# Stochastic Analysis and PDE

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# Preface

These are preliminary notes to the course “Stochastic Analysis and PDE” taught during the academic year 2023/24 in the frame of the Master of Mathematics program at the University of Luxembourg.

CHAPTER 1

# Motivation

Let $M$ be a differentiable manifold of dimension $n$ (e.g. $M = \mathbb{R}^n$, $S^n$, $\mathbb{T}^n$, $\mathbb{H}^n, \ldots$ or an open domain in such a space, ...) and denote by

$$TM \xrightarrow{\pi} M$$

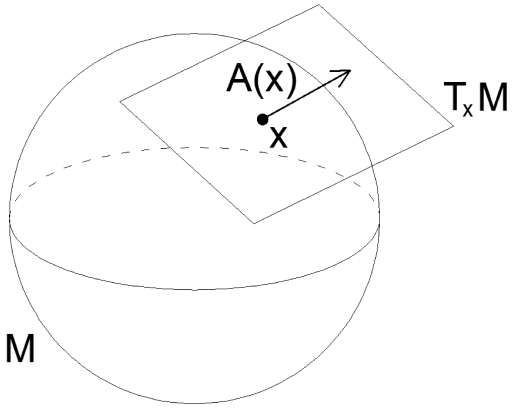
its *tangent bundle*. In particular, we have i.e.

$$TM = \dot{\cup}_{x \in M} T_x M, \quad \pi | T_x M = x.$$

The space of smooth sections of $TM$ is denoted by

$$\begin{aligned} \Gamma(TM) &= \{A \colon M \to TM \text{ smooth} \mid \pi \circ A = \mathrm{id}_M\} \\ &= \{A \colon M \to TM \text{ smooth} \mid A(x) \in T_x M \text{ for all } x \in M\}. \end{aligned}$$

Sections $A \in \Gamma(TM)$ are called (smooth) *vector fields* on $M$.




NOTE 1.1. As usual, we identify vector fields on $M$ and $\mathbb{R}$-derivations on $C^\infty(M)$ as follows:

$$\Gamma(TM) \mathrel{\widehat{=}} \left\{A \colon C^\infty(M) \to C^\infty(M) \text{ } \mathbb{R}\text{-linear} \mid A(fg) = fA(g) + gA(f) \ \ \forall f, g \in C^\infty(M)\right\}$$

where a vector field $A \in \Gamma(TM)$ is considered as $\mathbb{R}$-derivation via

$$A(f)(x) := df_x \, A(x) \in \mathbb{R}, \quad x \in M, \tag{1.1}$$

using the *differential* $df_x \colon T_x M \to \mathbb{R}$ of $f$ at $x$.

NOTE 1.2. Let $(h, U)$ be a *local chart* of $M$. Then

$$\frac{\partial}{\partial h_1}\Big|_x, \cdots, \frac{\partial}{\partial h_n}\Big|_x$$

is a basis of $T_x M$ for every $x \in U$, where for $f \in C^\infty(U)$,

$$\left(\frac{\partial}{\partial h_i} f\right)(x) := \partial_i \left(f \circ h^{-1}\right)(h(x)), \quad i = 1, \ldots, n.$$

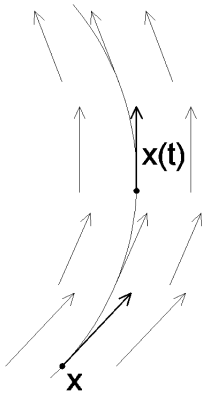
Hence $A \in \Gamma(TM)$ can be written locally as

$$A|U = \sum_{i=1}^{n} A_i \frac{\partial}{\partial h_i} \quad \text{where } A_i \in C^\infty(U).$$

## 1.1. Flow to a vector field

Given a vector field $A \in \Gamma(TM)$. For a fixed $x \in M$ we consider the smooth curve $t \mapsto x(t)$ in $M$ with the properties

$$x(0) = x \ \text{ and } \ \dot{x}(t) = A(x(t)).$$




We write $\phi_t(x) := x(t)$. In this way, we obtain for $A \in \Gamma(TM)$ the corresponding *flow to* $A$ given by

$$\begin{cases} \dfrac{d}{dt}\phi_t = A(\phi_t), \\ \phi_0 = \mathrm{id}_M. \end{cases} \tag{1.2}$$

System (1.2) means that for any $f \in C_c^\infty(M)$ (space of compactly supported smooth functions on $M$) the following conditions hold:

$$\begin{cases} \dfrac{d}{dt}(f \circ \phi_t) = A(f) \circ \phi_t \\ f \circ \phi_0 = f. \end{cases} \tag{1.3}$$

Indeed, by the chain rule along with definition (1.1), we have for each $f \in C_c^\infty(M)$,

$$\frac{d}{dt}(f \circ \phi_t) = (df)_{\phi_t} \frac{d}{dt}\phi_t = (df)_{\phi_t} A(\phi_t) = A(f)(\phi_t).$$

In integrated form, for each $f \in C_c^\infty(M)$, the conditions (1.3) write as:

$$f\big(\phi_t(x)\big) - f(x) - \int_0^t A(f)(\phi_s(x))\, ds = 0, \quad t \geq 0,\ x \in M. \tag{1.4}$$

NOTATION 1.3. As usual, the curve

$$\phi_\bullet(x)\colon\ t \mapsto \phi_t(x)$$

is called *flow curve* (or *integral curve*) to $A$ starting at $x$.

REMARK 1.4. Defining $P_t f := f \circ \phi_t$, we observe that $\frac{d}{dt} P_t f = P_t(A(f))$, in particular

$$\frac{d}{dt}\Big|_{t=0} P_t f = A(f). \tag{1.5}$$

In other words, from the knowledge of the flow $\phi_t$, the underlying vector field $A$ can be recovered by taking the derivative at zero as in Eq. (1.5).

In the same way as a vector field on a differentiable manifold induces a flow, second order differential operators induce stochastic flows with similar properties. In this sense, Brownian motion on $\mathbb{R}^n$ or on a Riemannian manifold appears as the stochastic flow associated to the canonical Laplace operator. The new feature of stochastic flows is that the flow curves depend on a random parameter and display an irregular behaviour as functions of time. This irregularity reveals an irreversibility in time which is inherent to stochastic phenomena.

## 1.2. Flow to a second order differential operator

Now let $L$ be a second order partial differential operator (PDO) on $M$, e.g. of the form

$$L = A_0 + \sum_{i=1}^{r} A_i^2, \tag{1.6}$$

where $A_0, A_1, \ldots, A_r \in \Gamma(TM)$ for some $r \in \mathbb{N}$. Note that $A_i^2 = A_i \circ A_i$ is understood as composition of derivations, i.e.

$$A_i^2(f) = A_i(A_i(f)), \quad f \in C^\infty(M).$$

EXAMPLE 1.5. Let $M = \mathbb{R}^n$ and consider

$$A_0 = 0 \text{ and } A_i = \frac{\partial}{\partial x_i} \text{ for } i = 1, \ldots, n.$$

Then $L = \Delta$ where $\Delta$ is the classical Laplace operator on $\mathbb{R}^n$.

Alternatively, we may consider partial differentiable operators $L$ on $M$ which locally in a chart $(h, U)$ can be written as

$$L|U = \sum_{i=1}^{n} b_i \partial_i + \sum_{i,j=1}^{n} a_{ij} \partial_i \partial_j, \tag{1.7}$$

where $b \in C^\infty(U, \mathbb{R}^n)$ and $a \in C^\infty(U, \mathbb{R}^n \otimes \mathbb{R}^n)$ such that $a_{ij} = a_{ji}$ for all $i, j$ ($a$ symmetric). Here we use the notation $\partial_i = \frac{\partial}{\partial h_i}$.

Motivated by the example of a flow to a vector field (vector fields can be seen as first order differential operators) we want to investigate the question whether an analogous concept of flow exists for second order PDOs.

QUESTION. Is there a notion of a flow to $L$ if $L$ is a second order PDO given by (1.6) or (1.7)?

DEFINITION 1.6. Let $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_t)_{t\geq 0})$ be a filtered probability space, i.e. a probability space equipped with increasing sequence of sub-$\sigma$-algebras $\mathscr{F}_t$ of $\mathscr{F}$. An adapted continuous process

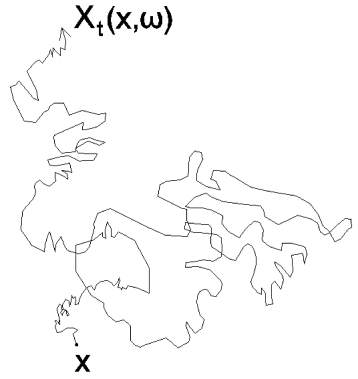
$$X_\bullet(x) \mathrel{\widehat{=}} (X_t(x))_{t\geq 0}$$

on $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_t)_{t\geq 0})$ taking values in $M$, is called *flow process* to $L$ (or *L-diffusion*) with starting point $x$ if $X_0(x) = x$ and if, for all test functions $f \in C_c^\infty(M)$, the process

$$N_t^f(x) := f(X_t(x)) - f(x) - \int_0^t (Lf)(X_s(x))\, ds, \quad t \geq 0, \tag{1.8}$$

is a martingale, i.e.

$$\mathbb{E}^{\mathscr{F}_s}\underbrace{\left[f(X_t(x)) - f(X_s(x)) - \int_s^t (Lf)(X_r(x))\,dr\right]}_{=N_t^f(x)-N_s^f(x)} = 0, \quad \text{for all } s \le t.$$




By definition, flow processes to a second order PDO depend on an additional random parameter $\omega \in \Omega$. For each $t \ge 0$, $X_t(x) \equiv (X_t(x,\omega))_{\omega\in\Omega}$ is an $\mathscr{F}_t$-measurable random variable. The defining equation (1.4) for flow curves translates to the martingale property of (1.8), i.e. the flow curve condition (1.4) only holds under conditional expectations. The theory of martingales gives a rigorous meaning to the idea of a process without systematic drift.

REMARK 1.7. Since $N_0^f(x) = 0$, we get from the martingale property of $N^f(x)$ that

$$\mathbb{E}\big[N_t^f(x)\big] = \mathbb{E}\big[N_0^f(x)\big] = 0.$$

Hence, defining $P_t f(x) := \mathbb{E}\left[f(X_t(x))\right]$, we observe that

$$P_t f(x) = f(x) + \int_0^t \mathbb{E}\left[(Lf)(X_s(x))\right]\,ds,$$

and thus

$$\frac{d}{dt}P_t f(x) = \mathbb{E}\left[(Lf)(X_t(x))\right] = P_t(Lf)(x),$$

in particular

$$\frac{d}{dt}\Big|_{t=0}\mathbb{E}\left[f(X_t(x))\right] \equiv \frac{d}{dt}\Big|_{t=0} P_t f(x) = Lf(x).$$

The last formula shows that as for deterministic flows we can recover the operator $L$ from its stochastic flow process. To this end however, we have to average over all possible trajectories starting from $x$.

EXAMPLE 1.8 (Brownian motion). Let $M = \mathbb{R}^n$ and $L = \frac{1}{2}\Delta$ where $\Delta$ is the Laplacian on $\mathbb{R}^n$. Let $X \equiv (X_t)$ be a standard Brownian motion on $\mathbb{R}^n$ starting at the origin. By Itô's formula, for $f \in C^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned} d(f\circ X_t) &= \sum_{i=1}^n \partial_i f(X_t)\,dX_t^i + \frac{1}{2}\sum_{i,j=1}^n \partial_i\partial_j f(X_t)\,dX_t^i dX_t^j \\ &= \langle(\nabla f)(X_t), dX_t\rangle + \frac{1}{2}(\Delta f)(X_t)\,dt. \end{aligned}$$

Thus, for each $f \in C_c^\infty(\mathbb{R}^n)$,

$$f(X_t) - f(X_0) - \int_0^t \frac{1}{2}(\Delta f)(X_s)\, ds, \quad t \geq 0,$$

is a martingale.

This means that

$$X_t(x) := x + X_t$$

is an $L$-diffusion to $\frac{1}{2}\Delta$ in the sense of Definition 1.6.

REMARKS 1.9. As deterministic flow curves may be defined only up to some finite maximal lifetime, we have the same phenomenon for flow processes: they also may explode in finite time.

1. We allow $X_\bullet(x)$ to be defined only up to some stopping time $\zeta(x)$, i.e.

$$X_\bullet(x)|[0, \zeta(x)[$$

where

$$\{\zeta(x) < \infty\} \subset \left\{ \lim_{t \uparrow \zeta(x)} X_t(\omega) = \infty \text{ in } \hat{M} := M \dot{\cup} \{\infty\} \right\} \quad \mathbb{P}\text{-a.s.} \tag{1.9}$$

Here $\hat{M}$ denotes the one-point-compactification of $M$. A stopping time $\zeta(x)$ with property (1.9) is called (maximal) *lifetime* for the process $X_\bullet(x)$ starting at $x$.

More precisely, let $U_n \subset M$ be open, relatively compact subsets exhausting $M$ in the sense that

$$U_n \subset \bar{U}_n \subset U_{n+1} \subset \ldots, \quad \bar{U}_n \text{ compact, and } \cup_n U_n = M.$$

Then $\zeta(x) = \sup_n \tau_n(x)$ for the maximal lifetime of $X_\bullet(x)$ where $\tau_n(x)$ is the family of stopping times (*first exit times* of $U_n$) defined by

$$\tau_n(x) := \inf\{t \geq 0 \colon\ X_t(x) \notin U_n\}.$$

2. For $f \in C^\infty(M)$ (not necessarily compactly supported), the process $N^f(x)$ will in general only be a *local* martingale in the sense that there exist stopping times $\tau_n \uparrow \zeta(x)$ such that

$$\forall n \in \mathbb{N}, \quad \left(N^f_{t \wedge \tau_n}(x)\right)_{t \geq 0} \text{ is a (true) martingale.}$$

3. The following two statements are equivalent (the proof will be given later):
   (a) The process
   $$f(X_\bullet(x)) = (f(X_t(x)))_{t \geq 0}$$
   is of locally bounded variation for all $f \in C_c^\infty(M)$.
   (b) The operator $L$ is of first order, i.e. $L$ is a vector field (in which case the flow is deterministic).

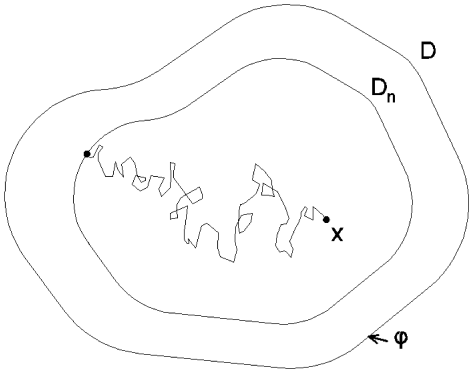
   In other words, flow processes have "nice paths" (for instance, paths of bounded variation) if and only if the corresponding operator is first order (i.e. a vector field).

## 1.3. What are $L$-diffusions good for?

Before discussing the problem of how to construct $L$-diffusions, we want to study some implications to indicate the usefulness and power of this concept. In the following two examples we only assume existence of an $L$-diffusion to a given operator $L$.

