states that g can be holomorphically extended on the polydisc Δ, i.e. g extends to a holomorphic function on $\varphi(V)$. By uniqueness of this extension we obtain that g must exist everywhere on Δ, which means that $f \circ \varphi^{-1}$ cannot have zeros on Δ and hence that f has no zeros in $A \cap V$: contradiction since $a \in A \cap V$ and $f(a) = 0$.

Figure 7.6: the local situation at the "endpoint"



Remark :
Such an argument does not apply if A is "not bounded" in U because if such a polydisc Δ exists, then $\varphi(V) \setminus \Delta$ is not connected and Hartog's Lemma does not apply (see figure 7.7).

Figure 7.7: $\varphi(V) \setminus \Delta$ is no longer connected



7.4.4 Exercise

Show that any submanifold of a complex manifold is an analytic set.

This directly follows from characterization (4.5). In fact, being a submanifold is a stronger condition than being an analytic set since the maximal rank condition also needs to be satisfied.
The converse is false : not any analytic set A defines a complex submanifold since A may have singularities, i.e. points at which the rank of the Jacobian matrix drops (hence the maximal rank condition is not satisfied).

7.4.5 Proposition

Let M be a complex manifold, $U \subseteq M$ be open and A be an analytic set in U. Then A is closed in U.

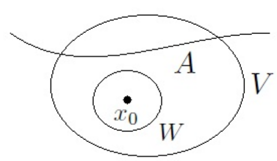
Proof. Recall that M, hence U, is by definition a topological space. However since we require that the coordinate charts are homeomorphisms, this topology must be locally homeomorphic to the standard topology of $\mathbb{C}^n$.

If $A = U$, then $A = A \cap U = \{ x \in U \mid 0(x) = 0 \}$: closed in U. Hence we may assume that $A \neq U$. We have to show that $U \setminus A$ is open in U.
Let $x_0 \in U \setminus A$. Since A is an analytic set, we know that there is a neighborhood V of x_0 in U such that

$$A \cap V = \left\{ x \in V \;\middle|\; f_1(x) = \ldots = f_k(x) = 0 \right\}$$

for some suitable $f_i \in \mathcal{O}_M(V)$, $i \in \{1, \ldots, k\}$. Not all i satisfy $f_i(x_0) = 0$ since otherwise $x_0 \in V \cap A \subseteq A$. Assume for example that $f_1(x_0) \neq 0$. Since f_1 is holomorphic on V, we get by continuity that $\exists\, W \subseteq V$ open such that $x_0 \in W$ and $f_{1|W} \neq 0$, i.e. $f_1(y) \neq 0$, $\forall\, y \in W$. Thus $y \notin V \cap A$, $\forall\, y \in W \subseteq V \;\Rightarrow\; y \notin A$, $\forall\, y \in W$. Hence $W \cap A = \emptyset$ (see figure 7.8), which precisely means that $W \subseteq U \setminus A$ with $x_0 \in W$. □

Figure 7.8: W is an open neighborhood of x_0 not intersecting A



7.4.6 Recall

Let X be a topological space and $A \subseteq X$. The topological *boundary* of A is given by

$$\partial A := \bar{A} \cap \overline{X \setminus A} = \left\{ x \in X \;\middle|\; \forall U \subseteq X \text{ open set containing } x,\ \exists\, y, z \in U \text{ such that } y \in A \text{ and } z \notin A \right\}$$

Moreover we have the relations

$$A^\circ \subset A \subset \bar{A} \quad , \quad \bar{A} = A \cup \partial A \quad , \quad \bar{A} = A^\circ \,\dot{\cup}\, \partial A \quad , \quad A^\circ = A \setminus \partial A \quad , \quad \overline{X \setminus A} = X \setminus A^\circ \quad , \quad \partial A^\circ \subseteq \partial A$$

where $A^\circ = \{ x \in X \mid \exists U \text{ open neighborhood of } x \text{ such that } U \subseteq A \}$ contains all interior points of A. A° is always open and ∂A is always closed, but may be different from $\partial A^\circ = \partial(A^\circ)$.
In particular : $\partial\emptyset = \emptyset$ and $\partial X = \emptyset$, but $\partial A \neq \emptyset$, $\forall\, A \notin \{\emptyset, X\}$. Any non-trivial subset of X has boundary points.

7.4.7 Proposition

Let A be an analytic set in M such that $A \neq M$. Then A has empty interior, i.e. $A^\circ = \emptyset$.

Proof. We already know that A is closed, hence $\bar{A} = A$. Assume that $A^\circ \neq \emptyset$. Then $\partial A^\circ \neq \emptyset$ too because

$$A \neq M \;\Rightarrow\; A^\circ \subsetneq M \;\Rightarrow\; A^\circ \subset A \subsetneq M \;\Rightarrow\; \overline{A^\circ} \subset \bar{A} = A \subsetneq M \quad \text{with } \overline{A^\circ} = (A^\circ)^\circ \,\dot{\cup}\, \partial(A^\circ) = A^\circ \,\dot{\cup}\, \partial A^\circ$$

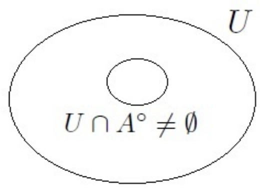
If $\partial A^\circ = \emptyset$, then $\overline{A^\circ} = A^\circ$, which means that A° is closed and open. Since $A^\circ \neq \emptyset$ and M is connected, this implies that $A^\circ = M \;\Rightarrow\; A = M$, which is a contradiction to our hypothesis. So let $x_0 \in \partial A^\circ \subset \overline{A^\circ} \;\Rightarrow\;$ any open neighborhood U_{x_0} of x_0 intersects $A^\circ : U_{x_0} \cap A^\circ \neq \emptyset$ and this intersection is still open.
Now take such a neighborhood $U = U_{x_0}$ which is connected and small enough (chart domain) such that

$$A \cap U = \left\{ x \in U \;\middle|\; f_1(x) = \ldots = f_k(x) = 0 \right\}$$

for some $f_i \in \mathcal{O}_M(U)$ (which is possible since A is an analytic set). Hence $\forall\, i \in \{1, \ldots, k\}$, we have

$$f_{i|A\cap U} = 0 \;\Rightarrow\; f_{i|U\cap A^\circ} = 0 \quad \text{with } U \cap A^\circ \text{ open, non-empty}$$

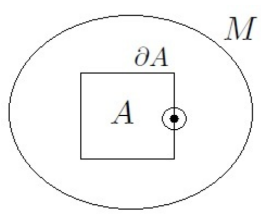
Since U is connected and contains $U \cap A^\circ \neq \emptyset$, we obtain by the Identity Theorem (via local coordinates) that $f_{i|U} = 0$, $\forall\, i$, i.e. $f_i = 0$ as functions in $\mathcal{O}_M(U)$.



This implies that $A \cap U = U \;\Rightarrow\; U \subseteq A$, which contradicts the fact that U contains elements which are not in A. Indeed, U is a neighborhood of $x_0 \in \partial A^\circ \subseteq \partial A$, hence $U \cap (M \setminus A) \neq \emptyset$ and this is not compatible with $U \subseteq A$. Therefore $A^\circ = \emptyset$. □

Consequence :
If A is an analytic set in a complex manifold M, then A is either equal to the whole space or has empty interior. In particular, not all closed sets in M are analytic sets, as e.g. in the following example :

Figure 7.9: A has a non-empty interior and is hence not an analytic set



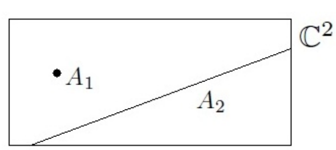
7.4.8 Corollary

If $A \neq M$ is analytic, then $M \setminus A$ is open and dense in M, i.e. $\overline{M \setminus A} = M$.

Proof. $M \setminus A$ is open since A is closed. Moreover $\overline{M \setminus A} = M \setminus A^\circ = M \setminus \emptyset = M$. □

Example :
If $M = \mathbb{C}^2$, A_1 is a point and A_2 is a line in $\mathbb{C}^2$ (both are analytic sets), then $\overline{\mathbb{C}^2 \setminus \{\text{pt}\}} = \overline{\mathbb{C}^2 \setminus \text{line}} = \mathbb{C}^2$.

Figure 7.10: the complement of a point or a line in $\mathbb{C}^2$ is dense



7.5 Application to meromorphic functions

Let M be a complex manifold and (A, f) be a meromorphic function on M, i.e. $f \in \mathcal{O}_M(M \setminus A)$, where the exceptional set A is an analytic set as shown in 7.4.3. Then it is possible to extend f to a bigger set (maybe f cannot be extended on the whole A, but at least as far as possible).

7.5.1 Lemma

Let $U \subseteq M$ be open, $x_0 \in U$ and $g, h \in \mathcal{O}_M(U)$ be relatively prime such that $h(x_0) = g(x_0) = 0$. Then in every neighborhood V of x_0 and for every $c \in \mathbb{C}$, $\exists\, x \in V$ such that $\frac{g(x)}{h(x)} = c$.

The proof of this lemma uses the Weierstrass preparation theorem. As an example, consider the function $f(z_1, z_2) = \frac{z_1}{z_2}$ at $(0,0)$. In 7.1.3 we saw that f takes all possible complex values, including ∞, around $(0,0)$.

7.5.2 Proposition

Let $Y \subset M$ be an open and dense subset of M and $f \in \mathcal{O}_M(Y)$ be holomorphic on Y. Assume that $\forall\, x_0 \in M \setminus Y$, $\exists\, U$ open set containing x_0 and $\exists\, g, h \in \mathcal{O}_M(U)$ such that $h(x) \cdot f(x) = g(x)$, $\forall\, x \in U$ and the germs g_{x_0} and h_{x_0} are relatively prime (as defined in 7.2.1). If we define

$$A := \left\{ x_0 \in M \setminus Y \mid \forall\, r \in \mathbb{R},\ \forall\, V \text{ open neighborhood of } x_0,\ \exists\, x \in V \cap Y \text{ such that } |f(x)| > r \right\} \tag{7.3}$$

then there exists a unique holomorphic extension $\hat{f}$ of f to $M \setminus A \supseteq Y$ such that $(A, \hat{f})$ is a meromorphic function.

<u>Remark</u> : The condition in (7.3) is not uniform since we only require that $\exists\, x \in V \cap Y$ instead of $\forall\, x \in V \cap Y$.

Proof. Let $x_0 \in M \setminus Y \ \Rightarrow\ \exists\, U$ small open neighborhood of x_0 and $\exists\, g_U, h_U \in \mathcal{O}_M(U)$ such that

$$g_U(x) = h_U(x) \cdot f(x)\,,\ \forall\, x \in U \tag{7.4}$$

If $h_U(x_0) \neq 0$, then by continuity $\exists\, W \subseteq U$ open such that $x_0 \in W$ and $h_{U|W} \neq 0$. Hence $\frac{g_U}{h_U}$ will be bounded in this neighborhood W around x_0, which means that $x_0 \notin A$.
If $h_U(x_0) = 0$ and $g_U(x_0) \neq 0$, i.e. $g_{U|W} \neq 0$ for some open neighborhood W of x_0, then (7.4) implies that $0 \cdot f(x_0) \neq 0$, i.e. "$f(x_0) = \infty$". More precisely, this means that $|f|$ will be uniformly unbounded in any small neighborhood W around x_0. It follows that $x_0 \in A$.
If $h_U(x_0) = g_U(x_0) = 0$ where g_U and h_U are relatively prime, lemma 7.5.1 shows that $x_0 \in A$. Hence

$$A \cap U = \left\{ x \in U \;\middle|\; h_U(x) = 0 \right\} \quad \Rightarrow \quad A = \bigcup_{U \ni x_0} (A \cap U) \quad \text{since } A \subseteq M \setminus Y \subseteq \bigcup_{U \ni x_0} U$$

This shows in particular that A is an analytic set.
Moreover this whole argument is independent of the choice of g_U and h_U since they are relatively prime, i.e. different representatives only differ by units on U (holomorphic functions which have no zeros on U), hence $A \cap U$ is always the same. Now one can extend f as follows : first we set

$$\hat{f}_U \ :\ (M \setminus A) \cap U \longrightarrow \mathbb{C}\ ,\ \hat{f}_U(x) := \frac{g_U(x)}{h_U(x)} = f_{|U}(x)$$

This exists since A (the zero set of all h_U) has been removed. Moreover $\hat{f}_U$ is well-defined on the intersections : if U_1, U_2 have a non-empty intersection $U_{12} := U_1 \cap U_2 \neq \emptyset$ with $\hat{f}_{U_1} = \frac{g_{U_1}}{h_{U_1}}$ and $\hat{f}_{U_2} = \frac{g_{U_2}}{h_{U_2}}$, then

$$\left(\hat{f}_{U_1}\right)_{|U_{12}} = \left(\frac{g_{U_1}}{h_{U_1}}\right)_{|U_{12}} = \left(f_{|U_1}\right)_{|U_{12}} = \left(f_{|U_2}\right)_{|U_{12}} = \left(\frac{g_{U_2}}{h_{U_2}}\right)_{|U_{12}} = \left(\hat{f}_{U_2}\right)_{|U_{12}}$$

Since $A \subseteq M \setminus Y \ \Rightarrow\ Y \subseteq M \setminus A$, we then can define

$$\hat{f} \ :\ M \setminus A \longrightarrow \mathbb{C}\ ,\ \hat{f}(x) := \begin{cases} f(x) & \text{if } x \in Y \\ \hat{f}_U(x) = \frac{g_U(x)}{h_U(x)} & \text{if } x \in M \setminus Y\,,\ x \in U \end{cases}$$

This gives indeed a holomorphic extension of f since $h_U \cdot f_{|U} = g_U$ by assumption $\Rightarrow$ $f_{|U} = \frac{g_U}{h_U}$ after A has been removed. It remains to show that the extension is unique.

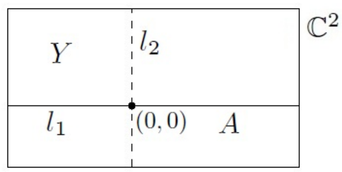
Let $\hat{f}'$ be any extension of f to $M \setminus A$. By definition, this means that $\hat{f}'_{|Y} = f = \hat{f}_{|Y}$, hence $\hat{f}$ and $\hat{f}'$ coincide on Y. Differences can thus only happen on $M \setminus Y$. But this is not the case since Y is dense in M. Every point in $M \setminus Y$ is a limit of sequences from Y, i.e.

$$\forall\, x_0 \notin Y \;:\; \exists\, (x_n)_n \subset Y \text{ such that } x_n \to x_0 \text{ as } n \to +\infty$$

In particular, if $x_0 \in M \setminus A$, then $\hat{f}(x_0)$ and $\hat{f}'(x_0)$ exist and by continuity, we have $\hat{f}'(x_n) \to \hat{f}'(x_0)$. $x_n \in Y$ implies that $\hat{f}'(x_n) = f(x_n) = \hat{f}(x_n) \to \hat{f}(x_0)$. By uniqueness of the limit we finally obtain that $\hat{f}(x_0) = \hat{f}'(x_0)$ too, i.e. $\hat{f}$ and $\hat{f}'$ coincide on $M \setminus A \supseteq Y$. □

7.5.3 Example

Consider $\mathbb{C}^2$ and denote the coordinate axes by $l_1 = \{\, z \in \mathbb{C}^2 \mid z_2 = 0 \,\}$ and $l_2 = \{\, z \in \mathbb{C}^2 \mid z_1 = 0 \,\}$.
Let $Y := \mathbb{C}^2 \setminus (l_1 \cup l_2)$ $\Rightarrow$ Y is open and dense in $\mathbb{C}^2$ and consider the function $f(z) = \frac{z_1}{z_2}$, which is well-defined on Y. Define A as in (7.3) $\Rightarrow$ $A \subseteq l_1 \cup l_2$. By 7.1.3, we have that $l_1 \subseteq A$ since $|f| \to \infty$ when approaching $z \in l_1$ where $z \neq (0,0)$; on $l_1 \setminus \{(0,0)\}$, $|f|$ even goes uniformly to ∞. However no point from $l_2 \setminus \{(0,0)\}$ belongs to A since f is not unbounded in any neighborhood of these points.



Moreover z_1 and z_2 are relatively prime, hence $A = l_1$ and f can be extended to a holomorphic function $\hat{f}$ on $\mathbb{C}^2 \setminus l_1$. This is the maximal extension of f.

7.5.4 Theorem

Let M be a (connected) complex manifold.
Then the set of all meromorphic functions $\mathcal{M}(M)$ is an algebra and a field extension of $\mathbb{C}$.

Proof. $\mathbb{C} \subset \mathcal{M}(M)$ since any constant $\in \mathbb{C}$ can be written as $c = \frac{c}{1}$ with $V(1) = \emptyset$. Now we have to define additive and multiplicative structures on $\mathcal{M}(M)$. Let (A, f) and (A', f') be 2 meromorphic functions on M.
Since A and A' are analytic sets (see 7.4.3), we know that A, A' are closed and that $M \setminus A$ and $M \setminus A'$ are dense in M. Thus $Y := M \setminus (A \cup A') = (M \setminus A) \cap (M \setminus A')$ is open and dense in M (as intersection of 2 dense subsets). Thus $f + f'$ and $f \cdot f'$ are defined on Y and can be extended by proposition 7.5.2 to some bigger set $M \setminus A_1$ where A_1 is again an analytic set. Hence $\mathcal{M}(M)$ is already a $\mathbb{C}$–algebra.
In order to construct the inverse of e.g. (A, f), we write f locally as $f = \frac{g}{h}$ with $A = V(h)$ and $B = V(g)$. This is independent of the choice of g and h because if $f = \frac{g_i}{h_i} = \frac{g_j}{h_j}$ on $U_i \cap U_j$ (where numerator and denominator are relatively prime), then $g_i \cdot h_j = g_j \cdot h_i$ on $U_i \cap U_j$. This implies that there cannot exist zeros on $U_i \cap U_j$ for one representative which are not zeros for the other one. Now, as zero sets of holomorphic functions, A and B are analytic sets and so is $A \cup B$. If $Y = M \setminus (A \cup B)$, then $\frac{1}{f} = \frac{h}{g}$ is defined on Y and we can extend it to a meromorphic function $(A_2, \frac{1}{f}) \in \mathcal{M}(M)$ such that $M \setminus A_2$ is the maximal domain of definition of $\frac{1}{f}$. □

7.5.5 Remarks

It is important to always extend the objects to the maximal domain. Consider e.g. the meromorphic functions (A, f) and $(A, -f)$. Then $f + (-f) = 0$, but 0 is not only defined on $M \setminus A$. Extending it will define 0 as a meromorphic function on M.
Hence by extending, one can (sometimes) reduce the exceptional set of a meromorphic function.

One can show that the extension $\mathcal{M}(M) \supset \mathbb{C}$ is a transcendental field extension. Moreover the dimension of the manifold M is closely related to the transcendence degree of this extension. In fact they are often, but not always, equal. We have for example equality in the case where $M = \mathbb{C}^n$ or if M is a projective variety.

Chapter 8

Analytic sets and singularities

8.1 Hypersurfaces

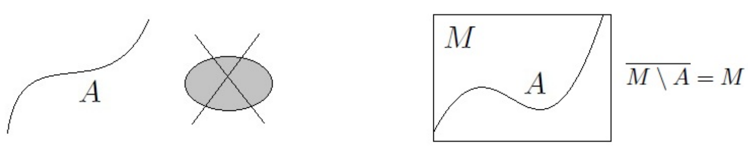
8.1.1 Recalls

Let M be a complex manifold. $A \subseteq M$ is called an analytic set if there is an open covering $\{U_i\}_{i\in J}$ of M such that $\forall\, i \in J$, $\exists\, f_1^i, \ldots, f_{k_i}^i \in \mathcal{O}_M(U_i)$ satisfying

$$A \cap U_i = \big\{\, x \in U_i \;\big|\; f_1^i(x) = \ldots = f_{k_i}^i(x) = 0 \,\big\} = \bigcap_{j=1}^{k_i} (f_j^i)^{-1}\big(\{0\}\big)$$

Let A be an analytic set. We have showed that :
a) A is closed in M.
b) $A = M$ or $A^\circ = \emptyset$ (see figure 8.1).
c) if $A \neq M$, then $M \setminus A$ is open and dense in M (see figure 8.1).

Figure 8.1: analytic sets have empty interior and their complement is dense



Submanifolds of M are analytic, but the converse is not true ; complex affine varieties can e.g. have singularities (points where the rank of the Jacobian drops). In order to make them analytic, one first has to remove all singularities . This will be the aim of this chapter.

8.1.2 Definition

A *hypersurface* is a non-empty analytic set that can locally be given by 1 non-constant function, i.e. $A \subseteq M$ is a hypersurface if there is an open covering $\{U_i\}_{i\in J}$ of M such that $\forall\, i \in J$, $\exists\, f_i \in \mathcal{O}_M(U_i)$ satisfying

$$A \cap U_i = \big\{\, x \in U_i \;\big|\; f_i(x) = 0 \,\big\} = f_i^{-1}\big(\{0\}\big)$$

Not every analytic set can be written under such a form. Note that this does not mean that $k_i = 1, \forall\, i \in J$ since k_i is not uniquely given (see section 7.4.2). Moreover it does not imply that A is of codimension 1 since A is not a manifold (we first need to define the codimension of an analytic set, see section 8.3.2).

By definition, M cannot be a hypersurface. Indeed if $A = M$, then $A \cap U_i = M \cap U_i = U_i = f_i^{-1}\big(\{0\}\big)$, so $f = 0$ on U_i, which is non-empty and open, i.e. $f = 0$ on M by the Identity Theorem and this was excluded.
In addition one has to take care that A is well-defined on the intersections, i.e. if U_i and U_j are open such that $(A \cap U_i) \cap U_j \neq \emptyset$ with $A \cap U_i = V(f_i)$ for $f_i \in \mathcal{O}_M(U_i)$ and $A \cap U_j = V(f_j)$ for $f_j \in \mathcal{O}_M(U_j)$, we need that

$$V\big(f_{i|U_i\cap U_j}\big) = V\big(f_{j|U_i\cap U_j}\big) \;\Leftrightarrow\; \big\{\, x \in U_i \cap U_j \;\big|\; f_i(x) = 0 \,\big\} = \big\{\, x \in U_i \cap U_j \;\big|\; f_j(x) = 0 \,\big\} \tag{8.1}$$

If $(A \cap U_i) \cap U_j = \emptyset$, then (8.1) is not a restriction (see example 8.1.3).

8.1.3 Example

Let $M = \mathbb{P}^1(\mathbb{C})$ and $\{U_0, U_1\}$ be the open covering of M given by the affine sets

$$U_0 = \{\, (z_0 : z_1) \mid z_0 \neq 0 \,\} = \{\, (1 : z_1) \mid z_1 \in \mathbb{C} \,\} \quad , \quad U_1 = \{\, (z_0 : z_1) \mid z_1 \neq 0 \,\} = \{\, (z_0 : 1) \mid z_0 \in \mathbb{C} \,\}$$

where e.g. $U_0 \cong \{\, \omega \mid \omega \in \mathbb{C} \,\}$ by $(z_0 : z_1) = (1 : \frac{z_1}{z_0}) = (1 : \omega)$. Let $f \in \mathcal{O}_M(U_1)$ be given by

$$f \ : \ U_1 \subsetneq \mathbb{P}^1(\mathbb{C}) \longrightarrow \mathbb{C} \ , \ f(z_0 : z_1) = \frac{z_0}{z_1}$$

f is well-defined on U_1 and independent of the representative since $\frac{\lambda z_0}{\lambda z_1} = \frac{z_0}{z_1}$. Moreover $f^{-1}\big(\{0\}\big) = \{(0:1)\}$. Let $A = \{(0:1)\}$ with $f \in \mathcal{O}_M(U_1)$ and $g \in \mathcal{O}_M(U_0)$ given by $g = 1$. With this we obtain

$$A \cap U_0 = \emptyset = g^{-1}\big(\{0\}\big) \quad , \quad A \cap U_1 = A = f^{-1}\big(\{0\}\big)$$

with $(A \cap U_0) \cap U_1 = \emptyset$, hence A is a well-defined analytic subset of $\mathbb{P}^1(\mathbb{C})$ and it is even a hypersurface since any $A \cap U_i$ is given by 1 holomorphic function.
Note that f cannot be extended to $U_0 \cup U_1$ by definition, but also since f is non-constant and there are no non-constant global holomorphic functions on $\mathbb{P}^1(\mathbb{C})$ since it is compact.

8.2 Regular and singular points

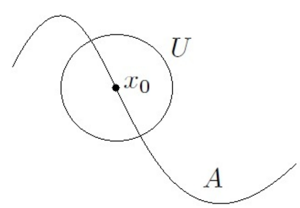
8.2.1 Definitions

Let M be a complex manifold with $\dim_{\mathbb{C}} M = n$ and $A \subseteq M$ be an analytic set.
a) $x_0 \in A$ is called a *regular point* (or a *smooth* point) of A if there is a chart (U, φ) around x_0 and there exist holomorphic functions $f_1, \ldots, f_k \in \mathcal{O}_M(U)$ such that (see figure 8.2)

$$U \cap A = \{\, x \in U \mid f_1(x) = \ldots = f_k(x) = 0 \,\} \quad , \quad J(x_0) = \left(\frac{\partial (f_i \circ \varphi^{-1})}{\partial z_j} \big(\varphi(x_0)\big) \right)_{\substack{i=1,\ldots,k \\ j=1,\ldots,n}} \tag{8.2}$$

where the $k \times n$–Jacobian matrix $J(x_0)$ evaluated at $\varphi(x_0) \in \mathbb{C}^n$ has maximal rank (i.e. rank k), if the local coordinates are $\varphi = (z_1, \ldots, z_n)$. We will see in section 8.3.2 that $k > n$ is not possible if this is the case.

Figure 8.2: x_0 is a regular point of the Jacobian matrix has maximal rank



Note that the condition of $U \cap A$ being a zero set of a finite number of holomorphic functions is always satisfied since A is analytic. We require that there are holomorphic functions which satisfy both conditions.
In the following, we may omit the composition with the chart map φ^{-1} and simply write $\frac{\partial f_i}{\partial z_j}(x_0)$ instead.

b) If $x_0 \in A$ is not a regular point, then it is called a *singular point* (or a *singularity*) of A.
c) We denote $S(A) := \{\, x \in A \mid x \text{ is singular} \,\}$.

8.2.2 Examples

Let $M = \mathbb{C}^2$ be the complex plane and consider $A = V_i = V(f_i)$ for $f_i = f_i(z_1, z_2)$ given by

$$f_1 = z_2^2 - 4z_1(z_1+1)(z_1-1) \quad , \quad f_2 = z_2^2 - z_1^3 \quad , \quad f_3 = z_1 \quad , \quad f_4 = z_1 z_2 \quad , \quad f_5 = z_1^2$$

Consider the trivial covering $U = \mathbb{C}^2$; all functions are holomorphic on $\mathbb{C}^2$. Hence any V_i is an analytic set as the zero set of a holomorphic function and using the notation in (8.2), we always have $k = 1$. This implies that the 1×2–Jacobian matrix has maximal rank at a point $x_0 \in V_i$ if and only if

$$\left(\frac{\partial f_i}{\partial z_1}(x_0) \quad \frac{\partial f_i}{\partial z_2}(x_0)\right) \neq (0\,,\,0)$$

a)

$$\frac{\partial f_1}{\partial z_1}(z_1, z_2) = -4(z_1+1)(z_1-1) - 4z_1(z_1-1) - 4z_1(z_1+1) \quad , \quad \frac{\partial f_1}{\partial z_2}(z_1, z_2) = 2z_2$$

Together with the condition that $(z_1, z_2) \in V_1$, the singular points of V_1 are given by the solutions of the system

$$\begin{cases} z_2^2 - 4z_1(z_1+1)(z_1-1) = 0 \\ (z_1+1)(z_1-1) + z_1(z_1-1) + z_1(z_1+1) = 0 \\ 2z_2 = 0 \end{cases} \Leftrightarrow \begin{cases} z_1(z_1+1)(z_1-1) = 0 & (1) \\ 3z_1^2 - 1 = 0 & (2) \\ z_2 = 0 \end{cases}$$

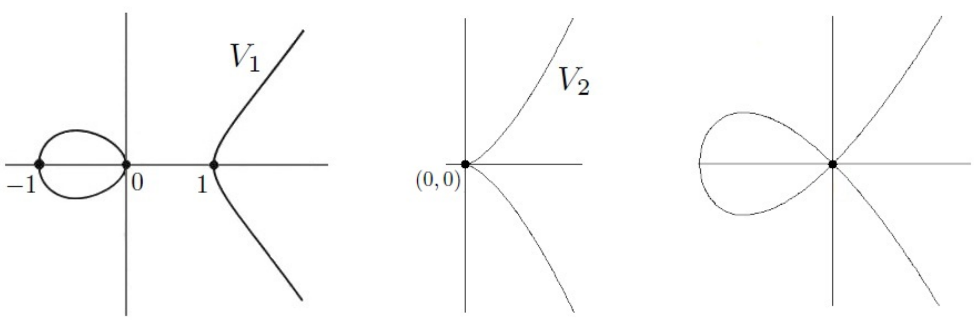
where (1) and (2) are not compatible since (1) $\Leftrightarrow$ $z_1 \in \{-1, 0, 1\}$ and none of these is a solution of (2). It follows that $S(V_1) = \emptyset$ since the system has no common solution : V_1 is a curve without singular points. It is actually given by a plane *elliptic curve*, which is a non-singular cubic. Figure 8.3 shows how V_1 looks like in the real case. In the complex case the curve will become connected.

b)

$$\begin{cases} f_2(z_1, z_2) = 0 \\ \frac{\partial f_2}{\partial z_1}(z_1, z_2) = 0 \\ \frac{\partial f_2}{\partial z_2}(z_1, z_2) = 0 \end{cases} \Leftrightarrow \begin{cases} z_2^2 - z_1^3 = 0 \\ -3z_1^2 = 0 \\ 2z_2 = 0 \end{cases} \Leftrightarrow \; z_1 = z_2 = 0 \text{ with } (0,0) \in V_2$$

Hence $(0,0)$ is a singular point of V_2 (the only one) and $S(V_2) = \{(0,0)\}$. V_2 is called a *cuspidal cubic* (see figure 8.3). Another type of such curves is the *nodal curve*.

