Analyse 1

HOMEWORK 6 : DUE THURSDAY DECEMBER 1, IN CLASS

This set of exercises ask you to find ***extremes*** of a differentiable function over some specified interval. Here we recall several concepts. Let a, b be two real numbers such that $a < b$, and $f : [a,b] \to \mathbb{R}$ be a continuous function. Then we know from some result in lecture that f attains its maximum and minimum in $[a,b]$.

1. We say f attains its ***maximum*** at $x \in [a,b]$, if $f(x) \geq f(y)$ for any $y \in [a,b]$.
2. We say f attains its ***minimum*** at $x \in [a,b]$, if $f(x) \leq f(y)$ for any $y \in [a,b]$.
3. We say f has its ***local maximum*** at a if there exists some $\delta > 0$ such that $a + \delta < b$ and $f(a) \geq f(y)$ for any $y \in [a, a+\delta]$.
4. We say f has its ***local minimum*** at a if there exists some $\delta > 0$ such that $a + \delta < b$ and $f(a) \leq f(y)$ for any $y \in [a, a+\delta]$.
5. We say f has its ***local minimum*** at $x \in (a,b)$ if there exists some $\delta > 0$ such that $a < x - \delta < x + \delta < b$ and $f(x) \leq f(y)$ for any $y \in [x-\delta, x+\delta]$.
6. We say f has its ***local maximum*** at $x \in (a,b)$ if there exists some $\delta > 0$ such that $a < x - \delta < x + \delta < b$ and $f(x) \geq f(y)$ for any $y \in [x-\delta, x+\delta]$.
7. We call (local) maximum and (local) minimum the extremes of the function f, and we call the points where these extremes are attained the ***critical points***.

Exercice I

Consider $f(x) = x^3 - 3x + 3$, $x \in [-2, 4]$. Find all the critical points and corresponding extremes and specify the maximum and minimum of f over $[-2, 4]$.

Exercice II

Consider $g(x) = (x-3)\sqrt{x}$, $x \in [0, +\infty)$. Please specify all the intervals on which g is increasing or g is decreasing. Then find all the critical points and corresponding extremes and specify the maximum and minimum of g over $[0, 100]$.

Exercice III

Consider the function $h : [0, 10] \to \mathbb{R}$ defined as : $h(0) = 0$ and $h(x) = x \log x$ for $x \in (0, 10]$.

1. Show that h is a continuous function on $[0, 10]$.
2. Find all the critical points and corresponding extremes and specify the maximum and minimum of h over $[0, 10]$.

Exercice IV

Prove the following inequalities for $x > 0$:

A)
$$x - \frac{x^3}{6} < \sin x < x.$$

B)
$$x - \frac{x^2}{2} < \log(1+x) < x.$$

Exercice V

Consider $p(x) = x/e^x$, $x \in [-1, +\infty)$. Find all the critical points and corresponding extremes and specify the maximum and minimum of p over $[-1, +\infty)$.