Analyse 1

HOMEWORK 4 : DUE THURSDAY NOVEMBER 10 IN CLASS

Exercice I

Let f, g be two functions from $\mathbb{R}$ to $\mathbb{R}$ such that $f(x) \to A$ and $g(x) \to B$, as $x \to x_0$. We assume that A, B are (strictly) positive real numbers. **Show** that

$$\lim_{x \to x_0} f(x)^{g(x)} = A^B.$$

Bonus questions :

(i) If we assume $f(x) \to A \in (0,1)$, and $g(x) \to +\infty$, as $x \to x_0$, then what is the limit

$$\lim_{x \to x_0} f(x)^{g(x)} \ ?$$

(ii) If we assume $f(x) \to A \in (1,+\infty)$, and $g(x) \to -\infty$, as $x \to x_0$, then what is the limit

$$\lim_{x \to x_0} f(x)^{g(x)} \ ?$$

(iii) What if we assume $f(x) \to 1$, and $g(x) \to +\infty$, as $x \to x_0$? For example,

$$\lim_{x \to 0} (\cos x)^{1/\sin^2(x)} = ?$$

Can you discuss the possible outcomes in this scenario ?

Note. You may need these results to ease your computation in **Exercise II**.

Exercice II

Find the limits of the following functions. Notice that for this question, you only need write down the main steps without too much details.

1.
$$\lim_{x \to 0} \frac{x - \sin(x)}{x}$$

2.
$$\lim_{x \to 0} \left(\frac{x+1}{2x+1} \right)^{x^2}$$

3.
$$\lim_{x \to +\infty} \left(\frac{x-1}{x+1} \right)^{x}$$

4.
$$\lim_{x \to 0} \left(\frac{\sin(2x)}{x} \right)^{1+x}$$

5.
$$\lim_{x \to 0} \left(\frac{x^2 - 2x + 3}{x^2 - 3x + 2} \right)^{\frac{\sin x}{x}}$$

Exercice III

Recall that for any $x \in \mathbb{R}$, the integer part of x, denoted by $\lfloor x \rfloor$, is defined to be the **unique** integer such that $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$. For example, $\lfloor 0.1 \rfloor = 0$, $\lfloor \sqrt{2} \rfloor = 1$, $\lfloor \pi \rfloor = 3$, $\lfloor -e \rfloor = -3$.

Define $f(x) := \lfloor x \rfloor$ for every $x \in \mathbb{R}$.

1. For each $n \in \mathbb{Z}$, compute

$$\lim_{x \to n+} \lfloor x \rfloor \quad \text{AND} \quad \lim_{x \to n-} \lfloor x \rfloor .$$

2. Compute

$$\lim_{x \to +\infty} \frac{\lfloor x \rfloor}{x} \quad \text{AND} \quad \lim_{x \to -\infty} \frac{\lfloor x \rfloor}{x} .$$

3. Compute

$$\lim_{x \to +\infty} \frac{\lfloor kx \rfloor}{\lfloor hx \rfloor}$$

for $k, h \in (0, +\infty)$.

4. Define $g(x) = x f(1/x)$ for $x \neq 0$ and $g(0) = 1$. This defines a function g. Is it continuous at zero ?