Figure 8.3: an elliptic, a cuspidal and a nodal curve



c)

$$\frac{\partial f_3}{\partial z_1}(z_1, z_2) = 1 \quad , \quad \frac{\partial f_3}{\partial z_2}(z_1, z_2) = 0 \qquad \Rightarrow \qquad (1\,,\,0) \neq (0\,,\,0)$$

Hence V_3 has no singular points (which is intuitively clear since it is just a line, see figure 8.4) : $S(V_3) = \emptyset$.

d)

$$f_4(z_1, z_2) = z_1 z_2 \quad , \quad \frac{\partial f_4}{\partial z_1}(z_1, z_2) = z_2 \quad , \quad \frac{\partial f_4}{\partial z_2}(z_1, z_2) = z_1$$

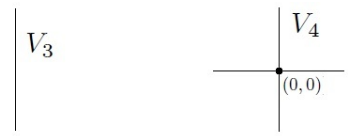
For V_4, $(0,0)$ is the only candidate for a singular point and it is also one (see figure 8.4) : $S(V_4) = \{(0,0)\}$.

e)

$$f_5(z_1, z_2) = z_1^2 \quad , \quad \frac{\partial f_5}{\partial z_1}(z_1, z_2) = 2z_1 \quad , \quad \frac{\partial f_5}{\partial z_2}(z_1, z_2) = 0$$

Hence the candidates for singularities are all points of the type $(0, z_2)$, $\forall\, z_2 \in \mathbb{C}$. We have $(0, z_2) \in V_5$, $\forall\, z_2 \in \mathbb{C}$. But $V_5 = \{\, (0, z_2) \mid z_2 \in \mathbb{C}\,\}$. Does this imply that all points of the curve are singularities ?

Figure 8.4: V_3 and V_4 only consist of the coordinate axes



No, because the definition of a singular point says that there does not exist any finite number of holomorphic functions which satisfy both of the conditions in (8.2). Indeed, f_5 is not the good choice, but f_3 works because

$$V_5 = V_3 = \mathbb{C}^2 \cap V_3 = f_3^{-1}\big(\{0\}\big)$$

and its Jacobian has rank 1. We took the wrong function to describe the analytic set V_5. In fact, the vanishing ideal of V_5 is given by $\langle\, z_1 \,\rangle$, and not by $\langle\, z_1^2 \,\rangle$ since z_1^2 is not irreducible. Finally, $S(V_5) = S(V_3) = \emptyset$.

Remark :
In practise, when dealing with a system of (polynomial) equations like

$$\begin{cases} f(z_1, z_2) = 0 & (1) \\ \frac{\partial f}{\partial z_1}(z_1, z_2) = 0 & (2) \\ \frac{\partial f}{\partial z_2}(z_1, z_2) = 0 & (3) \end{cases}$$

it is easier to solve (2) and (3) first and then check if (1) is also satisfied because (1) is of higher degree than (2) and (3). Solving (1) is in general much more complicated.

8.2.3 Proposition

The Jacobian matrix $J(x_0)$ in (8.2) obviously depends on the local coordinates $\varphi = (z_1, \ldots, z_n)$. However :
The rank of $J(x_0)$ does not depend on the chosen local coordinates.

Proof. This follows from the chain rule. Let (U, φ) and (V, ψ) be 2 coordinate charts around x_0 and set

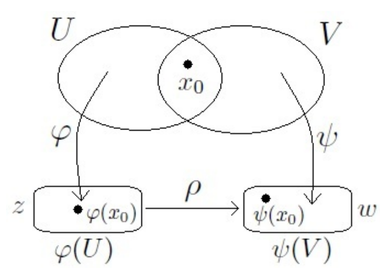
$$\varphi = (z_1, \ldots, z_n) \quad , \quad \psi = (w_1, \ldots, w_n) \quad , \quad \rho = \psi \circ \varphi^{-1} \quad , \quad g_i = f_i \circ \varphi^{-1} \quad , \quad h_i = f_i \circ \psi^{-1}$$

We have to show that the associated Jacobian matrices as given in (8.2) have the same rank. Note that

$$\forall\, i \in \{1, \ldots, k\} \;:\; g_i = f_i \circ \varphi^{-1} = f_i \circ \psi^{-1} \circ \psi \circ \varphi^{-1} = h_i \circ \rho$$

$$\Rightarrow\; J(g_i)\big(\varphi(x_0)\big) = J(h_i \circ \rho)\big(\varphi(x_0)\big) = J(h_i)\big(\rho(\varphi(x_0))\big) \circ J(\rho)\big(\varphi(x_0)\big) = J(h_i)\big(\psi(x_0)\big) \circ J(\rho)\big(\varphi(x_0)\big) \tag{8.3}$$

Figure 8.5: a coordinate change from (U, φ) to (V, ψ)



Equation (8.3) thus shows that for any <u>fixed</u> indices $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n\}$:

$$\Big(J_\varphi(x_0)\Big)_{ij} = \frac{\partial(f_i \circ \varphi^{-1})}{\partial z_j}\big(\varphi(x_0)\big) = \frac{\partial(h_i \circ \rho)}{\partial z_j}\big(\varphi(x_0)\big) = \sum_{k=1}^{n} \frac{\partial h_i}{\partial w_k}\big(\psi(x_0)\big) \cdot \frac{\partial w_k}{\partial z_j}\big(\varphi(x_0)\big)$$

$$= \sum_{k=1}^{n} \frac{\partial(f_i \circ \psi^{-1})}{\partial w_k}\big(\psi(x_0)\big) \cdot \frac{\partial \rho_k}{\partial z_j}\big(\varphi(x_0)\big) = \Big(J_\psi(x_0) \cdot J(\rho)\big(\varphi(x_0)\big)\Big)_{ij}$$

where $J(\rho)\big(\varphi(x_0)\big)$ denotes the Jacobian matrix of the coordinate change ρ evaluated at the point $\varphi(x_0) \in \mathbb{C}^n$.

Using matrix notation, this means that

$$J_\varphi(x_0) = J_\psi(x_0) \cdot J(\rho)\big(\varphi(x_0)\big) \;\Leftrightarrow\; J_\psi(x_0) = J_\varphi(x_0) \cdot \Big[J(\rho)\big(\varphi(x_0)\big)\Big]^{-1}$$

The $n \times n$–matrix $J(\rho)\big(\varphi(x_0)\big)$ is invertible at any point since ρ is a change of variables; in particular it has maximal rank. Hence $J_\varphi(x_0)$ and $J_\psi(x_0)$ have the same rank since they are related by an invertible matrix. $\square$

8.2.4 Theorem

Let M be a complex manifold and $U \subseteq M$ be non-empty and open. Let $A \subseteq M$ be an analytic set and $S(A)$ be the set of singular points in A. Assume that $S(A) \cap U = \emptyset$, i.e. U does not contain singular points of A. Then $A \cap U$ is a submanifold of U.
In particular, if $S(A) = \emptyset$ (A has no singular points), then the analytic set A is a submanifold of M.

Proof. Being a submanifold is a local statement, so let $x_0 \in U \cap A$. Since A is analytic, we know that A is locally at x_0 given by holomorphic functions $f_1, \ldots, f_k$. $x_0 \notin S(A)$ because $S(A) \cap U = \emptyset$, so the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial z_1}(x_0) & \frac{\partial f_1}{\partial z_2}(x_0) & \ldots & \frac{\partial f_1}{\partial z_n}(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial z_1}(x_0) & \frac{\partial f_k}{\partial z_2}(x_0) & \ldots & \frac{\partial f_k}{\partial z_n}(x_0) \end{pmatrix}$$

is of rank k. By permuting and renumbering the coordinates, we may assume that the first k columns are linearly independent, hence

$$\det \left[\left(\frac{\partial f_i}{\partial z_j}(x_0) \right)_{\substack{i=1,\ldots,k \\ j=1,\ldots,k}} \right] \neq 0$$

Let (U', φ) be a chart around x_0 and denote $z^{(0)} := \varphi(x_0)$. Then we define the map $F : \varphi(U') \subseteq \mathbb{C}^n \to \mathbb{C}^n$ by

$$F(z_1, \ldots, z_n) := \big((f_1 \circ \varphi^{-1})(z_1, \ldots, z_n), \ldots, (f_k \circ \varphi^{-1})(z_1, \ldots, z_n)\,,\, z_{k+1} - z_{k+1}^{(0)}, \ldots, z_n - z_n^{(0)} \big)$$

$$\Rightarrow\; J(F)\big(z^{(0)}\big) = \left(\begin{array}{c|c} \left(\frac{\partial f_i}{\partial z_j}(x_0) \right)_{\substack{i=1,\ldots,k \\ j=1,\ldots,k}} & \times \\ \hline \begin{matrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{matrix} & \begin{matrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{matrix} \end{array} \right)$$

In addition : $F\big(z^{(0)}\big) = 0$ since $\varphi^{-1}\big(z^{(0)}\big) = x_0 \in U' \cap A = \{\, x \in U' \mid f_1(x) = \ldots = f_k(x) = 0 \,\}$ and

$$\det\Big(J(F)\big(z^{(0)}\big)\Big) = \det \left[\left(\frac{\partial f_i}{\partial z_j}(x_0) \right)_{\substack{i=1,\ldots,k \\ j=1,\ldots,k}} \right] \cdot 1 \cdot \ldots \cdot 1 \neq 0$$

By the Inverse Function Theorem, we can thus invert the function F locally, i.e. there is an open neighborhood V' of $z^{(0)}$ in $\varphi(U') \subseteq \mathbb{C}^n$ and an open neighborhood W of 0 in $\mathbb{C}^n$ such that $F(V') \subseteq W$ is open and

$$F_{|V'} \;:\; V' \subseteq \varphi(U') \longrightarrow W \subseteq \mathbb{C}^n$$

is biholomorphic on V'. Since $\varphi : U' \to \varphi(U')$ is a homeomorphism, we may assume that $V' = \varphi(V)$ for some open neighborhood V of x_0 in $U' \subseteq U$. It follows that $x_0 \in V \subseteq U$ and

$$(F \circ \varphi)_{|V} \;:\; V \subseteq U \longrightarrow W \subseteq \mathbb{C}^n$$

is also biholomorphic (by definition). Moreover since $V \cap A \subseteq U' \cap A$, i.e. $f_i(V \cap A) = 0$, $\forall\, i$, we obtain that

$$(F \circ \varphi)(V \cap A) = \big\{\, w \in W \;\big|\; w_1 = \ldots = w_k = 0 \,\big\} = \bigcap_{i=1}^{k} \big(p_{i|W}\big)^{-1}\big(\{0\}\big)$$

where $p_i : \mathbb{C}^n \to \mathbb{C}$ is the projection onto the i^{th} coordinate, which is holomorphic.

We conclude that $(F \circ \varphi)(U \cap A)$ is a complex submanifold since it is locally a zero set of holomorphic functions and the rank of the Jacobian of the projections is obviously maximal. Finally $U \cap A$ is a complex submanifold as well since $F \circ \varphi$ is biholomorphic on V, i.e. the complex structure of $U \cap A$ is preserved under this map. $\square$

8.3 Irreducibility

8.3.1 Definition

An analytic set A is called *reducible* $\Leftrightarrow$ $\exists\, A_1, A_2$ analytic sets such that $A \neq A_1$, $A \neq A_2$ and $A = A_1 \cup A_2$.
A is called *irreducible* if it is not reducible.

Example :
Consider the examples in 8.2.2. We see that $V_4 = V(z_1 z_2) = V(z_1) \cup V(z_2)$ is reducible.
V_1, V_2 and V_3 are irreducible (where we recall that V_1 is not disconnected in the complex picture).

8.3.2 Results

In the following, we state a number of facts without (detailed) proof :

1) If $A \subseteq M$ is an analytic subset, there exists a countable set of irreducible analytic sets $\{A_j\}_{j \in J}$ such that

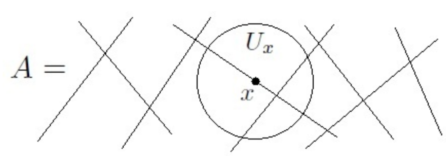
a) $A = \bigcup_{j \in J} A_i$

b) The system is locally finite, i.e. $\forall\, x \in M$, $\exists\, U_x$ open neighborhood of x such that $U_x \cap A_j \neq \emptyset$ only for finitely many $j \in J$ (see figure 8.6)

c) If $A_{j_1} \neq A_{j_2}$, then $A_{j_1} \nsubseteq A_{j_2}$. This is the maximality condition : the irreducible parts are maximal.

This "decomposition" of A is called the decomposition into irreducible components.

Figure 8.6: a locally finite decomposition of A

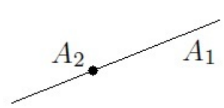


In particular : if M is a compact complex manifold, then J can be chosen to be finite because

$$M = \bigcup_{x \in M} U_x \quad \Rightarrow \quad M = \bigcup_{i \in I} U_i \ : \ I \text{ finite}$$

and every U_i satisfies $U_i \cap A_j \neq \emptyset$ only for finitely many $j \in J$, hence there can only be finitely many A_j since $\{U_i\}$ is a covering of M.

c) is for example not satisfied in the following case :



The line A_1 is irreducible and contains other irreducible subsets as e.g. the point A_2, hence $A_1 = A_1 \cup A_2$, but $A_2 \neq A_1$ and $A_2 \subset A_1$.

2) Let $A \subseteq M$ be an analytic set and $S(A)$ be the subset of singular points of A.
Then $S(A)$ is an analytic set which is nowhere dense in A ("nowhere" means that it is not dense in any component of A ; because "not dense" does not exclude that it may be dense in some component of A).

Intuitive idea : let J be a $k \times n$–Jacobian matrix. Then $\operatorname{rk} J(x) \leq k$ at any point $x \in A$. How to check that $\operatorname{rk} J$ drops at a point $x_0 \in A$?

$$J(x) = \left(\frac{\partial f_i}{\partial z_j}(x) \right)_{\substack{i=1,\dots,k \\ j=1,\dots,n}}$$

Recall that a *minor* of J is a submatrix of J obtained by removing one or more lines and columns from J. The rank of J is then given by the dimension of the biggest minor inside J which is a square-matrix and has non-zero determinant. Hence the rank at a point drops if all $k \times k$–subdeterminants in J are zero. Consider for example

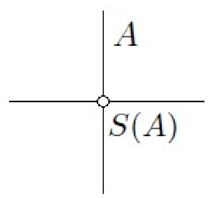
$$J = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \quad \Rightarrow \quad \operatorname{rk}(J) \leq 2$$

$$\operatorname{rk}(J) \leq 1 \;\Leftrightarrow\; \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = 0 \;\; , \quad \det \begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix} = 0 \quad \text{and} \quad \det \begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{pmatrix} = 0$$

Let now $x_0 \notin S(A)$ be a non-singular point $\;\Rightarrow\;$ $\operatorname{rk} J(x_0) = k$. Since determinants are polynomial functions, hence continuous, this implies that there is a neighborhood U of x_0 on which the rank is still equal to k, i.e. all point in U are non-singular points. It follows that $S(A)$ cannot be dense in A since we cannot approach $x_0 \in A$ by singular points.

3) If an analytic set $A \subseteq M$ is irreducible, then $A \setminus S(A)$ is connected.
Here we need the assumption that is A irreducible (see figure 8.7).

Figure 8.7: $A \setminus S(A)$ is no longer connected since A is not irreducible



4) If an analytic set $A \subseteq M$ is irreducible, then $A \setminus S(A)$ is a (connected) complex manifold.
This follows from **2)**, **3)** and 8.2.4 since $S(A)$ is analytic, hence closed, so by choosing $U = M \setminus S(A)$, we obtain that $A \cap U = A \setminus S(A)$ is a complex submanifold of M, which is in addition connected by **3)**.
In particular, $A \setminus S(A)$ has now a well-defined dimension given by the definition of a complex manifold. We set

$$\dim A := \dim \big(A \setminus S(A)\big) \qquad , \qquad \operatorname{codim} A := \dim M - \dim \big(A \setminus S(A)\big)$$

Conclusion :
A has well-defined dimension and codimension (after removing singular points) $\;\Leftrightarrow\;$ A is irreducible.

5) Let $n = \dim M$. If $A = V(f_1, \dots, f_l)$ is the vanishing set of a finite number of holomorphic functions such that A is irreducible, then $\dim A \geq n - l$. Hence by adding an additional function to the zero set, the dimension of A goes down by at most 1 (it may remain the same).

8.4 Divisors

8.4.1 Definitions

A *prime divisor* is an irreducible analytic subset of codimension 1, i.e. an irreducible hypersurface.
A *divisor* D is a formal sum

$$D = \sum_Y n_Y \cdot Y \;,\; n_Y \in \mathbb{Z} \tag{8.4}$$

where we "sum" over all prime divisors Y with the condition that the sum is locally finite, i.e. $\forall\, x \in M$, $\exists\, U_x$ open neighborhood of x such that only finitely many of the Y with $Y \cap U_x \neq \emptyset$ have a non-zero coefficient n_Y.

In other words this means that for all $x \in M$, the set

$$\left\{ Y \text{ prime divisor} \;\middle|\; U_x \cap Y \neq \emptyset \text{ and } n_Y \neq 0 \right\}$$

must be finite. All values for n_Y such that this condition is satisfied are allowed.

8.4.2 Examples

If M is compact, we can again replace "locally finite" by "finite", i.e. $D = \sum_Y n_Y \cdot Y$ with $n_Y \neq 0$ for at most finitely many Y. In general, this condition is too strong, as shows the example in section 8.4.3.

Let $M = \mathbb{C}$ (or in general : a complex manifold of dimension 1). In this case the prime divisors are the points because $\operatorname{codim}_M Y = 1 \;\Rightarrow\; \dim_M Y = 0$ and irreducibility implies that Y must be a single point (not many points). The set of all prime divisors is thus the set of all points in M.

8.4.3 Divisor associated to a meromorphic function

Let $h \in \mathcal{M}(M)$ be a meromorphic function and $p \in M$. The *order* or *multiplicity* of h at p is defined as

$$\operatorname{ord}_p(h) := \begin{cases} 0 & \text{if } h \text{ is holomorphic at } p \text{ with } h(p) \neq 0 \\ k & \text{if } p \text{ is a zero of multiplicity } k \text{ of } h \\ -k & \text{if } h \text{ has a singularity of order } k \text{ at } p \\ \infty & \text{if } h \text{ is identically zero} \end{cases}$$

Now consider $M = \mathbb{C}$ and let $f : \mathbb{C} \to \mathbb{C}$ be a meromorphic function. The *divisor associated to* f is given by

$$(f) := \sum_{p \in \mathbb{C}} \operatorname{ord}_p(f) \cdot \{p\}$$

The previous example shows that (f) is indeed a divisor (points are the only prime divisors). This sum is in addition locally finite since the zero set of a (non-zero) holomorphic function on $\mathbb{C}$ is discrete. For example

$$(z) = 1 \cdot \{0\} \quad , \quad (z^2) = 2 \cdot \{0\} \quad , \quad (z-a) = 1 \cdot \{a\}$$

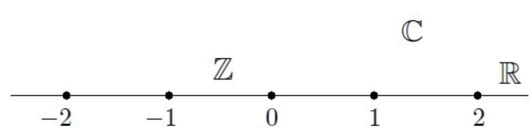
We also see that one can obtain all the prime divisors (all points) by choosing all $a \in \mathbb{C}$. Moreover we have :

$$\forall f, g \in \mathcal{M}(\mathbb{C}) \;:\; (f \cdot g) = (f) + (g) \quad \text{ and } \quad \left(\tfrac{1}{f}\right) = -(f)$$

since $\operatorname{ord}_p(f \cdot g) = \operatorname{ord}_p(f) + \operatorname{ord}_p(g)$ and $\operatorname{ord}_p\left(\frac{1}{f}\right) = -\operatorname{ord}_p(f)$ if $f \not\equiv 0$.

Example :
Let $\mathbb{R} : \mathbb{C} \to \mathbb{C}$, $f(z) = \sin(\pi z)$. Then $V(f) = \mathbb{Z}$ and $(f) = \sum_{k \in \mathbb{Z}} 1 \cdot \{k\}$, which is locally finite (see figure 8.8).
Hence we see that requiring a finite sum in (8.4) is too restrictive ($\mathbb{C}$ is not compact).

Figure 8.8: the divisor associated to $z \mapsto \sin(\pi z)$ is locally finite, but not finite



In general one can define (f) for any meromorphic function $f \in \mathcal{M}(M)$ where M is a 1-dimensional complex manifold (i.e. a Riemann surface) since points are the only prime divisors in this case. still locally finite?
A divisor D is called a *principal divisor* if $\exists\, f \in \mathcal{M}(M)$ such that $D = (f)$.

For $\dim M \geq 2$, (f) is in general not a divisor. However, one can then associate to a meromorphic function $f : M \to \mathbb{C}$ and a prime divisor Y the number $(f, Y) \mapsto n_Y \in \mathbb{N}_0$ such that f is holomorphic in Y and vanishes along Y with multiplicity n_Y.

Chapter 9

Holomorphic vector bundles

9.1 Biholomorphic functions

Theorem
Any bijective holomorphic function $f : U \subseteq \mathbb{C}^n \to f(U) \subseteq \mathbb{C}^n$ is biholomorphic.

We are going to discuss this result for $n = 1$. It also holds for $n > 1$, but this is much more difficult.

Open mapping theorem : (no proof)
If U is an open connected subset of $\mathbb{C}$ and $f : U \to \mathbb{C}$ is a non-constant holomorphic function, then f is open.

This already implies that the inverse of a bijective holomorphic map is continuous.
Hence by the Inverse Function Theorem it remains to show that $\frac{\partial f}{\partial z}(z_0) \neq 0,\ \forall\, z_0 \in U$ (since f^{-1} will then be holomorphic in a neighborhood of $f(z_0)$, i.e. in a neighborhood of any point in $f(U)$).

approach 1 :
Since f is holomorphic, it is also analytic and $\forall\, z_0 \in U$, it can be written in a neighborhood of z_0 as

$$f(z) = \sum_{k=0}^{\infty} a_k \cdot (z - z_0)^k = a_0 + a_1\,(z - z_0) + a_2\,(z - z_0)^2 + \dots \qquad \text{where } a_k = \frac{1}{k!} \cdot \frac{\partial^k f}{\partial z^k}(z_0)$$

Assume that $\frac{\partial f}{\partial z}(z_0) = 0$, then $a_1 = 0$ and f locally writes as $f(z) = a_0 + a_2(z - z_0)^2 + \dots$, which implies that f cannot be injective near z_0. Hence bijective holomorphic functions never have vanishing first derivatives.

approach 2 :
We know that f^{-1} is holomorphic around any point $f(z_0)$ such that $\frac{\partial f}{\partial z}(z_0) \neq 0$. Since $\frac{\partial f}{\partial z}$ is also holomorphic, we know that its zero set is discrete ($\frac{\partial f}{\partial z}$ is not identically zero because f is bijective). Let p be a zero of $\frac{\partial f}{\partial z}$. Then f^{-1} is continuous on $f(U_p)$ and holomorphic on $f(U_p) \setminus \{f(p)\}$, where U_p is a small open neighborhood of p not containing other zeros of $\frac{\partial f}{\partial z}$. It follows that f^{-1} is holomorphic on $f(U_p)$ (see recall in section 2.6.1). This holds for any point of the discrete zero set, hence f^{-1} is holomorphic on $f(U)$.

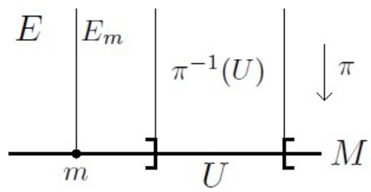
For $n > 1$, the task is more complicated since matrices are involved, i.e. non-vanishing conditions must be replaced by maximal rank and non-zero determinant conditions. Moreover the analytic expansion of a holomorphic function in several variables is more complicated (see section 2.3).

9.2 Vector bundles : basic notions

9.2.1 Definitions

Let M and E be complex manifold and $\pi : E \to M$ be a surjective holomorphic map. π is called a *foot-map*.
For any $m \in M$, we denote the *fiber over m of π* by $E_m := \pi^{-1}\big(\{m\}\big)$. By abuse of notation, we will denote $E_m = \pi^{-1}(m)$ in the following (π is not necessarily bijective, see figure 9.1).

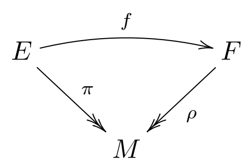
Figure 9.1: fibers correspond to preimages under the foot-map



If $U \subseteq M$ is open, then $\pi^{-1}(U)$ is open in E since π is holomorphic (hence continuous) and the assignment denoted by $\pi_{|U} : \pi^{-1}(U) \to U$ gives a "local situation". E is called a *fibration* over the *base manifold* M.

$\pi : E \to M$ is called a *family of vector spaces* (over $\mathbb{C}$) if $E_m = \pi^{-1}(m)$ is a vector space over $\mathbb{C}$ for all $m \in M$. We only consider finite-dimensional vector spaces because otherwise E will be a infinite-dimensional manifold.

Let $\pi : E \to M$ and $\rho : F \to M$ be two families of vector spaces (over the same base manifold M).
A *morphism* of families of vector spaces is a holomorphic map $f : E \to F$ such that the diagram



(9.1)

commutes, i.e. $\rho \circ f = \pi$, and such that for all $m \in M$, the function f_m given by

$$f_m := f_{|\pi^{-1}(m)} \; : \; E_m \longrightarrow F_m$$

is a linear map over $\mathbb{C}$ between the finite-dimensional vector spaces E_m and F_m.
Note that f_m is well-defined because of (9.1) : $\forall\, x \in \pi^{-1}(m)$, $\rho\big(f(x)\big) = \pi(x) = m \;\Rightarrow\; f(x) \in \rho^{-1}(m) = F_m$.
Hence f preserves the fibers of E and F : it maps a fiber over m of π to a fiber over m of ρ.

9.2.2 Example : the standard trivial family

Let M be a complex manifold of dimension n, $k \in \mathbb{N}_0$, $E := M \times \mathbb{C}^k$ and $\pi = p_1$ (first projection).
p_1 is holomorphic since M and $M \times \mathbb{C}^k$ (product) are manifolds. Moreover $\forall\, m \in M$:

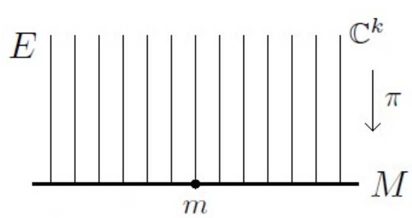
$$\pi^{-1}(m) = p_1^{-1}\big(\{m\}\big) = \{m\} \times \mathbb{C}^k \cong \mathbb{C}^k$$

where $\{m\} \times \mathbb{C}^k$ is a vector space with respect to the operations

$$(m, v_1) + (m, v_2) = (m, v_1 + v_2) \quad , \quad \lambda \cdot (m, v) = (m, \lambda\, v) \quad , \quad 0_m := (m, 0)$$

for $v_1, v_2, v, 0 \in \mathbb{C}^k$, $\lambda \in \mathbb{C}$. Hence $p_1 : M \times \mathbb{C}^k \to M$ is a family of vector spaces (see figure 9.2) and it is called the *standard trivial family of rank k*.