I. (*Dirichlet problem*) Let $\varnothing \neq D \subsetneq M$ be an open, connected, relatively compact domain, $\varphi \in C(\partial D)$ and let $L$ be a second order PDO on $M$. The *Dirichlet problem* (DP) is the problem to find a function $u \in C(\bar{D}) \cap C^2(D)$ such that

$$\text{(DP)} \qquad \begin{cases} Lu = 0 \text{ on } D \\ u|_{\partial D} = \varphi. \end{cases}$$




Suppose that there is an $L$-diffusion $(X_t(x))_{t\geq 0}$. We choose a sequence of open domains $D_n \uparrow D$ such that $\bar{D}_n \subset D$, and for each $n$ we consider the *first exit time* of $D_n$,

$$\tau_n(x) = \inf\{t \geq 0,\ X_t(x) \notin D_n\}.$$

Then $\tau_n(x) \uparrow \tau(x)$ where

$$\tau(x) = \sup_n \tau_n(x) = \inf\{t \geq 0,\ X_t(x) \notin D\}.$$

Now assume that $u$ is a solution to (DP). We may choose test functions $u_n \in C_c^\infty(M)$ such that $u_n|D_n = u|D_n$ and $\operatorname{supp} u_n \subset D$. Then, by the property of an $L$-diffusion,

$$N_t(x) := u_n(X_t(x)) - u_n(x) - \int_0^t (Lu_n)(X_r(x))\, dr$$

is a martingale. Suppose that $x \in D_n$. Then

$$\begin{aligned} N_{t\wedge\tau_n(x)}(x) &= u_n(X_{t\wedge\tau_n(x)}(x)) - u_n(x) - \int_0^{t\wedge\tau_n(x)} \underbrace{(Lu_n)(X_r(x))}_{=0}\, dr \\ &= u(X_{t\wedge\tau_n(x)}(x)) - u(x) \end{aligned} \tag{1.10}$$

is also a martingale; here we used that the integral in (1.10) is zero since $Lu_n = Lu = 0$ on $D_n$. Thus we get

$$\mathbb{E}\left[N_{t\wedge\tau_n(x)}(x)\right] = \mathbb{E}\left[N_0(x)\right] = 0$$

which shows that for each $n \in \mathbb{N}$,

$$u(x) = \mathbb{E}\left[u(X_{t\wedge\tau_n(x)}(x))\right]. \tag{1.11}$$

From Eq. (1.11) we conclude by dominated convergence and since $\tau_n(x) \uparrow \tau(x)$ that

$$u(x) = \lim_{n\to\infty} \mathbb{E}\left[u(X_{t\wedge\tau_n(x)}(x))\right] = \mathbb{E}\left[\lim_{n\to\infty} u(X_{t\wedge\tau_n(x)}(x))\right] = \mathbb{E}\left[u(X_{t\wedge\tau(x)}(x))\right].$$

We now make the *hypothesis* that

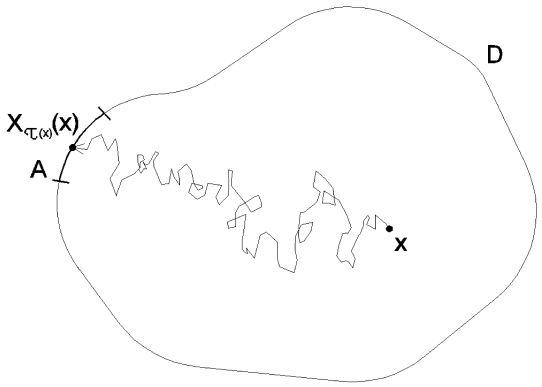
$$\tau(x) < \infty \quad \text{a.s.}$$

In other words, the process starting at $x \in D$ exits the domain $D$ in finite time. Then

$$\begin{aligned} u(x) &= \lim_{t\to\infty} \mathbb{E}\left[u(X_{t\wedge\tau(x)}(x))\right] = \mathbb{E}\left[\lim_{t\to\infty} u(X_{t\wedge\tau(x)}(x))\right] \\ &= \mathbb{E}\left[u(X_{\tau(x)}(x))\right] = \mathbb{E}\left[\varphi(X_{\tau(x)}(x))\right] \end{aligned}$$

where for the last equality we used the boundary condition $u|\partial D = \varphi$. Note that by passing to the image measure $\mu_x := \mathbb{P} \circ X_{\tau(x)}(x)^{-1}$ on the boundary we have

$$\mathbb{E}\left[\varphi(X_{\tau(x)}(x))\right] = \int_{\partial D} \varphi(z)\, \mu_x(dz).$$




Here we used that

$$\mathbb{E}\left[\varphi(X_{\tau(x)}(x))\right] = \int_{\partial D} \varphi\, d\underbrace{(\mathbb{P} \circ X_{\tau(x)}(x)^{-1})}_{=\mu_x} = \int_{\partial D} \varphi(z)\, \mu_x(dz).$$

NOTATION 1.10. *The measure $\mu_x$, defined on Borel sets $A \subset \partial D$,*

$$\mu_x(A) = \mathbb{P}\left\{X_{\tau(x)}(x) \in A\right\},$$

*is called exit measure from the domain $D$ of the diffusion $X_t(x)$. It represents the probability that the process $X_t$, when started at $x$ in $D$, exits the domain $D$ through the boundary set $A$.*

*Conclusions* From the discussion of the Dirichlet problem above we note the following observations.

(a) (Uniqueness) Under the hypothesis

$$\boxed{\tau(x) < \infty \text{ a.s. } \forall\, x \in D}$$

we have uniqueness of the solutions to the Dirichlet problem (DP).

(b) (Existence) Under the hypothesis

$$\boxed{\tau(x) \to 0 \text{ if } D \ni x \to a \in \partial D}$$

we have

$$\mathbb{E}\left[\varphi(X_{\tau(x)}(x))\right] \to \varphi(a), \quad \text{if } D \ni x \to a \in \partial D.$$

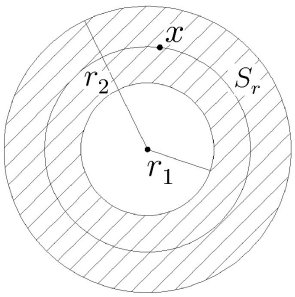
Thus we may define $u(x) := \mathbb{E}\left[\varphi(X_{\tau(x)}(x))\right]$. It can be shown that $u$ is always $L$-harmonic on $D$ if twice differentiable; thus under the hypothesis in (b), $u$ will then satisfy the boundary condition and hence solve (DP).

EXAMPLES 1.11.

(1) Let $M = \mathbb{R}\backslash\{0\}$ and $D = \{x \in \mathbb{R}^2 : r_1 < |x| < r_2\}$ with $0 < r_1 < r_2$. Consider the operator

$$L = \frac{1}{2}\frac{\partial^2}{\partial \vartheta^2}$$

where $\vartheta$ denotes the angle. If $u$ is a solution of (DP), then $u + v(r)$ is a solution of (DP) as well, for any radial function $v(r)$ satisfying $v(r_1) = v(r_2) = 0$. Hence, uniqueness of solutions fails.



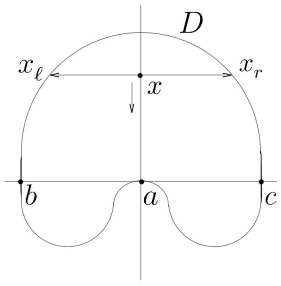

*Note:* For $x \in D$ with $|x| = r$, let $S_r = \{x \in \mathbb{R}^2 : |x| = r\}$. Then, the flow process $X_\bullet(x)$ to $L$ is easily seen to be the (one-dimensional) Brownian motion on $S_r$. In particular,

$$\tau(x) = +\infty \text{ a.s.}$$

(2) Let $M = \mathbb{R}^2$ and consider the operator

$$L = \frac{1}{2}\frac{\partial^2}{\partial x_1^2}$$

on a domain $D$ in $\mathbb{R}^2$ of the following shape:




Then, for $x = (x_1, x_2) \in D$, the flow process $X_\bullet(x)$ starting at $x$ is a (one-dimensional) Brownian motion on $\mathbb{R} \times \{x_2\}$. In other words, flow processes move on horizontal lines. In particular, when started at $x \in D$, the process can only exit at two points (e.g. $x_\ell$ and $x_r$ in the picture). Letting $x$ vertically approach $a$, by symmetry of the one-dimensional Brownian motion, we see that there exists a solution of (DP) if and only if

$$\varphi(a) = \frac{\varphi(b) + \varphi(c)}{2}.$$

II. (*Heat equation*) Let $L$ be a second order PDO on $M$ and fix $f \in C(M)$. The *heat equation* on $M$ with initial condition $f$ concerns the problem of finding a real-valued function $u = u(t,x)$ defined on $\mathbb{R}_+ \times M$ such that

$$\text{(HE)} \qquad \begin{cases} \frac{\partial u}{\partial t} = Lu \quad \text{on } ]0,\infty[ \times M, \\ u|_{t=0} = f. \end{cases}$$

Suppose now that there is an $L$-diffusion $X_\bullet(x)$. It is straightforward to see that the "time-space process" $(t, X_t(x))$ will then be a $\hat{L}$-diffusion for the parabolic operator

$$\hat{L} = \frac{\partial}{\partial t} + L$$

with starting point $(0,x)$. This means that for all $\varphi \in C^2(\mathbb{R}_+ \times M)$,

$$d\varphi(t, X_t(x)) - \big(\hat{L}\varphi\big)(t, X_t(x))\, dt \stackrel{\text{m}}{=} 0$$

where $\stackrel{\text{m}}{=}$ denotes equality modulo differentials of local martingales.

We adopt the *hypothesis* that $\zeta(x) = +\infty$ a.s. for all $x \in M$, i.e.

$$\boxed{\mathbb{P}\{\, X_t(x) \in M,\ \forall t \geq 0\} = 1, \quad \forall x \in M.}$$

Suppose now that $u$ is a *bounded* solution of (HE). We fix $t \geq 0$ and consider $u|[0,t] \times M$. Then

$$u(t-s, X_s(x)) - u(t,x) - \int_0^s \left[\left(\frac{\partial}{\partial r} + L\right) u(t-r,\cdot)\right](X_r(x))\, dr, \quad 0 \leq s < t,$$

is a local martingale. In other words, we have for $0 \leq s < t$,

$$u(t-s, X_s(x)) = u(t,x) + \int_0^s \underbrace{\left(\frac{\partial}{\partial r} + L\right) u(t-r,\cdot)(X_r(x))dr}_{=0,\ \text{since } u \text{ solves (HE)}} + (\text{local martingale})_s.$$

This shows that the local martingale in the last equation is actually a bounded local martingale (since $u(t-s, X_s(x)) - u(t,x)$ is bounded) and hence a true martingale equal to zero at time $0$.

Using the martingale property we first take expectations and then pass to the limit as $s \uparrow t$ to obtain

$$u(t,x) = \mathbb{E}\left[u(t-s, X_s(x))\right] \to \mathbb{E}\left[u(0, X_t(x))\right] = \mathbb{E}\left[f(X_t(x))\right], \quad \text{as } s \uparrow t,$$

where for the limit we used dominated convergence (recall that $u$ is bounded).

*Conclusion* Under the hypothesis

$$\zeta(x) = +\infty, \quad \text{for each } x \in M,$$

we have uniqueness of (bounded) solutions to the heat equation (HE). Solutions are necessarily of the form

$$\boxed{u(t,x) = \mathbb{E}\left[f(X_t(x))\right]}$$

*Interpretation* The solution $u(t,x)$ at time $t$ and at the point $x$ can be constructed as follows: run an $L$-diffusion process starting from $x$ up time $t$, apply the initial condition $f$ to the random position at time $t$ and average over all possible paths.

## 1.4. $\Gamma$-operators and quadratic variation

DEFINITION 1.12. Let $L\colon C^\infty(M) \to C^\infty(M)$ be a linear map (e.g. a second order PDO). The $\Gamma$-*operator associated to* $L$ ("*l'operateur carré du champ*") is the bilinear map

$$\Gamma\colon C^\infty(M) \times C^\infty(M) \to C^\infty(M) \text{ given as}$$

$$\boxed{\Gamma(f,g) := \frac{1}{2}[L(fg) - fL(g) - gL(f)].}$$

EXAMPLE 1.13. Let $L$ be a second order PDO on $M$ without constant term (i.e. $L\mathbb{1} = 0$). Suppose that in a local chart $(h, U)$ for $M$ the operator $L$ writes as

$$L|C_U^\infty(M) = \sum_{i,j=1}^n a_{ij}\,\partial_i\partial_j + \sum_{i=1}^n b_i\,\partial_i$$

where $C_U^\infty(M) = \{f \in C^\infty(M) :\ \operatorname{supp} f \subset U\}$ and $\partial_i = \frac{\partial}{\partial h_i}$. Then

$$\Gamma(f,g) = \sum_{i,j=1}^n a_{ij}(\partial_i f)(\partial_j g), \quad \forall f, g \in C_U^\infty(M).$$

In the special case that $M = \mathbb{R}^n$ and $L = \Delta$, we find

$$\Gamma(f,f) = |\nabla f|^2.$$

REMARK 1.14. Let $L$ be a second order PDO. Then $\Gamma(f,g) = 0$ for all $f, g \in C^\infty(M)$ if and only if $L$ is of first order, i.e. $L \in \Gamma(TM)$.

For instance, if $L = A_0 + \sum_{i=1}^r A_i^2$, then

$$\Gamma(f,g) = \sum_{i=1}^r A_i(f)A_i(g),$$

and in particular

$$\Gamma \equiv 0 \quad \text{if and only if} \quad A_1 = A_2 = \ldots = A_r = 0.$$

REMARK 1.15. A continuous real-valued stochastic process $(X_t)_{t\geq 0}$ is called a *semimartingale* if it can be decomposed as $X_t = X_0 + M_t + A_t$ where $M$ is a local martingale and $A$ is an adapted process of locally finite variation (with $M_0 = A_0 = 0$).

DEFINITION 1.16. Let $X$ be a continuous adapted process taking values in a manifold $M$. Then $X$ is called *semimartingale on* $M$ if

$$f(X) \equiv (f(X_t))_{t\geq 0}$$

is a real semimartingale for all $f \in C^\infty(M)$.

REMARK 1.17. If $X$ has maximal lifetime $\zeta$, i.e.,

$$\{\zeta < \infty\} \subset \left\{\lim_{t\uparrow\zeta} X_t = \infty \text{ in } \hat{M} = M\dot{\cup}\{\infty\}\right\} \text{ a.s.},$$

then $f(X)$ is a well-defined as a process globally on $\mathbb{R}_+$ for all $f \in C_c^\infty(M)$ (with the convention $f(\infty) = 0$). For $f \in C^\infty(M)$ however, in general,

$$f(X) \equiv (f(X_t))_{t<\zeta}$$

is only a semimartingale with lifetime $\zeta$.

PROPOSITION 1.18. *Let $L\colon C^\infty(M) \to C^\infty(M)$ be an $\mathbb{R}$-linear map and $X$ be a semimartingale on $M$ such that for all $f \in C^\infty(M)$,*

$$N_t^f := f(X_t) - f(X_0) - \int_0^t Lf(X_r)\,dr$$

*is a continuous local martingale (same lifetime as $X$) (i.e. $d(f(X)) - Lf(X)\,dt \stackrel{\mathrm{m}}{=} 0$ where $\stackrel{\mathrm{m}}{=}$ denotes equality modulo differentials of local martingales).*

*Then, for all $f, g \in C^\infty(M)$, the quadratic variation $[f(X), g(X)]$ of $f(X)$ and $g(X)$ is given by*

$$d\,[f(X), g(X)] \equiv d\big[N^f, N^g\big] = 2\Gamma(f,g)(X)\,dt.$$

*In particular, $\Gamma(f,f)(X) \geq 0$ a.s.*

PROOF. Let $f \in C^\infty(M, \mathbb{R}^r)$ and $\phi \in C^\infty(\mathbb{R}^r)$. Writing as above $\stackrel{\mathrm{m}}{=}$ for equality modulo differentials of local martingales, we have

$$d((\phi \circ f)(X)) \stackrel{\mathrm{m}}{=} L(\phi \circ f)(X)\,dt. \tag{1.12}$$

Developing the left-hand side in Eq. (1.12) by Itô's formula (applying the function $\phi$ to the semimartingale $f(X)$), we get

$$\begin{aligned} d((\phi \circ f)(X)) &= \sum_{i=1}^r (D_i\phi)\big(f(X)\big)\,d\big(f^i(X)\big) + \frac{1}{2}\sum_{i,j=1}^r (D_iD_j\phi)\big(f(X)\big)\,d[f^i(X), f^j(X)] \\ &\stackrel{\mathrm{m}}{=} \sum_{i=1}^r (D_i\phi)\big(f(X)\big)\,(Lf^i)(X)\,dt + \frac{1}{2}\sum_{i,j=1}^r (D_iD_j\phi)\big(f(X)\big)\,d[f^i(X), f^j(X)] \end{aligned}$$

where $D_i = \partial/\partial x_i$. By equating the drift parts we find

$$\Big(L(\phi\circ f) - \sum_{i=1}^r \left((D_i\phi)\circ f\right)(Lf^i)\Big)(X)\,dt = \tfrac{1}{2}\sum_{i,j=1}^r (D_iD_j\phi)\big(f(X)\big)\,d[f^i(X), f^j(X)].$$

Now taking $r = 2$ and considering the special case $\phi(x,y) = xy$, we get for $f = (f^1, f^2)$,

$$\underbrace{\left[L(f^1f^2) - f^1L(f^2) - f^2L(f^1)\right](X)\,dt}_{=\,2\,\Gamma(f^1,f^2)(X)\,dt} = d\left[f^1(X), f^2(X)\right].$$

□

LEMMA 1.19. *For an $\mathbb{R}$-linear map $L\colon C^\infty(M) \to C^\infty(M)$ the following statements are equivalent:*

(i) *$L$ is a second order* PDO *(without constant term)*
(ii) *$L$ satisfies the second order chain rule, i.e. for all $f \in C^\infty(M, \mathbb{R}^r)$ and $\phi \in C^\infty(\mathbb{R}^r)$,*

$$L(\phi\circ f) = \sum_{i=1}^r (D_i\phi\circ f)(Lf^i) + \sum_{i,j=1}^r (D_iD_j\phi\circ f)\,\Gamma(f^i,f^j).$$

PROOF. (i) ⇒ (ii): Write $L$ in local coordinates as

$$L|C_U^\infty(M) = \sum_{i,j=1}^n a_{ij}\,\partial_i\partial_j + \sum_{i=1}^n b_i\,\partial_i$$

and use that $\Gamma(f,g) = \sum_{i,j=1}^n a_{ij}\,(\partial_i f)(\partial_j g)$.

