Analyse 1

# HOMEWORK 5: DUE THURSDAY NOVEMBER 24 IN CLASS

We recall the following definitions:

- a function $f : A \to B$ is injective if for every $a \neq a' \in A$ we have $f(a) \neq f(a')$;
- a function $f : A \to B$ is surjective if for every $b \in B$ there exists $a \in A$ such that $f(a) = b$.
- a function $f : A \to B$ is bijective if it is both injective and surjective.

In this exercise sheet we denote with $\mathbb{N}$ the set $\{1, 2, 3, \dots, \}$.

**Exercise I**

Let $f : A \to A$ be a function such that $f(f(a)) = f(a)$. Define $Fix(f) = \{x \in A \mid f(a) = a\}$.

1. Prove that $\operatorname{im}(f) = Fix(f)$ ;
2. Prove that $f$ is surjective if and only if $f(a) = a$ for every $a \in A$ ;
3. Prove that $f$ is injective if and only if $f(a) = a$ for every $a \in A$.

**Exercise II**

1. Prove that a strictly monotonic function $f : \mathbb{R} \to \mathbb{R}$ is injective.
2. Find a function $f : \mathbb{R} \to \mathbb{R}$ which is strictly increasing but not surjective.
3. Find a function $f : \mathbb{R} \to \mathbb{R}$ which is injective but not monotonic.

**Exercise III**

Prove that for every $n \in \mathbb{N}$ we have

$$\sum_{k=1}^{n} \frac{1}{k} \geq 1 + \frac{1}{2} \lfloor \log_2(n) \rfloor \ .$$

Deduce that

$$\lim_{n\to\infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right) = +\infty \ .$$

**Exercise IV**

Prove that for every $x > 0$ we have

$$\arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2} \ .$$

**Exercise V**

Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function. Prove that there exist two constant $a, b > 0$ (indipendent from $x$) such that

$$|f(x)| \leq a|x| + b$$

for every $x \in [a,b]\mathbb{R}$.

**Exercise VI**

Let $f : (a,b) \to \mathbb{R}$ be a differentiable function. Suppose that the derivative of $f$ has exactly $n$ zeros. Prove that $f$ has at most $n+1$ zeros.

*We recall that $x_0 \in (a,b)$ is a zero of $f$ if $f(x_0) = 0$.*

**Exercise VII**

Prove that for every $n \in \mathbb{N}$, the equation

$$\tan(x) = (1-x)^n$$

has a unique solution $x_n \in (0,1)$.
Prove that the sequence $x_n$ is decreasing and compute

$$\lim_{n\to+\infty} x_n.$$

**Exercise VIII**

Consider the function $f(x) = x^3 + 5x + 2$. Prove that $f : \mathbb{R} \to \mathbb{R}$ is bijective.
Compute

$$\lim_{y\to 2} \frac{f^{-1}(y)}{y-2} \ .$$

Prove that there exists a unique real number $a \in \mathbb{R}$ such that

$$\lim_{y\to+\infty} \frac{f^{-1}(y)}{y^a} \in \mathbb{R} \setminus \{0\} \ .$$