Figure 9.2: the standard trivial family



The idea of a vector bundle is now that a bundle should be locally look like this standard trivial family.

9.2.3 Definition

A family of vector spaces $\pi : E \to M$ is called a *(holomorphic) vector bundle* $\Leftrightarrow$ there is an open covering $\{U_i\}_{i\in J}$ of M such that for all $i \in J$, the fibration $\pi_{|\pi^{-1}(U_i)} : \pi^{-1}(U_i) \to U_i$ is isomorphic (as family of vector spaces) to the standard trivial family $p_1 : U_i \times \mathbb{C}^k \to U_i$. In other words, for any U_i there is a morphism ψ_{U_i} such that

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\psi_{U_i}} & U_i \times \mathbb{C}^k \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle p_1} & \\ & U_i & \end{array} \tag{9.2}$$

where ψ_{U_i} is bijective and holomorphic ($\Leftrightarrow$ biholomorphic) and the map $\psi_{U_i|\pi^{-1}(m)} : E_m \to \{m\} \times \mathbb{C}^k$ is a linear isomorphism, $\forall\, m \in U_i$.
Note that $\pi_{|\pi^{-1}(U_i)}$ is still surjective since $\pi\big(\pi^{-1}(U_i)\big) = U_i$ (π is surjective). One calls $\{U_i\}_{i\in J}$ a *trivializing open covering* for the vector bundle E.
In the case where M is a compact complex manifold one can always achieve this with finitely many U_i.

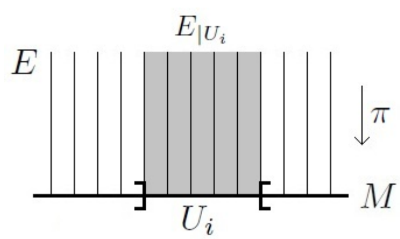
9.2.4 Rank of a vector bundle

For a trivializing open covering $\{U_i\}_{i\in J}$, we denote the vector bundle *restricted* to U_i by (see figure 9.3)

$$E_{|U_i} := \Big(\pi_{|\pi^{-1}(U_i)} : \pi^{-1}(U_i) \to U_i\Big) \cong \Big(p_1 : U_i \times \mathbb{C}^k \to \mathbb{C}^k\Big)$$

The same construction can be done for some arbitrary open set $U \subseteq M$, i.e. a vector bundle $\pi : E \to M$ can always be restricted to $\pi : E_{|U} = \pi^{-1}(U) \to U$.

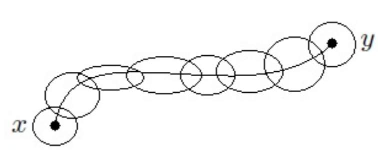
Figure 9.3: restriction of a vector bundle to a trivializing open set



Due to the isomorphism ψ_{U_i} in (9.2), every fiber in a vector bundle E has in particular dimension k and we define the *rank* of E as $\operatorname{rk} E := k$. By connectedness of M, this number k is the same for all $m \in M$:
Assume for example that for $m \in U_i$ and $m' \in U_j$ open with $U_i \cap U_j \neq \emptyset$, we have

$$\begin{gathered} E_{|U_i} \cong U_i \times \mathbb{C}^{k_i} \quad , \quad E_{|U_j} \cong U_j \times \mathbb{C}^{k_j} \qquad \Rightarrow \qquad \dim E_m = k_i \quad , \quad \dim E_{m'} = k_j \\ \text{but} : \ (U_i \cap U_j) \times \mathbb{C}^{k_i} \cong \big(E_{|U_i}\big)_{|U_j} = \big(E_{|U_j}\big)_{|U_i} \cong (U_i \cap U_j) \times \mathbb{C}^{k_j} \end{gathered} \tag{9.3}$$

The linear isomorphism implies that the dimensions are the same : $k_i = k_j$.
Let now $x, y \in M$ be arbitrary. By path-connectedness, there is continuous path from x to y which can be covered by finitely many chart domains. Repeating argument (9.3) for any of these finitely many open sets, we obtain that the fibers over x and y are isomorphic. In particular : $\dim E_x = \dim E_y = k$.



We only consider vector bundles of finite rank (rank equal to 0 is possible). In particular, a vector bundle of rank 1 is also called a *line bundle* (see section 10.5).

9.2.5 Example

The tangent bundle $\mathbb{T}M := \bigsqcup_{m\in M} \mathbb{T}_m M$ with foot-map $\pi : \mathbb{T}_m M \to M : v_m \mapsto m$ is a vector bundle of rank n. It is however not globally trivial, i.e. $\mathbb{T}M \ncong M \times \mathbb{C}^n$. In fact, globally trivial bundles are very rare (an example is $M = \mathbb{C}^n$ since there is just 1 chart in this case). Note that for any vector bundle E, we have :

$$E = \pi^{-1}(M) = \pi^{-1}\Big(\bigcup_{m\in M}\{m\}\Big) = \bigsqcup_{m\in M} \pi^{-1}\big(\{m\}\big) = \bigsqcup_{m\in M} E_m \cong \bigsqcup_{m\in M} \mathbb{C}^k \cong M \times \mathbb{C}^k$$

since all k-dimensional vector spaces over $\mathbb{C}$ are isomorphic to $\mathbb{C}^k$. BUT : this "isomorphism" is only set-theoretical, i.e. it is just a bijection between sets ; it is neither a (bi)holomorphic, nor a continuous identification.

Remark :
There are complex vector bundles over real differentiable manifolds which are not holomorphic.

9.3 Sections

For short, we say that E is a vector bundle over M if we mean that $\pi : E \to M$ is a vector bundle.

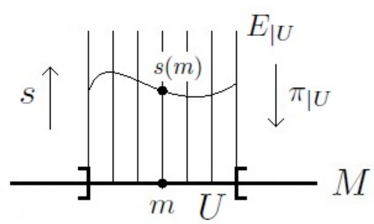
9.3.1 Definition

Let E be a vector bundle over M and $U \subseteq M$ be open. Then $E_{|U}$ is a vector bundle over U.
A holomorphic map $s : U \to E_{|U}$ is called a *(global) section of $E_{|U}$* or a *local section of E* if $\pi_{|U} \circ s = \mathrm{id}_U$.
Consider figure 9.4 : in particular, $s(m) \in E_m = \pi^{-1}(m)$, $\forall\, m \in U$ since $\pi\big(s(m)\big) = m$. Moreover s is injective since it has a left inverse, thus $s(U) \cong U$ and s is biholomorphic onto its image. We denote

$$\mathcal{V}(U) := \big\{ \text{(global) sections of the vector bundle } E_{|U} \big\}$$

We will see that $\mathcal{V}$ defines indeed a sheaf of local sections of E.

Figure 9.4: a local section of a vector bundle



9.3.2 Sheaf of sections

First of all, $\mathcal{V}(U) \neq \emptyset$, $\forall\, U \subseteq M$ open : it always contains the zero section $\theta : U \to E_{|U} : m \mapsto 0 \in E_m$ where 0 is the unique zero element in the vector space E_m (see figure 9.5). Indeed $\forall\, m \in U$:

$$\pi\big(\theta(m)\big) = \pi(0) = m \quad \text{because } 0 \in E_m = \pi^{-1}(m) \text{ and } \pi\big(\pi^{-1}(m)\big) = \{m\}$$

θ is in addition holomorphic on U since its local form on a small neighborhood of $m \in U$ is given by

$$\theta \;:\; U \longrightarrow U \times \mathbb{C}^k \;:\; m \longmapsto (m, 0)$$

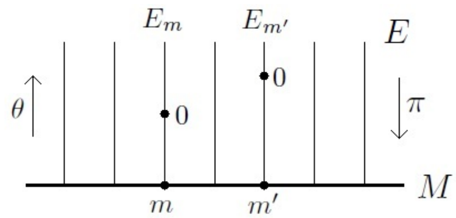
where the first coordinate indicates the fiber and the second one the element in the fiber to which m is mapped. This is obviously holomorphic, so finally $\theta \in \mathcal{V}(U)$.
Next we want to define a structure on $\mathcal{V}(U)$ for any open set $U \subseteq M$.

1) $\mathcal{V}(U)$ is a complex vector space with respect to the pointwise additional structure

$$s_1, s_2 \in \mathcal{V}(U) \;\Rightarrow\; s_1 + s_2 \in \mathcal{V}(U) \text{ by } (s_1 + s_2)(m) := s_1(m) + s_2(m) \in E_m$$
$$s \in \mathcal{V}(U),\, \lambda \in \mathbb{C} \;\Rightarrow\; \lambda \cdot s \in \mathcal{V}(U) \text{ by } (\lambda \cdot s)(m) := \lambda \cdot s(m) \in E_m$$

However we cannot say anything about its dimension (in fact $\dim \mathcal{V}(U) = \infty$ in most cases).

Figure 9.5: the zero section



2) $\mathcal{V}(U)$ is a module over $\mathcal{O}_M(U)$. Let $f \in \mathcal{O}_M(U)$ be holomorphic on U, $s \in \mathcal{V}(U)$ and define $f * s \in \mathcal{V}(U)$ by

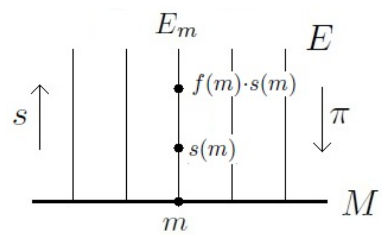
$$\forall\, m \in U \;:\; (f * s)(m) := \underbrace{f(m)}_{\in \mathbb{C}} \cdot \underbrace{s(m)}_{\in E_m} \in E_m \;\; \text{(vector space)}$$

This is still a section : $\forall\, m \in U$, $\pi\big((f * s)(m)\big) = m$ since $f(m) \cdot s(m) \in E_m = \pi^{-1}(m)$. Moreover the module axioms as e.g. $1 * s = s$, $(f+g) * s = f * s + g * s$, etc. are trivially satisfied.
It remains to check that $f * s$ is again holomorphic on U. Its local form is given by (see figure 9.6)

$$s \;:\; U \longrightarrow U \times \mathbb{C}^k \;:\; m \longmapsto \big(m, s(m)\big) \quad , \quad f * s \;:\; U \longrightarrow U \times \mathbb{C}^k \;:\; m \longmapsto \big(m\,,\, f(m) \cdot s(m)\big)$$

since $(f * s)(m)$ is still in the fiber over m of π. f and s being holomorphic, we get that $f * s$ is holomorphic too.

Figure 9.6: multiplication with complex numbers preserves the fiber E_m



3) $\mathcal{V} : U \mapsto \mathcal{V}(U)$ is a sheaf of complex vector spaces with the canonical restriction

$$\rho^U_V \;:\; \mathcal{V}(U) \longrightarrow \mathcal{V}(V) \;:\; s \longmapsto s_{|V}$$

if $V \subseteq U$. The gluing property 4) of a sheaf is satisfied since holomorphy is a local condition.

4) $\mathcal{V} : U \mapsto \mathcal{V}(U)$ is a sheaf of $\mathcal{O}_M$–modules by the operation

$$* \;:\; \mathcal{O}_M(U) \times \mathcal{V}(U) \longrightarrow \mathcal{V}(U) \;:\; (f, s) \longmapsto f * s$$

This structure is compatible with the restrictions : $\forall\, m' \in V \subseteq U$, $(f * s)(m') = f(m') \cdot s(m')$, thus as in (6.2) :

$$(f * s)_{|V} = f_{|V} * s_{|V} \;\;\Rightarrow\;\; \rho^U_V(f *_U s) = f_{|V} *_V \rho^U_V(s)$$

5) $\mathcal{V}$ is a locally free sheaf of $\mathcal{O}_M$–modules of rank $k = \operatorname{rk} E$, i.e. $\exists$ open covering $\{U_i\}_{i \in J}$ of M such that

$$\forall\, i \in J \;:\; \mathcal{V}_{|U_i} \cong \big(\mathcal{O}_{U_i}\big)^k \tag{9.4}$$

This follows from the fact that if $\{U_i\}$ is a trivialization cover of the bundle E, then $E_{|U_i} \cong U_i \times \mathbb{C}^k$ and this implies that $\mathcal{V}(U_i) \cong \mathcal{O}_{U_i}(U_i) \times \ldots \times \mathcal{O}_{U_i}(U_i)$ as $\mathcal{O}_{U_i}(U_i)$–modules because the trivialization allows us to see a holomorphic section locally as a usual holomorphic function whose image has k components. Since we are dealing with functions and compatible restrictions of functions, (9.4) now follows by the remark in section 6.4.2.

9.3.3 Example

For any vector bundle E with a trivialization $\{U_i\}_{i\in J}$ we have certain standard sections $\hat{e}_l$ given by

$$\hat{e}_l \ : \ U_i \longrightarrow \pi^{-1}(U_i) \cong p_1^{-1}(U_i) = U_i \times \mathbb{C}^k \ : \ m \longmapsto (m, e_l)$$

where $e_l = (0,\ldots,1,\ldots,0)$ is the l^{th} standard basis vector of $\mathbb{C}^k$. $\hat{e}_l$ is a holomorphic section with respect to p_1 since $p_1\big(\hat{e}_l(m)\big) = p_1(m, e_l) = m$ by definition. We want to show that $\{\hat{e}_l\}_{l=1,\ldots,k}$ is an $\mathcal{O}_M(U_i)$–basis of the module $\mathcal{V}(U_i)$, i.e. $\mathcal{V}(U_i)$ is a free $\mathcal{O}_M(U_i)$–module of rank k.

− linear independence : assume that $\sum_{l=1}^{k} f_l \cdot \hat{e}_l = 0$ for some $f_l \in \mathcal{O}_M(U_i)$, which means that

$$\forall\, m \in M \ : \ \sum_{l=1}^{k} f_l(m)\cdot \hat{e}_l(m) = 0 \ \Leftrightarrow \ \sum_{l=1}^{k} \overbrace{f_l(m)}^{\in\mathbb{C}} \cdot (m, e_l) = 0$$

$\{e_l\}_{l=1,\ldots,k}$ is a basis of $\mathbb{C}^k \ \Rightarrow \ \{(m,e_l)\}_{l=1,\ldots,k}$ is a basis of $\{m\}\times\mathbb{C}^k$, so $f_l(m) = 0$ for all $l \in \{1,\ldots,k\}$. This holds for all $m \in M$, which means that $f_l = 0$ (zero function), $\forall\, l \in \{1,\ldots,k\}$.

− generating set : can all sections in $\mathcal{V}(U_i)$ be written as an $\mathcal{O}_M(U_i)$–linear combination of the $\hat{e}_l$? Let

$$s \ : \ U_i \longrightarrow U_i \times \mathbb{C}^k \ : \ m \longmapsto \big(m, \tilde{s}(m)\big) \quad \text{where } \tilde{s} : U_i \to \mathbb{C}^k$$

Since $\{e_l\}_{l=1,\ldots,k}$ is a basis of $\mathbb{C}^k$, we get that $\forall\, m \in U_i$, $\exists\, g_l(m) \in \mathbb{C}$ such that $\tilde{s}(m) = \sum_{l=1}^{k} g_l(m)\cdot e_l$.
Hence as a candidate we consider the functions $g_l : m \mapsto g_l(m) \ \Rightarrow \ s = \sum_{l=1}^{k} g_l \cdot \hat{e}_l$ is clear.
It remains to check that these functions are holomorphic on U_i, i.e. $g_l \in \mathcal{O}_M(U_i)$. Consider the sequence

$$U_i \xrightarrow{\ s\ } U_i \times \mathbb{C}^k \xrightarrow{\ p_2\ } \mathbb{C}^k \xrightarrow{\ p_l\ } \mathbb{C}$$

where $p_l : (\alpha_1,\ldots,\alpha_k) \mapsto \alpha_l$. The map $p_l \circ p_2 \circ s : U_i \to \mathbb{C}$ is holomorphic since s, p_2 and p_l are.
But $p_2 \circ s = \tilde{s}$ and $p_l \circ p_2 \circ s = g_l$ by definition of $g_l(m)$:

$$\big(p_l \circ p_2 \circ s\big)(m) = p_l\big(p_2\big(s(m)\big)\big) = p_l\Big(p_2\big(m, \tilde{s}(m)\big)\Big) = p_l\big(\tilde{s}(m)\big) = p_l\Big(\sum_{l=1}^{k} g_l(m)\cdot e_l\Big) = g_l(m)$$

Hence every section on a trivializing open set can (uniquely) be written in such a form.

9.3.4 Examples

Sections of the tangent bundle $\mathbb{T}M$ of a complex manifold M are nothing but holomorphic vector fields on M :

$$X \ : \ M \longrightarrow \mathbb{T}M \ : \ m \longmapsto X(m) = X_m \in T_mM$$

Similarly the sections of the cotangent bundle $\mathbb{T}^*M$ are the differential 1-forms.

9.3.5 Definitions

A vector bundle E over M is called a *trivial vector bundle* if it is globally isomorphic to the standard trivial family, i.e. $E \cong M \times \mathbb{C}^k$ as isomorphism of family of vector spaces. In this case, we even get that $\mathcal{V} \cong \mathcal{O}_M^k$, i.e. $\mathcal{V}$ will be a free sheaf of $\mathcal{O}_M$–modules of rank k. The proof in the same as in **5)**.

Two vector bundles E and F over the same base manifold M are said to be *isomorphic* if there exist two morphisms of families of vector spaces $f : E \to F$ and $g : F \to E$ such that $f \circ g = \mathrm{id}_F$ and $g \circ f = \mathrm{id}_E$.
An *isoclass* is just the set of equivalence classes of the equivalence relation $E \sim F \ \Leftrightarrow \ E \cong F$.

Our next goal is to prove the following theorem :

9.3.6 Theorem

The category of isoclasses of holomorphic vector bundles of rank k over a manifold M is equivalent to the category of locally free sheaves of $\mathcal{O}_M$–modules of rank k.
(Saying that the categories are equivalent means that there is a 1-to-1 correspondence between the objects.)
As a corollary, the isoclass of a trivial vector bundle is uniquely given by $\mathcal{V} \cong \mathcal{O}_M^k$.

By the above, we already showed that to any vector bundle E of rank k one can associated a locally free sheaf of rank k given by $\mathcal{V}$, the sheaf of local sections of E.
It remains the question : How to assign a vector bundle of rank k to a locally free sheaf $\mathcal{F}$ of rank k ?
For this, we have to introduce the notion of cocycles.

9.4 Cocycles

9.4.1 Construction

Let $\pi : E \to M$ be a vector bundle of rank k with a trivializing covering $\mathcal{U} = \{U_i\}_{i \in J}$ and let $U, V \in \mathcal{U}$ be trivializing open subsets of M such that $U \cap V \neq \emptyset$. By definition, we have

$$\pi^{-1}(U) = E_{|U} \overset{\varphi_U}{\cong} U \times \mathbb{C}^k \quad , \qquad \pi^{-1}(U) = E_{|V} \overset{\varphi_V}{\cong} V \times \mathbb{C}^k$$

where $\varphi_U : E_{|U} \xrightarrow{\sim} U \times \mathbb{C}^k$ and $\varphi_V : E_{|V} \xrightarrow{\sim} V \times \mathbb{C}^k$ are morphisms of families of vector spaces as in (9.2).

$$\begin{aligned} &(U \cap V) \times \mathbb{C}^k \overset{\varphi_{V|U\cap V}}{\cong} E_{|U\cap V} \overset{\varphi_{U|U\cap V}}{\cong} (U \cap V) \times \mathbb{C}^k \\ \Rightarrow \quad &\varphi_{U|U\cap V} \circ \big(\varphi_{V|U\cap V}\big)^{-1} \ : \ (U \cap V) \times \mathbb{C}^k \xrightarrow{\sim} (U \cap V) \times \mathbb{C}^k \end{aligned} \tag{9.5}$$

Let $x \in U \cap V$ be a base point. φ_U and φ_V being morphisms, they have to respect the base point, i.e.

$$\varphi_U(E_x) = \varphi_{U|U\cap V}\big(E_x\big) \subseteq \{x\} \times \mathbb{C}^k \qquad \text{and} \qquad \varphi_V(E_x) = \varphi_{V|U\cap V}\big(E_x\big) \subseteq \{x\} \times \mathbb{C}^k$$

Let $v \in \mathbb{C}^k$. We can express the map in (9.5) as

$$\big(\varphi_{U|U\cap V} \circ (\varphi_{V|U\cap V})^{-1}\big)(x, v) = \big(x \, , \, g_{UV}(x)(v)\big) = (x \, , \, g_{UV}(x) \cdot v)$$

where $(\varphi_{V|U\cap V})^{-1}(x, v) \in E_x$ (fiber over x) and g_{UV} is a function depending on x and v induced by the restrictions

$$\varphi_{U|\pi^{-1}(x)} \ : \ E_x \to \{x\} \times \mathbb{C}^k \qquad \text{and} \qquad \varphi_{V|\pi^{-1}(x)} \ : \ E_x \to \{x\} \times \mathbb{C}^k$$

which are linear isomorphisms by definition (9.2) of a trivialization. Indeed $g_{UV}(x) : \mathbb{C}^k \to \mathbb{C}^k$ is given by

$$g_{UV}(x)(v) = p_2\Big(\varphi_{U|\pi^{-1}(x)}\Big(\big(\varphi_{V|\pi^{-1}(x)}\big)^{-1}(x, v)\Big)\Big) \tag{9.6}$$

Hence $g_{UV}(x) : \mathbb{C}^k \to \mathbb{C}^k$ is a linear isomorphism for any fixed $x \in U \cap V$ ($p_2 : \{x\} \times \mathbb{C}^k \to \mathbb{C}^k$ is linear and bijective) and since all vector spaces are finite-dimensional, it can thus be represented by an invertible $k \times k$–matrix :

$$\forall\, x \in U \cap V \ : \ g_{UV}(x) \in \mathrm{GL}(k, \mathbb{C})$$

Moreover g_{UV} is holomorphic with respect to x since the morphisms φ_U, φ_V are holomorphic, hence so is the composition $\varphi_{U|U\cap V} \circ (\varphi_{V|U\cap V})^{-1}$. g_{UV} is the second projection of this expression, thus it is holomorphic by definition of the product manifold. Finally g_{UV} is a holomorphic matrix-valued function on $(U \cap V) \times \mathbb{C}^k$.
Equivalently, one can consider $g_{UV} \in \mathrm{GL}\left(k \, , \, \mathcal{O}_M(U \cap V)\right)$: matrix whose entries are holomorphic functions.

$$g_{UV} = \begin{pmatrix} g_{UV_{11}} & g_{UV_{12}} & \cdots & g_{UV_{1k}} \\ g_{UV_{21}} & g_{UV_{22}} & \cdots & g_{UV_{2k}} \\ \vdots & \vdots & \ddots & \vdots \\ g_{UV_{k1}} & g_{UV_{k2}} & \cdots & g_{UV_{kk}} \end{pmatrix} \quad \Rightarrow \quad g_{UV}(x) = \begin{pmatrix} g_{UV_{11}}(x) & g_{UV_{12}}(x) & \cdots & g_{UV_{1k}}(x) \\ g_{UV_{21}}(x) & g_{UV_{22}}(x) & \cdots & g_{UV_{2k}}(x) \\ \vdots & \vdots & \ddots & \vdots \\ g_{UV_{k1}}(x) & g_{UV_{k2}}(x) & \cdots & g_{UV_{kk}}(x) \end{pmatrix}$$

This matrix is invertible for all x, i.e. $\det\big(g_{UV}(x)\big) \neq 0$, $\forall\, x \in U \cap V$, hence one can also consider

$$g_{UV}^{-1} = \frac{1}{\det(g_{UV})} \cdot \big(\text{transpose of the adjoint of } g_{UV}\big) \in \mathrm{GL}\,\big(k\,,\, \mathcal{O}_M(U \cap V)\big)$$

Since the determinant is non-zero everywhere on $U \cap V$ and taking transpose and adjoint are polynomial operations, g_{UV}^{-1} is also a holomorphic function on $(U \cap V) \times \mathbb{C}^k$.
Such a construction of g_{UV} and g_{UV}^{-1} can be done for any trivializing sets $U, V \in \mathcal{U}$.
Note in addition that g_{UV} is uniquely determined by the vector bundle E and the covering $\mathcal{U}$ since is made up of the trivialization morphisms φ_U and φ_V.

9.4.2 Properties

Let $\mathcal{U} = \{U_i\}_{i \in J}$ be a trivializing covering of a vector bundle E over M. For all $U, V, W \in \mathcal{U}$, we have
1) $g_{UU} = \mathrm{id}$, i.e. $g_{UU}(x) = \mathrm{Id} \in \mathrm{GL}(k, \mathbb{C})$, $\forall\, x \in U$
2) $g_{UV} = (g_{VU})^{-1}$ if $U \cap V \neq \emptyset$
3) If $U \cap V \cap W \neq \emptyset$, then $g_{UV} \cdot g_{VW} = g_{UW}$ on $U \cap V \cap W$ (matrix multiplication), i.e.

$$g_{UV}(x) \cdot g_{VW}(x) = g_{UW}(x)\,,\ \forall\, x \in U \cap V \cap W$$

Proof. 2) and 3) are true because

$$\varphi_{U|\pi^{-1}(x)} \circ \big(\varphi_{V|\pi^{-1}(x)}\big)^{-1} = \Big(\varphi_{V|\pi^{-1}(x)} \circ \big(\varphi_{U|\pi^{-1}(x)}\big)^{-1}\Big)^{-1}$$

$$\varphi_{U|\pi^{-1}(x)} \circ \big(\varphi_{W|\pi^{-1}(x)}\big)^{-1} = \Big(\varphi_{U|\pi^{-1}(x)} \circ \big(\varphi_{V|\pi^{-1}(x)}\big)^{-1}\Big) \circ \Big(\varphi_{V|\pi^{-1}(x)} \circ \big(\varphi_{W|\pi^{-1}(x)}\big)^{-1}\Big)$$

By using 1) and 2), condition 3) is equivalent to :

$$3')\ \text{ if } U \cap V \cap W \neq \emptyset,\ \text{ then } g_{UV} \cdot g_{VW} \cdot g_{WU} = g_{UU} = \mathrm{id}_{U \cap V \cap W}$$

□

1), 2), 3) are called the *cocycle conditions* for the family $(g_{UV})_{U,V \in \mathcal{U}}$.

9.4.3 Definition

Let M be a complex manifold and $\mathcal{U}$ be a set of open subset of M which is a covering of M. An object g is called a *cocycle* if $\forall\, U, V \in \mathcal{U}$, we have $g_{UV} \in \mathrm{GL}\,\big(k\,,\, \mathcal{O}_M(U \cap V)\big)$ with the convention $g_{UV} = \mathrm{id}$ if $U \cap V = \emptyset$ such that the cocycle conditions 1), 2), 3) are satisfied.

Examples :
1) M : complex manifold with $\mathcal{U} = \{M\}$ and $g_{MM} = \mathrm{id}$
2) $\mathcal{U}$ is given by all open sets in M and $g_{UV} = \mathrm{id}$, $\forall\, U, V \in \mathcal{U}$
Setting everything equal to identity is of course always true. The important fact is to see that cocycles always come together with an open covering of M.

9.4.4 Inverse construction

In 9.4.1 we started with a vector bundle E and constructed, after choice of a trivialization covering $\mathcal{U}$, a cocycle $(g_{UV})_{U,V \in \mathcal{U}}$ with $g_{UV} \in \mathrm{GL}\,\big(k\,,\, \mathcal{O}_M(U \cap V)\big)$, $\forall\, U, V \in \mathcal{U}$. This cocycle depends on the chosen trivialization.
Now we take the opposite direction : Given a cocycle, we want to construct a (unique) vector bundle such that its associated cocycles exactly correspond to the starting cocycle. More precisely :

Theorem :
Let M be a complex manifold. Suppose that we are given an open coordinate covering $\{U_\alpha\}_{\alpha \in J}$ of M together with a cocycle $(g_{\alpha\beta})_{\alpha,\beta \in J}$. Then there exist a unique holomorphic vector bundle E over M such that $\{U_\alpha\}_{\alpha \in J}$ is a trivializing open covering for E and the canonical cocycles of E are exactly given by the $(g_{\alpha\beta})_{\alpha,\beta \in J}$.