(ii) $\Rightarrow$ (i): Determine the action of $L$ on functions $\varphi$ written in local coordinates $(h, U)$ via

$$L(\varphi)|U = L(\varphi \circ h^{-1} \circ h) \equiv L(\phi \circ f)$$

where $\phi = \varphi \circ h^{-1}$ and $f = h$. Details are left to the reader as an exercise. □

PROPOSITION 1.20. *Let $L\colon C^\infty(M) \to C^\infty(M)$ be an $\mathbb{R}$-linear mapping. Suppose that for each $x \in M$ there exists a semimartingale $X$ on $M$ such that $X_0 = x$ and such that for each $f \in C^\infty(M)$,*

$$f(X_t) - f(x) - \int_0^t Lf(X_r)\, dr$$

*is a local martingale. Then $L$ is necessary a* PDO *of order at most* 2.

*In addition, $X$ has "nice" trajectories (e.g. in the sense that $[f(X), f(X)] = 0$ for all $f \in C^\infty(M)$) if and only if $L$ is first order.*

PROOF. As in the proof of Proposition 1.18, for all $f \in C^\infty(M, \mathbb{R}^r)$ and $\phi \in C^\infty(\mathbb{R}^r)$, we have

$$\Big(L(\phi \circ f) - \sum_{i=1}^r (D_i\phi \circ f)(Lf^i) + \sum_{i,j=1}^r (D_iD_j\phi \circ f)\, \Gamma(f^i, f^j)\Big)(X) = 0,$$

from where we get

$$L(\phi \circ f) - \sum_{i=1}^r (D_i\phi \circ f)(Lf^i) + \sum_{i,j=1}^r (D_iD_j\phi \circ f)\, \Gamma(f^i, f^j) = 0.$$

Thus $L$ satisfies the second order chain rule. By Lemma 1.19, $L$ is therefore a second order PDO. The additional claim follows from

$$d[f(X), g(X)] = 2\Gamma(f, g)(X)\, dt, \quad f, g \in C^\infty(M).$$ □

CHAPTER 2

# SDE and $L$-diffusions

## 2.1. Stochastic differential equations on Euclidean space

EXAMPLE 2.1 (ODE on $\mathbb{R}^n$). Given $\beta\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ continuous. One wants to find a differentiable function $t \mapsto y(t) \in \mathbb{R}^n$ such that

$$\dot{y}(t) = \beta(t, y(t)),$$

i.e.

$$\boxed{dy(t) = \beta(t, y(t))\, dt}$$

REMARK 2.2. Solutions to an ODE may explode in finite time, e.g. the solution to

$$\dot{y}(t) = y(t)^2 \text{ with } y(0) > 0$$

is given by $y(t) = \left(\frac{1}{y(0)} - t\right)^{-1} = \frac{y(0)}{1 - t y(0)}$.

EXAMPLE 2.3 (SDE on $\mathbb{R}^n$). Given $\beta$ as above and in addition a function

$$\sigma\colon \mathbb{R}_+ \times \mathbb{R}^n \to \text{Hom}(\mathbb{R}^r, \mathbb{R}^n) \equiv \text{Matr}(n \times r, \mathbb{R}).$$

Let $B$ be a Brownian motion on $\mathbb{R}^r$. One wants to find a continuous semimartingale $Y$ on $\mathbb{R}^n$ such that

$$\boxed{dY_t = \beta(t, Y_t)\, dt + \sigma(t, Y_t) dB_t}$$

in the sense of Itô, i.e.

$$Y_t = Y_0 + \int_0^t \beta(s, Y_s)\, ds + \int_0^t \sigma(s, Y_s)\, dB_s. \tag{2.1}$$

In Eq. (2.1) the first integral describes the "systematic part" (*drift term*) in the evolution of $Y$, whereas the second integral represents the "fluctuating part" (*diffusion term*).

DEFINITION 2.4. An $\mathbb{R}^n$-valued stochastic process $(Y_t)_{t \geq 0}$ is called *Itô process* if it has a representation as

$$Y_t = Y_0 + \int_0^t K_s\, ds + \int_0^t H_s\, dB_s$$

where

- $Y_0$ is $\mathscr{F}_0$-measurable;
- $K_s$ and $H_s$ are adapted processes taking values in $\mathbb{R}^n$, resp. $\text{Hom}(\mathbb{R}^r, \mathbb{R}^n)$;
- $\mathbb{E}\big[\int_0^t |K_s|\, ds\big] < \infty$ and $\mathbb{E}\big[\int_0^t H_s^2\, ds\big] < \infty$ for each $t \geq 0$.

PROPOSITION 2.5. *Let $\beta\colon\ \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma\colon\ \mathbb{R}_+ \times \mathbb{R}^n \to \mathrm{Hom}(\mathbb{R}^r, \mathbb{R}^n)$ be continuous. For a continuous semimartingale $Y$ on $\mathbb{R}^n$, defined up to some predictable stopping time $\tau$ (i.e. $\exists$ a sequence of stopping times $\tau_n < \tau$ with $\tau_n \uparrow \tau$), the following conditions are equivalent:*

(a) *$Y$ is a solution of the* SDE

$$dY_t = \beta(t, Y_t)\,dt + \sigma(t, Y_t)\,dB_t \quad \text{on } [0, \tau[, \tag{2.2}$$

*i.e.,*

$$Y_\rho = Y_0 + \int_0^\rho \beta(t, Y_t)\,dt + \int_0^\rho \sigma(t, Y_t)\,dB_t$$

*for each stopping time $\rho$ such that $\rho < \tau$ a.s.*

(b) *For each $f \in C^\infty(\mathbb{R}^n)$, it holds that*

$$d\big(f(Y)\big) = (Lf)(t, Y)\,dt + \sum_{k=1}^n \sum_{i=1}^r \sigma_{ki}(t, Y)\,(D_k f)(Y)\,dB_i \quad \text{on } [0, \tau[\,,$$

*where*

$$L = \sum_{k=1}^n \beta_k D_k + \frac{1}{2} \sum_{k,\ell=1}^n (\sigma\sigma^*)_{k\ell} D_k D_\ell,$$

*where $\sigma^*$ is a transpose of $\sigma$, and $(\sigma\sigma^*)_{k\ell} = \sum_{i=1}^r \sigma_{ki}\sigma_{\ell i}$. In particular, every solution of* (2.2) *is a $L$-diffusion on $[0, \tau[$ in the sense that*

$$d\big(f(Y)\big) - Lf(t, Y)\,dt = d(\textit{local martingale}) \text{ on } [0, \tau[.$$

PROOF. (a) $\Rightarrow$ (b) Let $Y$ be a solution of SDE (2.2). Then

$$dY^k dY^\ell \equiv d[Y^k, Y^\ell] = (\sigma\sigma^*)_{k\ell}(t, Y)\,dt$$

where $[Y^k, Y^\ell]$ represents the quadratic covariation of $Y^k$ and $Y^\ell$. By Itô's formula we get

$$\begin{aligned} d\big(f(Y)\big) &= \sum_{k=1}^n (D_k f)(Y)\Big(\beta_k(t, Y)\,dt + \sum_{i=1}^r \sigma_{ki}(t, Y)\,dB^i\Big) \\ &\qquad + \frac{1}{2} \sum_{k,\ell=1}^n (D_k D_\ell f)(Y)\,\underbrace{(\sigma\sigma^*)_{k\ell}(t, Y)\,dt}_{=d[Y^k, Y^\ell]} \\ &= Lf(t, Y)\,dt + \sum_{k=1}^n \sum_{i=1}^r \sigma_{ki}(t, Y)\,(D_k f)(Y)\,dB_i \\ &= Lf(t, Y)\,dt + d(\text{local martingale}). \end{aligned}$$

(b) $\Rightarrow$ (a) Take $f(x) = x_\ell$. Then $D_k f = \delta_{k\ell}$ and $Lf = \beta_\ell$, thus

$$dY^\ell = \beta_\ell(t, Y)\,dt + \sum_{i=1}^r \sigma_{\ell i}(t, Y) dB^i \quad \forall \ell = 1, \dots, n.$$

This shows that $Y$ solves SDE (2.2) on $[0, \tau[$. □

PROPOSITION 2.6 (Itô SDEs on $\mathbb{R}^n$; case of global Lipschitz conditions). *Let $Z$ be a continuous semimartingale on $\mathbb{R}^r$ and*

$$\alpha\colon \mathbb{R}^n \to \mathrm{Hom}(\mathbb{R}^r, \mathbb{R}^n)\ (= \mathrm{Matr}(n \times r; \mathbb{R}))$$

*such that*

$$\exists K > 0,\ \ |\alpha(y) - \alpha(z)| \le K|y - z|\ \ \forall y, z \in \mathbb{R}^n \quad (\textit{global Lipschitz conditions}).$$

*Then, for each $\mathscr{F}_0$-measurable $\mathbb{R}^n$-valued random variable $x_0$, there exists a unique continuous semimartingale $(X_t)_{t\in\mathbb{R}_+}$ on $\mathbb{R}^n$ such that*

$$dX = \alpha(X)\, dZ \ \textit{ and }\ X_0 = x_0.$$

*Uniqueness holds in the following sense: if $Y$ is another continuous semimartingale such that $dY = \alpha(Y)\, dZ$ and $Y_0 = x_0$, then $X_t = Y_t$ for all $t$ a.s.*

EXAMPLE 2.7. Let $B_t$ be a standard Brownian motion on $\mathbb{R}^r$. Then the space-time process $Z_t = (t, B_t)$ is a semimartingale on $\mathbb{R}^{r+1}$ and

$$dX = \beta(X)\, dt + \sigma(X)\, dB = \underbrace{(\beta(X)|\sigma(X))}_{=:\,\alpha(X)} \begin{pmatrix} dt \\ dB \end{pmatrix} = \alpha(X)\, dZ$$

where $\alpha(X)$ now takes values in $\mathrm{Matr}(n \times (1 + r))$.

## 2.2. Stratonovich differentials

DEFINITION 2.8. For continuous real semimartingales $X$ and $Y$ we define the *Stratonovich differential* of $X$ with respect to $Y$ as

$$X \circ dY := X\, dY + \frac{1}{2} d[X, Y]$$

where on the right-hand side, $X\,dY$ denotes the classical Itô differential of $X$ with respect to $Y$, and as usual, $d[X, Y] \equiv dX\,dY$. The integral

$$\int_0^t X \circ\, dY = \int_0^t X\, dY + \frac{1}{2}[X, Y]_t \tag{2.3}$$

is called *Stratonovich integral* of $X$ with respect to $Y$.

Formula (2.3) gives the relation between the Stratonovich integral and the usual Itô integral. Note that if we consider the semimartingale $\left(\int_0^\bullet X \circ\, dY\right)_t := \int_0^t X \circ\, dY$, then

$$d\left(\int_0^\bullet X \circ dY\right) = X\, dY + \frac{1}{2} dX\, dY = X \circ dY.$$

REMARK 2.9.

1. (*Associativity*) $X \circ (Y \circ dZ) = (XY) \circ dZ$, i.e.,

$$X \circ d\left(\int_0^\bullet Y \circ dZ\right) = (XY) \circ dZ.$$

Indeed, we have

$$\begin{aligned} X \circ (Y \circ dZ) &= X \circ d\left(\int_0^\bullet Y \circ dZ\right) \\ &= X\, d\left(\int_0^\bullet Y \circ dZ\right) + \frac{1}{2} dX\, d\left(\int_0^\bullet Y \circ dZ\right) \\ &= X(Y dZ) + \frac{1}{2}\, X\, dY dZ + \frac{1}{2} dX \left(Y dZ + \frac{1}{2}\, dY dZ\right) \\ &= (XY) dZ + \frac{1}{2}\,(X dY + Y dX + dX dY) dZ \\ &= (XY) dZ + \frac{1}{2}\, d(XY) dZ \\ &= (XY) \circ dZ. \end{aligned}$$

2. (*Product rule*) $d(XY) = X \circ dY + Y \circ dX$

PROOF. By Itô's formula we have

$$\begin{aligned} d(XY) &= X dY + Y dX + dX dY \\ &= X \circ dY + Y \circ dX. \end{aligned}$$

□

PROPOSITION 2.10 (Itô-Stratonovich formula). *Let $X$ be a continuous $\mathbb{R}^n$-valued semimartingale and $f \in C^3(\mathbb{R}^n)$. Then*

$$d(f(X)) = \sum_{i=1}^n (D_i f)(X) \circ dX^i \equiv \langle \nabla f(X), \circ\, dX \rangle.$$

PROOF. By Itô's formula, we have

$$d((D_i f)(X)) = \sum_{k=1}^n (D_i D_k f)(X)\, dX^k + \tfrac{1}{2} \sum_{k,\ell=1}^n (D_i D_k D_\ell f)(X)\, dX^k dX^\ell.$$

Hence we get

$$\begin{aligned} \sum_{i=1}^n (D_i f)(X) \circ dX^i &= \sum_{i=1}^n (D_i f)(X)\, dX^i + \tfrac{1}{2} \sum_{i=1}^n d(D_i f(X)) dX^i \\ &= \sum_{i=1}^n (D_i f)(X)\, dX^i + \tfrac{1}{2} \sum_{k=1}^n (D_i D_k f(X))\, dX^k dX^i \\ &= d(f(X)). \end{aligned}$$

□

PROPOSITION 2.11. *Let $\beta \colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous, $\sigma \colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathrm{Hom}(\mathbb{R}^r, \mathbb{R}^n)$ be $C^1$ and $B$ be a Brownian motion on $\mathbb{R}^r$. For a semimartingale $Y$ on $\mathbb{R}^n$ (defined up to some predictable stopping time $\tau$) the following conditions are equivalent:*

(i) *$Y$ is a solution of the Stratonovich* SDE

$$dY = \beta(t, Y)\, dt + \sigma(t, Y) \circ dB, \tag{2.4}$$

*i.e.*

$$Y_\sigma = Y_0 + \int_0^\sigma \beta(t, Y_t)\, dt + \int_0^\sigma \sigma(t, Y_t) \circ dB_t$$

*for every stopping time $\sigma$ such that $0 \le \sigma < \tau$ a.s.*

(ii) *For all* $f \in C^\infty(\mathbb{R}^n)$,

$$d\big(f(Y)\big) = (Lf)(t,Y)\,dt + \sum_{k=1}^{r}(A_k f)(t,Y)dB^k \quad \textit{on } [0,\tau[$$

*where*

$$L = A_0 + \frac{1}{2}\sum_{i=1}^{r} A_k^2,$$

*with the (time-dependent) vector fields* $A_i \in \Gamma(T\mathbb{R}^n)$ *defined as*

$$A_0 = \sum_{i=1}^{n}\beta_i D_i, \quad A_k = \sum_{i=1}^{n}\sigma_{ik}D_i, \quad k = 1,\dots,r.$$

PROOF. (i) $\Rightarrow$ (ii) By the Itô-Stratonovich formula we have

$$\begin{aligned}
d\big(f(Y)\big) &= \sum_{i=1}^{n}(D_i f)(Y)\circ dY^i \\
&= \sum_{i=1}^{n}(D_i f)(Y)\,\beta_i(t,Y)\,dt + \sum_{i=1}^{n}(D_i f)(Y)\circ\left(\sum_{k=1}^{r}\sigma_{ik}(t,Y)\circ dB^k\right) \\
&= \sum_{i=1}^{n}\beta_i(t,Y)(D_i f)(Y)\,dt + \sum_{k=1}^{r}\left(\sum_{i=1}^{n}\sigma_{ik}(t,Y)(D_i f)(Y)\right)\circ dB^k \\
&= (A_0 f)(t,Y)\,dt + \sum_{k=1}^{r}(A_k f)(t,Y)\circ dB^k \\
&= (A_0 f)(t,Y)\,dt + \sum_{k=1}^{r}(A_k f)(t,Y)\,dB_k + \frac{1}{2}\sum_{k=1}^{r} d\big((A_k f)(t,Y)\big)dB^k.
\end{aligned}$$

Since

$$d(A_k f(t,Y)) = \partial_t(A_k f)(t,Y)\,dt + (A_0 A_k f)(t,Y)\,dt + \sum_{\ell=1}^{r}(A_\ell A_k f)(t,Y)\circ dB^\ell,$$

we observe that

$$d(A_k f(t,Y))\,dB^k = (A_k^2 f)(t,Y)\,dt,$$

and hence

$$\begin{aligned}
d\big(f(Y)\big) &= \left((A_0 f)(t,Y) + \frac{1}{2}\sum_{k=1}^{r}(A_k^2 f)(t,Y)\right)dt + \sum_{k=1}^{r}(A_k f)(t,Y)\,dB^k \\
&= (Lf)(t,Y)\,dt + \sum_{k=1}^{r}(A_k f)(t,Y)\,dB^k.
\end{aligned}$$

(ii) $\Rightarrow$ (i) It is sufficient to take $f(x) = x_\ell$. □

COROLLARY 2.12. *Solutions to the Stratonovich* SDE

$$dY = \beta(t,Y)\,dt + \sigma(t,Y)\circ dB$$

*define $L$-diffusions for the operator*

$$L = A_0 + \tfrac{1}{2}\sum_{i=1}^{r} A_i^2,$$

*in the sense that*

$$d\big(f(Y)\big) - (Lf)(t,Y)\,dt \overset{\text{m}}{=} 0$$

*for all $f \in C^\infty(\mathbb{R}^n)$.*

## 2.3. Stochastic differential equations on manifolds

DEFINITION 2.13. Let $M$ be a differentiable manifold,

$$\pi\colon TM \to M$$

its tangent bundle and let $E$ a finite dimensional vector space (without restrictions $E = \mathbb{R}^r$). A *stochastic differential equation on $M$* is a pair $(A, Z)$ where

(1) $Z$ is a semimartingale taking values in $E$;
(2) $A\colon M \times E \to TM$ is a smooth homomorphism of vector bundles over $M$, i.e.

$$(x,e) \;\longmapsto\; A(x)e := A(x,e)$$

$$\begin{array}{ccc} M\times E & \xrightarrow[A]{} & TM \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow[\mathrm{id}]{} & M \end{array}$$

such that $A(x)\colon E \to T_xM$ is linear for each $x \in M$.

