Analyse 1

HOMEWORK 7 : DUE THURSDAY DECEMBER 8, IN CLASS

Fix $-\infty < a < b < +\infty$, $f : (a,b) \to \mathbb{R}$ twice-differentiable.

1. We say f is convex around $x_0 \in (a,b)$ if $f''(x_0) > 0$; we say f is concave around x_0 if $f''(x_0) < 0$.
2. We say $x_0 \in (a,b)$ is a *point of inflection* of f, if for some $\delta > 0$ such that $a < x_0 - \delta < x_0 + \delta < b$, $f''(y), f''(z)$ have different sign for any $y \in (x_0 - \delta, x_0)$ and any $z \in (x_0, x_0 + \delta)$. In particular, at such a point of inflection, $f''(x_0) = 0$.

Exercice I

Find the points of inflection of the following functions, then specify the intervals on which these functions are convex, concave, increasing and/or decreasing.

1. $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}}$, $x \in \mathbb{R}$.
2. $g(x) = x \log(x^\beta)$, $x \in (0, +\infty)$ (here $\beta > 0$ is a given constant)
3. $h(x) = \frac{e^x - e^{-x}}{2}$, $x \in \mathbb{R}$.

Exercice II

Compute the following indeterminate forms. Note that you need to provide enough details/reason to support your computation.

1. $$\lim_{x\to 0} \frac{\log x}{\cot x} \,.$$
2. $$\lim_{x\to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{x^2} \right) \,.$$
3. $$\lim_{x\to 0} \left(\cos 2x \right)^{3/x^2}$$
4. $$\lim_{x\to 1} \left[\frac{1}{2(1-\sqrt{x})} - \frac{1}{3(1-\sqrt[3]{x})} \right]$$
5. $$\lim_{x\to 0} x^x$$
6. $$\lim_{x\to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x}$$

Exercice III

1. Let $u_n = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!}$. Compute $\lim_{n\to+\infty} u_n$.

   Hint. *Let $f(x) = e^x$ and write the Taylor expansion of f at $x = -1$. Then try to control the error.*

2. Compute
$$\lim_{n\to+\infty} \sum_{k=0}^{n} \frac{(-1)^k}{k+1}$$

   Hint. *Let $g(x) = \log(1+x)$ and write the Taylor expansion of f at $x = +1$.*