Proof. Let $(g_{\alpha\beta})_{\alpha,\beta\in J}$ be a cocycle such that every U_α is a chart domain in M. We define

$$E' := \bigsqcup_{\alpha\in J} \big(U_\alpha \times \mathbb{C}^k\big)$$

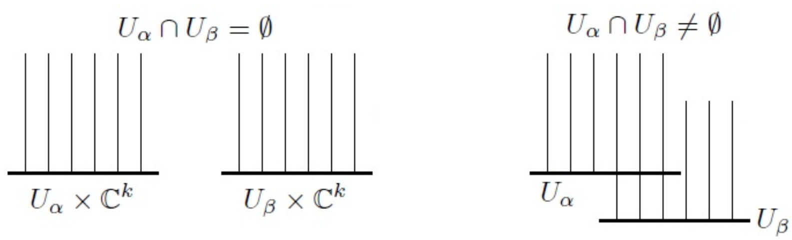
Being defined as a disjoint union, E' is not connected and therefore not a vector bundle (since it's not a manifold).

In order to make E' a vector bundle, we first have to connected the different components $U_\alpha \times \mathbb{C}^k$ (which are all open). There is nothing we can do if $U_\alpha \cap U_\beta = \emptyset$ (see figure 9.7). So assume that $U_\alpha \cap U_\beta \neq \emptyset$; we have to glue the overlapping parts on the intersection (figure 9.7). If $g_{\alpha\beta} = \mathrm{id}$, this is not a problem. Otherwise we do the following :
In order to identify the fibers, we introduce the equivalence relation $\sim$ on E' given by

$$(x,v) \in U_\alpha \times \mathbb{C}^k\,,\ (y,w) \in U_\beta \times \mathbb{C}^k\ :\ (x,v) \sim (y,w)\ \Leftrightarrow\ x = y\ \text{ and }\ v = g_{\alpha\beta}(y)\cdot w$$

Note that $x = y$ already implies that $U_\alpha \cap U_\beta \neq \emptyset$ since $x \in U_\alpha$ and $y \in U_\beta$.

Figure 9.7: gluing the individuals parts of E is only possible on non-empty intersections



Exercise : Show that $\sim$ is an equivalence relation on E'.
This follows from the conditions $1), 2), 3)$ of being a cocycle. It suffices to check it on the second argument.
− reflexivity : $(x,v) \sim (x,v)$ since $g_{\alpha\alpha} = \mathrm{id}$, thus $v = g_{\alpha\alpha}(x)\cdot v = \mathrm{Id}\cdot v$
− symmetry : if $(x,v) \sim (y,w)$ with $v = g_{\alpha\beta}(y)\cdot w\ \Rightarrow\ w = g_{\alpha\beta}^{-1}(y)\cdot v = g_{\beta\alpha}(y)\cdot v$
− transitivity : if $v = g_{\alpha\beta}(y)\cdot w$ and $w = g_{\beta\gamma}(z)\cdot u$, then $v = g_{\alpha\beta}(y)\cdot g_{\beta\gamma}(z)\cdot u = g_{\alpha\gamma}(z)\cdot u$ since $x = y = z$.

Hence we can glue the disjoint union E' via the cocycle g. In general, this gluing is done in a non-trivial way since $g_{\alpha\beta}$ is not necessarily given by the identity. Since $\{U_\alpha\}_{\alpha\in J}$ is a coordinate covering of M, i.e. every chart domain intersects at least one other chart domain (M being connected), we obtain that

$$E := E'/\sim$$

is connected and induces a projection map $p : E \to M : [x,v] \mapsto x$ (well-defined since $(x,v) \sim (y,w)\ \Rightarrow\ x = y$).

In order to show that E is a complex manifold, we introduce the following notation :
Let $[x,v] \in E$ with a (unique) first representative $x \in M$. Consider all the open chart domains U_α of M such that $x \in U_\alpha$. We denote $v_\alpha :=$ the unique second representative of $[x,v]$ such that $(x,v_\alpha) \in U_\alpha \times \mathbb{C}^k$. v_α is unique since E' is a disjoint union of open sets of the type $U_\alpha \times \mathbb{C}^k$. Hence we have a well-defined bijective map

$$\varphi_{U_\alpha}\ :\ E \longrightarrow U_\alpha \times \mathbb{C}^k\ :\ [x,v] \longmapsto (x,v_\alpha) \quad \text{for } \alpha \in J \text{ fixed}$$

We denote the quotient map by $\nu : E' \to E$ and endow E with the quotient topology with respect to ν. As chart domains of E, we then take $V_\alpha := \nu(U_\alpha \times \mathbb{C}^k)$. V_α is open in E for any α because

$$\nu^{-1}(V_\alpha) = \nu^{-1}\big(\nu(U_\alpha \times \mathbb{C}^k)\big) = \bigsqcup_{U_\beta \cap U_\alpha \neq \emptyset} \big(U_\beta \times \mathbb{C}^k\big)\ :\ \text{open in } E'$$

Hence $\{V_\alpha\}_{\alpha\in J}$ is an open covering of E and ν is an open continuous map. As chart maps, we take

$$\phi_\alpha\ :\ V_\alpha \longrightarrow \mathbb{C}^{n+k}\ :\ [x,v] \longmapsto \big(\varphi_\alpha(x), v_\alpha\big) \in \varphi_\alpha(U_\alpha) \times \mathbb{C}^k$$

where $n = \dim M$ and $(U_\alpha, \varphi_\alpha)$ is the corresponding chart of M at x.

ϕ_α is well-defined since we are restricted to V_α and $U_\alpha \times \mathbb{C}^k$, i.e. they only cocycles which may appear are the $g_{\alpha\alpha} = \mathrm{id}$. Moreover ϕ_α is a topological homeomorphism since the chart $(\varphi_\alpha \times \mathrm{id})$ of the product manifold is one. The transition maps are holomorphic because

$$\phi_\alpha \circ \phi_\beta^{-1} \;:\; \varphi_\beta(U_\beta) \times \mathbb{C}^k \longrightarrow \varphi_\alpha(U_\alpha) \times \mathbb{C}^k \;:\; (x,v) \longmapsto \Big(\varphi_\alpha\big(\varphi_\beta^{-1}(x)\big)\,,\, g_{\alpha\beta}(x)\cdot v\Big)$$

where $\varphi_\alpha \circ \varphi_\beta^{-1}$ and $g_{\alpha\beta}$ are holomorphic. It follows that $(V_\alpha, \phi_\alpha)_{\alpha\in J}$ is an atlas of E and $\dim E = \dim M + k$.

With respect to this atlas, p is surjective (as projection) and holomorphic because

$$\varphi_\alpha \circ p \circ \phi_\alpha^{-1} \;:\; \varphi_\alpha(U_\alpha) \times \mathbb{C}^k \longrightarrow \varphi_\alpha(U_\alpha) \;:\; (x,v) \longmapsto \varphi_\alpha\big(\varphi_\alpha^{-1}(x)\big) = x \tag{9.7}$$

is holomorphic (the local form of the projection is again a projection).
Moreover the restricted bijective maps $\varphi_{U_\alpha|V_\alpha} : V_\alpha \to U_\alpha \times \mathbb{C}^k$ are biholomorphic (as defined in 4.3.1) because

$$(\varphi_\alpha \times \mathrm{id}) \circ \varphi_{U_\alpha} = \phi_\alpha \qquad \text{and} \qquad (\varphi_{U_\alpha})^{-1} = \nu_{|U_\alpha \times \mathbb{C}^k}$$

Now let $x \in U_\alpha \subseteq M$ be fixed. We want to find the fiber over x of p. For this, note that

$$(p \circ \varphi_{U_\alpha}^{-1})(x,v) = x,\ \forall\, v \in \mathbb{C}^k \quad \Rightarrow \quad \varphi_{U_\alpha}^{-1}(x, \mathbb{C}^k) \subseteq p^{-1}(x)$$

And if $u \in p^{-1}(x)$, then $p(u) = x \;\Rightarrow\; u \in V_\alpha$, hence $\exists\, v \in \mathbb{C}^k$ such that $u = \varphi_{U_\alpha}^{-1}(x,v)$ since φ_{U_α} is bijective on V_α. It follows that

$$p^{-1}(x) = \varphi_{U_\alpha}^{-1}(x, \mathbb{C}^k)$$

This space carries a complex vector space structure with respect to the **definition**

$$\forall\, v,w \in \mathbb{C}^k,\ \forall\, \lambda \in \mathbb{C} \;:\; \varphi_{U_\alpha}^{-1}(x,v) + \varphi_{U_\alpha}^{-1}(x,w) := \varphi_{U_\alpha}^{-1}(x, v+w) \quad , \quad \lambda \cdot \varphi_{U_\alpha}^{-1}(x,v) := \varphi_{U_\alpha}^{-1}(x, \lambda\, v)$$

This definition moreover implies that the restriction to the fiber $\varphi_{U_\alpha|p^{-1}(x)}$ is trivially linear and hence

$$\varphi_{U_\alpha|p^{-1}(x)} \;:\; p^{-1}(x) \overset{\sim}{\longrightarrow} \{x\} \times \mathbb{C}^k \tag{9.8}$$

is a linear isomorphism between k-dimensional vector spaces. We also have to add that this definition does not depend on the choice of α since if $x \in U_\alpha \cap U_\beta$, then $\varphi_{U_\alpha}^{-1}(x, \mathbb{C}^k) \cong \varphi_{U_\beta}^{-1}(x, \mathbb{C}^k)$. This follows from the fact that

$$\begin{array}{ccc} \varphi_{U_\alpha}^{-1}(x,\mathbb{C}^k) & & \varphi_{U_\beta}^{-1}(x,\mathbb{C}^k) \\ \Big\downarrow{\scriptstyle \varphi_{U_\alpha}} & & \Big\downarrow{\scriptstyle \varphi_{U_\beta}} \\ \{x\}\times\mathbb{C}^k & \xrightarrow{\varphi_{U_\alpha}\circ\varphi_{U_\beta}^{-1}} & \{x\}\times\mathbb{C}^k \end{array}$$

where the considered restrictions of φ_{U_α} and φ_{U_β} are linear isomorphisms as shown in (9.8) and

$$(\varphi_{U_\alpha} \circ \varphi_{U_\beta}^{-1})(x,v) = \big(x, g_{\alpha\beta}(x)\cdot v\big)\ ,\ \forall\, v \in \mathbb{C}^k$$

with $g_{\alpha\beta}(x) \in \mathrm{GL}(k, \mathbb{C})$. Thus $\varphi_{U_\beta}^{-1} \circ (\varphi_{U_\alpha} \circ \varphi_{U_\beta}^{-1}) \circ \varphi_{U_\alpha}$, restricted to $\varphi_{U_\alpha}^{-1}(x, \mathbb{C}^k)$, is an isomorphism too.

Similarly as in (9.8), one also shows that for all $\alpha \in J$,

$$E_{|U_\alpha} = p^{-1}(U_\alpha) \overset{\varphi_{U_\alpha}}{\cong} U_\alpha \times \mathbb{C}^k \tag{9.9}$$

Thus $\{U_\alpha\}_{\alpha\in J}$ is a trivializing open covering for E and it finally follows that φ_{U_α} is a morphism of families of vector spaces and that $p: E \to M$ is indeed a vector bundle. Now let $U_\alpha \cap U_\beta \neq \emptyset$ and consider its cocyles $\tilde{g}_{U_\alpha U_\beta}$ as given in (9.6) :

$$\tilde{g}_{U_\alpha U_\beta}(x)(v) = p_2\bigg(\varphi_{U_\alpha|p^{-1}(x)}\Big(\big(\varphi_{U_\beta|p^{-1}(x)}\big)^{-1}(x,v)\Big)\bigg) = p_2\Big(\varphi_{U_\alpha}\big(\varphi_{U_\beta}^{-1}(x,v)\big)\Big) = g_{\alpha\beta}(x)\cdot v = g_{\alpha\beta}(x)(v)$$

We hence recover the initial cocycles : $\tilde{g}_{U_\alpha U_\beta} = g_{\alpha\beta}$.
And uniqueness of E follows now from the fact that the cocycles of a vector bundle are uniquely given once a trivialization has been fixed, which as shown in (9.9) is the case here. ☐

9.5 Isomorphic bundles

Let E and F be vector bundles over the same base manifold M such that $E \cong F$ (as defined in 9.3.5). We choose a common trivialization covering $\{U_\alpha\}_{\alpha\in J}$ of E and F (take intersections of the 2 individual trivializations).

$$E \overset{\phi}{\cong} F \quad \Rightarrow \quad E_{|U_\alpha} \overset{\psi_\alpha}{\cong} U_\alpha \times \mathbb{C}^k \overset{\varphi_\alpha}{\cong} F_{|U_\alpha} \,,\ \forall\, \alpha \in J$$

where $\varphi_\alpha : F_{|U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{C}^k$ and $\psi_\alpha : E_{|U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{C}^k$. φ_α and ψ_α are morphisms of families of vector spaces and their restrictions to the fibers $\varphi_{\alpha|F_x}$ and $\psi_{\alpha|E_x}$ are linear isomorphisms onto $\{x\}\times\mathbb{C}^k$, $\forall\, x \in U_\alpha$. Let also

$$\phi \ : E \longrightarrow F \quad \Rightarrow \quad \phi_\alpha := \phi_{|U_\alpha} : \ E_{|U_\alpha} \longrightarrow F_{|U_\alpha}$$
$$\Rightarrow \ \varphi_\alpha \circ \phi_\alpha \circ \psi_\alpha^{-1} \ : \ U_\alpha \times \mathbb{C}^k \longrightarrow U_\alpha \times \mathbb{C}^k \ : \ (x,v) \longmapsto \big(x \,,\, c_\alpha(x)\cdot v\big)$$

The last map being a biholomorphic morphism of families of vector spaces (hence the base point x is preserved), we obtain that $c_\alpha(x)$ is an invertible $k\times k$–matrix for all $x \in U_\alpha$. Moreover c_α is holomorphic with respect to x on U_α, thus

$$c_\alpha \in \mathrm{GL}\big(k \,,\, \mathcal{O}_M(U_\alpha)\big) \,,\ \forall\, \alpha \in J \tag{9.10}$$

One can show that the opposite direction is also true, i.e. if we start with a collection $(c_\alpha)_{\alpha\in J}$ satisfying (9.10), one can associate isomorphic vector bundles E and F to this collection whose c_α defined as above exactly correspond to the initial ones.

9.5.1 Definition

Given a trivialization $\{U_\alpha\}_{\alpha\in J}$, we introduce a relation on cocycles $g_{\alpha\beta}$ given by

$$g'_{\alpha\beta} \sim g_{\alpha\beta} \ \Leftrightarrow \ \exists\,(c_\alpha)_{\alpha\in J} \text{ with } c_\alpha \in \mathrm{GL}\big(k \,,\, \mathcal{O}_M(U_\alpha)\big) \text{ such that } g'_{\alpha\beta} = c_\alpha \circ g_{\alpha\beta} \circ c_\beta^{-1} \,,\ \forall\, \alpha,\beta \in J$$

We call 2 cocycles *cohomologous* if they are equivalent with respect to the relation $\sim$.

Exercise : Show that $\sim$ is an equivalence relation.
– reflexivity : choose a collection with $c_\alpha = \mathrm{id}$, $\forall\, \alpha \in J \ \Rightarrow \ g_{\alpha\alpha} = \mathrm{id}\circ g_{\alpha\alpha}\circ \mathrm{id}^{-1}$
– symmetry : since the c_α are invertible, we get $g'_{\alpha\beta} = c_\alpha \circ g_{\alpha\beta} \circ c_\beta^{-1} \ \Rightarrow \ g_{\alpha\beta} = c_\alpha^{-1} \circ g'_{\alpha\beta} \circ (c_\beta^{-1})^{-1}$
– transitivity : if $g'_{\alpha\beta} = c_\alpha \circ g_{\alpha\beta} \circ c_\beta^{-1}$ and $g''_{\beta\gamma} = c'_\beta \circ g'_{\beta\gamma} \circ c'^{-1}_\gamma$, then

$$g''_{\beta\gamma} = c'_\beta \circ \big(c_\beta \circ g_{\beta\gamma} \circ c_\gamma^{-1}\big) \circ c'^{-1}_\gamma = (c'_\beta \circ c_\beta) \circ g_{\beta\gamma} \circ \big(c'_\gamma \circ c_\gamma\big)^{-1} \,,\ \forall\, \beta,\gamma \in J$$

9.5.2 Theorem

We state without proof :
If we start with 2 cohomologous cocycles, the associated vector bundles constructed as in 9.4.4 are isomorphic. This implies that the isoclasses of vector bundles are in 1-to-1 correspondence with the cohomology classes of cocycles (after the choice of a common trivialization of the bundles).

9.5.3 Proof of theorem 9.3.6

Let $\mathcal{F}$ be a locally free sheaf of $\mathcal{O}_M$–modules of rank k. We have to find an associated vector bundle such that the sheaf of sections $\mathcal{V}$ of this vector bundle is again given by $\mathcal{F}$ (up to isomorphism).

Proof. Since $\mathcal{F}$ is locally free, there is an open covering $\{U_\alpha\}_{\alpha\in J}$ of M such that $\mathcal{F}(U_\alpha) \overset{\psi_\alpha}{\cong} \big(\mathcal{O}_M(U_\alpha)\big)^k$, $\forall\, \alpha \in J$, where ψ_α is a module isomorphism. If $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$, we hence obtain

$$\big(\mathcal{O}_M(U_\alpha\cap U_\beta)\big)^k \xleftarrow{\psi_{\alpha|U_{\alpha\beta}}} \mathcal{F}(U_\alpha\cap U_\beta) \xrightarrow{\psi_{\beta|U_{\alpha\beta}}} \big(\mathcal{O}_M(U_\alpha\cap U_\beta)\big)^k$$
$$\Rightarrow \ \psi_{\alpha|U_{\alpha\beta}} \circ \big(\psi_{\beta|U_{\alpha\beta}}\big)^{-1} \ : \ \big(\mathcal{O}_M(U_\alpha\cap U_\beta)\big)^k \longrightarrow \big(\mathcal{O}_M(U_\alpha\cap U_\beta)\big)^k$$

Since $\big(\mathcal{O}_M(U_\alpha \cap U_\beta)\big)^k$ corresponds to a vector of k holomorphic functions, the isomorphism $\psi_{\alpha|U_{\alpha\beta}} \circ \big(\psi_{\beta|U_{\alpha\beta}}\big)^{-1}$ can uniquely be given by a cocycle $g_{\alpha\beta} \in \mathrm{GL}\big(k\,,\, \mathcal{O}_M(U_\alpha \cap U_\beta)\big)$ (same construction in 9.4.1).

Let F be the vector bundle over M which is constructed by these cocycles as in 9.4.4 and $\mathcal{V}$ be its sheaf of local sections, i.e. $\forall\, U \subseteq M$ open : $s \in \mathcal{V}(U) \;\Rightarrow\; p \circ s = \mathrm{id}_U$ where $p : F \to M : [x, v] \mapsto x$. Since both sheaves are locally free, it suffices to consider the case where U is a trivializing open set. The local form (9.7) of p is then

$$p \;:\; U \times \mathbb{C}^k \longrightarrow U \;:\; (x, v) \longmapsto x$$

and a local section $s \in \mathcal{V}(U)$ is necessarily of the form $s(x) = \big(x, \tilde{s}(x)\big)$, where $\tilde{s} : U \to \mathbb{C}^k$ is holomorphic. This form is also sufficient, hence s can be identified with $\tilde{s}$, which can again be seen as a k-tuple of holomorphic function $U \to \mathbb{C}$. It follows that

$$\mathcal{V}(U) \cong \big(\mathcal{O}_M(U)\big)^k \cong \mathcal{F}(U)$$

for any trivializing open subset $U \subseteq M$, so $\mathcal{V} \cong \mathcal{F}$ as sheaves of $\mathcal{O}_M$–modules. Hence the vector bundle F satisfies all the assumptions of the theorem. □

9.5.4 Bundle maps

Let $\pi : E \to M$ and $p : F \to N$ be 2 arbitrary vector bundles (not necessarily over the same base manifold). A *bundle map* is a pair (f, g) of holomorphic maps $f : E \to F$ and $g : M \to N$ such that $p \circ f = g \circ \pi$ and the restriction $f_{|\pi^{-1}(x)} : E_x \to F_{g(x)}$ is a linear map for all $x \in M$.

$$\begin{array}{ccc} E & \xrightarrow{\;f\;} & F \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p} \\ M & \xrightarrow{\;g\;} & N \end{array}$$

A bundle map is in fact the generalization of a morphism of families of vector spaces and it allows us to form the category of vector bundles since bundle maps are precisely the morphisms in this category.

9.6 Frame of a vector bundle

9.6.1 Definition

Let E be a holomorphic vector bundle of rank k over a complex manifold M. A *frame* of E over an open set $U \subseteq M$ is a set of sections $\Sigma \subset \mathcal{V}(U)$ such that $\forall\, x \in U$, the set of sections evaluated at x is a basis of the fiber E_x. This immediately implies that Σ must have exactly k elements, i.e. $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ for $\sigma_i \in \mathcal{V}(U)$, $\forall\, i$. Thus the set $\{\sigma_1(x), \sigma_2(x) \ldots, \sigma_k(x)\}$ is a basis of the vector space E_x. In particular $\sigma_i(x) \in E_x$, $\forall\, i$.

Remarks :
1) This does not imply that $\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ is a $\mathbb{C}$–basis of $\mathcal{V}(U)$ (since $\dim \mathcal{V}(U) = \infty$ in general).
2) Frames do not exist over all open sets $U \subseteq M$.
3) But : given a trivialization covering $\{U_\alpha\}_{\alpha \in J}$ and a local trivialization $E_{|U_\alpha} \cong U_\alpha \times \mathbb{C}^k$, there always exists a frame over U_α given by $\{\hat{e}_1, \ldots, \hat{e}_k\}$ where $\hat{e}_i(x) = (x, e_i)$, $\forall\, i \in \{1, \ldots, k\}$. Since $\{(x, e_i)\}_{i=1,\ldots,k}$ is a basis of the vector space $\{x\} \times \mathbb{C}^k \cong E_x$, $\{\hat{e}_1(x), \ldots, \hat{e}_k(x)\}$ is thus a basis of E_x.

We conclude that, since any vector bundle is locally of this form (locally trivial) :
For any vector bundle E over M, there is a covering (the trivializing covering) such that there exists a frame over any open set in this covering. In particular, frames of vector bundles depend on the chosen covering of M. Furthermore there is even a stronger statement about this fact :

9.6.2 Theorem

There exists a frame of over an open set $U \subseteq M \;\Leftrightarrow\; E_{|U}$ is a trivial vector bundle (i.e. $E_{|U} \cong U \times \mathbb{C}^k$).
In particular, if there is a global frame of E, then E must be the trivial bundle. In other words, only the trivial vector bundle has a global frame.

Proof. $\Leftarrow$: This was shown above : the frame is given by $\{\hat{e}_1, \dots, \hat{e}_k\}$.

$\Rightarrow$: Let $U \subseteq M$ be open and assume that a frame $\{\sigma_1, \dots, \sigma_k\}$ of E over U is given. We want to construct a morphism of families of vector spaces $\varphi : E_{|U} \xrightarrow{\sim} U \times \mathbb{C}^k$. Let $u \in E_{|U}$; since $E_{|U}$ is equal to the disjoint union of its fibers E_x for $x \in U$, we know that there is a unique element $x \in U$ such that $u \in E_x$, hence

$$u = \sum_{i=1}^{k} \lambda_i \cdot \sigma_i(x)$$

and the coefficients λ_i are uniquely given since we have a basis. Then we set $\varphi(u) := \big(x, (\lambda_1, \dots, \lambda_k)\big)$. φ is obviously surjective and injective since the coefficients are unique and it is (bi)holomorphic since it's a projection. Moreover $\varphi_{|E_x}$ is a linear isomorphism for all $x \in U$ (once x is fixed, E_x and $\{x\} \times \mathbb{C}^k$ are two k-dimensional vector spaces over $\mathbb{C}$, hence isomorphic). So all conditions in (9.2) are satisfied and φ defines a trivialization. $\square$

9.6.3 Frames, sections and cocycles

Another construction that one can do by using frames is the following :
Let F be a vector bundle of rank k and assume that a frame $\{\sigma_1, \dots, \sigma_k\}$ of F over an open subset $U \subseteq M$ is given. Let $s \in \mathcal{V}(U)$ be a holomorphic section. Since $s(x) \in E_x$, $\forall\, x \in U$, we can decompose

$$\forall\, x \in U \ : \ s(x) = \sum_{i=1}^{k} \lambda_i^x \cdot \sigma_i(x) \quad \Rightarrow \quad \exists!\, f_1, \dots, f_k \in \mathcal{O}_M(U) \text{ such that } s = \sum_{i=1}^{k} f_i \cdot \sigma_i \tag{9.11}$$

where the functions $f_i(x) = \lambda_i^x$ are holomorphic by a similar argument as in example 9.3.3. Thus one can assign

$$s \longmapsto \sigma := \big(f_1, f_2, \dots, f_k\big) \in \big(\mathcal{O}_M(U)\big)^k$$

Hence there is a 1-to-1 correspondence between holomorphic sections and k-tuples of holomorphic functions :

$$\mathcal{V}(U) \overset{1:1}{\longleftrightarrow} \big(\mathcal{O}_M(U)\big)^k$$

because the f_i are uniquely determined by the section. Note that this identification requires the fixing of a frame.

Let U and V now be 2 trivializing open sets with $U \cap V \neq \emptyset$, so there is a frame $\{\sigma_i^U\}_i$ over U and a frame $\{\sigma_j^V\}_j$ over V. We want to know what happens on $U \cap V$.
Let g_{UV} be the cocycle given as in (9.6) which defines the vector bundle, i.e. we have $g_{UV} =$ "$\varphi_U \circ \varphi_V^{-1}$" where

$$\begin{aligned} \varphi_U \ &: \ E_{|U} \xrightarrow{\sim} U \times \mathbb{C}^k \quad , \quad \varphi_{U|\pi^{-1}(x)} \ : \ E_x \xrightarrow{\sim} \{x\} \times \mathbb{C}^k \\ \varphi_V \ &: \ E_{|V} \xrightarrow{\sim} V \times \mathbb{C}^k \quad , \quad \varphi_{V|\pi^{-1}(x)} \ : \ E_x \xrightarrow{\sim} \{x\} \times \mathbb{C}^k \end{aligned}$$

We want to show that the 2 frames are related via g_{UV}.
Let $s \in \mathcal{V}(U \cap V)$; using (9.11), s can now be decomposed in both frames as

$$s \longmapsto f := \big(f_1, f_2, \dots, f_k\big)\,,\ f_i \in \mathcal{O}_M(U \cap V) \quad \text{and} \quad s \longmapsto f' := \big(f'_1, f'_2, \dots, f'_k\big)\,,\ f'_i \in \mathcal{O}_M(U \cap V)$$

9.6.4 Theorem

$f = g_{UV} \cdot f'$, where $\cdot$ denotes the matrix multiplication.

Proof. From linear algebra, we know that the transformation law for the coefficients is $f = M \cdot f'$ where f are the old coordinates, f' the new ones and M is the basis change matrix from $\{\sigma_i^U\}_i$ to $\{\sigma_j^V\}_j$. Hence it remains to find the matrix M. Indeed the basis transformation is given by the cocycles because $\forall\, x \in U \cap V$:

$$\mathbb{C}^k \cong \{x\} \times \mathbb{C}^k \overset{\varphi_{U|\pi^{-1}(x)}}{\longleftarrow} E_x = \langle\, \sigma_1^U(x), \dots, \sigma_k^U \,\rangle = \langle\, \sigma_1^V(x), \dots, \sigma_k^V \,\rangle = E_x \overset{\varphi_{V|\pi^{-1}(x)}}{\longrightarrow} \{x\} \times \mathbb{C}^k \cong \mathbb{C}^k$$

where the basis change $\{\sigma_i^U\}_i \to \{\sigma_j^V\}_j$ induces a basis change in $\mathbb{C}^k$. Hence we know that the coordinate change is given by $\varphi_{U|\pi^{-1}(x)} \circ \big(\varphi_{V|\pi^{-1}(x)}\big)^{-1}$. But this is exactly the definition of the cocycles, so $M = g_{UV}$. $\square$

9.6.5 Conclusion

Sections of E correspond to local vectors of functions with respect to certain frames which transform in a certain manner, essentially given by the defining cocycles of the vector bundles.