REMARK 2.14. Formally we consider $A$ as section $A \in \Gamma(E^* \otimes TM)$, i.e.

$$\begin{cases} \forall x \in M \text{ fixed}, & A(x) \in \mathrm{Hom}(E, T_xM), \\ \forall e \in E \text{ fixed}, & A(\cdot)e \in \Gamma(TM). \end{cases}$$

NOTATION 2.15. *For the* SDE $(A, Z)$ *we also write*

$$\boxed{dX = A(x)\circ dZ} \quad \textit{or} \quad \boxed{dX = \textstyle\sum_{i=1}^r A_i(X)\circ dZ^i}$$

*where $A_i = A(\cdot)e_i \in \Gamma(TM)$ and $e_1, \dots, e_r$ is a basis of $E$.*

DEFINITION 2.16. Let $(A, Z)$ be an SDE on $M$ and let $x_0\colon \Omega \to M$ be $\mathscr{F}_0$-measurable. An adapted continuous process $X|[0,\zeta[ \equiv (X_t)_{t<\zeta}$ taking values in $M$, defined up to the stopping time $\zeta$, is called solution to the SDE

$$dX = A(X)\circ dZ \tag{2.5}$$

with initial condition $X_0 = x_0$, if for all $f \in C_c^\infty(M)$ the following conditions are satisfied:

(i) $f(X)$ is a semimartingale on $[0,\zeta[$ with $f(X_0) = f(x_0)$;
(ii) $f(X)$ satisfies

$$d(f(X_t)) = (df)_{X_t} A(X_t)\circ dZ_t. \tag{2.6}$$

We call $X$ *maximal solution* of the SDE (2.5) if

$$\{\zeta < \infty\} \subset \left\{ \lim_{t \uparrow \zeta} X_t = \infty \text{ in } \hat{M} = M \dot{\cup} \{\infty\} \right\} \text{ a.s.}$$

*Note:* For the definition of the r.h.s. in (2.6) we use that for each $x \in M$,

$$E \xrightarrow{A(x)} T_x M \xrightarrow{(df)_x} \mathbb{R}$$

is a linear map.

REMARK 2.17. We adopt the convention $X_t(\omega) := \infty$ for $\zeta(\omega) \le t < \infty$ and $f(\infty) = 0$ for $f \in C_c^\infty(M)$. Thus we may write, for all $t \ge 0$,

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t (df)_{X_s} A(X_s) \circ dZ_s \\ &= f(X_0) = \sum_{i=1}^r \int_0^t \underbrace{(df)_{X_s} A_i(X_s)}_{=(A_i f)(X_s)} \circ dZ_s^i \quad \text{with } A_i = A(\cdot) e_i. \end{aligned}$$

EXAMPLE 2.18. Let $E = \mathbb{R}^{r+1}$ and $Z = (t, Z^1, \dots, Z^r)$ where $(Z_1, \dots, Z_r)$ is a Brownian motion on $\mathbb{R}^r$. Denote the standard basis of $\mathbb{R}^{r+1}$ by $(e_0, e_1, \dots, e_r)$. Let

$$A \colon M \times E \to TM$$

be a homomorphism of vector bundles over $M$, and consider the vector fields

$$A_i := A(\cdot) e_i \in \Gamma(TM), \quad i = 0, 1, \dots, r.$$

Then the SDE

$$\boxed{dX = A(X) \circ dZ} \tag{2.7}$$

writes as

$$\boxed{dX = A_0(X)\, dt + \sum_{i=1}^r A_i(X) \circ dZ^i}.$$

For each $f \in C_c^\infty(M)$ we have

$$\begin{aligned} d\big(f(X)\big) &= (df)_X A(X) \circ dZ \\ &= \sum_{i=0}^r (df)_X A(X) e_i \circ dZ^i \\ &= \sum_{i=0}^r \underbrace{(df)_X A_i(X)}_{(A_i f)(X)} \circ dZ^i \\ &= (A_0 f)(X)\, dt + \sum_{i=1}^r (A_i f)(X) \circ dZ^i \\ &= (A_0 f)(X)\, dt + \sum_{i=1}^r \left[ (A_i f)(X)\, dZ^i + \frac{1}{2} d\big((A_i f)(X)\big)\, dZ^i \right]. \end{aligned}$$

Taking into account that

$$d\big((A_i f)(X)\big) = \sum_{j=1}^r (A_j A_i f)(X)\, dZ^j + d(\text{bounded variation}),$$

we see that

$$d\big((A_i f)(X)\big)\, dZ^i = (A_i^2 f)(X)\, dt,$$

where we used that $dZ^i dZ^j = \delta_{ij}\, dt$ for $1 \le i, j \le r$. Hence we get

$$\begin{aligned} d\big(f(X)\big) &= (A_0 f)(X)\, dt + \frac{1}{2}\sum_{j=1}^{r}(A_i^2 f)(X)\, dt + \sum_{i=1}^{r}(A_i f)(X)\, dZ^i \\ &= (Lf)(X)\, dt + \sum_{i=1}^{r}(A_i f)(X)\, dZ^i. \end{aligned}$$

COROLLARY 2.19. *Let $L = A_0 + \frac{1}{2}\sum_{i=1}^{r} A_i^2$ and let $X$ be a solution to Eq.* (2.7)*. Then*

$$d\big(f(X)\big) - (Lf)(X)\, dt \overset{\mathrm{m}}{=} 0, \quad \forall f \in C_c^\infty(M),$$

*where $\overset{\mathrm{m}}{=}$ denotes equality modulo differentials of martingales.*

*In other words, maximal solutions to the* SDE

$$\boxed{dX = A(X) \circ dZ}$$

*are $L$-diffusions for the operator*

$$\boxed{L = A_0 + \frac{1}{2}\sum_{i=1}^{r} A_i^2}$$

THEOREM 2.20 (SDE: Existence and Uniqueness of solutions; $M = \mathbb{R}^n$). *Let $(A, Z)$ be an* SDE *on $M = \mathbb{R}^n$ and $x_0$ an $\mathscr{F}_0$-measurable random variable taking values in $\mathbb{R}^n$. Then there exists a unique maximal solution $X$ (with maximal lifetime $\zeta > 0$ a.s.) to the* SDE

$$dX = A(X) \circ dZ \tag{2.8}$$

*with initial condition $X_0 = x_0$. Uniqueness holds in the following sense: if $Y|[0, \xi[$ is another solution of* (2.8) *to the same initial condition, then $\xi \le \zeta$ a.s. and $X|[0, \xi[ = Y$ a.s.*

PROOF. Let $B(0, R) = \{x \in \mathbb{R}^n \colon |x| \le R\}$ where $R = 1, 2, \ldots$ and choose test functions $\phi_R \in C_c^\infty(\mathbb{R}^n)$ such that $\phi_R | B(0, R) \equiv 1$. Since

$$A \in \Gamma(\mathrm{Hom}(\mathbb{R}^r, TM)),$$

we have

$$\forall x \in \mathbb{R}^n, \quad A(x) \colon \mathbb{R}^r \to T_x M \quad \text{is linear,}$$

in other words, $A \in C^\infty(\mathbb{R}^n, \mathrm{Matr}(n \times r; \mathbb{R}))$.

Consider the "truncated SDE"

$$dX^R = A^R(X^R) \circ dZ \tag{2.9}$$

where $A^R = \phi_R\, A$. By Proposition 2.6, the truncated SDE (2.9) has a unique global solution $X^R$ with initial condition $X_0^R = x_0$, i.e. for each $R$ there exists a continuous $\mathbb{R}^n$-valued semimartingale $(X_t^R)_{t \in \mathbb{R}_+}$ satisfying $X_0^R = x_0$ such that (2.9) holds in the Itô-Stratonovich sense.

Considering the stopping times

$$\tau_R := \inf\left\{t \ge 0 : X_t^R \notin B(0, R)\right\},$$

we have for $R < R'$,

$$X^{R'}|[0, \tau_R[ = X^R|[0, \tau_R[ \quad \text{a.s.}$$

Hence a stochastic process $X$ (with lifetime $\zeta = \lim_{R\uparrow\infty} \tau_R$) is well-defined via

$$X|[0,\tau_R[ = X^R|[0,\tau_R[.$$

For each $f \in C_c^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}(f) \subset B(0,R)$ (with $R$ sufficiently large), we have on $[0,\tau_R[$,

$$\begin{aligned}
d\big(f(X)\big) &= d\big(f(X^R)\big) \\
&= \sum_{k=1}^n (D_k f(X^R)) \circ d(X^R)^k \quad \text{(using Itô-Stratonovich formula)} \\
&= \langle \nabla f(X^R), \circ\, dX^R \rangle \\
&= \langle \nabla f(X^R), \phi_R(X^R)\, A(X^R) \circ dZ \rangle \\
&= \langle \nabla f(X), A(X) \circ dZ \rangle \\
&= \sum_{i=1}^r \langle \nabla f(X), A_i(X) \circ dZ^i \rangle \\
&= \sum_{i=1}^r (df)_X A_i(X) \circ dZ^i \\
&= (df)_X A(X) \circ dZ.
\end{aligned}$$

Hence, since $R$ is arbitrary, the semimartingale $X$ is the unique solution to Eq. (2.8) with initial condition $X_0 = x_0$. □

REMARK 2.21. The proof above shows in particular that $X$ is a solution to the SDE

$$dX = A(X) \circ dZ$$

on $\mathbb{R}^n$ in the Itô-Stratonovich sense if and only if $X$ is a solution to this SDE in the sense of Definition 2.16, i.e. for each $f \in C_c^\infty(\mathbb{R}^n)$,

$$d\big(f(X)\big) = (df)_X A(X) \circ dZ.$$

THEOREM 2.22 (SDE: Existence and Uniqueness of solutions; general case). *Let $(A, Z)$ be an* SDE *on a differentiable manifold $M$ and let $x_0 \colon \Omega \to M$ be $\mathscr{F}_0$-measurable. There exists a unique maximal solution $X|[0,\zeta[$ (where $\zeta > 0$ a.s.) to the* SDE

$$dX = A(X) \circ dZ$$

*with initial condition $X_0 = x_0$. Uniqueness holds in the sense that if $Y|[0,\xi[$ is another solution with $Y_0 = x_0$, then $\xi \le \zeta$ a.s. and $X|[0,\xi[ = Y$ a.s.*

WHITNEY'S EMBEDDING THEOREM. *Each manifold $M$ of dimension $n$ can be embedded into $\mathbb{R}^{n+k}$ as a closed submanifold (for $k$ sufficiently large, e.g. $k = n+1$), i.e.,*

$$M \hookrightarrow \iota(M) \subset \mathbb{R}^{n+k}$$

*where $\iota : M \to \iota(M)$ is a diffeomorphism and $\iota(M) \subset \mathbb{R}^{n+k}$ a closed submanifold.*

PROOF OF THEOREM 2.22. We choose a Whitney embedding (in general not intrinsic)

$$M \underset{\text{diffeom.}}{\overset{\iota}{\hookrightarrow}} \iota(M) \subset \mathbb{R}^{n+k}$$

and identify $M$ and $\iota(M)$; in particular for each $x \in M$ the tangent space $T_xM$ is then a linear subspace of $\mathbb{R}^{n+k}$ according to

$$T_xM \overset{d\iota_x}{\hookrightarrow} T_x\mathbb{R}^{n+k} \equiv \mathbb{R}^{n+k}.$$

Vector fields $A_1, \dots, A_r \in \Gamma(TM)$ can be extended to vector fields

$$\bar{A}_1, \dots, \bar{A}_r \in \Gamma(T\mathbb{R}^{n+k}) \equiv C^\infty(\mathbb{R}^{n+k}; \mathbb{R}^{n+k}) \quad \text{with } \bar{A}_i | M = A_i,$$

i.e. $\bar{A}_i \circ \iota = d\iota \circ A_i$. Hence a given bundle map

$$A : M \times \mathbb{R}^r \to TM, \quad (x, z) \mapsto A(x)z = \sum_{i=1}^r A_i(x) z^i$$

has a continuation

$$\bar{A} : \mathbb{R}^{n+k} \times \mathbb{R}^r \to \mathbb{R}^{n+k} \times \mathbb{R}^{n+k}, \quad (x, z) \mapsto \bar{A}(x)z = \sum_{i=1}^r \bar{A}_i(x) z^i.$$

The idea is to consider in place of the original SDE

$$(*) \qquad dX = A(X) \circ dZ \ \text{ on } M$$

the SDE

$$(\bar{*}) \qquad dX = \bar{A}(X) \circ dZ \ \text{ on } \mathbb{R}^{n+k}.$$

It is clear that any solution of $(*)$ in $M$ provides a solution of $(\bar{*})$ in $\mathbb{R}^{d+k}$. More precisely: If $X$ is a solution to $(*)$ with starting value $X_0 = x_0$, then $\bar{X} := \iota \circ X$ solves equation $(\bar{*})$ with starting value $\bar{X}_0 = \iota \circ x_0$. Indeed if $\bar{f} \in C_c^\infty(\mathbb{R}^{d+k})$, then $f := \bar{f}|M = \bar{f} \circ \iota \in C_c^\infty(M)$, and we have:

$$\begin{aligned} d(\bar{f} \circ \bar{X}) = d(f \circ X) &= \sum_{i=1}^r (df)_X \, A_i(X) \circ dZ^i = \sum_{i=1}^r (d\bar{f})_{\bar{X}} \, (d\iota)_X \, A_i(X) \circ dZ^i \\ &= \sum_{i=1}^r (d\bar{f})_{\bar{X}} \, \bar{A}_i(\iota \circ X) \circ dZ^i = \sum_{i=1}^r (d\bar{f})_{\bar{X}} \, \bar{A}_i(\bar{X}) \circ dZ^i. \end{aligned}$$

This implies in particular uniqueness of solutions to $(*)$, since equation $(\bar{*})$ has a unique solution to a given initial condition.

To establish existence of solutions to $(*)$ we remark that any test function $f \in C_c^\infty(M)$ has a continuation $\bar{f} \in C_c^\infty(\mathbb{R}^{n+k})$ such that $\bar{f}|M \equiv \bar{f} \circ \iota = f$. We have the following important observation.

*Each solution $X|[0, \zeta[$ of $(\bar{*})$ in $\mathbb{R}^{n+k}$ with $X_0 = x_0$ which stays on $M$ for $t < \zeta$ (where $x_0$ is an $M$-valued $\mathscr{F}_0$-measurable random variable) gives a solution of $(*)$.*

To complete the proof it is hence sufficient to show the following lemma. □

LEMMA 2.23. *If $X|[0, \zeta[$ is the maximal solution of $(\bar{*})$ in $\mathbb{R}^{n+k}$ with $X_0 = x_0$, then*

$$\{t < \zeta\} \subset \{X_t \in M\}, \quad \forall t \ \textit{a.s.}$$

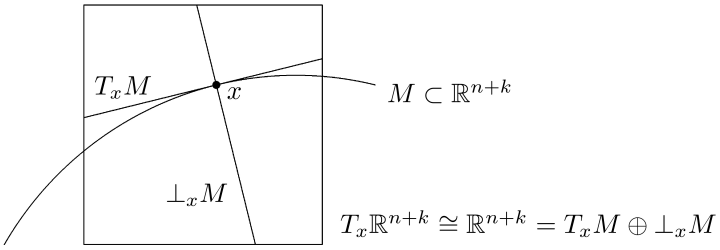
Observe that it is enough to verify Lemma 2.23 for one specific continuation $\bar{A}$ of $A$.

PROOF OF LEMMA 2.23. Let

$$\perp M = \left\{ (x, v) \in M \times \mathbb{R}^{n+k} \mid v \in (T_x M)^\perp \right\},$$

be the normal bundle of $M$ and consider $M$ embedded into $\perp M$ as zero section:

$$M \hookrightarrow \perp M, \quad x \mapsto (x, 0)$$




*Fact:* There is a smooth function $\varepsilon\colon M \to ]0,\infty[$ such that the map

$$\tau_\varepsilon(M) := \big\{(x,v) \in \perp M\colon\ |v| < \varepsilon(x)\big\} \xrightarrow{\cong} \bigcup_{x\in M} \{y \in \mathbb{R}^{n+k} : |y-x| < \varepsilon(x)\}\,,$$
$$(x,v) \longmapsto x+v\,,$$

is a diffeomorphism from the tubular neighbourhood $\tau_\varepsilon(M)$ of $M$ of radius $\varepsilon$ onto the indicated part in $\mathbb{R}^{n+k}$. Note that both

$$\pi\colon \tau_\varepsilon(M) \to M, \quad (x,v) \mapsto x$$
$$\operatorname{dist}^2(\cdot, M)\colon \tau_\varepsilon(M) \to \mathbb{R}, \quad (x,v) \mapsto |v|^2,$$

are smooth maps.