Chapter 10

Operations on vector bundles

10.1 Induced operations on vector bundles

Let E and M be complex manifold and $\pi : E \to M$ be a surjective holomorphic map. We know that a vector bundle of rank k is defined by

1) an open trivialization covering $\{U_\alpha\}_{\alpha\in J}$ of M
2) local trivialization morphisms $\varphi_{U_\alpha} : E_{|U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{C}^k$ that are biholomorphic and linear isomorphisms when restricted to the fibers
3) the cocycles (also called *patching functions*) $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{C})$ that are holomorphic such that $g_{\alpha\beta} = $ "$\varphi_{U_\alpha} \circ \varphi_{U_\beta}^{-1}$" and satisfying the cocycle relations

Problem :
Given a vector bundle, we want to construct a new one by using these data.
In the following, let $\pi : E \to M$ and $p : F \to M$ be 2 complex vector bundles of rank k and l respectively over the same complex base manifold M. We choose a common trivialization $\{U_\alpha\}_{\alpha\in J}$ and denote the cocycles of E by $g_{\alpha\beta}$ and those of F by $h_{\alpha\beta}$.
Let also V and W be complex vector spaces of finite dimension $k = \dim V$, $l = \dim W$ with bases $B = \{e_i\}_{i=1,\dots,k}$ and $B' = \{e'_i\}_{i=1,\dots,l}$ respectively. The goal is to show that operations on vector spaces induce the same operations on vector bundles. The idea is to apply the operations in each fiber (which is a vector space) of the bundle.

10.1.1 Dual bundle

Recall that the *dual space* of V is given by

$$V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) = \big\{\, \varphi : V \to \mathbb{C} \mid \varphi \text{ linear and continuous} \,\big\} \quad\Rightarrow\quad \dim V^* = k$$

It is in fact not necessary to require continuity since any linear map in finite dimension is also continuous. We also recall that for a linear map $T : V \to W$, one defines its *dual map* ${}^tT : W^* \to V^*$ (which is also linear) by

$$\forall\, \alpha \in W^*,\ {}^tT(\alpha) := \alpha \circ T \ \Leftrightarrow\ {}^tT(\alpha)(v) = \alpha\big(T(v)\big) \in \mathbb{C},\ \forall\, v \in V$$

Whenever well-defined we have the relations $({}^tT)^{-1} = {}^t(T^{-1})$ and ${}^t(T_1 \circ T_2) = {}^tT_2 \circ {}^tT_1$.
Moreover if T is given by the matrix A (after choice of a basis), then the matrix associated to ${}^tT$ is ${}^tA$ (transpose).

Proof. The matrix $A = (a_{ij})$ is defined by the relation $T(e_i) = \sum_j a_{ji} e'_j$, $\forall\, i \in \{1, \dots, k\}$. Let $\{\varepsilon^i\}_{i=1,\dots,k}$ and $\{\varepsilon'^i\}_{i=1,\dots,k}$ be the standard bases of V^* and W^* associated to B and B', i.e. $\varepsilon^i(e_j) = \varepsilon'^i(e'_j) = \delta^i_j$, $\forall\, i, j$. Then

$${}^tT(\varepsilon'^i) = \varepsilon'^i \circ T = \sum_j b_{ji}\, \varepsilon^j \in V^* \quad\Rightarrow\quad {}^tT(\varepsilon'^i)(e_k) = \sum_j b_{ji}\, \varepsilon^j(e_k) = b_{ki}$$

$${}^tT(\varepsilon'^i)(e_k) = (\varepsilon'^i \circ T)(e_k) = \varepsilon'^i\big(T(e_k)\big) = \varepsilon'^i\Big(\sum_j a_{jk} e'_j\Big) = \sum_j a_{jk}\, \varepsilon'^i(e'_j) = a_{ik}$$

Hence the coefficients of the matrix associated to ${}^tT$ satisfy $b_{ij} = a_{ji} = {}^ta_{ij}$: we get the matrix ${}^tA$. □

Now we want to define the dual bundle $E^* \to M$. We know that $E = \bigsqcup_{x\in M} E_x$ is locally given by

$$E_{|U_\alpha} \overset{\varphi_{U_\alpha}}{\cong} U_\alpha \times \mathbb{C}^k$$

Each fiber E_x being a vector space, we can take its dual space $(E_x)^*$ and we define the *dual bundle* E^* as

$$E^* := \bigsqcup_{x\in M} (E_x)^*$$

i.e. we take the same foot-map $\pi : E^* \to M$ with $\pi^{-1}(x) = (E^*)_x = (E_x)^*$ (take the dual in each fiber). In order to describe the structure of E^* we have to find its trivialization and its cocycles. For this, we consider the map

$$\varphi_x := \varphi_{U_\alpha|E_x} \ : \ E_x \overset{\sim}{\longrightarrow} \{x\} \times \mathbb{C}^k$$

which is a linear isomorphism and a basis of E_x is given by $\{\varphi_x^{-1}(x, e_i)\}_{i=1,\dots,k}$. Thus ${}^t\varphi_x : \{x\} \times (\mathbb{C}^k)^* \overset{\sim}{\longrightarrow} (E_x)^*$ is an isomorphism too and a basis of $(E_x)^*$ is given by $\{{}^t\varphi_x(x,\varepsilon^i)\}_{i=1,\dots,k}$. Finally we obtain that

$$\left({}^t\varphi_x\right)^{-1} \ : \ (E_x)^* \overset{\sim}{\longrightarrow} \{x\} \times (\mathbb{C}^k)^*$$

This holds for any $x \in M$, hence a trivialization of E^* is given by $\psi_{U_\alpha} : (E^*)_{|U_\alpha} \overset{\sim}{\longrightarrow} U_\alpha \times (\mathbb{C}^k)^*$ where $\{U_\alpha\}_{\alpha\in J}$ is the same trivializing open covering as for E and $\psi_{U_\alpha} = \left({}^t\varphi_{U_\alpha}\right)^{-1}$. For the cocycles this now implies that

$$\psi_{U_\alpha} \circ \psi_{U_\beta}^{-1} = \left({}^t\varphi_{U_\alpha}\right)^{-1} \circ {}^t\varphi_{U_\beta} = \left(\left({}^t\varphi_{U_\beta}\right)^{-1} \circ {}^t\varphi_{U_\alpha}\right)^{-1} = \left({}^t\left(\varphi_{U_\beta}^{-1}\right) \circ {}^t\varphi_{U_\alpha}\right)^{-1} = \left({}^t\left(\varphi_{U_\alpha} \circ \varphi_{U_\beta}^{-1}\right)\right)^{-1}$$

$$\Rightarrow \ j_{\alpha\beta} \ : \ U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(k,\mathbb{C}) \ : \ j_{\alpha\beta}(x) = \left({}^t\left(g_{\alpha\beta}(x)\right)\right)^{-1} = {}^t\left(\left(g_{\alpha\beta}(x)\right)^{-1}\right) = {}^t\left(g_{\beta\alpha}(x)\right)$$

These $j_{\alpha\beta}$ are in addition holomorphic and satisfy the cocycle relations : $j_{\alpha\alpha} = \mathrm{id}$, $j_{\beta\alpha} = (j_{\alpha\beta})^{-1}$ and

$$j_{\alpha\beta}(x) \circ j_{\beta\gamma}(x) = {}^t\left(g_{\beta\alpha}(x)\right) \circ {}^t\left(g_{\gamma\beta}(x)\right) = {}^t\left(g_{\gamma\beta}(x) \circ g_{\beta\alpha}(x)\right) = {}^t\left(g_{\gamma\alpha}(x)\right) = j_{\alpha\gamma}(x)$$

In particular E^* is again a vector bundle of rank k.

10.1.2 Direct sum bundle

The "direct sum" of V and W is defined as

$$V \oplus W \cong V \times W = \left\{ (v,w) \mid v \in V, \ w \in W \right\}$$

Hence $\dim(V \oplus W) = k + l$ and a basis is given by $B \cup B'$ (here we do not mean the direct sum of 2 vector subspaces of some bigger vector space). If V' and W' are 2 other vector spaces and we are given the linear maps $S : V \to V'$ and $T : W \to W'$, we can define the *direct sum* $S \oplus T$ by

$$S \oplus T \ : \ V \oplus W \longrightarrow V' \oplus W' \ , \ (S \oplus T)(v,w) := \left(S(v), T(w)\right)$$

If S and T are represented by matrices A and B (after choice of a basis), the matrix associated to $S \oplus T$ in the induced basis given as above is

$$S \oplus T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \tag{10.1}$$

since S only acts on V and T only acts on W.
We are now able to define the direct sum of 2 vector bundles E and F. As in 10.1.1 we have the local situation

$$E_{|U_\alpha} \overset{\varphi_{U_\alpha}}{\cong} U_\alpha \times \mathbb{C}^k \quad , \quad F_{|U_\alpha} \overset{\psi_{U_\alpha}}{\cong} U_\alpha \times \mathbb{C}^l$$

E_x and F_x being vector spaces for any $x \in M$, we can take their direct sum $E_x \oplus F_x$ and hence define

$$E \oplus F := \bigsqcup_{x\in M} (E_x \oplus F_x)$$

i.e. the fibers of the direct sum $E \oplus F$ are given by $(E \oplus F)_x = E_x \oplus F_x$. We again have to find the trivialization and the cocycles of this new vector bundle.

Consider the linear isomorphisms

$$\varphi_x := \varphi_{U_\alpha|E_x} \ : \ E_x \xrightarrow{\sim} \{x\} \times \mathbb{C}^k \quad , \quad \psi_x := \psi_{U_\alpha|F_x} \ : \ F_x \xrightarrow{\sim} \{x\} \times \mathbb{C}^l$$

They induce the map $\varphi_x \oplus \psi_x : E_x \oplus F_x \to \{x\} \times (\mathbb{C}^k \oplus \mathbb{C}^k)$ which is again an isomorphism because of (10.1). It follows that the local trivialization of $E \oplus F$ is given by

$$\rho_{U_\alpha} \ : \ (E \oplus F)_{|U_\alpha} \xrightarrow{\sim} U_\alpha \times (\mathbb{C}^k \oplus \mathbb{C}^l)$$

with $\rho_{U_\alpha} = \varphi_{U_\alpha} \oplus \psi_{U_\alpha}$ and $E \oplus F$ is a vector bundle of rank $k + l$. For the cocycles note that

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} \ \Rightarrow \ \rho_{U_\alpha}^{-1} = (\varphi_{U_\alpha} \oplus \psi_{U_\alpha})^{-1} = \varphi_{U_\alpha}^{-1} \oplus \psi_{U_\alpha}^{-1}$$

and hence $\rho_{U_\alpha} \circ \rho_{U_\beta}^{-1} = (\varphi_{U_\alpha} \oplus \psi_{U_\alpha}) \circ (\varphi_{U_\beta}^{-1} \oplus \psi_{U_\beta}^{-1}) = (\varphi_{U_\alpha} \circ \varphi_{U_\beta}^{-1}) \oplus (\psi_{U_\alpha} \circ \psi_{U_\beta}^{-1})$ so that finally

$$j_{\alpha\beta} \ : \ U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(k+l, \mathbb{C}) \ : \ j_{\alpha\beta}(x) = g_{\alpha\beta}(x) \oplus h_{\alpha\beta}(x) = \begin{pmatrix} g_{\alpha\beta}(x) & 0 \\ 0 & h_{\alpha\beta}(x) \end{pmatrix}$$

This is obviously holomorphic and $j_{\alpha\beta}$ satisfies the cocycle conditions since $g_{\alpha\beta}$ and $h_{\alpha\beta}$ do.

10.1.3 Tensor product bundle

The *tensor product* of V and W is given by

$$V \otimes W = \mathcal{L}_2(V^* \times W^*, \mathbb{C}) = \left\{ \varphi \ : \ V^* \times W^* \to \mathbb{C} \mid \varphi \text{ bilinear } \right\}$$

i.e. elements of the tensor product are bilinear forms defined on the corresponding dual spaces. They write as a finite linear combination of terms of the form $v \otimes w$ for some $v \in V$, $w \in W$, defined by the condition

$$(v \otimes w)(f, g) := f(v) \cdot g(w) \in \mathbb{C}$$

and extended by linearity. Moreover a basis of $V \otimes W$ is given by $\{e_i \otimes e'_j\}_{i,j}$, thus $\dim(V \otimes W) = k \cdot l$.
If V' and W' are again 2 other vector spaces and $S : V \to V'$ and $T : W \to W'$ are linear maps, they induce the *tensor map*

$$S \otimes T \ : \ V \otimes W \longrightarrow V' \otimes W' \ , \ (S \otimes T)(v \otimes w) := S(v) \otimes T(w) \tag{10.2}$$

again extended by linearity. We will not discuss the matrix representation of this map. Moreover it must be said that the representation of an element of the tensor product as a finite linear combination is not uniquely given, hence (10.2) may a priori not be well-defined since it must be linearly extended. One can however show that the map is independent of this decomposition and that (10.2) is always well-defined.

Concerning the tensor product of 2 vector bundles E and F, we again have the local situation

$$E_{|U_\alpha} \overset{\varphi_{U_\alpha}}{\cong} U_\alpha \times \mathbb{C}^k \quad , \quad F_{|U_\alpha} \overset{\psi_{U_\alpha}}{\cong} U_\alpha \times \mathbb{C}^l$$

and define the *tensor bundle* $E \otimes F$ again as the disjoint union of the tensor products of all individual fibers, i.e.

$$E \otimes F := \bigsqcup_{x \in M} (E_x \otimes F_x)$$

The constructions being exactly the same as in 10.1.2, we will omit the details and end up with the trivialization

$$\rho_{U_\alpha} \ : \ (E \otimes F)_{|U_\alpha} \xrightarrow{\sim} U_\alpha \times (\mathbb{C}^k \otimes \mathbb{C}^l)$$

where $\rho_{U_\alpha} = \varphi_{U_\alpha} \otimes \psi_{U_\alpha}$, thus $E \otimes F$ is a vector bundle of rank $k \cdot l$. And the cocycles are given by

$$j_{\alpha\beta} \ : \ U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(k \cdot l, \mathbb{C}) \ : \ j_{\alpha\beta}(x) = g_{\alpha\beta}(x) \otimes h_{\alpha\beta}(x)$$

10.1.4 Exterior power bundles

For any $r \in \{0,1,2,\dots,k\}$ we can define the r^{th} *exterior powers* of V, given by

$$\otimes^r V = \mathcal{L}_r(V^* \times \dots \times V^*, \mathbb{C}) = \left\{ \varphi : V^* \times \dots \times V^* \to \mathbb{C} \;\middle|\; \varphi \text{ linear in each argument} \right\}$$
$$\Lambda^r V = \mathcal{A}_r(V^* \times \dots \times V^*, \mathbb{C}) = \left\{ \varphi \in \mathcal{L}_r(V^* \times \dots \times V^*, \mathbb{C}) \;\middle|\; \varphi \text{ alternating} \right\}$$

i.e. $\varphi \in \Lambda^r V \;\Leftrightarrow\; \varphi \in \otimes^r V$ and $\varphi(\alpha^1,\dots,\alpha^r) = \operatorname{sign}(\sigma)\cdot\varphi(\alpha^{\sigma(1)},\dots,\alpha^{\sigma(r)})$, $\forall\,\alpha^1,\dots,\alpha^r \in V^*$, $\forall\,\sigma \in \mathcal{S}_r$.
This implies that $\Lambda^r V = \{0\}$ if $r > k$. By convention, we set $\otimes^0 V = \Lambda^0 V = \mathbb{C}$.

A basis for $\otimes^r V$ is given by $\{e_{j_1} \otimes \dots \otimes e_{j_r}\}_{j_1,\dots,j_r}$ where $\{e_i\}_{i=1,\dots,k}$ is the standard basis of V and

$$(e_{j_1} \otimes \dots \otimes e_{j_r})(\alpha^1,\dots,\alpha^r) := \alpha^1(e_1)\cdot\ldots\cdot\alpha^r(e_{j_r}) \in \mathbb{C}\,,\ \forall\,\alpha^1,\dots,\alpha^r \in V^*$$

It follows that $\dim(\otimes^r V) = k^r$ and that any element from $\otimes^r V$ writes as a linear combination of such functions.
For $T \in \Lambda^r V$ and $S \in \Lambda^s V$, we define the *wedge product* $T \wedge S \in \Lambda^{r+s} V$ pointwise by

$$(T \wedge S)(\alpha^1,\dots,\alpha^{r+s}) := \frac{1}{r!\,s!}\cdot\sum_{\sigma\in\mathcal{S}_{r+s}} \operatorname{sign}(\sigma)\cdot T\big(\alpha^{\sigma(1)},\dots,\alpha^{\sigma(r)}\big)\cdot S\big(\alpha^{\sigma(r+1)},\dots,\alpha^{\sigma(r+s)}\big)$$

One can show that this product is associative, distributive and anti-commutative. Moreover a basis of $\Lambda^r V$ is given by $\{e_{i_1} \wedge \dots \wedge e_{i_r}\}_{1\le i_1<\dots<i_r\le k}$, hence $\dim(\Lambda^r V) = \binom{k}{r}$.

If $T : V \to W$ is a linear map, it induces the 2 following *exterior power maps* :

$$\otimes^r T \;:\; \otimes^r V \longrightarrow \otimes^r W\,,\ \otimes^r T(e_{j_1}\otimes\dots\otimes e_{j_r}) := T(e_{j_1})\otimes\dots\otimes T(e_{j_r})$$
$$\Lambda^r T \;:\; \Lambda^r V \longrightarrow \Lambda^r W\,,\ \Lambda^r T(e_{i_1}\wedge\dots\wedge e_{i_r}) := T(e_{i_1})\wedge\dots\wedge T(e_{i_r})$$

extended by linearity. One can again show that this is indeed well-defined. If $S : Z \to V$ is another linear map and whenever T is a linear isomorphism, then $\otimes^r T$ and $\Lambda^r T$ are again isomorphisms and they satisfy

$$(\otimes^r T)^{-1} = \otimes^r(T^{-1})\quad,\quad \otimes^r(T\circ S) = \otimes^r T\circ\otimes^r S\quad,\quad (\Lambda^r T)^{-1} = \Lambda^r(T^{-1})\quad,\quad \Lambda^r(T\circ S) = \Lambda^r T\circ\Lambda^r S \tag{10.3}$$

Let E now be a vector bundle with local trivialization $\varphi_{U_\alpha} : E_{|U_\alpha} \xrightarrow{\sim} U_\alpha\times\mathbb{C}^k$. As before we define the *exterior power bundles* $\otimes^r E$ and $\Lambda^r E$ by taking the exterior powers in each fiber :

$$\otimes^r E := \bigsqcup_{x\in M}(\otimes^r E_x)\qquad,\qquad \Lambda^r E := \bigsqcup_{x\in M}(\Lambda^r E_x)$$

Similarly as in the previous examples, we then obtain the local trivializations

$$\psi_{U_\alpha} \;:\; (\otimes^r E)_{|U_\alpha} \xrightarrow{\sim} U_\alpha\times(\otimes^r\mathbb{C}^k)\qquad,\qquad \rho_{U_\alpha} \;:\; (\Lambda^r E)_{|U_\alpha} \xrightarrow{\sim} U_\alpha\times(\Lambda^r\mathbb{C}^k)$$

showing that $\otimes^r E$ and $\Lambda^r E$ are holomorphic vector bundles of rank k^r and $\binom{k}{r}$ respectively. The cocycles

$$j_{\alpha\beta} \;:\; U_\alpha\cap U_\beta \longrightarrow \mathrm{GL}(k^r,\mathbb{C}) \;:\; j_{\alpha\beta}(x) = \otimes^r\big(g_{\alpha\beta}(x)\big)$$
$$l_{\alpha\beta} \;:\; U_\alpha\cap U_\beta \longrightarrow \mathrm{GL}\left(\tbinom{k}{r},\mathbb{C}\right) \;:\; l_{\alpha\beta}(x) = \Lambda^r\big(g_{\alpha\beta}(x)\big)$$

are holomorphic and satisfy the cocycle conditions because of (10.3).

10.1.5 Particular case

For $k = \dim V$, $\Lambda^k V$ is a vector space of dimension $\binom{k}{k} = 1$, hence $\Lambda^k E$ will be a vector bundle of rank 1. It is a line bundle, called the *top exterior power* of E. It is also called the *determinant bundle* and denoted $\Lambda^k E = \det E$.
The reason is the following :
Let $\mathbf{Vect}_f(\mathbb{C})$ be the category of finite-dimensional vector spaces over $\mathbb{C}$ and consider the functor

$$\Lambda^r \;:\; \mathbf{Vect}_f(\mathbb{C}) \longrightarrow \mathbf{Vect}_f(\mathbb{C}) \;:\; V \longmapsto \Lambda^r V$$
$$\mathrm{Hom}_{\mathbb{C}}(V,W) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\Lambda^r V,\Lambda^r W) \;:\; T \longmapsto \Lambda^r T$$

Take $W = V$. Since $\Lambda^k V$ is 1-dimensional, we obtain that $\Lambda^r T : \Lambda^r V \to \Lambda^r V$, i.e. $\Lambda^r T$ is a linear map from a 1-dimensional space to another 1-dimensional space. Hence $\Lambda^r T \in \mathrm{Mat}(1,\mathbb{C}) \cong \mathbb{C}$ and $\Lambda^r T$ is just given by the multiplication with a complex number. It actually turns out that $\Lambda^r T(\lambda) = \det T\cdot\lambda$, $\forall\,\lambda\in\Lambda^r V$.

10.1.6 Generalization

We have seen that the construction of these new vector bundles is essentially always the same; all you need is an operation on vector spaces that induces an associated operation on linear maps between vector spaces. Thus we have the following generalization :
If $\mathbf{Vect}_f(\mathbb{C})$ denotes again the category of finite-dimensional vector spaces over $\mathbb{C}$, the morphisms of this category are given by linear maps between these vector spaces. Let $\mathcal{F} : \mathbf{Vect}_f(\mathbb{C}) \to \mathbf{Vect}_f(\mathbb{C})$ be a vector space functor, for example :

$$\mathcal{F} = {}^* \;\Rightarrow\; \mathcal{F}(V) = V^* \text{ and } \mathcal{F}(T) = {}^tT$$
$$\mathcal{F} = \Lambda^r \;\Rightarrow\; \mathcal{F}(V) = \Lambda^r V \text{ and } \mathcal{F}(T) = \Lambda^r T$$

where * is a contravariant and Λ^r is a covariant functor. Then $\mathcal{F}$ induces a functor $\mathcal{F}'$ on the category of vector bundles given by the following data :
To a vector bundle $\pi : E \to M$ with trivialization $E_{|U_\alpha} \cong U_\alpha \times \mathbb{C}^k$ and cocycles $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{C})$, $\mathcal{F}'$ associates the vector bundle $\pi : \mathcal{F}(E) \to M$ defined by

$$\mathcal{F}(E) := \bigsqcup_{x \in M} \mathcal{F}(E_x)$$

with the trivialization $\mathcal{F}(E)_{|U_\alpha} \cong U_\alpha \times \mathcal{F}(\mathbb{C}^k)$ and cocycles $\mathcal{F}(g_{\alpha\beta}) : U_\alpha \cap U_\beta \to \mathrm{GL}(k', \mathbb{C})$ where $k' = \dim \mathcal{F}(\mathbb{C}^k)$.

Remark :
The same of course also applies for constructions involving bifunctors of vector spaces, as e.g. $\oplus$ and $\otimes$.

10.2 Sub-bundles and quotient bundles

10.2.1 Definition

Let $\pi : E \to M$ be a holomorphic vector bundle of rank k.
A *vector sub-bundle* of E is a family of vector subspaces $\{F_x \subseteq E_x\}_{x \in M}$, indexed by M, such that

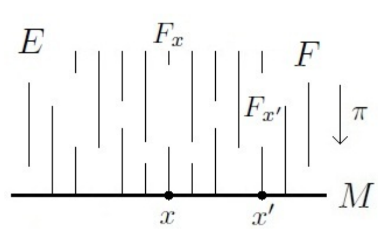
$$F := \bigsqcup_{x \in M} F_x$$

is a complex submanifold of the manifold E and $\pi_{|F} : F \to M$ is itself a vector bundle of rank $l \leq k$. This implies that $\forall\, x \in M$ there is an open neighborhood U of x in M and a trivialization $\varphi_U : E_{|U} \xrightarrow{\sim} U \times \mathbb{C}^k$ such that the restriction to $F_{|U}$ satisfies $\varphi_{U|F_{|U}} : F_{|U} \xrightarrow{\sim} U \times \mathbb{C}^l \subseteq U \times \mathbb{C}^k$ for $l \leq k$.
The intuitive idea of a vector sub-bundle is that $F \subseteq E$ such that for any point in M there is an open neighborhood on which both bundles E and F are trivial and the trivialization maps for F are nothing but the restrictions to F of the trivialization maps of E.

10.2.2 Examples

1) For an open set $U \subsetneq M$, $E_{|U}$ is not a sub-bundle of E since it is not indexed over M. In a sub-bundle F the fiber over any point $x \in M$ must be non-empty and of dimension l (see figure 10.1).

Figure 10.1: F is a sub-bundle of E



2) Any vector bundle E is a sub-bundle over itself.
3) The *zero-bundle* given by $F_x = \{0\}$, $\forall\, x \in M$ is a sub-bundle of any vector bundle E.
Moreover it is in 1-to-1 correspondence with M since $\bigsqcup_{x \in M} \{0\} \cong M$.

10.2.3 Vector space quotients

Let V and W be vector spaces of dimension k and l and consider the vector subspaces $V' \subseteq V$ and $W' \subseteq W$ of dimension $k' \leq k$ and $l' \leq l$. Assume that bases for V and W are given by $\{e_i\}_{i=1,\dots,k}$ and $\{e'_i\}_{i=1,\dots,l}$ such that $\{e_i\}_{i=1,\dots,k'}$ and $\{e'_i\}_{i=1,\dots,l'}$ are bases of V' and W'.
The *quotient space* V/V' is defined by the equivalence relation $x \sim y \;\Leftrightarrow\; \exists\, v \in V'$ such that $x = y + v'$. We denote the equivalence class of $x \in V$ by $\bar{x}$. It follows that $\dim(V/V') = k - k'$ and a basis of V/V' is given by $\{\bar{e}_i\}_{i=k'+1,\dots,k}$. Any element $v \in V'$ satisfies $\bar{v} = 0$.
Let now $f : V \to W$ be a linear map such that $f(V') \subseteq W'$, i.e. f preserves the vector subspaces. In the bases of V and W, f is then given by a matrix of the type

$$f = \big(a_{ij}\big)_{i,j} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \quad \text{for } A : l' \times k' \ , \ B : l' \times (k - k') \ , \ D : (l - l') \times (k - k')$$

This happens because for any e_i with $i \in \{1, \dots, k'\}$, we have $e_i \in V' \;\Rightarrow\; f(e_i) \in W'$, hence

$$\forall\, i \in \{1, \dots, k'\} \ : \ f(e_i) = \sum_{j=1}^{l'} a_{ji} e'_j + \sum_{j=l'+1}^{l} 0 \cdot e'_j \quad \Rightarrow \quad a_{ji} = 0, \ \forall\, j \in \{l'+1, \dots, l\}$$

or in other words : $a_{ij} = 0$ if i is "big" and j is "small". We moreover see that $f_{|V'} : V' \to W'$ is represented by the matrix A. f now induces the map $\bar{f} : V/V' \to W/W'$ defined by

$$\bar{f}(\bar{x}) := \overline{f(x)}$$

This is well-defined since f is linear and satisfies $f(V') \subseteq W'$: $\overline{f(x+v)} = \overline{f(x) + f(v)} = \overline{f(x)}$ for any $v \in V'$ because $f(v) \in W'$. Moreover $\bar{f}$ is then given by the matrix D since all first basis vectors vanish in the quotient.

10.2.4 Definition

Let E be a vector bundle of rank k and F be a sub-bundle of E of rank l. The *quotient bundle* E/F is defined by

$$E/F := \bigsqcup_{x \in M} (E_x/F_x)$$

i.e. the fibers of E/F are given by the quotients of the individual fibers : $(E/F)_x = E_x/F_x$.

10.2.5 Cocycles for sub-bundles and quotients

Let E be a vector bundle of rank k, F a sub-bundle of E of rank l and E/F be the quotient bundle. If $x \in M$ and U, V are 2 trivializing open sets for E with $U \cap V \neq \emptyset$, then $\varphi_U : E_{|U} \xrightarrow{\sim} U \times \mathbb{C}^k$, $\varphi_V : E_{|V} \xrightarrow{\sim} V \times \mathbb{C}^k$ and

$$g_{UV}(x) = \text{"}\varphi_U \circ \varphi_V^{-1}\text{"} = \begin{pmatrix} h_{UV}(x) & k_{UV}(x) \\ i_{UV}(x) & j_{UV}(x) \end{pmatrix} \in \mathrm{GL}(k, \mathbb{C})$$

with $h_{UV}(x) : l \times l$, $k_{UV}(x) : l \times (k - l)$, $i_{UV}(x) : (k - l) \times l$ and $j_{UV}(x) : (k - l) \times (k - l)$. First of all we need that $i_{UV} = 0$ since F is a sub-bundle of E, hence the subspace $\mathbb{C}^l \subset \mathbb{C}^k$ must be preserved. This is true because

$$\psi_U = \varphi_{U|F_{|U}} \ : \ F_{|U} \xrightarrow{\sim} U \times \mathbb{C}^l \quad \Rightarrow \quad g_{UV}(x) = \begin{pmatrix} h_{UV}(x) & k_{UV}(x) \\ 0 & j_{UV}(x) \end{pmatrix} \ : \ \mathbb{C}^k \longrightarrow \mathbb{C}^k$$

The cocycles of F are induced by $\psi_U \circ \psi_V^{-1}$, i.e. by a simple restriction of $\varphi_U \circ \varphi_V^{-1}$ and hence given by the first l components of g_{UV} in order to preserve the subspace $\mathbb{C}^l \subset \mathbb{C}^k$. Finally the cocycles of F are the

$$h_{UV} \ : \ U \cap V \longrightarrow \mathrm{GL}(l, \mathbb{C})$$

In order to find a trivialization and cocycles of E/F, let $\{U_\alpha\}_{\alpha \in J}$ be a trivializing open covering for E with

$$\varphi_{U_\alpha} \ : \ E_{|U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{C}^k \quad , \quad \varphi_{U_\alpha|E_x} = \varphi_x \ : \ E_x \xrightarrow{\sim} \{x\} \times \mathbb{C}^k \quad , \quad \varphi_{U_\alpha|F_x} = \varphi_{x|F_x} \ : \ F_x \xrightarrow{\sim} \{x\} \times \mathbb{C}^l$$

This induces the map $\bar{\varphi}_x : E_x/F_x \xrightarrow{\sim} \{x\} \times \mathbb{C}^k/\mathbb{C}^l \cong \mathbb{C}^{k-l}$ and we construct similarly

$$\psi_{U_\alpha} \;:\; (E/F)_{|U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{C}^{k-l}$$

$\bar{\varphi}_x$ arising from the lower-right $(k-l) \times (k-l)$ matrix describing φ_x, ψ_{U_α} is given by a similar expression and the cocycles are induced by the $\psi_{U_\alpha} \circ \psi_{U_\beta}^{-1}$, i.e. the cocycles of E/F are the lower-right part of g_{UV} :

$$j_{UV} \;:\; U \cap V \longrightarrow \mathrm{GL}(k-l, \mathbb{C})$$

10.3 Pull-back bundles

10.3.1 Definition

Let $\pi : E \to M$ be a holomorphic vector bundle of rank k and $f : N \to M$ be a holomorphic map of complex manifolds. The *pull-back* of E along f is the vector bundle over N defined by

$$f^*E := \big\{ (x,u) \in N \times E \mid f(x) = \pi(u) \big\}$$

with the first projection $p_1 : f^*E \to N$ as foot-map. Hence the following diagram commutes by definition :

$$\begin{array}{ccc} f^*E & \xrightarrow{p_2} & E \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{f} & M \end{array}$$

The fiber of p_1 over $x \in N$ is therefore given by

$$\begin{aligned} (f^*E)_x = p_1^{-1}(x) &= \big\{ (y,u) \in f^*E \mid y = x \big\} = \big\{ (x,u) \mid u \in E,\ f(x) = \pi(u) \big\} \\ &= \big\{ (x,u) \mid u \in \pi^{-1}\big(f(x)\big) \big\} = \{x\} \times \pi^{-1}\big(f(x)\big) = \{x\} \times E_{f(x)} \cong E_{f(x)} \end{aligned}$$

It follows that the rank of f^*E is also equal to k. Using the Implicit Function Theorem one can moreover show that f^*E is a complex submanifold of $N \times E$. In order to show that f^*E defines indeed a vector bundle, we have to find a trivialization and the associated cocycles.
Let $\varphi_{U_\alpha} : E_{|U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{C}^k$ be a trivialization for E with cocycles $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{C})$ and U_α, U_β open sets in M such that $U_\alpha \cap U_\beta \neq \emptyset$. f being continuous, the sets $V_\alpha = f^{-1}(U_\alpha)$ are an open covering of N (note however that V_α can be empty even if $U_\alpha \neq \emptyset$). Now consider

$$\begin{aligned} &\forall\, y \in M,\ \varphi_y \;:\; E_y \xrightarrow{\sim} \{y\} \times \mathbb{C}^k \\ \Rightarrow\ &\forall\, x \in N,\ \varphi_{f(x)} \;:\; E_{f(x)} \xrightarrow{\sim} \{f(x)\} \times \mathbb{C}^k \cong \{x\} \times \mathbb{C}^k \end{aligned} \tag{10.4}$$

where $f(x) \in U_\alpha$ if $x \in V_\alpha$. This is not really a composition ($\varphi_{f(x)} \neq \varphi_x \circ f$), but we can nevertheless take

$$\psi_{V_\alpha} \;:\; (f^*E)_{|V_\alpha} \xrightarrow{\sim} V_\alpha \times \mathbb{C}^k$$

where $\psi_{V_\alpha} =$ "$\varphi_{U_\alpha} \circ f$" as a trivialization for f^*E. But (10.4) suggests that the cocycles are given by an expression where any x has been replaced by $f(x)$. And this is indeed true because we can take

$$j_{\alpha\beta} \;:\; V_\alpha \cap V_\beta \longrightarrow \mathrm{GL}(k, \mathbb{C}) \;:\; j_{\alpha\beta}(x) = g_{\alpha\beta}\big(f(x)\big)$$

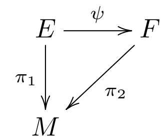
i.e. $j_{\alpha\beta} = g_{\alpha\beta} \circ f$, and this composition is well-defined. The relations can be summarized as follows :

$$\begin{array}{ccc} U_\alpha \cap U_\beta & \xrightarrow{g_{\alpha\beta}} & \mathrm{GL}(k, \mathbb{C}) \\ \uparrow{\scriptstyle f} & \nearrow{\scriptstyle j_{\alpha\beta}} & \\ V_\alpha \cap V_\beta & & \end{array}$$

Remark :
In the case where $i : N \hookrightarrow M$ is the inclusion, i^*E will be the restriction of the vector bundle E to $E_{|N}$.

10.3.2 Problem

Let $\pi_1 : E \to M$ and $\pi_2 : F \to M$ be 2 vector bundles over the same base manifold M and $\psi : E \to F$ be a holomorphic morphism of families of vector spaces, i.e. $\pi_2 \circ \psi = \pi_1$ such that ψ preserves the fibers.



Moreover the map $\psi_x : E_x \to F_x$ is linear (but not an isomorphism in general), hence we can consider the finite-dimensional vector spaces $\ker\psi_x$ and $\operatorname{im}\psi_x$. The idea is to define

$$\ker\psi := \bigsqcup_{x\in M} \ker\psi_x \subseteq E \qquad \text{and} \qquad \operatorname{im}\psi := \bigsqcup_{x\in M} \operatorname{im}\psi_x \subseteq F$$

There are however a certain number of problems with these definitions; in particular $\ker\psi$ and $\operatorname{im}\psi$ do not necessarily define sub-bundles of E and F. First of all $\operatorname{im}\psi_x$ is in general not always closed, hence $\ker\psi$ is not closed and cannot be a complex submanifold of F. But the bigger problem is that, for both of them, the dimensions of the fibers can "jump" : $\dim(\ker\psi_x)$ and $\dim(\operatorname{im}\psi_x)$ are not uniquely determined by E and F, but strongly depend on the linear maps ψ_x. Hence $\ker\psi$ is only a sub-bundle of E if the maps ψ_x all have the same rank. And for $\operatorname{im}\psi$ it is in addition required that all images $\operatorname{im}\psi_x$ are closed.

Conclusion :
There are no kernels or cokernels in the category of vector bundles. In particular, since the category of (isoclasses of) vector bundles of rank k is equivalent to the category of locally free sheaves of rank k, the locally free sheaves do not admit kernels or cokernels neither. If one wants to consider kernels and cokernels, one has to pass to the so-called *coherent sheaves* (sheaves where the rank can "jump").

10.4 Associated frames

Let M be a complex manifold and $\pi : E \to M$ be a holomorphic vector bundle of rank k with local trivialization

$$\varphi_U \;:\; E_{|U} \xrightarrow{\sim} U \times \mathbb{C}^k$$

We recall that a frame for E over U (which always exists since U is a trivializing open set, see section 9.6.1) is a collection $\{\sigma_1, \dots, \sigma_k\}$ of local sections of E over U such that for all $x \in U$, $\{\sigma_1(x), \dots, \sigma_k(x)\}$ is a basis for the fiber E_x (k-dimensional vector space).
Let F be another holomorphic vector bundle over M with the same trivialization as E, i.e. locally

$$E_{|U} \cong U \times \mathbb{C}^k \quad , \quad F_{|U} \cong U \times \mathbb{C}^l$$

and assume that $\{\sigma_1, \dots, \sigma_k\}$ is a frame for E over U and $\{\tau_1, \dots, \tau_l\}$ is a frame for F over U. We want to find expressions for the frames over U of the associated vector bundles constructed in 10.1.

1) The dual bundle E^* locally looks like $(E^*)_{|U} \cong U \times (\mathbb{C}^k)^*$. We have to find a basis of $(E_x)^*$, $\forall\, x \in U$.
Since a basis of E_x is already given, we simply take the canonical basis of the dual space. Thus a frame of E^* over U is given by $\{\sigma_1^*, \dots, \sigma_k^*\}$, where $\sigma_i^* : U \to (E_x)^* \;\Rightarrow\; \sigma_i^*(x) : E_x \to \mathbb{C}$ and $\sigma_i^*(x)$ is linear, $\forall\, x \in U$. And to define the $\sigma_i^*(x)$ on E_x, it is by linearity sufficient to give its value on each basis vector $\sigma_j(x)$ of E_x. We set

$$\sigma_i^*(x) \;:\; E_x \to \mathbb{C} \;,\; \sigma_i^*(x)\big(\sigma_j(x)\big) := \delta_{ij}$$

2) The direct sum $E \oplus F$ locally looks like $(E \oplus F)_{|U} \cong U \times (\mathbb{C}^k \oplus \mathbb{C}^l)$ and a frame over U is given by

$$\{\sigma_1, \dots, \sigma_k, \tau_1, \dots, \tau_l\}$$

Indeed, $\forall\, x \in U$, $\{\sigma_1(x), \dots, \sigma_k(x)\}$ is a basis for $E_x \cong \mathbb{C}^k$ and $\{\tau_1(x), \dots, \tau_l(x)\}$ is a basis for $F_x \cong \mathbb{C}^l$, hence we know from linear algebra that $\{\sigma_1(x), \dots, \sigma_k(x), \tau_1(x), \dots, \tau_l(x)\}$ is a basis for $E_x \oplus F_x \cong \mathbb{C}^k \oplus \mathbb{C}^l$.

3) The tensor product $E \otimes F$ locally looks like $(E \otimes F)_{|U} \cong U \times (\mathbb{C}^k \otimes \mathbb{C}^l)$ and a frame over U is given by

$$\{\sigma_i \otimes \tau_j\}_{i=1,\dots,k;j=1,\dots,l}$$

where $\sigma_i \otimes \tau_j(x) := \sigma_i(x) \otimes \tau_j(x) \in E_x \otimes F_x \cong \mathbb{C}^k \otimes \mathbb{C}^l$, $\forall\, x \in U$. Again this is the case because $\{\sigma_1(x), \dots, \sigma_k(x)\}$ and $\{\tau_1(x), \dots, \tau_l(x)\}$ are bases for E_x and F_x, hence $\{\sigma_i(x) \otimes \tau_j(x)\}_{i,j}$ is a basis for $E_x \otimes F_x = (E \otimes F)_x$.

4) The exterior powers $\otimes^k E$ and $\Lambda^r E$ locally look like $(\otimes^r E)_{|U} \cong U \times (\otimes^r \mathbb{C}^k)$ and $(\Lambda^r E)_{|U} \cong U \times (\Lambda^r \mathbb{C}^k)$. Hence as in 10.1.4, frames over U are given by $\{\sigma_{j_1} \otimes \dots \otimes \sigma_{j_r}\}_{j_1,\dots,j_r}$ and $\{\sigma_{i_1} \wedge \dots \wedge \sigma_{i_r}\}_{i_1<\dots<i_r}$, where

$$(\sigma_{j_1} \otimes \dots \otimes \sigma_{j_r})(x) := \sigma_{j_1}(x) \otimes \dots \otimes \sigma_{j_r}(x) \quad \text{and} \quad (\sigma_{i_1} \wedge \dots \wedge \sigma_{i_r})(x) := \sigma_{i_1}(x) \wedge \dots \wedge \sigma_{i_r}(x)$$

In the case of the determinant bundle $\det E = \Lambda^k E$ of rank 1, this frame becomes

$$(\sigma_1 \wedge \dots \wedge \sigma_k)(x) := \sigma_1(x) \wedge \dots \wedge \sigma_k(x) \in \Lambda^k(E_x) \cong \Lambda^k(\mathbb{C}^k) \cong \mathbb{C}$$

and actually corresponds to taking the volume form in each fiber.

10.5 Line bundles

10.5.1 Definition

A *line bundle* over a complex manifold M is a holomorphic vector bundle of rank 1, i.e. locally on a trivializing open covering $\{U_\alpha\}_{\alpha \in J}$ it looks like $U_\alpha \times \mathbb{C}$ and all fibers are 1-dimensional.
An advantage of line bundles is that a lot of the vector bundle properties simplify in this case.

a) The cocycles are given by $g_{UV} : U \cap V \to \mathrm{GL}(1, \mathbb{C}) \cong \big(\mathbb{C} \setminus \{0\}, \cdot\big)$ with invertibility condition $g_{UV}^{-1} = g_{VU}$, so we need that $g_{UV}(x) \neq 0$, $\forall\, x \in U \cap V$. More precisely, g_{UV} is a map that associates to any $x \in U \cap V$ the linear map given by multiplication with the non-zero complex number $g_{UV}(x)$, i.e. g_{UV} is holomorphic with respect to x and can be identified with a map $U \cap V \to \mathbb{C}$. Hence $g_{UV} \in \mathcal{O}^*(U \cap V)$, where $\mathcal{O}^*$ is the sheaf of nowhere-vanishing holomorphic functions to ensure that $\frac{1}{g_{UV}}$ is well-defined and holomorphic again.

b) A frame for a line bundle E over an open set $U \subseteq M$ is just a holomorphic function $\sigma \in \mathcal{O}^*(U)$. Since the fibers are 1-dimensional (hence isomorphic to $\mathbb{C}$), we just need 1 non-zero vector (which can thus be identified with a non-zero complex number) to define a basis. This must be satisfied for any $x \in U$, hence the function $\sigma : U \to \mathbb{C}$ should not vanish on U (since if $\sigma(x) = 0$ for some $x \in U$, then $\{\sigma(x)\}$ is not a basis of $E_x \cong \mathbb{C}$).

c) The dual bundle E^* is again a line bundle and its cocycles are $j_{UV} = ({}^t g_{UV})^{-1} = g_{UV}^{-1} = g_{VU} = \frac{1}{g_{UV}}$, i.e.

$$j_{UV}(x) = \frac{1}{g_{UV}(x)}\,,\ \forall\, x \in U \cap V$$

The reason why the dual map has no affect on g_{UV} is that a linear map T can be identified with its dual map ${}^tT$ in dimension 1 (because if T is represented by the matrix A, then the matrix of ${}^tT$ is ${}^tA = A$ too) ; T and ${}^tT$ are not the same maps, but they have the same associated matrix and are hence equivalent.

d) The tensor product of 2 line bundle E and F over M is again a line bundle since $\mathrm{rk}(E \otimes F) = \mathrm{rk}\, E \cdot \mathrm{rk}\, F = 1$. By choosing a common trivialization for E and F, we have locally $E_{|U} \cong U \times \mathbb{C}$ and $F_{|U} \cong U \times \mathbb{C}$, hence

$$(E \otimes F)_{|U} \cong U \times (\mathbb{C} \otimes \mathbb{C}) \cong U \times \mathbb{C}$$

If the cocycles of E and F are g_{UV} and h_{UV}, then the cocycles of the tensor bundle $E \otimes F$ are given by

$$j_{UV}(x) = g_{UV}(x) \otimes h_{UV}(x) = \big(v \mapsto g_{UV}(x) \cdot h_{UV}(x) \cdot v\big) \tag{10.5}$$

where $g_{UV}(x) \in \mathbb{C}$, $h_{UV}(x) \in \mathbb{C}$ and the tensor product $\otimes$ in dimension 1 is nothing but the usual multiplication. After identification we thus can write that $j_{UV} \in \mathcal{O}^*(U \cap V)$ is given by $j_{UV}(x) = g_{UV}(x) \cdot h_{UV}(x)$, $\forall\, x \in U \cap V$.

10.5.2 Theorem

The set of isoclasses of line bundles over a complex manifold M is an abelian group under the $\otimes$-operation. This group is called the *Picard group*.

Proof. Recall that $E \sim F \;\Leftrightarrow\; E \cong F$ as vector bundles as defined in 9.3.5
For associativity and commutativity we have to show that $E \otimes (F \otimes G) \cong (E \otimes F) \otimes G$ and $E \otimes F \cong F \otimes E$. By theorem 9.5.2, we know that cohomologous cocycles define isomorphic vector bundles. Hence in our case it already suffices to show that both vector bundles have the same cocycles (hence cohomologous cocycles). If the cocycles of E, F, G are g_{UV}, h_{VU}, j_{UV}, then (10.5) shows that the cocycles of the tensor products are

$$E \otimes (F \otimes G) \longrightarrow g_{UV} \cdot (h_{VU} \cdot j_{UV}) \quad , \quad (E \otimes F) \otimes G \longrightarrow (g_{UV} \cdot h_{VU}) \cdot j_{UV}$$
$$E \otimes F \longrightarrow g_{UV} \cdot h_{VU} \quad , \quad F \otimes E \longrightarrow h_{VU} \cdot g_{UV}$$

Hence these 2 properties follow from associativity and commutativity of the complex numbers.
For the neutral element we have to find a line bundle E_0 such that $E_0 \otimes E \cong E \otimes E_0 \cong E$ for any line bundle E. We take $E_0 := M \times \mathbb{C}$, the trivial bundle, whose cocycles are given by $g_{UV}(x) = 1$, $\forall\, x \in U \cap V$ since we have

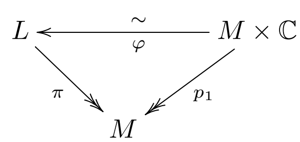
$$\varphi_U \;:\; E_{0|U} \xrightarrow{\sim} U \times \mathbb{C} \;\Rightarrow\; \varphi_U = \mathrm{id},\; \forall\, U \subseteq M \text{ open} \quad\Rightarrow\; g_{UV} = \text{"}\varphi_U \circ \varphi_V^{-1}\text{"} = \mathrm{id} \text{ as well}$$

Concerning the inverse, given a line bundle E, we want to find E^{-1} such that $E \otimes E^{-1} \cong E^{-1} \otimes E \cong E_0$. We take the dual bundle $E^* = E^{-1}$ because the cocycles of E^* are $\frac{1}{g_{UV}}$, so $g_{UV}(x) \cdot \frac{1}{g_{UV}(x)} = 1$, $\forall\, x \in U \cap V$. $\square$

10.5.3 Proposition

A line bundle is trivial $\Leftrightarrow$ it admits a nowhere-vanishing global section.

Proof. By theorem 9.6.2, a trivialization on an open set $U \subseteq M$ is equivalent to the existence of a frame over U.
$\underline{\Rightarrow}$: If a line bundle E is trivial, it admits a frame over M. But since the fibers are 1-dimensional, there must exists a global section $\sigma : M \to E$ such that $\{\sigma(x)\}$ is a basis for E_x, $\forall\, x \in M$. Moreover σ must be nowhere-vanishing since $\sigma(x) = 0$ would not define a basis.
$\underline{\Leftarrow}$: If there exists a nowhere-vanishing global section σ, then $\{\sigma\}$ defines a frame over M since any non-zero vector defines a basis of a 1-dimensional vector space : $\sigma(x) \neq 0 \;\Rightarrow\; \{\sigma(x)\}$ is a basis for E_x, $\forall\, x \in M$. Hence we can see $\sigma \in \mathcal{O}^*(M)$ by considering $\sigma(x) \in E_x \cong \mathbb{C}$. Theorem 9.6.2 then implies that the line bundle is trivial. We recall that the morphism of families of vector spaces is given by $\varphi : M \times \mathbb{C} \xrightarrow{\sim} L$, $\varphi(x, \lambda) := \lambda \cdot \sigma(x)$.



$\square$

Remark :
Note that this is only true for line bundles and does not hold for vector bundles of higher rank.

10.5.4 Definition

Recall that vector bundles over a complex manifold M corresponds to locally free sheaves over M, given by the sheaf of holomorphic sections. Hence the line bundles over M correspond to locally free sheaves of rank 1, also called *invertible sheaves*. Finally we have seen in section 9.3.6 that the trivial vector bundle $M \times \mathbb{C}^k$ is uniquely given by the free sheaf $\mathcal{O}_M^k$. Hence the trivial line bundle $M \times \mathbb{C}$ corresponds to the sheaf $\mathcal{O}_M$ of holomorphic functions, in particular because sections of the trivial bundle satisfy $p_1 \circ s = \mathrm{id} \;\Rightarrow\; s(x) = (x, \tilde{s}(x)) \in M \times \mathbb{C}$, where $\tilde{s} : M \to \mathbb{C}$ can be any holomorphic function. Therefore sections of the trivial line bundle can be identified with holomorphic functions. $f \in \mathcal{O}_M \;\Leftrightarrow\; f : M \to \mathbb{C}$ is a holomorphic section of the trivial bundle $M \times \mathbb{C}$.
In the following we denote by $\mathcal{L}$ the sheaf of holomorphic sections of a line bundle (i.e. $\mathcal{L} = \mathcal{V}$, but we want to point out that we are dealing with a line bundle).

10.5.5 Proposition

Let $\pi : L \to M$ be a line bundle over M and $s \in \mathcal{L}(M)$ be a global holomorphic section such that $s \not\equiv 0$, i.e. s is a globally holomorphic map

$$s \ : \ M \longrightarrow L = \bigsqcup_{x\in M} L_x \ : \ x \longmapsto v_x \in L_x$$

Then the set $(s) := \{\, x \in M \mid s(x) = 0 \,\}$ (where $0 \notin \mathbb{C}$, but $0 \in L_x$) is an analytic subset of M.

Proof. Recall that analytic subsets of M are locally given by zero sets of finitely many holomorphic functions (i.e. sections of the trivial line bundle). Note however that (s) is not an analytic set by definition since $s \notin \mathcal{O}_M(M)$; s is not a map $M \to \mathbb{C}$ and $(s) \neq V(s)$. But s locally corresponds to a holomorphic $\mathbb{C}$-valued function. If $U \subseteq M$ is a trivializing open set for L, then $L_{|U} \cong U \times \mathbb{C}$ and

$$s_{|U} \equiv \tilde{s} \ \text{ where } \ s_{|U} \ : U \to L_{|U} \ \Rightarrow \ s(x) = (x, \tilde{s}(x)), \ \tilde{s} \in \mathcal{O}_M(U), \ \forall\, x \in U$$

Now if $x \in U$ is such that $s(x) = 0$, the local form is given by $x \mapsto (x, 0) \in U \times L_x \cong U \times \mathbb{C}$. U being a trivializing open set, we know by theorem 9.6.2 that there is a frame $\{\sigma\}$ over U. Since $s(x) \in L_x$ for all x and $\{\sigma(x)\}$ is a basis for the 1-dimensional fiber, we obtain that

$$\forall\, x \in U, \ \exists\, \lambda_x \in \mathbb{C} \text{ such that } s_{|U}(x) = \lambda_x \cdot \sigma(x)$$

Define $f : U \to \mathbb{C}, \ f(x) = \lambda_x \ \Rightarrow \ s_{|U}(x) = f(x) \cdot \sigma(x), \ \forall\, x \in U$ where f is holomorphic in U by a similar argument as in example 9.3.3. Since σ is nowhere-vanishing on U, we have

$$s_{|U}(x) = 0 \in L_x \ \Leftrightarrow \ f(x) = 0 \in \mathbb{C}$$
$$\Rightarrow \ (s) \cap U = \big\{\, x \in U \mid s(x) = 0 \,\big\} = \big\{\, x \in U \mid s_{|U}(x) = 0 \,\big\} = \big\{\, x \in U \mid f(x) = 0 \,\big\} = V(f)$$

with $f \in \mathcal{O}_M(U)$, hence the trivializing covering for L defines a covering of M such that (s) is given by the zero set of a holomorphic function on each open set. It follows that (s) is an analytic subset of M.
We also have to show that this description of (s) is independent of the chosen frame, i.e. independent of the trivialization of L. If $\{\tau\}$ is another frame over U, we can write

$$s_{|U}(x) = f(x) \cdot \sigma(x) = g(x) \cdot \tau(x), \ f, g \in \mathcal{O}_M(U), \ \forall\, x \in U$$

and $x \in U$ is a zero of $s \ \Leftrightarrow \ f(x) = 0 = g(x)$. Since $\{\sigma(x)\}$ and $\{\tau(x)\}$ are bases of L_x, there is a basis change matrix $A = A(x) \in \mathrm{GL}(1, \mathbb{C})$ such that $\tau(x) = A(x) \cdot \sigma(x), \forall\, x \in U$. $A(x)$ being invertible for all $x \in U$, meaning that $A(x) \neq 0, \forall\, x \in U$, we get $A \in \mathcal{O}^*(U)$ and it follows that

$$\forall\, x \in U \ : \ s_{|U}(x) = f(x) \cdot \sigma(x) = g(x) \cdot \tau(x) = g(x) \cdot A(x) \cdot \sigma(x) \text{ with } A(x) \neq 0$$

Hence $f(x) = 0 \ \Leftrightarrow \ g(x) = 0$ and $(s) \cap U = V(f) = V(g)$: we get the same local zero set independently of the choice of the frame. ☐

Remark :
Note that there exist line bundles which do not admit non-zero global holomorphic sections.

10.5.6 Proposition

Let $\pi : L \to M$ be a line bundle over M and $s \in \mathcal{L}(M)$ be a global holomorphic section such that $s \not\equiv 0$. Then

$$\text{either} \ : \ 1) \ (s) = \emptyset \ \Rightarrow \ L = M \times \mathbb{C} \text{ and } \mathcal{L} \cong \mathcal{O}_M$$
$$\text{or} \ : \ 2) \ (s) \neq \emptyset \ \Rightarrow \ \text{the irreducible components of } (s) \text{ have codimension 1}$$

Hence if (s) is non-empty and irreducible, then it is a connected complex submanifold of codimension 1 after removing the singular points in (s) (see section 8.3.2).

Proof. 1) is just a consequence of proposition 10.5.3 and theorem 9.3.6

2) If $(s) \neq \emptyset$, we already showed that it is analytic, hence closed. If we supposed that it is irreducible, we know by 8.3.2 and theorem 8.2.4 that $(s) \setminus S((s))$ is in addition a connected submanifold of M. Moreover (s) is locally defined by the zero set of 1 holomorphic function f, hence its codimension is 1.