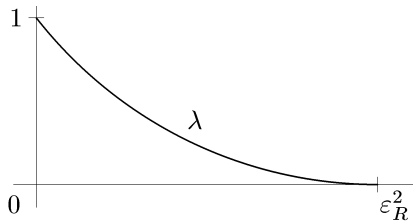
Now letting $R > 0$ be sufficiently large such that

$$M \cap B(0, R+1) \neq \varnothing,$$

then

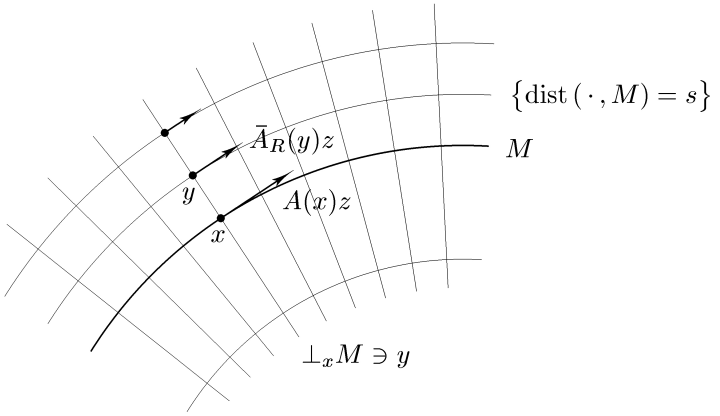
$$\varepsilon_R = \inf\{\varepsilon(x) \mid x \in M \cap B(0,R+1)\} > 0.$$

We choose a decreasing smooth function $\lambda : [0,\infty[ \to [0,1]$ of the form




and a test function $0 \le \varphi \in C_c^\infty(\mathbb{R}^{n+k})$ such that $\varphi | B(0,R) \equiv 1$ and $\operatorname{supp}(\varphi) \subset B(0,R+1)$. Consider

$$\bar{A}^R\colon \mathbb{R}^{n+k} \times \mathbb{R}^r \to \mathbb{R}^{n+k} \times \mathbb{R}^{n+k},$$
$$\bar{A}^R(y,z) := \begin{cases} \varphi(y)\,\lambda(\operatorname{dist}^2(y,M))\,A(\pi(y))\,z & \text{if } y \in \tau_\varepsilon(M), \\ 0 & \text{if } y \notin \tau_\varepsilon(M). \end{cases}$$




Now let $X$ be the maximal solution to

$$dX = \bar{A}^R(X) \circ dZ, \quad X_0 = x_0. \tag{2.10}$$

Consider $f \in C_c^\infty(\mathbb{R}^{n+k})$ given as

$$f(y) = \varphi(y)\, \lambda(\mathrm{dist}^2(y, M)).$$

Then

$$\begin{aligned} d(f \circ X) &= (df)_X \bar{A}^R(X) \circ dZ \\ &= \langle \nabla f(X), \bar{A}^R(X) \circ dZ \rangle \\ &= 0 \quad \text{on } [0, \tau_R[\,, \end{aligned}$$

where $\tau_R := \inf\{t \geq 0 \colon X_t \notin B(0, R)\}$. Indeed, $f$ is constant on submanifolds of the form

$$\{\mathrm{dist}(\cdot, M) = s\} \cap B(0, R), \quad s < \varepsilon_R,$$

whereas $\bar{A}^R(y, z)$ is tangent to such submanifolds. Thus

$$\nabla f(y) \perp \bar{A}^R(y)z, \quad \forall y \in B(0, R),\ z \in \mathbb{R}^r.$$

Hence, for the maximal solution $X$ to (2.10), we obtain

$$f(X) \equiv \text{constant on } [0, \tau_R[ \text{ a.s.}$$

Since $R$ is arbitrary, this completes the proof of the Lemma. □

With the usual localization method Proposition 2.6 can be generalized to the case of Itô SDEs with local Lipschitz coefficients.

PROPOSITION 2.24 (Itô SDEs on $\mathbb{R}^n$: case of the *local Lipschitz coefficients*). *Let $Z$ be a continuous semimartingale on $\mathbb{R}^n$ and let*

$$\alpha : \mathbb{R}^n \to \mathrm{Hom}(\mathbb{R}^r, \mathbb{R}^n),$$

*be locally Lipschitz, i.e. $\forall K \subset \mathbb{R}^n$ compact there exists a constant $L_K > 0$ such that*

$$\forall y, z \in K, \quad |\alpha(y) - \alpha(z)| \leq L_K |y - z|.$$

*Then, for any $x_0$ $\mathscr{F}_0$-measurable, there exists a unique maximal solution $X|[0, \zeta[$ of the SDE*

$$dX = \alpha(X)\, dZ, \quad X_0 = x_0.$$

*Uniqueness holds in the sense that if $Y|[0, \xi[$ is another solution and $Y_0 = x_0$, then $\xi \leq \zeta$ a.s. and $X|[0, \xi[ = Y$.*

PROOF. We adopt again the method of truncating the coefficients. For $R > 0$ let

$$dX^R = \alpha^R(X^R)\, dZ, \quad X_0^R = x_0,$$

where $\alpha^R := \phi_R \cdot \alpha$ and $\phi_R \in C_c^\infty(\mathbb{R}^n)$ is a test function such that $\phi_R | B(0,R) \equiv 1$. For each $R > 0$, this equation has a unique global solution since the coefficients are now globally Lipschitz. Thus

$$X|[0, \tau_R[ := X^R|[0, \tau_R[$$

is well-defined by uniqueness, where

$$\tau_R = \inf\{t \geq 0 : X_t^R \notin B(0,R)\},$$

which finally gives

$$X|[0, \zeta[ \quad \text{where } \zeta = \sup_R \tau_R. \qquad \square$$

EXAMPLE 2.25. Consider the following Itô SDE on $\mathbb{R}^n$ (with $b$ and $\sigma$ locally Lipschitz):

$$dX = \underbrace{b(X)}_{n\times 1}\, dt + \underbrace{\sigma(X)}_{n\times r}\, \underbrace{dW}_{r\times 1} \tag{2.11}$$

where $W$ is Brownian motion on $\mathbb{R}^r$. Then maximal solutions to Eq. (2.11) are $L$-diffusions to the operator

$$L = \sum_{i=1}^n b_i \partial_i + \frac{1}{2} \sum_{i,j=1}^n (\sigma\sigma^*)_{ij} \partial_i \partial_j,$$

where $\partial_i = \partial/\partial x_i$ is the derivative in direction $i$.

CHAPTER 3

# Some probabilistic formulas for solutions of PDEs

Let $L$ be a second order PDO on $M$. If $M$ is a general differentiable manifold then $L$ may be taken of the form

$$L = A_0 + \frac{1}{2}\sum_{i=1}^{r} A_i^2,$$

where $A_0, \dots, A_r \in \Gamma(TM)$, or if $M = \mathbb{R}^n$ then

$$L = \sum_{i=1}^{n} b_i \partial_i + \frac{1}{2}\sum_{i,j=1}^{n} (\sigma\sigma^*)_{ij}\partial_i\partial_j.$$

where $b\colon \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma\colon \mathbb{R}^n \to \mathrm{Matr}(n \times r; \mathbb{R})$.

For $x \in M$, let $X_t(x)$ be an $L$-diffusion, starting from $x$ at time $t = 0$, i.e. $X_0(x) = x$. Recall that $X_t(x)$ can be constructed as the solution to the Stratonovich SDE

$$\begin{cases} dX = A_0(X)\,dt + \sum_{i=1}^{r} A_i(x) \circ dW^i, \\ X_0 = x, \end{cases}$$

on $M$, resp. on $\mathbb{R}^n$ as solution to the Itô SDE

$$\begin{cases} dX = b(X)\,dt + \sigma(X)\,dW, \\ X_0 = x, \end{cases}$$

on $\mathbb{R}^n$, where in both cases the driving process $W$ is a standard Brownian motion on $\mathbb{R}^r$.

*Suppose that the lifetime of $X_t(x)$ is infinite a.s. for all $x \in M$.*

## 3.1. Feynman-Kac formula

PROPOSITION 3.1 (Feynman-Kac formula). *Let $f\colon M \to \mathbb{R}$ be continuous and bounded and $V\colon M \to \mathbb{R}$ be continuous and bounded above, i.e. $V \le K$ for some $K \in \mathbb{R}_+$.*

*Let $u\colon \mathbb{R}_+ \times M \to \mathbb{R}$ be a bounded solution of the following "initial value problem"*

$$\begin{cases} \frac{\partial}{\partial t}u = Lu + Vu \\ u|_{t=0} = f, \end{cases} \tag{3.1}$$

*i.e.*

$$\begin{cases} \frac{\partial}{\partial t}u(t,\cdot) = Lu(t,\cdot) + V(\cdot)\,u(t,\cdot) \\ u(0,\cdot) = f(\cdot). \end{cases}$$

*Then $u$ is given by the formula*

$$u(t,x) = \mathbb{E}\left[\exp\left(\int_0^t V(X_s(x))\,ds\right) f(X_t(x))\right].$$

REMARK 3.2. Operators of the form

$$H = L + V$$

(where $V$ is the multiplication operator by $V$) are called *Schrödinger operators*, for instance, $H = \frac{1}{2}\triangle + V$. The function $V$ is called *potential*. If $H$ is (essentially) self-adjoint, then by *semigroup theory*,

$$u(t, \cdot) = e^{tH} f$$

where the right-hand side then is defined by the spectral theorem.

PROOF. Let $u$ be a bounded solution to (3.1). Fix $t > 0$ and consider the process $(Y_s)_{0 \le s \le t}$ where $Y_s := A_s U_s$ and

$$\begin{cases} U_s := u(t-s, X_s(x)), \\ A_s := \exp\left( \int_0^s V(X_r(x))\, dr \right). \end{cases}$$

We claim that $(Y_s)_{0 \le s \le t}$ is a martingale under our assumptions.

*Indeed:* First note that by Itô's formula

$$dU_s = \Big( \partial_s u(t-s, \cdot) + Lu(t-s, \cdot) \Big)(X_s(x))\, ds + dM_s$$

where $M_s$ is local martingale. Thus, since $A_s$ is of bounded variation, we have

$$\begin{aligned} dY_s &= U_s dA_s + A_s dU_s \\ &= U_s A_s V(X_s(x))\, ds + A_s \Big( \partial_s u(t-s, \cdot) + Lu(t-s, \cdot) \Big)(X_s(x))\, ds + A_s dM_s \\ &= A_s \underbrace{\big( -\partial_t u + Lu + Vu \big)}_{=0}(t-s, X_s(x))\, ds + A_s dM_s \\ &= A_s dM_s. \end{aligned}$$

Hence $(Y_s)_{0 \le s \le t}$ is a local martingale, and as it is bounded, $(Y_s)_{0 \le s \le t}$ is a true martingale. In particular, by taking expectations we obtain

$$\begin{aligned} u(t,x) = \mathbb{E}\,[Y_0] = \mathbb{E}[Y_t] &= \mathbb{E}\left[ \exp\left( \int_0^t V(X_r(x))\, dr \right) u(0, X_t(x)) \right] \\ &= \mathbb{E}\left[ \exp\left( \int_0^t V(X_r(x))\, dr \right) f(X_t(x)) \right]. \end{aligned}$$

□

REMARK 3.3 (*Interpretation* for $M = \mathbb{R}^n$ and $L = \frac{1}{2}\triangle$).
Physicists think of the solution to

$$\begin{cases} \dfrac{\partial}{\partial t} u = Hu \quad \text{where } H = \dfrac{1}{2}\Delta - V \quad (\textit{Hamiltonian}) \\ u|_{t=0} = f \end{cases}$$

formally as follows:

$$u(t, x_0) = \int_{\Omega_{x_0}} e^{-S(t, x(\cdot))} f(x(t))\, \mathcal{D}x(\cdot) \tag{3.2}$$

where

- $\Omega_{x_0}$ is the space of (differentiable) paths $x(\cdot)$ starting from $x_0$,

$$\Omega_{x_0} = \{ x \colon [0,t] \to \mathbb{R}^n \,|\, x(0) = x_0 \},$$

- $S(t, x(\cdot))$ denotes the *classical action* of the path $x(\cdot)$ on $[0, t]$, i.e.

$$S(t, x(\cdot)) = \int_0^t \left( \frac{1}{2} |\dot{x}(s)|^2 + V(x(s)) \right) ds,$$

- $\mathcal{D}x(\cdot)$ is thought of a kind of "Lebesgue measure" on $\Omega_{x_0}$ (in the sense of a uniform distribution on $\Omega_{x_0}$).

The heuristic expression (3.2) is called a *Feynman path integral*.

*Claim.* The expression

$$u(t, x_0) = \mathbb{E}\left[ \exp\left( -\int_0^t V(X_s)\, ds \right) f(X_t) \right] \tag{3.3}$$

where $X$ is a Brownian motion on $\mathbb{R}^n$ starting from $x_0$, gives a rigorous meaning to (3.2).

*Indeed:* We approximate the integral in (3.3) by a Riemann sum as follows. Let $t_i := i\Delta_m$ (for $i = 0, 1, \ldots, m$) be a partition of the interval $[0, t]$ where $\Delta_m := \frac{t}{m}$ is the step length of the partition, and consider

$$u_m(t, x_0) = \mathbb{E}\underbrace{\left[ \exp\left( -\Delta_m \sum_{i=1}^{m-1} V(X_{t_i}) \right) f(X_{t_m}) \right]}_{=:\ F_m(X_{t_0}, X_{t_1}, \ldots, X_{t_m})}.$$

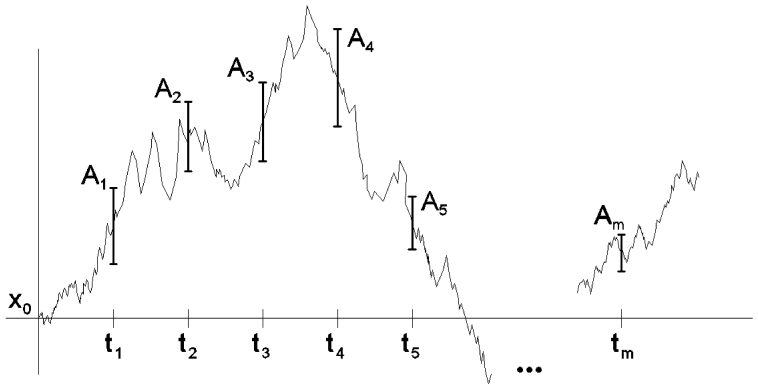
Note that $F_m(X_{t_0}, X_{t_1}, \ldots, X_{t_m}) = F_m(x_0, X_{t_1}, \ldots, X_{t_m})$. Then

$$u_m(t, x_0) \xrightarrow{m\to\infty} u(t, x_0) = \mathbb{E}\left[ \exp\left( -\int_0^t V(X_s)\, ds \right) f(X_t) \right].$$

Recall that Brownian motion on $\mathbb{R}^n$ has independent and normally distributed stationary increments, i.e. for any $A \in \mathcal{B}(\mathbb{R}^n)$ and $s < t$,

$$\mathbb{P}\{X_t - X_s \in A\} = \nu_{t-s}(A) := \int_A g_{t-s}(x)\, dx,$$

where $g_t(x) = (2\pi t)^{-n/2} \exp(-|x|^2/2t)$. In particular, if $A_1, \ldots, A_m \in \mathcal{B}(\mathbb{R}^n)$, then




$$\begin{aligned}
&\mathbb{P}\left\{X_{t_1} \in A_1, \ldots, X_{t_m} \in A_m\right\} = \\
&\quad = \int_{A_1} \nu_{t_1-t_0}(dx_1 - x_0) \int_{A_2} \nu_{t_2-t_1}(dx_2 - x_1) \cdot \ldots \cdot \int_{A_m} \nu_{t_m-t_{m-1}}(dx_m - x_{m-1}) \\
&\quad = \int_{A_1} \ldots \int_{A_m} g_{t_1-t_0}(x_1 - x_0) \cdot \ldots \cdot g_{t_m-t_{m-1}}(x_m - x_{m-1})\, dx_1 \ldots dx_m,
\end{aligned}$$

which gives the probability for a Brownian motion (starting from $x_0$) to pass through $A_i$ at time $t_i$ for $i = 1, \dots, m$ ("probability of a *successful slalom*").