Intuitively this can also be interpreted as follows : in local coordinates $z_1, \ldots, z_n$ where $n = \dim M$, (s) is defined by the condition $f(z_1, \ldots, z_n) = 0$. So we have n coordinates with 1 constraint, which yields $n-1$ degrees of freedom.
In general (s) is neither irreducible nor connected ; it may have different irreducible components, but all components have codimension 1 (after removing singularities) by the previous argument. And then one can define

$$\dim\big((s)\big) := \text{ maximum of the dimensions of the components} = 1$$

□

10.5.7 Sections and divisors

If $\pi : L \to M$ is a line bundle over M and $s \in \mathcal{L}(M)$ a global non-zero holomorphic section, we want to assign a divisor D to the analytic set (s) :

$$D = \sum_Y n_Y \cdot Y$$

where $n_Y \in \mathbb{Z}$, Y are irreducible analytic subsets of M of codimension 1 and the sum is locally finite. In order to consider (s) as a divisor, we set n_Y to be the vanishing order of s along Y. This is possible since L is a line bundle, i.e. the section s can be represented by a frame $\{\sigma\}$ and a holomorphic function $f \in \mathcal{O}_M(U)$ such that $s_{|U}(x) = f(x) \cdot \sigma(x)$, $\forall\, x \in U$ where $U \subseteq M$ is a trivializing open set. Then we set

$$\mathrm{ord}_x(s) := \mathrm{ord}_x(f)\,,\ \forall\, x \in U$$

This is independent of the chosen frame since sections defining frames do not have zeros in the trivializing set :

$$s_{|U}(x) = f(x) \cdot \sigma(x) = g(x) \cdot \tau(x) \ \Rightarrow\ \mathrm{ord}_x(s) = \mathrm{ord}_x(f) = \mathrm{ord}_x(g)$$

where $\{\tau\}$ is another frame over U and $g \in \mathcal{O}_M(U)$. This is why we consider line bundles for this construction.
In the end we have defined a map $\mathcal{L}(M) \to \mathrm{Div}(M)$: $s \mapsto (s)^d := \sum_Y \mathrm{ord}_Y(s) \cdot Y$. Note that Y is in general not given by a point ; this is only the case if $\dim M = 1$.

Chapter 11

Tangent vectors and differentials

11.1 The real picture

Let M be a complex manifold of complex dimension n. For the moment we forget about the complex structure, i.e. we consider M as a real differentiable manifold of real dimension $2n$. For this, we have the coordinates

$$\forall\, k \in \{1, \dots, n\} \ : \ z_k = x_k + i\, y_k \quad \Rightarrow \ (x_1, y_1, \dots, x_n, y_n) \ : \text{ real coordinates}$$

11.1.1 Definitions

We denote by C^∞_M the sheaf of $\mathbb{R}$-valued real differentiable functions on the manifold M. It is a sheaf of rings. A *point derivation* of $C^\infty(M)$ *at* $m \in M$ is a linear map $D_m : C^\infty(M) \to \mathbb{R}$ which satisfies the Leibniz rule :

$$\begin{aligned} \forall\, f, g \in C^\infty(M),\ \forall\, \lambda \in \mathbb{R} \ : \ & D_m(\lambda \cdot f + g) = \lambda \cdot D_m(f) + D_m(g) \\ & D_m(f \cdot g) = D_m(f) \cdot g(m) + f(m) \cdot D_m(g) \end{aligned}$$

A *tangent vector at* m is a point derivation at m, i.e. given $m \in M$, a tangent vector is a derivation rule ∂_m for functions which are locally defined at m (in the germ of m) such that $\partial_m : C^\infty_{M,m} \to \mathbb{R}$ is linear and

$$\forall\, f, g \in C^\infty_{M,m} \ : \ \partial_m(f \cdot g) = (\partial_m f) \cdot g(m) + f(m) \cdot (\partial_m g)$$

The set of all tangent vectors at m is called the *(real) tangent space at* m and is denoted by $\mathbb{T}_m M$. We know that $\mathbb{T}_m M$ is a $2n$-real dimensional vector space and a basis is given by

$$\left\{ \frac{\partial}{\partial x_1}\,|m \,,\, \frac{\partial}{\partial y_1}\,|m \,, \dots ,\, \frac{\partial}{\partial x_n}\,|m \,,\, \frac{\partial}{\partial y_n}\,|m \right\}$$

(partial derivatives evaluated at m). For short, we denote them in the following by $\partial_{x_1|m}$, $\partial_{y_1|m}$, etc.

11.1.2 The tangent bundle

The *(real) tangent bundle* of the manifold M, denoted by $\mathbb{T}_\mathbb{R} M$, is defined as $\mathbb{T}_\mathbb{R} M := \bigsqcup_{m \in M} \mathbb{T}_m M$.

Since $\mathbb{T}_m M \cong \mathbb{R}^{2n}$, the foot-map $\pi : \mathbb{T}_\mathbb{R} M \to M : u = \partial_m \mapsto m$ defines a real vector bundle of rank $2n$ and its fibers are $\pi^{-1}(m) = \mathbb{T}_m M$, $\forall\, m \in M$. Moreover $\mathbb{T}_\mathbb{R} M$ is a $4n$-real dimensional manifold with chart maps

$$\begin{aligned} \psi_\alpha \ & : \pi^{-1}(U_\alpha) \supset \mathbb{T}_m M \xrightarrow{\sim} \varphi_\alpha(U_\alpha) \times \mathbb{R}^{2n} \subset \mathbb{R}^{4n} \\ & u = \partial_m = \textstyle\sum_i v_i\, \partial_{x_i|m} + \sum_j w_j\, \partial_{y_j|m} \longmapsto \big(\varphi_\alpha(m)\,,\, (v_1, w_1, \dots, v_n, w_n)\big) \end{aligned} \tag{11.1}$$

where $(U_\alpha, \varphi_\alpha)_{\alpha \in J}$ is a $2n$-real dimensional atlas of M. A local frame is

$$\Sigma = \{\, \partial_{x_1} \,,\, \partial_{y_1} \,,\, \dots \,,\, \partial_{x_n} \,,\, \partial_{y_n} \,\}$$

where e.g. $\partial_{x_i} : U_\alpha \to \mathbb{T}_\mathbb{R} M_{|U_\alpha} : m \mapsto \partial_{x_i|m}$. Note that partial derivatives only exist in the charts since, strictly-speaking, we do not calculate $\partial_{x_i} f$, but $\partial_{x_i}(f \circ \varphi_\alpha^{-1})$ if $f \in C^\infty_M(U_\alpha)$.

$p \circ \partial_{x_i} = \mathrm{id}_{U_\alpha}$, so ∂_{x_i} and ∂_{y_i} are sections of $\mathbb{T}_\mathbb{R} M$. And it is indeed a frame since

$$\{\partial_{x_1}(m), \ldots, \partial_{y_n}(m)\} = \{\partial_{x_1|m}, \ldots, \partial_{y_n|m}\}$$

is a basis of $\mathbb{T}_m M$ for all $m \in U_\alpha$. By theorem 9.6.2 this shows in addition that a trivializing covering for $\mathbb{T}_\mathbb{R} M$ is indeed given by a coodinate covering of M : $\mathbb{T}_\mathbb{R} M_{|U_\alpha} \cong U_\alpha \times \mathbb{R}^{2n}$.

Finally we want to compute the cocycles defining the tangent bundle. For this, consider the trivialization

$$\mathbb{T}_\mathbb{R} M_{|U_\alpha} \cong U_\alpha \times \mathbb{R}^{2n} \quad \text{via} \quad \phi_\alpha \ : \ u \mapsto \big(m, (v_1, w_1, \ldots, v_n, w_n)\big)$$

where m is the unique $m \in U_\alpha$ such that $u \in \mathbb{T}_m M$ and $(v_1, w_1, \ldots, v_n, w_n) \in \mathbb{R}^{2n}$ are the coefficients of the decomposition of u in the basis of $\mathbb{T}_m M$ as in (11.1).
Let now U_α and U_β be 2 chart domains with non-empty intersection and associated charts φ_α, φ_β and the respective coordinates $(x_1', y_1', \ldots, x_n', y_n')$ and $(x_1, y_1, \ldots, x_n, y_n)$. If $\big(m, (v_1, w_1, \ldots, v_n, w_n)\big) \in U_\beta \times \mathbb{R}^{2n}$, then

$$\phi_\alpha\Big(\phi_\beta^{-1}\big(m, (v_1, w_1, \ldots, v_n, w_n)\big)\Big) = \phi_\alpha\Big(\sum_{i=1}^n v_i \cdot \partial_{x_i|m} + \sum_{j=1}^n w_j \cdot \partial_{y_j|m}\Big)$$
$$= \phi_\alpha\Big(\sum_{i=1}^n v_i' \cdot \partial_{x_i'|m} + \sum_{j=1}^n w_j' \cdot \partial_{y_j'|m}\Big) = \big(m, (v_1', w_1', \ldots, v_n', w_n')\big)$$

Thus $g_{U_\alpha U_\beta}(m) : (v_1, w_1, \ldots, v_n, w_n) \mapsto (v_1', w_1', \ldots, v_n', w_n')$ and the relation between these coefficients is given by the basis change $\{\partial_{x_i|m}, \partial_{y_j|m}\} \to \{\partial_{x_i'|m}, \partial_{y_j'|m}\}$ in the tangent space $\mathbb{T}_m M$. But we have the formula :

$$\nu^j = \sum_{i=1}^n J_{ji} \cdot \mu^i \quad \text{where } J_{ji} = \partial_{u^i}(v^j)_{|m} \ : \ \text{Jacobian matrix}$$

when changing the local coordinates $\{u^i\} \to \{v^j\}$ and the coefficients vary $\{\mu^i\} \to \{\nu^j\}$, i.e. as vectors $\nu = J \cdot \mu$. The same happens here by changing the coordinates $\{x_1, y_1, \ldots, x_n, y_n\} \to \{x_1', y_1', \ldots, x_n', y_n'\}$, hence

$$g_{U_\alpha U_\beta}(m) \ : \ {}^t(v_1, w_1, \ldots, v_n, w_n) \longmapsto {}^t(v_1', w_1', \ldots, v_n', w_n') = \left(\frac{\partial \psi}{\partial(x, y)}\right) \cdot {}^t(v_1, w_1, \ldots, v_n, w_n)$$

where $\psi = \varphi_\alpha \circ \varphi_\beta^{-1}$ is the transition function $(x', y') = \psi(x, y)$. The cocycle $g_{U_\alpha U_\beta}$ is therefore given by the $2n \times 2n$–Jacobian matrix of the transition map ψ. The coordinate change ψ being bijective, the determinant of the Jacobian matrix is in addition non-zero.

11.1.3 Vector fields

A *local vector field* on M is a local section of the tangent bundle $\mathbb{T}_\mathbb{R} M$. We denote the sheaf of local sections of $\mathbb{T}_\mathbb{R} M$ (i.e. the sheaf of local vector fields on M) by $\mathcal{T}_\mathbb{R}$. Hence X is a vector field on $U \Leftrightarrow X \in \mathcal{T}_\mathbb{R}(U)$. As sheaf of local sections, $\mathcal{T}_\mathbb{R}$ is thus a locally free sheaf of $\mathcal{O}_M$–modules of rank $2n$.
In a coordinate neighborhood U_α, a local vector field $X \in \mathcal{T}_\mathbb{R}(U_\alpha)$ (a local section) hence writes uniquely as

$$X = \sum_{i=1}^n f_i \cdot \partial_{x_i} + \sum_{j=1}^n g_j \cdot \partial_{y_j} \quad \text{for some } f_i, g_j \in C_M^\infty(U_\alpha) \text{ as in (9.11)}$$

A *global vector field* on M is a differentiable map $X : M \to \mathbb{T}_\mathbb{R} M$ such that $\forall\, m \in M$, $X(m) \in \mathbb{T}_m M$ and there is an open neighborhood $U \subseteq M$ around m such that $X_{|U}$ is a local vector field as defined above.

11.2 The complex picture

Now we introduce the complex structure into the tangent space and the tangent bundle. Recall that

$$z_k = x_k + i\, y_k \quad \Rightarrow \quad \frac{\partial}{\partial z_k} = \frac{1}{2} \cdot \Big(\frac{\partial}{\partial x_k} - i \cdot \frac{\partial}{\partial y_k}\Big) \ , \ \frac{\partial}{\partial \bar{z}_k} = \frac{1}{2} \cdot \Big(\frac{\partial}{\partial x_k} + i \cdot \frac{\partial}{\partial y_k}\Big) \ , \ \forall\, k \in \{1, \ldots, n\} \tag{11.2}$$

This transformation is biholomorphic and hence defines a holomorphic change of coordinates on M.

11.2.1 The complex tangent bundle

Let $n = \dim_{\mathbb{C}} M$. We define the *complex tangent space at* $m \in M$ as $\mathbb{T}_{\mathbb{C},m}M := \mathbb{T}_m M \otimes \mathbb{C}$. Intuitively this just means that we now also allow complex-valued functions. It is indeed the complexified version of the real tangent space. The dimension of this space now strongly depends on whether we consider it as a real or a complex vector space. First of all : $\dim_{\mathbb{R}}(\mathbb{T}_m M) = \dim_{\mathbb{C}}(\mathbb{T}_m M) = 2n$.

Proof. We know that $\dim_{\mathbb{R}}(\mathbb{T}_m M) = 2n$ and a basis over $\mathbb{R}$ is given by $\{\partial_{x_1|m}, \partial_{y_1|m}, \dots, \partial_{x_n|m}, \partial_{y_n|m}\}$. Hence $\mathbb{T}_m M$ only contains "real values" since any $u \in \mathbb{T}_m M$ writes as a real linear combination of these basis vectors. Now we want to find a basis of $\mathbb{T}_m M$ over $\mathbb{C}$. And it is not given by $\{\partial_{z_1|m}, \dots, \partial_{z_n|m}\}$ since even complex linear combinations of the $\partial_{z_i|m}$ do not generate all the real partial derivatives $\partial_{x_i|m}$ and $\partial_{y_j|m}$. In order to satisfy (11.2), we need to add at least all the $\partial_{\bar{z}_j|m}$. This is also sufficient, hence a basis of $\mathbb{T}_m M$ over $\mathbb{C}$ is given by $\{\partial_{z_1|m}, \dots, \partial_{z_n|m}, \partial_{\bar{z}_1|m}, \dots, \partial_{\bar{z}_n|m}\} \;\Rightarrow\; \dim_{\mathbb{C}}(\mathbb{T}_m M) = 2n$ as well. □

Note that this is not the only possibility : $\{\partial_{x_1|m}, \partial_{y_1|m}, \dots, \partial_{x_n|m}, \partial_{y_n|m}\}$ of course also defines a $\mathbb{C}$-basis of $\mathbb{T}_m M$ (since it is already a basis over $\mathbb{R}$). The converse however is not true! Inverting relation (11.2) yields

$$\forall\, k \in \{1, \dots, n\} \;:\; \frac{\partial}{\partial x_k} = \frac{\partial}{\partial z_k} + \frac{\partial}{\partial \bar{z}_k} \quad , \quad \frac{\partial}{\partial y_k} = i \cdot \Big(\frac{\partial}{\partial z_k} - \frac{\partial}{\partial \bar{z}_k}\Big) \tag{11.3}$$

Since only real linear combinations are allowed it is not possible to generate all the $\partial_{x_i|m}$ and $\partial_{y_j|m}$ only with the set $\{\partial_{z_1|m}, \dots, \partial_{z_n|m}, \partial_{\bar{z}_1|m}, \dots, \partial_{\bar{z}_n|m}\}$ which is therefore not an $\mathbb{R}$-basis of $\mathbb{T}_m M$.

The complex tangent space is obtained by tensorizing the real tangent space with $\mathbb{C}$, hence

$$\begin{aligned}
\dim_{\mathbb{C}}\big(\mathbb{T}_{\mathbb{C},m}M\big) &= \dim_{\mathbb{C}}\big(\mathbb{T}_m M \otimes \mathbb{C}\big) = \dim_{\mathbb{C}}(\mathbb{T}_m M) \cdot \dim_{\mathbb{C}}(\mathbb{C}) = 2n \cdot 1 = 2n \\
\dim_{\mathbb{R}}\big(\mathbb{T}_{\mathbb{C},m}M\big) &= \dim_{\mathbb{R}}\big(\mathbb{T}_m M \otimes \mathbb{C}\big) = \dim_{\mathbb{R}}(\mathbb{T}_m M) \cdot \dim_{\mathbb{R}}(\mathbb{C}) = 2n \cdot 2 = 4n
\end{aligned}$$

The occurrence of 4 can be interpreted as "dividing" a complex $\mathbb{C}$-valued function into real and imaginary part. By the tensor property we moreover conclude that bases of $\mathbb{T}_{\mathbb{C},m}M$ are

$$\begin{aligned}
&\text{over } \mathbb{C} \;:\; \big\{\partial_{z_1|m}, \dots, \partial_{z_n|m}, \partial_{\bar{z}_1|m}, \dots, \partial_{\bar{z}_n|m}\big\} \\
&\text{over } \mathbb{R} \;:\; \big\{\partial_{x_1|m}\,,\, \partial_{y_1|m}\,, \dots,\, \partial_{x_n|m}\,,\, \partial_{y_n|m}\,,\, i \cdot \partial_{x_1|m}\,,\, i \cdot \partial_{y_1|m}\,, \dots,\, i \cdot \partial_{x_n|m}\,,\, i \cdot \partial_{y_n|m}\big\}
\end{aligned}$$

Now the *complex tangent bundle* is defined as $\mathbb{T}_{\mathbb{C}}M := \bigsqcup_{m \in M} \mathbb{T}_{\mathbb{C},m}M$.
As in the real case, we obtain that $\pi : \mathbb{T}_{\mathbb{C}}M \to M$ is a complex vector bundle over M of complex rank $2n$ and real rank $4n$. But : it is NOT a holomorphic vector bundle! This happens because the second part of the basis

$$\big\{\partial_{z_1|m}, \dots, \partial_{z_n|m}, \partial_{\bar{z}_1|m}, \dots, \partial_{\bar{z}_n|m}\big\}$$

of $\mathbb{T}_{\mathbb{C},m}M$ is not holomorphic ($\bar{z}_k$ is involved).
More precisely, if U is a chart domain with local coordinates $(z_1, \dots, z_n)$, then a frame for $\mathbb{T}_{\mathbb{C}}M_{|U}$ is

$$\big\{\partial_{z_1}, \dots, \partial_{z_n}, \partial_{\bar{z}_1}, \dots, \partial_{\bar{z}_n}\big\} \cong \big\{\partial_{x_1}, \partial_{y_1}, \dots, \partial_{x_n}, \partial_{y_n}\big\}$$

One can however define the *holomorphic tangent bundle* $\mathbb{T}_{\text{hol}}M$, which is of rank n and where a local frame is given by $\{\partial_{z_1}, \dots, \partial_{z_n}\}$.

11.2.2 Complex vector fields

A *local complex vector field* X is a local section of the bundle $\mathbb{T}_{\mathbb{C}}M$ and is hence locally of the form

$$X = \sum_{i=1}^{n} \gamma_i \cdot \partial_{z_i} + \sum_{j=1}^{n} \varepsilon_j \cdot \partial_{\bar{z}_j} \tag{11.4}$$

for some $\mathbb{C}$-valued real differentiable functions γ_i and ε_j. But they are not necessarily holomorphic since $\mathbb{T}_{\mathbb{C}}M$ is not a holomorphic vector bundle, hence local sections do not need to be holomorphic neither.

We denote the sheaf of local complex vector fields by $\mathcal{T}_{\mathbb{C}}$. Such a vector field on an open set $U \subseteq M$ is of

- the type $(1,0)$ if $\varepsilon_j = 0, \forall j \in \{1,\dots,n\}$ (only derivatives in the holomorphic directions).
- the type $(0,1)$ if $\gamma_i = 0, \forall i \in \{1,\dots,n\}$ (only derivatives in the anti-holomorphic directions).
- *holomorphic type* if it is of the type $(1,0)$ and the γ_i are holomorphic in U, i.e. $\frac{\partial \gamma_i}{\partial \bar{z}_k} = 0, \forall k,i \in \{1,\dots,n\}$.

A *global complex vector field* on M is a differentiable map $X : M \to \mathbb{T}_{\mathbb{C}}M$ such that $\forall m \in M$, $X(m) \in \mathbb{T}_{\mathbb{C},m}M$ and there is an open neighborhood $U \subseteq M$ around m such that $X_{|U}$ is a local complex vector field as in (11.4). And a *holomorphic vector field* is a global complex vector field that it locally of holomorphic type.

As usual, we want that vector fields do not depend on the chosen coordinates. But this is satisfied here because M is a complex manifold, i.e. all coordinate changes $w = \psi(z)$ are holomorphic :

$$\left(\frac{\partial w}{\partial \bar{z}}\right) = 0 \quad \Rightarrow \quad J(\psi) = \begin{pmatrix} \frac{\partial w}{\partial z} & \frac{\partial w}{\partial \bar{z}} \\ \frac{\partial \bar{w}}{\partial z} & \frac{\partial \bar{w}}{\partial \bar{z}} \end{pmatrix} = \begin{pmatrix} \frac{\partial w}{\partial z} & 0 \\ 0 & \frac{\partial \bar{w}}{\partial \bar{z}} \end{pmatrix} \;:\; 2n \times 2n\text{–matrix}$$

Hence when changing coordinates, permutations can only happen within the individual parts of X in (11.4). It follows that the type of a vector field is independent under holomorphic coordinate transformations.
Note that this does not hold for real coordinate changes.

11.3 Differential forms

11.3.1 The cotangent bundle

The *cotangent bundle* is the dual bundle of the tangent bundle. Here again we can define the real, the complex and the holomorphic cotangent bundle. We quickly go through some constructions :

Consider the dual space $\mathbb{T}^*_m M$, which is again of real and complex dimension $2n$. A basis (over $\mathbb{R}$ and $\mathbb{C}$) is given by $\{dx_{1|m}, dy_{1|m}, \dots, dx_{n|m}, dy_{n|m}\}$, whereas $\{dz_{1|m}, \dots, dz_{n|m}, d\bar{z}_{1|m}, \dots, , d\bar{z}_{n|m}\}$ is a $\mathbb{C}$-basis only. We recall that $dx_{i|m}$ is the dual element associated to $\partial_{x_i|m}$, i.e.

$$dx_{i|m}\big(\partial_{x_j|m}\big) = dy_{i|m}\big(\partial_{y_j|m}\big) = dz_{i|m}\big(\partial_{z_j|m}\big) = d\bar{z}_{i|m}\big(\partial_{\bar{z}_j|m}\big) = \delta_{ij}$$

Similarly the dual space $\mathbb{T}^*_{\mathbb{C},m}M$ is of complex dimension $2n$ and admits the same $\mathbb{C}$-basis. Then we set

$$\mathbb{T}^*_{\mathbb{C}}M := \big(\mathbb{T}_{\mathbb{C}}M\big)^* = \bigsqcup_{m \in M} \mathbb{T}^*_{\mathbb{C},m}M$$

and it follows that a local frame over a chart domain $U_\alpha \subseteq M$ is given by $\{dz_1, \dots, dz_n, d\bar{z}_1, \dots, , d\bar{z}_n\}$, where

$$dz_i \;:\; U_\alpha \longrightarrow \mathbb{T}^*_{\mathbb{C}}M_{|U_\alpha} \;:\; m \longmapsto dz_{i|m}$$

This is again not a holomorphic vector bundle, but one can define the *holomorphic cotangent bundle* $\mathbb{T}^*_{\text{hol}}M$, which is of rank n and where a local frame is given by $\{dz_1, \dots, dz_n\}$.

Finally one can also consider the exterior powers $\Lambda^r\big(\mathbb{T}^*_{\mathbb{C}}M\big)$ for $r \in \{0,1,\dots,2n\}$ which have the local frame

$$\big\{dz_{i_1} \wedge \ldots \wedge dz_{i_s} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_t}\big\}_{s+t=r}$$

11.3.2 Definitions

We denote by $\mathcal{E}$ the sheaf of $\mathbb{C}$-valued functions on M which are differentiable with respect to the real structure. Let $V \subseteq M$ be open. A *1-differential* on V is a differentiable map $\omega : V \to \mathbb{T}^*_{\mathbb{C}}M$ such that $\forall m \in V$, $\omega(m) \in \mathbb{T}^*_{\mathbb{C},m}M$ and there is an open neighborhood $U \subseteq V$ of m such that $\omega_{|U}$ defines a local section of the cotangent bundle over U. Hence a 1-differential is locally of the form

$$\omega_{|U} = \sum_{i=1}^n a_i \cdot dx_i + \sum_{j=1}^n b_j \cdot dy_j = \sum_{i=1}^n \varepsilon_i \cdot dz_i + \sum_{j=1}^n \delta_j \cdot d\bar{z}_j$$

where $a_i, b_j, \varepsilon_i, \delta_j \in \mathcal{E}(U)$ and by using the decomposition $z_k = x_k + i\, y_k \;\Rightarrow\; dz_k = dx_k + i\, dy_k, \forall k \in \{1,\dots,n\}$.

The sheaf of *1-differentials* is denoted by $\mathcal{E}^1$. Moreover we set :

- $\omega \in \mathcal{E}^{(1,0)}(V) \;\Leftrightarrow\; \delta_j = 0, \forall\, j \in \{1,\dots,n\}$, i.e. $\omega_{|U} = \sum_i \varepsilon_i \cdot dz_i$ for $\varepsilon_i \in \mathcal{E}(U)$
- $\omega \in \mathcal{E}^{(0,1)}(V) \;\Leftrightarrow\; \varepsilon_i = 0, \forall\, i \in \{1,\dots,n\}$, i.e. $\omega_{|U} = \sum_j \delta_j \cdot d\bar{z}_j$ for $\delta_j \in \mathcal{E}(U)$

The type of these 1-differentials is respected by holomorphic coordinate changes, but not by real ones. We have

$$\mathcal{E}^1 = \mathcal{E}^{(1,0)} \oplus \mathcal{E}^{(0,1)}$$

i.e. any element $\omega \in \mathcal{E}^1(V)$ can uniquely be decomposed as $\omega = \omega_1 + \omega_2$ with $\omega_1 \in \mathcal{E}^{(1,0)}(V)$, $\omega_2 \in \mathcal{E}^{(0,1)}(V)$ and this decomposition is independent of the chosen coordinates.
Note that $\mathcal{E}^1$, $\mathcal{E}^{(1,0)}$ and $\mathcal{E}^{(0,1)}$ are locally free sheaves of $\mathcal{E}$–modules of rank $2n$, n, n respectively and hence correspond to the sheaf of sections of a certain **real** vector bundle (for complex manifolds, we need $\mathcal{O}_M$–modules).

The sheaf of *holomorphic 1-differentials* is denoted by Ω^1_M and it consists of 1-differentials which are locally over some open set $U \subseteq M$ of the form

$$\omega_{|U} = \sum_{i=1}^n \gamma_i \cdot dz_i \qquad \text{with } \gamma_i \in \mathcal{O}_M(U)$$

i.e. the coefficient functions γ_i have to be holomorphic in U (not only differentiable).
Ω^1_M is not a sheaf of $\mathcal{E}$–modules since $f \cdot \gamma$ is not necessarily holomorphic if $f \in \mathcal{E}(U)$ and $\gamma \in \mathcal{O}_M(U)$. But it is a locally free sheaf of $\mathcal{O}_M$–modules of rank n and hence corresponds to the sheaf of sections of some **holomorphic** vector bundle of rank n. And this is not the cotangent bundle $\mathbb{T}^*_{\mathbb{C}}M$, but the holomorphic bundle $\mathbb{T}^*_{\text{hol}}M$.

11.3.3 Concept of a differential : real picture

Let M be a complex manifold of complex dimension n with complex coordinates $(z_1,\dots,z_n)$ and real coordinates $\{x_1,y_1,\dots,x_n,y_n\}$. Let also $V \subseteq M$ be open and $f \in \mathcal{E}(V)$ be a real differentiable function on V. To f we want to associate a 1-differential, denoted by df and called the *differential* of f.
Let $\{U_\alpha\}_{\alpha\in J}$ be a coordinate covering of M. Then df is defined by the local condition

$$\forall\, \alpha \in J \;:\; df_{|U_\alpha \cap V} = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\, dx_i + \sum_{j=1}^n \frac{\partial f}{\partial y_j}\, dy_j$$

where $\{\partial_{x_1}, \partial_{y_1}, \dots, \partial_{x_n}, \partial_{y_n}\}$ and $\{dx_1, dy_1, \dots, dx_n, dy_n\}$ are frames over U_α of the tangent and cotangent bundle respectively. Thus df is not a function any more : $df \notin \mathcal{E}(V)$, but $df \in \mathcal{E}^1(V)$.