Taking into account that

$$g_{t_k - t_{k-1}}(x_k - x_{k-1}) = \frac{1}{(2\pi\Delta_m)^{n/2}} \exp\left\{-\frac{|x_k - x_{k-1}|^2}{2\Delta_m}\right\},$$

we obtain

$$u_m(t,x) = \underbrace{\int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n}}_{m \text{ times}} g_{t_1 - t_0}(x_1 - x_0) \dots g_{t_m - t_{m-1}}(x_m - x_{m-1})\, F_m(x_0, x_1, \dots, x_m)\, dx_1 \dots dx_m$$

where

$$F_m(x_0, x_1, \dots, x_m) = \exp\left(-\Delta_m \sum_{i=0}^{m-1} V(x_i)\right) f(x_m).$$

Thus, letting $A = (2\pi\Delta_m)^{n/2}$, we have

$$u_m(t,x) = \frac{1}{A^m} \underbrace{\int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n}}_{m \text{ times}} \exp\left(-S_m(t; x_1, \dots, x_m)\right) f(x_m)\, dx_1 \dots dx_m$$

where

$$S_m(t, x_1, \dots, x_m) = \Delta_m \sum_{i=0}^{m-1} \left(\frac{1}{2}\left(\frac{|x_{i+1} - x_i|}{\Delta_m}\right)^2 + V(x_i)\right)$$

represents the "Riemann sum" for the classical action $S(t, x(\cdot))$ of a particle of unit mass under the potential $V$ for a trajectory $x(\cdot)$ with $x(t_i) = x_i$ $(i = 0, 1, \dots, m)$.

Note that $\mathcal{D}x(\cdot)$ corresponds to the (mathematically not well-defined) infinite product measure

$$\lim_{m\to\infty} (2\pi\Delta_m)^{-nm/2}\, dx_1 \dots dx_m.$$

## 3.2. Elliptic boundary value problems

Let $L$ be a second order PDO on a differentiable manifold $M$, e.g.,

$$L = A_0 + \frac{1}{2}\sum_{i=1}^{r} A_i^2 \quad \text{on a general differential manifold } M,$$

or

$$L = \sum_{i=1}^{n} b_i \partial_i + \frac{1}{2}\sum_{i,j=1}^{n} (\sigma\sigma^*)_{ij} \partial_i \partial_j \quad \text{on } \mathbb{R}^n.$$

REMARK 3.4 (Ellipticity).

(1) The "*diffusion vector fields*" $A_1, \dots, A_r$ define for each $x \in M$ a linear map

$$A(x)\colon\ \mathbb{R}^r \to T_x M, \quad z \mapsto \sum_{i=1}^{r} A_i(x) z_i.$$

Recall that, denoting by $e_i$ the standard coordinate vectors of $\mathbb{R}^r$, then

$$A(\cdot)e_i = A_i, \quad i = 1, 2, \dots, r.$$

The operator

$$L = A_0 + \frac{1}{2}\sum_{i=1}^{r} A_i^2,$$

is called *elliptic* on some subset $D \subset M$, if the map $A(x)$ is *surjective* for each $x \in D$.

(2) Similarly, an operator of the form

$$L = \sum_{i=1}^{n} b_i \partial_i = \frac{1}{2}\sum_{i,j=1}^{n} (\sigma\sigma^*)_{ij}\partial_i\partial_j,$$

is called *elliptic* on some subset $D \subset M$ if the linear map

$$\sigma(x)\colon\ \mathbb{R}^r \to \mathbb{R}^n, \quad z \mapsto \underbrace{\sigma(x)}_{n\times r}\, z,$$

is *surjective* for each $x \in D$.

REMARK 3.5. The following conditions are equivalent:

$$\begin{aligned}\sigma(x) \text{ is surjective} &\iff \sigma^*(x) \text{ is injective}\\ &\iff a(x) := \sigma(x)\sigma^*(x) \text{ is invertible}\\ &\iff \langle a(x)v, v\rangle > 0, \quad \forall\, 0 \neq v \in \mathbb{R}^n.\end{aligned}$$

EXAMPLE 3.6 (Expected hitting time of a boundary). Let $\varnothing \neq D \subsetneq M$ be some open, relatively compact domain with boundary $\partial D$. Suppose that there exists a solution $u \in C^2(D)\cap C(\bar{D})$ to the problem

$$\begin{cases} Lu = -1 \ \text{ on } D,\\ u|_{\partial D} = 0. \end{cases} \tag{3.4}$$

(For instance, if $L$ is elliptic on $\bar{D}$ and the boundary $\partial D$ is smooth, it is well-known by classical PDE theory that such a solution exists).

Let $X_t(x)$ be an $L$-diffusion such that $X_0(x) = x$ and denote by

$$\tau_D(x) = \inf\{t > 0 : X_t(x) \in \partial D\}$$

its first exit time from $D$. Then, for each $x \in D$,

$$\boxed{u(x) = \mathbb{E}\big[\tau_D(x)\big]}$$

In particular, we see that $u > 0$ on $D$.

PROOF. For $x \in D$, let $X_t = X_t(x)$ and $\tau_D = \tau_D(x)$. We know that the process

$$u(X_{t\wedge\tau_D}) - u(x) - \int_0^{t\wedge\tau_D} Lu(X_s)\, ds, \quad t \geq 0,$$

is a martingale (starting at 0), and hence

$$\mathbb{E}[u(X_{t\wedge\tau_D})] - u(x) = \mathbb{E}\left[\int_0^{t\wedge\tau_D} \underbrace{Lu(X_s)}_{=-1}\, ds\right].$$

This shows that

$$\mathbb{E}[t\wedge\tau_D] = u(x) - \mathbb{E}[u(X_{t\wedge\tau_D})]. \tag{3.5}$$

Recall that $u$ is bounded, since $u \in C(\bar{D})$ with $\bar{D}$ compact, and hence by Beppo-Levi,

$$\mathbb{E}[\tau_D] = \lim_{t\to\infty} \mathbb{E}[t \wedge \tau_D] < +\infty.$$

Thus, by letting $t \uparrow +\infty$ in (3.5), we obtain

$$\mathbb{E}[\tau_D] = u(x) - \underbrace{\mathbb{E}[u(X_{\tau_D})]}_{=0},$$

where we used that $u|\partial D = 0$. □

COROLLARY 3.7. *If the boundary value problem* (3.4) *has a solution, then*

$$\mathbb{E}[\tau_D(x)] < \infty,$$

*and hence* $\tau_D(x) < \infty$ *a.s. for all* $x \in D$. *Thus* $L$*-diffusions starting at any point* $x \in D$ *eventually hit* $\partial D$ *with probability* 1.

REMARK 3.8. The property of an $L$-diffusion of hitting the boundary with probability 1 is a "non-degeneracy" condition on the operator $L$.

EXAMPLE 3.9. Consider an operator of the form

$$L = \sum_{i=1}^{n} b_i \partial_i + \frac{1}{2} \sum_{i,j=1}^{n} \underbrace{(\sigma\sigma^*)_{ij}}_{a_{ij}} \partial_i \partial_j \quad \text{on } \mathbb{R}^n$$

and let $D \subset \mathbb{R}^n$ be relatively compact. Suppose that the following "weak elliptic condition" is valid: For some $1 \le \ell \le n$ there holds

$$\boxed{\min_{x\in\bar{D}} a_{\ell\ell}(x) > 0}$$

Then $\mathbb{E}[\tau_D(x)] < \infty$ for every $x \in D$.

PROOF. Set

$$A := \min_{x\in\bar{D}} a_{\ell\ell}(x) \quad \text{and} \quad B := \max_{x\in\bar{D}} |b(x)|.$$

For $\mu, \nu > 0$ consider the smooth function

$$h(x) = -\mu e^{\nu x_\ell}, \quad x \in D.$$

Then

$$\begin{aligned} -Lh(x) &= \mu e^{\nu x_\ell} \left( \frac{\nu^2}{2} a_{\ell\ell}(x) + \nu b_\ell(x) \right) \\ &\ge \frac{1}{2} \mu\nu A \, e^{\nu x_\ell} \left( \nu - \frac{2B}{A} \right) \\ &\ge \frac{1}{2} \nu\mu A e^{\nu K} \left( \nu - \frac{2B}{A} \right) \quad \text{(take } \nu > 2B/A \text{ and let } K := \min_{x\in\bar{D}} x_\ell\text{)} \\ &\ge 1 \quad \text{for } \mu \text{ sufficiently large.} \end{aligned}$$

Thus

$$Lh \le -1 \quad \text{on } D.$$

As above, we may proceed as follows. The process

$$N_t^h := h(X_{t\wedge\tau_D}) - h(x) - \int_0^{t\wedge\tau_D} Lh(X_s)\, ds, \quad t \ge 0,$$

is a martingale (where again $X_t = X_t(x)$ and $\tau_D = \tau_D(x)$). By taking expectations we obtain

$$h(x) - \mathbb{E}[h(X_{t\wedge\tau_D})] = -\mathbb{E}\bigg[\int_0^{t\wedge\tau_D} \underbrace{Lh(X_s)}_{\leq -1}\, ds\bigg] \geq \mathbb{E}[t\wedge\tau_D].$$

Hence,

$$\begin{aligned}\mathbb{E}[\tau_D] &= \mathbb{E}\left[\liminf_{t\to\infty} t\wedge\tau_D\right]\\ &\leq \liminf_{t\to\infty}\mathbb{E}[t\wedge\tau_D]\\ &\leq 2\max_{y\in\bar{D}}|h(y)| < \infty.\end{aligned}$$

□

DEFINITION 3.10 (Generalized Dirichlet problem). Let $\varnothing \neq D \subsetneq M$ be an open, relatively compact domain and let $L$ be a second order PDO on $M$ as above. Assume to be given $g, k \in C(\bar{D}), k \geq 0$ and $\varphi \in C(\partial D)$. The generalized Dirichlet problem consists in finding $u \in C^2(D)\cap C(\bar{D})$ such that

$$\text{(GDP)}\qquad \begin{cases} -Lu + ku = g \quad \text{on } D\\ u|_{\partial D} = \varphi. \end{cases}$$

THEOREM 3.11 (Stochastic representation of solutions to the generalized Dirichlet problem). *Assume that $u$ is a solution to* (GDP)*. For $x \in D$, let $X_t(x)$ be an $L$-diffusion, starting from $x$, and assume that*

$$\mathbb{E}[\tau_D(x)] < \infty \quad \forall x \in D.$$

*Then*

$$u(x) = \mathbb{E}\left[\varphi(X_{\tau_D})\exp\left\{-\int_0^{\tau_D} k(X_s)\,ds\right\} + \int_0^{\tau_D} g(X_s)\exp\left\{-\int_0^s k(X_r)\,dr\right\}ds\right]$$

*where $\tau_D = \tau_D(x)$ and $X_t = X_t(x)$. In particular, solutions to* (GDP) *are unique.*

PROOF. Consider the semimartingale

$$N_t := u(X_t)\exp\left\{-\int_0^t k(X_s)\,ds\right\} + \int_0^t g(X_s)\exp\left\{-\int_0^s k(X_r)\,dr\right\}ds.$$

We find that

$$\begin{aligned} dN_t &= \exp\left\{-\int_0^t k(X_s)\,ds\right\}\Big(\underbrace{d\big(u(X_t)\big)}_{\overset{\mathrm{m}}{=}(Lu)(X_t)\,dt} - u(X_t)k(X_t)\,dt + g(X_t)\,dt\Big)\\ &\overset{\mathrm{m}}{=} \exp\left\{-\int_0^t k(X_s)\,ds\right\}(Lu - uk + g)(X_t)\,dt = 0,\end{aligned}$$

where as before $\overset{\mathrm{m}}{=}$ denotes equality modulo differentials of (local) martingales. Thus, the process

$$(N_{t\wedge\tau_D})_{t\geq 0}$$

is a martingale. In particular, by dominated convergence,

$$u(x) = \mathbb{E}[N_0] = \mathbb{E}[N_{t\wedge\tau_D}] \to \mathbb{E}[N_{\tau_D}],$$

and thus

$$u(x) = \mathbb{E}\left[u(X_{\tau_D})\exp\left\{-\int_0^{\tau_D} k(X_s)\,ds\right\} + \int_0^{\tau_D} g(X_s)\exp\left\{-\int_0^s k(X_s)\,dr\right\}ds\right].$$

Since $u|_{\partial D} = \varphi$, we have $u(X_{\tau_D}) = \varphi(X_{\tau_D})$ which gives the claim. □

We shall consider the result of Theorem 3.11 in some special cases.

I. (*Classical Feynman-Kac formula*) Consider the boundary problem

$$\begin{cases} -Lu + ku = g \ \text{ on } D, \\ u|_{\partial D} = 0. \end{cases}$$

Its solution is given by

$$u(x) = \mathbb{E}\left[\int_0^{\tau_D(x)} g(X_t(x)) \exp\left\{-\int_0^t k(X_r)\, dr\right\} dt\right], \quad x \in D.$$

In particular, if $k \equiv 0$ then

$$u(x) = \mathbb{E}\left[\int_0^{\tau_D(x)} g(X_t(x))\, dt\right] \quad \text{(Green's function)}$$

*Note* that $-Lu = g$ which means that $u = -L^{-1}g$. Thus the Green's function gives an "inverse" to $-L$.

II. (*Classical Dirichlet Problem*) Consider the problem

$$\text{(DP)} \qquad \begin{cases} Lu = 0 \ \text{ on } D, \\ u|_{\partial D} = \varphi. \end{cases}$$

If $X_t(x)$ is an $L$-diffusion, then

$$u(x) = \mathbb{E}\left[\varphi(X_{\tau_D(x)}(x))\right] = \int_{\partial D} \varphi\, d\mu_x$$

where the exit measure $\mu_x$ is given by

$$\mu_x(B) := \mathbb{P}\{X_{\tau_D(x)}(x) \in B\}, \quad B \subset \partial D \text{ measurable.}$$

Note that $u(x) = \int_{\partial D} \varphi d\mu(x)$ makes sense also for boundary functions $\varphi$ which are just bounded and measurable.

EXAMPLE 3.12. Assume that $\partial D = A \cup B$ where $A \cap B = \varnothing$. In Physics the solution to the Dirichlet problem

$$\begin{cases} Lu = 0 \text{ on } D, \\ u|A = 1, \\ u|B = 0, \end{cases}$$

is called *equilibrium potential* for the capacitor $(A, B)$.

Let $\varphi|_{\partial D}$ be defined as

$$\varphi(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \in B. \end{cases}$$

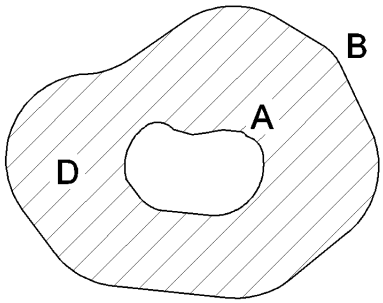
Then

$$u(x) = \mathbb{E}\left[\varphi(X_{\tau_D(x)}(x))\right] = \mathbb{P}\big\{\tau_A(x) < \tau_B(x)\big\}$$

where

$$\tau_A(x) = \inf\{t > 0,\ X_t(x) \in A\},$$
$$\tau_B(x) = \inf\{t > 0,\ X_t(x) \in B\}.$$

Thus $u(x)$ corresponds to *the probability that an $L$-diffusion, starting from $x$, hits $A$ before hitting $B$.*




EXAMPLE 3.13. Let $(W_s)_{s\geq 0}$ be a standard Brownian motion on the real line starting at $0$. We consider the decreasing function

$$F(t) := \mathbb{P}\left\{\sup_{0\leq s\leq t} |W_s| < 1\right\}$$

and want to show that

$$F(t) \sim e^{-\pi^2 t/8} \quad \text{as } t \to \infty.$$

In terms of

$$T = \inf\{s \geq 0\colon |W_s| = 1\},$$

we have

$$F(t) = \mathbb{P}\{T > t\}.$$

Note that $X_t(x) := x + W_t$ is a real-valued $L$-diffusion for the operator $L = \frac{1}{2}\partial_x^2$.

For $x \in \,]-1,1[$ let

$$T(x) = \inf\{s \geq 0\colon |X_s(x)| = 1\}$$

be the first exit time of the interval when starting at $x$. Then by definition $T(0) = T$.

Let us consider the following *eigenvalue problem* for $L := \frac{1}{2}\partial_x^2$ on $[-1,1]$. Given $\lambda \leq 0$, find $u \in C^2(]-1,1[) \cap C([-1,1])$ such that

$$\text{(EVP)} \qquad \begin{cases} Lu + \lambda u = 0 & \text{on } \{|x| < 1\}, \\ u = 1 & \text{on } \{|x| = 1\}. \end{cases}$$

We know that if a solution to (EVP) exists, then

$$u(x) = \mathbb{E}\left[\underbrace{\varphi(X_{T(x)})}_{=1} \exp\left(\int_0^{T(x)} \lambda\, dr\right)\right] = \mathbb{E}\left[e^{\lambda T(x)}\right].$$

However, as can be checked directly, (EVP) has the following explicit solution

$$u(x) = \frac{\cos(\sqrt{2\lambda}x)}{\cos(\sqrt{2\lambda})}.$$

For $\lambda < 0$ note that $\sqrt{2\lambda}$ is imaginary, but cosine of an imaginary is real:

$$\cos(iy) = \frac{e^y + e^{-y}}{2} = \cosh(y), \quad y \in \mathbb{R}.$$

Thus, for $T = T(0)$, we obtain

$$\mathbb{E}\left[e^{\lambda T}\right] = \frac{1}{\cos(\sqrt{2\lambda})}, \quad \lambda \leq 0 \quad (\textit{Laplace transform of the law of } T). \tag{3.6}$$

On the other hand Eq. (3.6) holds up to the first singularity of the function

$$\lambda \to \frac{1}{\cos(\sqrt{2\lambda})}$$

which is at $\lambda = \pi^2/8$. By monotone convergence we obtain

$$\mathbb{E}\left[\exp\left(\frac{\pi^2 T}{8}\right)\right] = \infty.$$

Hence

$$\mathbb{E}\left[e^{\lambda T}\right] = \begin{cases} < \infty, & \text{for } \lambda < \pi^2/8, \\ = \infty, & \text{for } \lambda \geq \pi^2/8. \end{cases}$$

Observe that

$$\mathbb{E}\left[e^{\lambda T}\right] = \int_0^\infty \mathbb{P}\{e^{\lambda T} > t\}\, dt < \infty \iff \int_0^\infty e^{\lambda t}\, \mathbb{P}\{T > t\}\, dt < \infty,$$

thus

$$F(t) = \mathbb{P}\{T > t\} \sim e^{-\lambda^* t}, \quad \text{as } t \to \infty,$$

where $\lambda^* = \pi^2/8$.

COROLLARY 3.14. *Let*

$$\lambda^* = \sup\left\{\lambda : \mathbb{E}[e^{\lambda T}] < \infty\right\}.$$

*Then* (EVP) *has a solution if and only if* $\lambda < \lambda^*$.

### 3.3. Parabolic boundary value problems

Let $D \subset M$ be an open and relatively compact domain. Consider a second order PDO $L$ on $M$ as above and let $(X_t(x))_{t\geq 0}$ be an $L$-diffusion. Fix $T > 0$ and let $V$ be a measurable function on $D$ such that

$$\mathbb{E}\left[\exp\left(\int_0^{T\wedge\tau_D(x)} V_-(X_s(x))\, ds\right)\right] < \infty, \quad \forall x \in D,$$

where $V_- := (-V) \vee 0$ denotes the negative part of $V$ and $\tau_D(x) = \inf\{t \geq 0 : X_t(x) \in \partial D\}$. Furthermore, let $f, g \in C(\bar{D})$ and $\varphi \in C(\partial D)$ be given.

*Problem.* Find a solution to the following parabolic boundary value problem:

$$\text{(BVP)} \qquad \begin{cases} \frac{\partial}{\partial t} u = Lu - Vu + g \quad \text{on } [0,T] \times D, \\ u(t,\cdot)|_{\partial D} = \varphi \quad \text{for } t \in [0,T], \\ u|_{t=0} = f. \end{cases}$$

Note that necessarily $f|_{\partial D} = \varphi$.

THEOREM 3.15. *Every solution* $u \in C^2([0,T] \times D) \cap C([0,T] \times \bar{D})$ *of* (BVP) *is of the form*

$$u(t,x) = \mathbb{E}\left[f(X_{t\wedge\tau_D}) \exp\left(-\int_0^{t\wedge\tau_D} V(X_s)\, ds\right) + \int_0^{t\wedge\tau_D} g(X_s) \exp\left(-\int_0^s V(X_r)\, dr\right) ds\right],$$

*where* $X_t = X_t(x)$ *and* $\tau_D = \tau_D(x)$.

PROOF. For $0 < t_0 \leq T$, we check by Itô's formula that

$$N_t := u(t_0 - t, X_t) \exp\left(-\int_0^t V(X_s) ds\right) + \int_0^t g(X_s) \exp\left(-\int_0^s V(X_r) dr\right) ds, \quad t \leq t_0 \wedge \tau_D,$$

is a martingale. Then it suffices to evaluate $u(t_0, x) = \mathbb{E}[N_0] = \mathbb{E}[N_{t_0\wedge\tau_D}] = \ldots$ □

EXAMPLE 3.16. Let $\varnothing \neq D \subsetneq \mathbb{R}^d$ be an open and relatively compact domain. Consider the space-time boundary problem

$$\begin{cases} \frac{\partial}{\partial t}u = Lu & \text{on } ]0,\infty[ \times D, \\ u = 0 & \text{on } ]0,\infty[ \times \partial D, \\ u|_{t=0} = 1. \end{cases}$$

By *classical PDE theory*, if $L$ is elliptic and self-adjoint, then (since the domain $D$ is bounded), the spectrum of $-L$ (with Dirichlet boundary conditions) is discrete. Let

$$(\lambda_n, \psi_n)_{n\geq 1}$$

be the sequence of eigenvalues to $-L$ with the corresponding normalized eigenfunctions $\psi_n$, i.e. $\|\psi_n\|_{L^2(D,dx)} = 1$. We have

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ...$$

Recall that $(\psi_n)_{n\geq 1}$ forms an $L^2$-orthonormal basis of $L^2(D, dx)$. Then in particular

$$u(t,x) = \sum_{n\geq 1} c_n\, e^{-\lambda_n t}\, \psi_n(x), \quad c_n = \langle \psi_n, 1\rangle_{L^2(D)}.$$

On the other hand,

$$N_s := u(t-s, X_s(x)), \quad s \leq t \wedge \tau_D := \sigma,$$

is a martingale, and hence

$$\begin{aligned} u(t,x) &= \mathbb{E}\left[u(t-\sigma, X_\sigma(x))\right] \\ &= \mathbb{P}\{t \leq \tau_D(x)\} \sim c_1\, e^{-\lambda_1 t}\psi_1(x), \quad \text{as } t \to \infty. \end{aligned}$$

COROLLARY 3.17 (Large deviations of exit times). *Let*

$$\tau_D(x) := \inf\{t \geq 0 : X_t(x) \in \partial D\}.$$

*Then*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\{\tau_D(x) \geq t\} = -\lambda_1,$$

*where*

$$\begin{aligned} \lambda_1 &= \inf\left\{ -\int_D (\psi L\psi)(x)\, dx \,\middle|\, \psi \in C_c^\infty(D),\ \|\psi\|_{L^2(D)} = 1 \right\} \\ &= \inf\left\{ -\langle \psi, L\psi\rangle_{L^2(D)} \,\middle|\, \psi \in C_c^\infty(D),\ \|\psi\|_{L^2(D)} = 1 \right\} \end{aligned}$$

*is the smallest eigenvalue of $-L$ with Dirichlet boundary conditions on $D$.*

EXAMPLE 3.18 (Kac 1951). Let $M = \mathbb{R}^d$ and $L = \frac{1}{2}\Delta$. Consider $X_t(x) = x + W_t$ where $W$ is a standard Wiener process (Brownian motion) on $\mathbb{R}^d$. For $V \in C(\mathbb{R}^d), V \geq 0$, one would like to find the asymptotics of

$$\mathbb{E}\left[\exp\left(-\int_0^t V(X_s(x))\, ds\right)\right], \quad \text{as } t \to \infty.$$

Assume that the spectrum of $\frac{1}{2}\Delta - V$ is discrete (this requires "mild" growth conditions on $V(x)$ as $|x| \to \infty$). Denote then by $(\lambda_n)_{n\geq 1}$,

$$0 < \lambda_1 \leq \lambda_2 \leq \dots,$$

the eigenvalues of $-\frac{1}{2}\Delta + V$, i.e.

$$\Big(\frac{1}{2}\Delta - V\Big)\psi_n = -\lambda_n \psi_n,$$

with $(\psi_n)$ an orthonormal basis of $L^2(\mathbb{R}^d, dx)$. For the *ground state* $\psi_1$, it is a standard fact that $\psi_1 > 0$. Then, in particular,

$$u_n(t,x) := \exp(-\lambda t)\psi_n(x)$$

solves

$$\begin{cases} \frac{\partial}{\partial t} u_n = \frac{1}{2}\Delta u_n - V u_n, \\ u_n|_{t=0} = \psi_n. \end{cases}$$

For $f \in C_b(\mathbb{R}^d) \cap L^2(\mathbb{R}^d, dx)$, we have

$$f(x) = \sum_{n \geq 1} c_n \psi_n(x) \quad \text{(convergence uniform and in } L^2\text{)}$$

where

$$c_n = \langle f, \psi_n \rangle_{L^2(\mathbb{R}^d, dx)}.$$

This implies that

$$u(t,x) := \sum_{n \geq 1} c_n \, e^{-\lambda t} \, \psi_n(x),$$

solves

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\Delta u - V u, \\ u|_{t=0} = f. \end{cases}$$

On the other hand, by Feynman-Kac,

$$u(t,x) = \mathbb{E}\left[ f(X_t(x)) \exp\left( - \int_0^t V(X_s(x))\, ds \right) \right].$$

Now let $f \uparrow 1$. Then $c_n = \int \psi_n(x) dx$ (where $c_1 > 0$ since $\psi_1 > 0$), and we get

$$\begin{aligned} u(t,x) &= \mathbb{E}\left[ \exp\left( - \int_0^t V(X_s(x))\, ds \right) \right] \\ &= \sum_{n \geq 1} c_n \, e^{-\lambda_n t} \, \psi_n(x) \\ &\sim c_1 \, e^{-\lambda_1 t} \, \psi_1(x), \quad \text{as } t \to \infty. \end{aligned}$$

COROLLARY 3.19. *Assume that spec$\big(\frac{1}{2}\Delta - V\big)$ is discrete and let*

$$u(t,x) = \mathbb{E}\left[ \exp\left( - \int_0^t V(X_s(x))\, ds \right) \right].$$

*Then*

$$\lim_{t \uparrow \infty} \frac{1}{t} \log u(t,x) = -\lambda_1,$$

*where*

$$\begin{aligned}\lambda_1 &= \inf \operatorname{spec}\Big(-\frac{1}{2}\Delta + V\Big) \\ &= \inf\left\{\Big\langle \psi, (-\frac{1}{2}\Delta + V)\psi\Big\rangle_{L^2(dx)} \,\Big|\, \psi \in C_c^\infty(\mathbb{R}^d),\ \|\psi\|^2_{L^2(dx)} = 1\right\} \\ &= \inf\left\{\int_{\mathbb{R}^d} \Big(\frac{1}{2}|\nabla\psi|^2 + V\psi^2\Big)(x)\,dx \,\Big|\, \psi \in C_c^\infty(\mathbb{R}^d),\ \|\psi\|^2_{L^2(dx)} = 1\right\}.\end{aligned}$$

CHAPTER 4

# Brownian motion and harmonic/holomorphic functions

For simplicity, let $M = \mathbb{R}^n$ and $L = \frac{1}{2}\Delta$. Let $X_t(x) = x + W_t$ be a Brownian motion on $\mathbb{R}^n$ starting from $x$.

## 4.1. Mean value property of harmonic functions

REMARK 4.1. Let $D \subset \mathbb{R}^n$ be a bounded open domain and $h \in C^2(D) \cap C(\bar{D})$ be harmonic on $D$ (i.e. $\Delta h = 0$ on $D$). Then, for each $x \in D$,

$$h(x) = \mathbb{E}\left[h(X_{\tau_D}(x))\right] = \int_{\partial D} h|_{\partial D}\, d\mu_x$$

where $\mu_x := \mathbb{P} \circ X_{\tau_D}(x)^{-1}$ denotes the exit measure of $X_t(x)$ from $D$ and

$$\tau_D = \inf\{t > 0 : X_t(x) \in \partial D\},$$

i.e. $\mu_x(A) = \mathbb{P}\{X_{\tau_D}(x) \in A\}$ for $A \subset \partial D$ measurable.

REMARK 4.2 (*Lévy's characterization of Brownian Motion*). Let $X$ be a continuous semi-martingale taking values in $\mathbb{R}^n$. Then $X$ is a Brownian motion if and only if $X$ is a local martingale with

$$d[X^i, X^j] = \delta_{ij}\, dt \quad \forall i, j.$$

PROOF. The necessity of the condition is obvious. To prove its sufficiency it is enough to check that for each $\xi \in \mathbb{R}^n$ the complex-valued process

$$M_t^\xi := \exp\left(i\langle \xi, X_t\rangle + \frac{1}{2}|\xi|^2 t\right)$$

is a martingale in $\mathbb{C}$. This is straight-forward by means of Itô's formula. □

COROLLARY 4.3. *Let $X$ be a Brownian motion on $\mathbb{R}^n$ and let $G$ be a continuous adapted process taking values in* $\mathrm{Matr}(n \times n; \mathbb{R})$. *Consider the local martingale defined as*

$$(G \cdot X)_t = \int_0^t G_s\, dX_s, \quad t \geq 0.$$

*Then $G \cdot X$ is a Brownian motion on $\mathbb{R}$ if and only if $G$ takes its values in the orthogonal group $O(n)$ almost surely.*

PROOF. Indeed,

$$\begin{aligned} d[(G\cdot X)^i, (G\cdot X)^j] &= \sum_{k,\ell} d(G_{ik}X^k)\, d(G_{j\ell}X^\ell) \\ &= \sum_{k,\ell} G_{ik}G_{j\ell}\, \underbrace{dX^k dX^\ell}_{=\delta_{k\ell}\, dt} = (GG^*)_{ij}\, dt. \end{aligned}$$

□

NOTE 4.4. If $(X_t)_{t\geq 0}$ is a Brownian motion on $\mathbb{R}^n$ and $G \in O(n)$, then $(GX_t)_{t\geq 0}$ is a Brownian motion as well.

COROLLARY 4.5 (Consequences for harmonic functions). *Let*

$$D = B(x,r) = \{y \in \mathbb{R}^n \colon\ |y-x| < r\}$$

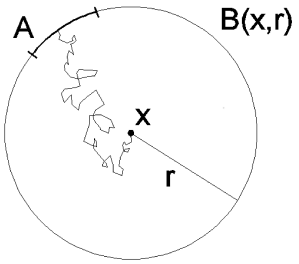
*be the open ball in* $\mathbb{R}^n$ *about* $x$ *of radius* $r$*. Then the exit measure*

$$\mu_x(A) := \mathbb{P}\{X_{\tau_{B(x,r)}}(x) \in A\}, \quad A \subset \partial B(x,r) \textit{ measurable},$$

*satisfies*

$$\mu_x(A) := \mathbb{P}\{x + W_{\tau_{B(x,r)}} \in A\} = \mathbb{P}\{x + GW_{\tau_{B(x,r)}} \in A\}, \quad G \in O(n),$$

*and is hence a rotationally invariant measure on* $\partial B(x,r) = S(x,r) = \{y \in \mathbb{R}^n \colon\ |y-x| = r\}$*. Thus, up to a normalizing constant,* $\mu_x$ *equals the surface measure on the sphere* $S(x,r)$*.*




REMARK 4.6 (Mean value property of the harmonic functions).

Let $h \in C^2((B(x,r)) \cap C(\bar{B}(x,r))$ be harmonic on $B(x,r)$. Then

$$h(x) = \frac{1}{\text{area of } \partial B(x,r)} \int_{\partial B(x,r)} h \, dS$$

where $dS$ is the surface measure on $\partial B(x,r)$, and hence

$$h(x) = \frac{1}{\text{vol}B(x,r)} \int_{B(x,r)} h(y) \, dy.$$

*Question* Why are positive harmonic functions on $\mathbb{R}^n$ constant?

PROPOSITION 4.7 (Liouville property). *Let* $h \in C^2(\mathbb{R}^n)$*. If* $h \geq 0$ *and* $\Delta h = 0$ *on* $\mathbb{R}^n$*, then* $h$ *is constant.*

PROOF. Indeed for $x, y \in \mathbb{R}^n$ fixed, let $R = |x-y| + r$ where $r > 0$. Then

$$\begin{aligned} h(x) &= \frac{1}{\text{vol}B(x,r)} \int_{B(x,r)} h \leq \frac{1}{\text{vol}B(x,r)} \int_{B(y,R)} h \\ &= \frac{\text{vol}B(y,R)}{\text{vol}B(x,r)} h(y) = \frac{\text{vol}B(0,R)}{\text{vol}B(0,r)} h(y) \to h(y), \quad \text{as } r \to \infty. \end{aligned}$$

By exchanging the roles of $x$ and $y$, we obtain $h(y) \leq h(x)$. □

## 4.2. Conformal invariance of Brownian motion

DEFINITION 4.8. Let $D \subset \mathbb{R}^n$ be an open and connected (not necessarily bounded) domain. A function $f\colon D \to \mathbb{R}^n$ is called BPP (*Brownian path preserving*) if for every Brownian motion $X$ on $\mathbb{R}^n$ (starting at $x \in D$) there exists another Brownian motion $\hat{X}$ on $\mathbb{R}^n$ and a continuous non-decreasing $\mathbb{R}_+$-valued process $T_t$ such that

$$f(X_t) = \hat{X}_{T_t}, \quad \forall t < \tau_D(x) \text{ a.s.}$$

This means that $f$ maps Brownian motions to Brownian motions modulo a possible change of the time scale (the paths of $t \mapsto f(X_t)$ are Brownian paths with a different parametrization). The two Brownian motions $X$ and $\hat{X}$ are allowed to run at a different clock.