Properties :
1) df is well-defined, i.e. it is independent of the chosen local coordinates.
2) The differential satisfies the Leibniz rule : $\forall\, f,g \in \mathcal{E}(V)$, $d(f\cdot g) = df \cdot g + f \cdot dg$.

Proof. 2) is clear since partial derivatives are linear and satisfy the Leibniz rule
1) For simplicity, assume that we are given a real differentiable manifold of real dimension m with a coordinate covering $\{U_\alpha\}_{\alpha\in J}$. Choose U_α and U_β such that $U_\alpha \cap U_\beta \neq \emptyset$ with local coordinates $x = (x_1,\dots,x_m)$ and $y = (y_1,\dots,y_m)$ respectively, related by the differentiable coordinate change $y = \psi(x)$ with invertible Jacobian matrix $J(\psi) = (J_{ij})$. Let also $f \in \mathcal{E}(V)$ for some open set V. We have to show that locally

$$\sum_{i=1}^m \frac{\partial f}{\partial x_i}\, dx_i = \sum_{j=1}^m \frac{\partial f}{\partial y_j}\, dy_j$$

Given the respective covariant and contravariant transformation laws, we obtain :

$$\frac{\partial}{\partial x_i} = \sum_j J_{ji} \cdot \frac{\partial}{\partial y_j} = \sum_j \frac{\partial y_j}{\partial x_i} \cdot \frac{\partial}{\partial y_j} \qquad , \qquad dx_i = \sum_k (J^{-1})_{ik}\, dy_k = \sum_k \frac{\partial x_i}{\partial y_k}\, dy_k$$

$$\Rightarrow \sum_i \frac{\partial f}{\partial x_i}\, dx_i = \sum_{i,j,k} \frac{\partial y_j}{\partial x_i} \cdot \frac{\partial f}{\partial y_j} \cdot \frac{\partial x_i}{\partial y_k}\, dy_k = \sum_{k,j,i} \Big(\frac{\partial y_j}{\partial x_i} \cdot \frac{\partial x_i}{\partial y_k}\Big) \cdot \frac{\partial f}{\partial y_j}\, dy_k$$

$$= \sum_{k,j} \Big(\sum_i J_{ji} \cdot (J^{-1})_{ik}\Big) \cdot \frac{\partial f}{\partial y_j}\, dy_k = \sum_{k,j} \delta_{jk} \cdot \frac{\partial f}{\partial y_j}\, dy_k = \sum_j \frac{\partial f}{\partial y_j}\, dy_j$$

□

11.3.4 The complex picture

Under the same assumptions as in 11.3.3, let $f \in \mathcal{E}(V)$ be a $\mathbb{C}$-valued differentiable function on V. We set locally

$$df := \sum_{i=1}^{n} \frac{\partial f}{\partial z_i}\, dz_i + \sum_{i=1}^{n} \frac{\partial f}{\partial \bar{z}_i}\, d\bar{z}_i =: \partial f + \bar{\partial} f$$

Hence $d = \partial + \bar{\partial}$ where ∂ consists of the derivatives in the holomorphic direction and $\bar{\partial}$ contains all derivatives in the anti-holomorphic direction. Note that ∂ and $\bar{\partial}$ are well-defined (i.e. independent of the chosen coordinates) because we already showed in 11.2.2 that holomorphic coordinate changes do not mix the holomorphic and anti-holomorphic parts. For $V \subseteq M$ open we thus constructed the maps

$$d \;:\; \mathcal{E}(V) \longrightarrow \mathcal{E}^1(V) \qquad , \qquad \partial \;:\; \mathcal{E}(V) \longrightarrow \mathcal{E}^{(1,0)}(V) \qquad , \qquad \bar{\partial} \;:\; \mathcal{E}(V) \longrightarrow \mathcal{E}^{(0,1)}(V)$$

We see that $f \in \mathcal{E}(V)$ is holomorphic in $V \;\Leftrightarrow\; \bar{\partial} f = 0$.
This is the case because $\bar{\partial} f = 0$ on $V \;\Rightarrow\; \bar{\partial} f_{|U} = 0$ for any open set $U \subseteq V$, hence if U is small enough :

$$0 = \bar{\partial} f_{|U} = \sum_i \frac{\partial f}{\partial \bar{z}_i}\, d\bar{z}_i \quad\Rightarrow\quad \sum_i \frac{\partial f}{\partial \bar{z}_i}(m) \cdot d\bar{z}_i(m) = \sum_i \frac{\partial f}{\partial \bar{z}_i}(m) \cdot d\bar{z}_{i|m} = 0,\ \forall\, m \in U$$

$$\Rightarrow\ \frac{\partial f}{\partial \bar{z}_i}(m) = 0,\ \forall\, m \in U,\ \forall\, i \in \{1, \ldots, n\} \text{ since } d\bar{z}_{1|m}, \ldots, d\bar{z}_{n|m} \text{ are linearly independent}$$

$$\Rightarrow\ \frac{\partial f}{\partial \bar{z}_i} = 0 \text{ in } U,\ \forall\, i \in \{1, \ldots, n\} \;\Rightarrow\; f \text{ is holomorphic in } U$$

which means that f is holomorphic in a small neighborhood of any point in V, i.e. f is holomorphic in V.
Hence $\mathcal{O}_M = \ker \bar{\partial}$ and this equality even holds in the sheaf-theoretical sense : $\mathcal{O}_M(V) = \ker \bar{\partial}(V)$ for any open set $V \subseteq M$. Moreover $\ker d$ is given by the sheaf of locally constant functions because (without details) :

$$0 = df = \sum_i \frac{\partial f}{\partial z_i}\, dz_i + \sum_i \frac{\partial f}{\partial \bar{z}_i}\, d\bar{z}_i \overset{\text{basis}}{\Rightarrow} \frac{\partial f}{\partial z_i} = \frac{\partial f}{\partial \bar{z}_i} = 0,\ \forall\, i \in \{1, \ldots, n\}$$

In particular : $\ker d(V) = \mathbb{C}$ if V is a connected open set.
Finally one can also consider d, ∂ and $\bar{\partial}$ as morphisms of sheaves, i.e. we have the sequences of sheaves

$$\mathcal{E} \xrightarrow{d} \mathcal{E}^1 \qquad , \qquad \mathcal{E} \xrightarrow{\partial} \mathcal{E}^{(1,0)} \qquad , \qquad \mathcal{E} \xrightarrow{\bar{\partial}} \mathcal{E}^{(0,1)}$$

The idea is now to continue these sequences to higher orders. In particular we are interested in the questions :
− Does any 1-differential come from a function, i.e. is d surjective ?
− How to integrate a differential form ?

$$\mathcal{E} \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \ldots \qquad , \qquad \mathcal{E} \xrightarrow{\partial} \mathcal{E}^{(1,0)} \xrightarrow{\partial} \ldots$$

11.3.5 Differentials of higher order

We recall that if W is an n-dimensional vector space, we can define its dual space W^*, consisting of all linear forms on W, and then the exterior power $\Lambda^p(W^*)$, consisting of all p-linear alternating maps on $W \times \ldots \times W$. In the case of a complex manifold M we hence obtain the tangent bundle $\mathbb{T}_{\mathbb{C}} M$, the cotangent bundle $\mathbb{T}^*_{\mathbb{C}} M$ and the exterior power bundle $\Lambda^p(\mathbb{T}^*_{\mathbb{C}} M)$, all of them not being a holomorphic vector bundle. If U is a coordinate neighborhood with local coordinates $(z_1, \ldots, z_n) = (x_1, y_1, \ldots, x_n, y_n)$, a frame over U of these bundles is

$$\begin{aligned} \mathbb{T}_{\mathbb{C}} M \;&:\; \left\{\partial_{x_1}, \partial_{y_1}, \ldots, \partial_{x_n}, \partial_{y_n}\right\} \cong \left\{\partial_{z_1}, \ldots, \partial_{z_n}, \partial_{\bar{z}_1}, \ldots, \partial_{\bar{z}_n}\right\} \\ \mathbb{T}^*_{\mathbb{C}} M \;&:\; \left\{dx_1, dy_1, \ldots, dx_n, dy_n\right\} \cong \left\{dz_1, \ldots, dz_n, d\bar{z}_1, \ldots, d\bar{z}_n\right\} \\ \Lambda^p(\mathbb{T}^*_{\mathbb{C}} M) \;&:\; \left\{dx_{i_1} \wedge \ldots \wedge dx_{i_s} \wedge dy_{j_1} \wedge \ldots \wedge dy_{j_t}\right\} \cong \left\{dz_{i_1} \wedge \ldots \wedge dz_{i_s} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_t}\right\} \end{aligned}$$

for $s + t = p$, $1 \leq i_1 < \ldots < i_s \leq n$, $1 \leq j_1 < \ldots < j_t \leq n$.
We already know that local vector fields and 1-differentials are local sections of the tangent bundle and the cotangent bundle respectively, the associated sheaves being denoted by $\mathcal{T}_{\mathbb{C}}$ and $\mathcal{E}^1$.

Let $V \subseteq M$ be open. A *p-differential* (or *differential p-form*) on V is a differentiable map $\omega : M \to \Lambda^p(\mathbb{T}^*_{\mathbb{C}}M)$ such that $\forall\, m \in V$, $\omega(m) \in \Lambda^p(\mathbb{T}^*_{\mathbb{C},m}M)$ and there is an open neighborhood $U \subseteq V$ of m such that $\omega_{|U}$ is a local section of $\Lambda^p(\mathbb{T}^*_{\mathbb{C}}M)$ over U. As a local section it is thus locally given of the form

$$\omega_{|U} = \sum_I \alpha_I\ dx_{i_1} \wedge \ldots \wedge dx_{i_s} \wedge dy_{j_1} \wedge \ldots \wedge dy_{j_t} = \sum_I \beta_I\ dz_{i_1} \wedge \ldots \wedge dz_{i_s} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_t} \tag{11.5}$$

where $\alpha_I, \beta_I \in \mathcal{E}(U)$ and we denoted for short

$$I = \left\{ (i_1, \ldots, i_s, j_1, \ldots, j_t) \mid s + t = p,\ 1 \leq i_1 < \ldots < i_s \leq n,\ 1 \leq j_1 < \ldots < j_t \leq n \right\}$$

Since the individual parts dx_i, dy_j, dz_i and $d\bar{z}_j$ are 1-differentials and the wedge product is anti-commutative, we obtain that

$$dx_i \wedge dx_i = dy_j \wedge dy_j = dz_i \wedge dz_i = d\bar{z}_j \wedge d\bar{z}_j = 0,\ \forall\, i, j \in \{1, \ldots, n\}$$
$$dx_i \wedge dx_j = -dx_j \wedge dx_i\ ,\ dy_i \wedge dy_j = -dy_j \wedge dy_i\ ,\ dz_i \wedge dz_j = -dz_j \wedge dz_i\ ,\ d\bar{z}_i \wedge d\bar{z}_j = -d\bar{z}_j \wedge d\bar{z}_i$$

Moreover the descriptions with real coordinates x_i, y_j and complex coordinates $z_i, \bar{z}_j$ are equivalent (but only over $\mathbb{C}$!), so both notations can be used without risk of confusion. The relation between both of them is :

$$dz \wedge d\bar{z} = (dx + i\, dy) \wedge (dx - i\, dy) = dx \wedge dx - i\, dx \wedge dy + i\, dy \wedge dx + dy \wedge dy = -2i\, dx \wedge dy$$

We can even see the change $(x_1, y_1, \ldots, x_n, y_n) \leftrightarrow (z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n)$ as a usual coordinate change on M since they are related by (11.2) and (11.3), which define a differentiable (but not a holomorphic) change of coordinates.

The *sheaf of p-differentials* (local sections) is denoted by $\mathcal{E}^p$. Moreover if such a form has s holomorphic differentials dz_i and t anti-holomorphic differentials $d\bar{z}_j$ as in the second part of (11.5), then we write $\mathcal{E}^{s,t}$. Hence

$$\mathcal{E}^p = \bigoplus_{s+t=p} \mathcal{E}^{s,t}$$

$\mathcal{E}^p$ is given by all the possibilities to decompose $p \in \{0, \ldots, 2n\}$ into s holomorphic and t anti-holomorphic parts. As an example consider $\mathcal{E}^2 = \mathcal{E}^{2,0} \oplus \mathcal{E}^{1,1} \oplus \mathcal{E}^{0,2}$, so all possible complex types of 2-differentials locally write as

$$\sum_{1 \leq i_1 < i_2 \leq n} \left(\alpha_{i_1 i_2}\ dz_{i_1} \wedge dz_{i_2} \right) + \sum_{i=1}^n \sum_{j=1}^n \left(\beta_{ij}\ dz_i \wedge d\bar{z}_j \right) + \sum_{1 \leq i_1 < i_2 \leq n} \left(\gamma_{i_1 i_2}\ d\bar{z}_{i_1} \wedge d\bar{z}_{i_2} \right)$$

for some locally differentiable functions $\alpha_{i_1 i_2}, \beta_{ij}, \gamma_{i_1 i_2}$.

11.3.6 The exterior derivative

The goal is now to extend the definition of the differential of a function to all differential forms. Let $\omega \in \mathcal{E}^p(V)$. Then locally over U, ω looks like $\sum_I \alpha_I\, dx_I = \sum_I \beta_I\, dz_I$ for $\alpha_I, \beta_I \in \mathcal{E}(U)$. Here we used the short-hand notation

$$dx_I := dx_{i_1} \wedge \ldots \wedge dx_{i_s} \wedge dy_{j_1} \wedge \ldots \wedge dy_{j_t} \qquad , \qquad dz_I := dz_{i_1} \wedge \ldots \wedge dz_{i_s} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_t}$$

Then we define $d\omega \in \mathcal{E}^{p+1}(V)$ by the local condition (over some small open set $U \subseteq V$)

$$d\omega_{|U} := \sum_{i=1}^n \sum_I \frac{\partial \alpha_I}{\partial x_i}\ dx_i \wedge dx_I + \sum_{i=1}^n \sum_I \frac{\partial \alpha_I}{\partial y_i}\ dy_i \wedge dx_I = \sum_{i=1}^n \sum_I \frac{\partial \beta_I}{\partial z_i}\ dz_i \wedge dz_I + \sum_{i=1}^n \sum_I \frac{\partial \beta_I}{\partial \bar{z}_i}\ d\bar{z}_i \wedge dz_I \tag{11.6}$$

One can show that this is always well-defined (not only for holomorphic changes) and independent of the chosen local coordinates, in particular the 2 above definitions coincide. Hence a global definition for $d\omega$ is also given. The map $d : \mathcal{E}^p(V) \to \mathcal{E}^{p+1}(V) : \omega \mapsto d\omega$ is called the *exterior derivative* and is a generalization of df.

Properties :
1) The map d is linear and satisfies $d \circ d = 0$.
2) $\forall\, \omega \in \mathcal{E}^p(V), \eta \in \mathcal{E}^q(V) :\ d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \cdot \omega \wedge d\eta$.

Proof. 2) is admitted
1) We only show how the computation is done in the sum involving dx_i ; all other terms are similar. Locally :

$$\begin{aligned} d(d\omega) &= d\Big(\sum_i \sum_I \frac{\partial \alpha_I}{\partial x_i}\, dx_i \wedge dx_I\Big) = \sum_{i,j} \sum_I \frac{\partial^2 \alpha_I}{\partial x_i \partial x_j}\, dx_j \wedge dx_i \wedge dx_I \\ &= \sum_I \Big(\sum_{i\neq j} \frac{\partial^2 \alpha_I}{\partial x_i \partial x_j}\, dx_j \wedge dx_i \wedge dx_I\Big) \quad \text{since } dx_i \wedge dx_i = 0 \\ &= \sum_I \Big(\sum_{i<j} \frac{\partial^2 \alpha_I}{\partial x_i \partial x_j}\, dx_j \wedge dx_i \wedge dx_I + \sum_{i>j} \frac{\partial^2 \alpha_I}{\partial x_i \partial x_j}\, dx_j \wedge dx_i \wedge dx_I\Big) \\ &= \sum_I \sum_{i<j} \Big(\frac{\partial^2 \alpha_I}{\partial x_i \partial x_j}\, dx_j \wedge dx_i \wedge dx_I + \frac{\partial^2 \alpha_I}{\partial x_j \partial x_i}\, dx_i \wedge dx_j \wedge dx_I\Big) \\ &= \sum_I \sum_{i<j} \Big(\frac{\partial^2 \alpha_I}{\partial x_i \partial x_j}\, dx_j \wedge dx_i \wedge dx_I - \frac{\partial^2 \alpha_I}{\partial x_i \partial x_j}\, dx_j \wedge dx_i \wedge dx_I\Big) = 0 \end{aligned}$$

where the last equality follows from Schwartz' Theorem (partial derivatives commute) and anti-commutativity of the wedge product. □

11.4 De Rham and Dolbeault cohomology

11.4.1 Definitions

Let M be a complex manifold of complex dimension n and $V \subseteq M$ be open. The vector space of all *differential forms* on V is given by

$$E(V) := \bigoplus_{p=0}^{2n} \mathcal{E}^p(V)$$

hence any $\omega \in E(V)$ writes uniquely as $\omega = \omega_0 + \omega_1 + \ldots + \omega_{2n}$ with $\omega_p \in \mathcal{E}^p(V)$, $\forall\, p \in \{0, \ldots, 2n\}$. Note that $\mathcal{E}^p(V) = \{0\}$ for $p > 2n$ because a linear alternating form with more than $2n$ arguments in always zero.
By linearity we can thus see the exterior derivative as a map $d : E(V) \to E(V)$ with the additional condition $d\big(\mathcal{E}^p(V)\big) \subset \mathcal{E}^{p+1}(V)$, $\forall\, p \in \{0, \ldots, 2n\}$. So we can consider the sequence of sheaves

$$0 \longrightarrow \mathcal{E} \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \xrightarrow{d} \ldots \xrightarrow{d} \mathcal{E}^{2n} \xrightarrow{d} 0 \tag{11.7}$$

which means that for any open set $V \subseteq M$ (not necessarily a coordinate neighborhood), we have the sequence of finite-dimensional vector spaces

$$\{0\} \longrightarrow \mathcal{E}(V) \xrightarrow{d} \mathcal{E}^1(V) \xrightarrow{d} \mathcal{E}^2(V) \xrightarrow{d} \ldots \xrightarrow{d} \mathcal{E}^{2n}(V) \xrightarrow{d} \{0\} \tag{11.8}$$

Since $d \circ d = 0$, we hence know that (11.7) is a *complex* of sheaves.

Let $\omega \in E(V)$ be a differential form. We say that ω is *closed* if $\omega \in \ker d$, i.e. $d\omega = 0$. ω is called *exact* if $\omega \in \operatorname{im} d$, i.e. there is a form $\eta \in E(V)$ such that $\omega = d\eta$. We also define the two vector spaces :

$$Z^p_{\mathrm{dR}}(V) := \ker d \cap \mathcal{E}^p(V) \qquad , \qquad B^p_{\mathrm{dR}}(V) := \operatorname{im} d \cap \mathcal{E}^p(V) = d\big(\mathcal{E}^{p-1}(V)\big)$$

Hence $Z^p_{\mathrm{dR}}(V)$ contains all closed p-forms, called *cocycles* or *cochains*, and $B^p_{\mathrm{dR}}(V)$ contains all exact p-forms, called *coboundaries*. The *p^{th} de Rham cohomology* of V is then given by

$$H^p_{\mathrm{dR}}(V) := Z^p_{\mathrm{dR}}(V) \,/\, B^p_{\mathrm{dR}}(V)$$

This now allows us to compute the cohomology of the sequence (11.8), which encodes the topology of M on V. (11.8) is called an *exact sequence* if $H^p_{\mathrm{dR}}(V) = \{0\}$ for all p, i.e. if $Z^p_{\mathrm{dR}}(V) \cong B^p_{\mathrm{dR}}(V)$, $\forall\, p \in \{0, \ldots, 2n\}$. Hence the cohomology "measures" the exactness of the sequence.

11.4.2 Results

Since $d \circ d = 0$, we obtain that any exact form is closed. The converse however is not always true and depends on the considered open set V. Actually :

$$\text{the sequence (11.8) is exact} \;\Leftrightarrow\; Z^p_{\mathrm{dR}}(V) \cong B^p_{\mathrm{dR}}(V),\ \forall\, p \;\Leftrightarrow\; \forall\, p : \text{closed } p\text{-forms are exact on } V$$

In other words, the sequence is exact if and only if we can integrate the differential forms on V. And we have the following important result :

Lemma of Poincaré :
If $V \subseteq M$ is a contractible open set, then $H^p_{\mathrm{dR}}(V) = \{0\}$, $\forall\, p \geq 1$.

An open set $V \subseteq M$ is called *contractible* if there is an $x_0 \in V$ and a differentiable maps $\phi : [0,1] \times V \to V$ such that $\phi(0,x) = x_0$ and $\phi(1,x) = x$, $\forall\, x \in V$.
Examples of contractible sets are e.g. $\mathbb{R}^n$, $\mathbb{C}^n$ open balls and star-shaped sets.

Remark :
Since the kernel of $d : \mathcal{E}(V) \to \mathcal{E}^1(V)$ is given by the locally constant functions on V, one usually extends the sequence (11.7) to the *augmented* complex

$$0 \longrightarrow \mathbb{C}_M \xrightarrow{\ i\ } \mathcal{E} \xrightarrow{\ d\ } \mathcal{E}^1 \xrightarrow{\ d\ } \mathcal{E}^2 \xrightarrow{\ d\ } \ldots \xrightarrow{\ d\ } \mathcal{E}^{2n} \xrightarrow{\ d\ } 0$$

where i is the inclusion map and $\mathbb{C}_M$ denotes the sheaf of locally constant functions on M.

11.4.3 Holomorphic p-forms

Recall that Ω^1_M is the sheaf of holomorphic 1-differentials which are locally of the form $\omega = \sum_i f_i \cdot dz_i$ where the coefficient-functions f_i are holomorphic whenever defined.
In a similar way one can also define holomorphic differentials of higher order. Let $V \subseteq M$ be open. A *holomorphic p-form* on V is a differential p-form that is locally over U of the form

$$\omega_{|U} = \sum_I f_I \; dz_{i_1} \wedge dz_{i_2} \wedge \ldots \wedge dz_{i_p}$$

for $f_I \in \mathcal{O}_M(U)$ and $1 \leq i_1 < \ldots < i_p \leq n$. The vector space of holomorphic p-forms on V is denoted by $\Omega^p(V)$, with $\Omega^0(V) = \mathcal{O}_M(V)$. Then we set

$$\Omega(V) := \bigoplus_{p=0}^{n} \Omega^p(V)$$

Since $\Omega^p(V) \subset \mathcal{E}^p(V)$ for all p, the exterior derivative d also applies to such forms, but the result may no longer be a holomorphic form : $d\big(\Omega(V)\big) \nsubseteq \Omega(V)$. Thus we have to consider the decomposition $d = \partial + \bar{\partial}$ and extend the operators ∂ and $\bar{\partial}$ as well. Let $\omega \in \mathcal{E}^p(V) = \bigoplus \mathcal{E}^{s,t}(V)$. We denote again

$$dz_I := dz_{i_1} \wedge dz_{i_2} \wedge \ldots \wedge dz_{i_s} \wedge d\bar{z}_{j_1} \wedge d\bar{z}_{j_2} \wedge \ldots \wedge d\bar{z}_{j_t}$$

Hence if ω locally writes as $\omega = \sum_I \alpha_I \, dz_I$, we define $\partial\omega$ and $\bar{\partial}\omega$ locally as

$$\partial\omega = \sum_{i=1}^{n} \sum_I \frac{\partial \alpha_I}{\partial z_i} \; dz_i \wedge dz_I \qquad , \qquad \bar{\partial}\omega = \sum_{j=1}^{n} \sum_I \frac{\partial \alpha_I}{\partial \bar{z}_j} \; d\bar{z}_j \wedge dz_I$$

i.e. it is just the decomposition of definition (11.6). Similarly as for d, one also shows that $\partial \circ \partial = \bar{\partial} \circ \bar{\partial} = 0$. This implies a certain number of properties :

$$\partial \;:\; \Omega^p(V) \longrightarrow \Omega^{p+1}(V) \qquad , \qquad \partial \;:\; \mathcal{E}^{s,t}(V) \longrightarrow \mathcal{E}^{s+1,t}(V)$$
$$\bar{\partial}\big(\Omega(V)\big) = \{0\} \qquad , \qquad \bar{\partial} \;:\; \mathcal{E}^{s,t}(V) \longrightarrow \mathcal{E}^{s,t+1}(V)$$

In particular we can now see the map as $\partial : \Omega(V) \to \Omega(V)$ with $\partial\big(\Omega^p(V)\big) \subset \Omega^{p+1}(V)$.

We thus again have the (augmented) complex of sheaves

$$0 \longrightarrow \mathbb{C}_M \xrightarrow{\ i\ } \mathcal{O}_M \xrightarrow{\ \partial\ } \Omega^1 \xrightarrow{\ \partial\ } \Omega^2 \xrightarrow{\ \partial\ } \dots \xrightarrow{\ \partial\ } \Omega^n \xrightarrow{\ \partial\ } 0$$

If for $V \subseteq M$ open we denote $d_p := \partial_{|\Omega^p(V)}$, we can also define the holomorphic de Rham cohomology by

$$H^p(V, \Omega^p_M) := \ker d_p \,/\, \operatorname{im} d_{p-1}$$

11.4.4 Dolbeault cohomology

We already know that if $\omega \in \mathcal{E}^{s,t}(V)$ is a (s,t)-form, then $\partial\omega$ is a $(s+1,t)$-form and $\bar{\partial}\omega$ is a $(s,t+1)$-form. Now fix $p \in \{0, \dots, n\}$ and consider the (augmented) complex of sheaves

$$0 \longrightarrow \Omega^p \xrightarrow{\ i\ } \mathcal{E}^{p,0} \xrightarrow{\ \bar{\partial}\ } \mathcal{E}^{p,1} \xrightarrow{\ \bar{\partial}\ } \mathcal{E}^{p,2} \xrightarrow{\ \bar{\partial}\ } \dots \xrightarrow{\ \bar{\partial}\ } \mathcal{E}^{p,n} \xrightarrow{\ \bar{\partial}\ } 0$$

where the kernel of $\bar{\partial} : \mathcal{E}^{p,0}(V) \to \mathcal{E}^{p,1}(V)$ is equal to $\Omega^p(V)$ since $(p,0)$-forms locally write as

$$\omega = \sum_I \alpha_I \; dz_{i_1} \wedge dz_{i_2} \wedge \dots \wedge dz_{i_p} \quad \Rightarrow \quad \bar{\partial}\omega = \sum_{j=1}^n \sum_I \frac{\partial \alpha_I}{\partial \bar{z}_j} \; d\bar{z}_j \wedge dz_{i_1} \wedge dz_{i_2} \wedge \dots \wedge dz_{i_p}$$

and $\bar{\partial}\omega = 0$ means that the derivatives of the coefficient-functions α_I vanish with respect to all $\bar{z}_j$, i.e. the α_I are holomorphic and $\omega \in \Omega^p(V)$. The *q^{th} Dolbeault cohomology* is then defined by

$$H^{p,q}_{\bar{\partial}}(V) := \frac{\ker\left(\bar{\partial} \,:\, \mathcal{E}^{p,q}(V) \to \mathcal{E}^{p,q+1}(V)\right)}{\operatorname{im}\left(\bar{\partial} \,:\, \mathcal{E}^{p,q-1}(V) \to \mathcal{E}^{p,q}(V)\right)}$$

Result :
If $\Delta \subseteq \mathbb{C}^n$ is a polydisc, then $H^{p,q}_{\bar{\partial}}(\Delta) = \{0\}$, $\forall\, q \geq 1$.

Bibliography

[1] Martin Schlichenmaier, *An Introduction to Riemann Surfaces, Algebraic Curves and Module Spaces*. Springer, 2007.

[2] Martin Schlichenmaier, lecture course *Complex manifolds*, academic year 2011.

[3] Aleksey Zinger, *Notes on Vector Bundles*, 2010.
webpage : http://www.math.sunysb.edu/~azinger/mat531-spr10/vectorbundles.pdf

[4] Wikipedia