THEOREM 4.9 (PaulLévy; conformal invariance of Brownian motion). *Let $D \subset \mathbb{R}^2 \mathrel{\widehat{=}} \mathbb{C}$ be an open connected domain. For a non-constant $C^2$-function $f\colon D \to \mathbb{C}$ the following conditions are equivalent:*

(1) *$f$ is* BPP*;*
(2) *$f$ is holomorphic or anti-holomorphic (i.e. $f$ or $\bar{f}$ is holomorphic).*

*In this case, for a Brownian motion $X$ starting in $D$ (i.e. $X_0 \in D$), the time change is given by*

$$T_t := \int_0^t |f'(X_s)|^2 \, ds.$$

Note that $T_t$ is strictly increasing a.s. (Indeed: if $T_\cdot(\omega)$ is constant in some interval $[a, b]$, then $f'(X_\cdot(\omega)|[a, b] = 0$ is $0$; however a holomorphic function which is constant along some non-constant curve must be constant).

PROOF. Write $f = u + iv$ and $X = (X^1, X^2)$. Then by Itô's formula we have

$$\begin{aligned} d(f \circ X) &= \big((\partial_1 f) \circ X\big) dX^1 + \big((\partial_2 f) \circ X\big) dX^2 + \frac{1}{2}\big((\Delta f) \circ X\big)\big) dt \\ &= \underbrace{\left[\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}(X)\right]}_{=: G(X)} \begin{pmatrix} dX^1 \\ dX^2 \end{pmatrix} + \frac{1}{2}\,\Delta f(X)\, dt, \end{aligned}$$

where $u_x$ and $u_y$ denote $\partial_1 u$ and $\partial_2 u$, respectively.

$(1) \Rightarrow (2)$: If $f$ is BPP then, by Lévy's characterization of Brownian motion, $\Delta f(X) = 0$ and $G(X_t)$ takes its values in $\mathrm{O}(2)$ (up to multiplication with an $\mathbb{R}_+$-valued process), i.e.

$$G(X_t) = \Lambda_t \cdot O_t$$

with $O_t \in \mathrm{O}(2)$ a.s. Thus $f$ must be holomorphic or anti-holomorphic (depending on whether $\det O_t \equiv 1$ or $\det O_t \equiv -1$ a.s.). In addition,

$$\Lambda_t^2 = |f'(X_t)|^2.$$

$(2) \Rightarrow (1)$: Now let $f$ be holomorphic, resp. anti-holomorphic. Then

$$\begin{aligned} d\big(f(X)\big) &= \left[\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}(X)\right] \begin{pmatrix} dX^1 \\ dX^2 \end{pmatrix} \\ &= \left[\begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix}(X)\right] \begin{pmatrix} dX^1 \\ \pm dX^2 \end{pmatrix} \end{aligned}$$

where we used the Cauchy-Riemann equations for the partial derivatives of $f$. Note that if $X = X^1 + iX^2$ is a Brownian motion in $\mathbb{C}$, then $\bar{X} = X^1 - iX^2$ is a Brownian motion in $\mathbb{C}$ as well. Hence, without restrictions, we may assume that $f$ is holomorphic. Then

$$d(f \circ X) = (f' \circ X)\, dX = |f'(X)|\, dX^*,$$

where

$$dX^* = \exp(i \arg f'(X))\, dX.$$

Thus $X^*$ is a Brownian motion. Let $t \mapsto \tau_t$ be the inverse of the (a.s. strictly increasing) function

$$t \mapsto T_t := \int_0^t |f'(X_s)|^2\, ds.$$

Note that

$$\tau_t := \inf\{s > 0 : T_s \geq t\}$$

are stopping times for $t < T_\infty = \lim_{t \uparrow \tau_D} T_t$ and that $t \mapsto \tau_t$ is continuous. Then

$$(\hat{X}_t)_{t<T_\infty} := \big(f(X_{\tau_t})\big)_{t<T_\infty}$$

is a Brownian motion. Indeed, for $i, j \in \{1, 2\}$, we have

$$[\hat{X}^i, \hat{X}^j]_t = \delta_{ij}\, T_{\tau_t} = \delta_{ij}\, t.$$

(If $T_\infty < \infty$ with positive probability, the process $\hat{X}$ can be extended to a Brownian motion defined for all times by attaching a piece of an independent Brownian motion). Thus we find

$$\hat{X}_{T_t} = f(X_t), \quad \forall t < \tau_D,$$

as wanted. □

COROLLARY 4.10. *A Brownian motion on $\mathbb{C} \mathrel{\widehat{=}} \mathbb{R}^2$ does not hit any point given in advance (for $t > 0$) with probability* 1.

PROOF. We may assume that the given point is 0 and that the Brownian motion $X$ starts from 1 in $\mathbb{C}$. We consider the image of $X$ under the holomorphic map $\exp\colon \mathbb{C} \to \mathbb{C}\backslash\{0\}$. By Lévy's theorem (Theorem 4.9) there exists a Brownian motion $\hat{X}$ in $\mathbb{C}$ such that

$$\exp(X_t) = \hat{X}_{T_t}$$

with the new clock

$$T_t = \int_0^t |\exp(X_s)|^2\, ds.$$

Hence $\hat{X}_t$ does not hit 0 before time $T_\infty$. We want to show that $T_\infty = \infty$ a.s. Suppose that

$$\int_0^\infty |\exp(X_s)|^2\, ds < \infty$$

with positive probability (then already $< \infty$ a.s. by the 0/1-law). Thus, almost surely,

$$\int_0^\infty \exp(2\mathrm{Re}X_s)\, ds < \infty,$$

and since with $X_t$ also $-X_t$ is a Brownian motion, we get as well $\int_0^\infty \exp(-2\mathrm{Re}X_s)\, ds < \infty$, almost surely. But then

$$\int_0^\infty \big(\exp(2\mathrm{Re}X_s) + \exp(-2\mathrm{Re}X_s)\big)\, ds < \infty, \text{ almost surely,}$$

which is a contradiction, since $e^r + e^{-r} \geq 2$. □

PROPOSITION 4.11. *The planar Brownian motion $(X_t)_{t\geq 0}$ is recurrent, i.e. $X_t$ enters every open disc infinitely often, as $t \to \infty$.*

PROOF. Without restrictions $X_0 = 1$ and $D = \{x \in \mathbb{R}^2 \colon |x| < \delta\}$ for some $\delta < 1$. We show that

$$P := \mathbb{P}\{X_t \text{ enters } D\} = 1.$$

(This gives the claim since Brownian motion starts anew at any stopping time).

Assume that $P < 1$ and consider the holomorphic mapping

$$f : \mathbb{C}\backslash\{0\} \to \mathbb{C}, \quad z \mapsto \frac{1}{z}.$$

By Lévy's theorem (Theorem 4.9), there exists a Brownian motion $\hat{X}$ such that

$$\frac{1}{X_t} = \hat{X}_{T_t} \quad \text{with the new clock } T_t = \int_0^t \frac{1}{|X_s|^4}\, ds.$$

But then, since $\hat{X}$ is also a Brownian motion,

$$1 > P \geq \mathbb{P}\{\hat{X} \text{ enters } D \text{ before } T_\infty\} = \mathbb{P}\left\{1/X_t \text{ enters } D\right\},$$

and thus

$$\mathbb{P}\left\{1/X_t \text{ does not enter } D\right\} > 0$$

which implies

$$\mathbb{P}\left\{|X_t| \text{ bounded by } 1/\delta\right\} > 0.$$

This is in contradiction to Lemma 4.12 below. Hence, we have

$$P = \mathbb{P}\{X_t \text{ enters } D\} = 1. \qquad \square$$

LEMMA 4.12. *Complex Brownian motion almost surely leaves every bounded set.*

PROOF. Fix $R > 0$ and let $X$ be a Brownian motion starting at $0$. Then, for each $t > 0$,

$$\begin{aligned}\mathbb{P}\{|X_t| \leq R\} &= \int_{r=0}^{R} \frac{1}{2\pi t} e^{-r^2/2t} 2\pi r\, dr \\ &= \left[-e^{-r^2/2t}\right]_{r=0}^{r=R} \\ &= 1 - e^{-R^2/2t} \leq R^2/2t.\end{aligned}$$

Let $t_n := n^2$. Then

$$\sum_{n=1}^{\infty} \mathbb{P}\{|X_{t_n}| \leq R\} \leq \frac{R^2}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

and hence by the Borel-Cantelli lemma, with probability one, for all but finitely many $n$ one has $|X_{t_n}| > R$. $\square$

Note that Brownian motion on $\mathbb{R}^2$ hits any disc infinitely often with probability $1$, but will almost surely never touch a point with rational coordinates.

COROLLARY 4.13 (New proof of the Liouville property for $\mathbb{R}^2$). *Let $h \colon \mathbb{R}^2 \to \mathbb{R}$ be harmonic and suppose that $h \geq 0$. Then $h$ is constant.*

PROOF. If not, then for any Brownian motion $X$ on $\mathbb{R}^2$, the image process $h(X_t)$ is a non-negative (local) martingale which is bounded below, hence a supermartingale. Therefore $h(X_t)$ converges almost surely, as $t \to \infty$ (by the martingale convergence theorem, non-negative supermartingales have a limit almost surely). Unless $h$ is a constant function, this is in contradiction to the recurrence of $X$ (the process $h(X_t)$ can only converge if $h$ is constant). □

APPLICATION 4.14 (A Brownian motion proof of $\zeta(2) = \pi^2/6$). Recall that

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_{p \text{ prime}} (1 - p^{-z})^{-1}, \quad z \in \mathbb{C}, \ \operatorname{Re} z > 1.$$

Let $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ be the open unit disc and suppose that $f \colon \mathbb{C} \to \mathbb{C}$ is analytic on a neighbourhood of $\bar{\mathbb{D}}$. This function maps the unit disk $\mathbb{D}$ to $f(\mathbb{D})$ with boundary $\partial f(\mathbb{D}) = f(\partial\mathbb{D})$ where $\partial\mathbb{D} = \{e^{i\theta} : 0 \le \theta < 2\pi\}$.

We develop $f$ on $\mathbb{D}$ as

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{where } a_n \in \mathbb{C}.$$

Without restriction we suppose that $f(0) = a_0 = 0$.

Let $X$ be a two-dimensional Brownian motion starting at $0$. We know that there is another Brownian motion $\hat{X}$ starting at $f(0) = 0$ such that

$$f(X_t) = \hat{X}_{T_t}. \tag{4.1}$$

Let

$$\tau = \inf\{t > 0 \colon \hat{X}_t \in \partial f(\mathbb{D})\}$$

be the first hitting time of $\hat{X}_t$ of the boundary $\partial f(\mathbb{D})$ of $f(\mathbb{D})$. Note that $\partial f(\mathbb{D}) = f(\partial\mathbb{D})$. We claim that

$$\mathbb{E}[\tau] = \frac{1}{2} \sum_{n=1}^{\infty} |a_n|^2. \tag{4.2}$$

Indeed, since $M_t = |\hat{X}_t|^2 - 2t$ is a martingale, the optional stopping theorem gives

$$\mathbb{E}\big[|\hat{X}_\tau|^2\big] = 2\,\mathbb{E}[\tau].$$

By (4.1), up to a time change, the trajectories of $f(X_t)$ are the same as the trajectories of $\hat{X}_t$. In particular, we find that

$$f(X_\sigma) = \hat{X}_\tau$$

where $\sigma = \inf\{t > 0 \colon X_t \in \partial\mathbb{D}\}$ is the first hitting time of $X_t$ of the boundary $\partial\mathbb{D} = S^1$. This shows that

$$\mathbb{E}\big[|\hat{X}_\tau|^2\big] = \mathbb{E}\big[|f(X_\sigma)|^2\big] = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} |f(e^{i\theta})|^2 \, d\theta = \sum_{n \ge 1} |a_n|^2,$$

where for the last equality we used Parseval's theorem. Thus the claim (4.2) follows.

Interesting identities result if we can find domains where the mean Brownian exit time $\mathbb{E}[\tau]$ can be explicitly determined. For instance, consider strips of the type

$$S_a = \{x + iy \colon -a < y < a\}, \quad a > 0.$$

Brownian motion starting at 0 takes on average $\mathbb{E}[\tau] = a^2$ to exit the strip $S_a$. Since

$$f(z) = \log\left(\frac{1-z}{1+z}\right) = -2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + \dots\right)$$

maps the unit disk $\mathbb{D}$ to the strip $S_{\pi/2}$, we obtain

$$\frac{\pi^2}{4} = 2\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots\right).$$

In other words,

$$\frac{\pi^2}{8} = \sum_{r=0}^{\infty} \frac{1}{(2r+1)^2}. \tag{4.3}$$

Since

$$\frac{3}{4}\,\zeta(z) = \zeta(z) - \frac{1}{4}\,\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} - \sum_{n=1}^{\infty} \frac{1}{(2n)^z} = \sum_{r=0}^{\infty} \frac{1}{(2r+1)^2},$$

the formula $\zeta(2) = \pi^2/6$ follows from identity (4.3). □

REMARK 4.15. The two-dimensional Brownian motion is an object with interesting path properties.

i. For any fixed $t \geq 0$, almost surely, a two-dimensional Brownian path contains a closed loop around $X_t$ in every interval of the form $]t, t+\varepsilon[$ (i.e. before time $t+\varepsilon$ the path will perform a closed loop).
ii. (*Cut points*) Almost surely, there exists $t \in ]0,1[$ such that

$$X([0,t[) \cap X(]t,1]) = \varnothing,$$

i.e. the paths of Brownian motion $X|[0,1]$ can be cut into two parts which do not intersect. The time $t$ is called a *cut time* and $X_t$ a *cut point*.

## 4.3. Picard's Little Theorem and the winding of Brownian motion

A holomorphic function $f \colon \mathbb{C} \to \mathbb{C}$ can miss at most one point, otherwise it is constant. In other words, a non-constant holomorphic function $f \colon \mathbb{C} \to \mathbb{C}$ satisfies

$$\#\big(\mathbb{C}\backslash f(\mathbb{C})\big) \leq 1.$$

THEOREM 4.16 (Picard's little theorem). *A holomorphic function $f \colon \mathbb{C} \to \mathbb{C}\backslash\{0,1\}$ must be a constant.*

In the sequel, we want to give a probabilistic proof of Picard's Little Theorem. To this end we study the winding behavior of planar Brownian motion.

I. *Winding of Brownian motion about one point in the plane.*
Consider the "punctured plane" $\mathbb{C}\backslash\{0\}$ and its the fundamental group

$$\pi_1(\mathbb{C}\backslash\{0\}) = \pi_1(\mathbb{R}^2\backslash\{0,0\}) = \pi_1(S^1) = \mathbb{Z}.$$

The exponential function

$$\exp \colon \mathbb{C} \to \mathbb{C}\backslash\{0\}, \quad z \mapsto e^z,$$

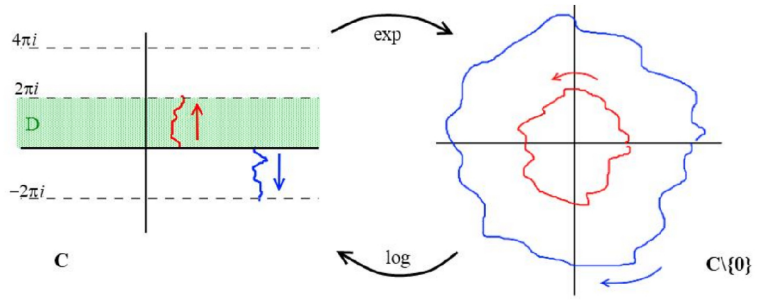
is the universal covering of $\mathbb{C}\backslash\{0\}$. Since exp is $2\pi$-periodic, i.e.

$$e^{z+2\pi ik} = e^z, \quad k \in \mathbb{Z},$$

it divides the complex plane into strips (fundamental domains of the covering) of the form

$$D_k := \{z \in \mathbb{C} \colon 2\pi k \leq \mathrm{Im} z < 2\pi(k+1)\}, \quad k \in \mathbb{Z}.$$

The exponential function maps each strip $D = D_k$ to $\mathbb{C}\backslash\{0\}$. Given a point $x$ in the punctured




plane $\mathbb{C}\backslash\{0\}$, say $x = r\exp(i\varphi)$ with $r > 0$, the preimage of $x$ under the exponential function is

$$\exp^{-1}(x) = \{\log r + i(\varphi + 2k\pi),\ k \in \mathbb{Z}\},$$

where each "fundamental domain" $D_k$ contains exactly one point of the preimage. Moreover if a curve in $\mathbb{C}$ crosses one of the fundamental domains $D_k$ the corresponding path in $\mathbb{C}\backslash\{0\}$ makes a "cycle" about the origin.

Now let $X$ be a Brownian motion in $\mathbb{R}^2$. By Lévy's theorem on conformal invariance of Brownian motion (Theorem 4.9) there exists a Brownian motion $\hat{X}$ on $\mathbb{R}^2$ such that

$$\exp(X_t) = \hat{X}_{T_t} \quad \text{where } T_t = \int_0^t |\exp(X_s)|^2\, ds \uparrow \infty.$$

In particular, $\hat{X}$ does not hit 0 a.s. If $X_t$ upcrosses [downcrosses] $D_k$, then $\hat{X}_t$ turns once counter-clockwise [clockwise] around the origin. Since however $X$ is recurrent, it will enter and exit each $D_k$ infinitely often.

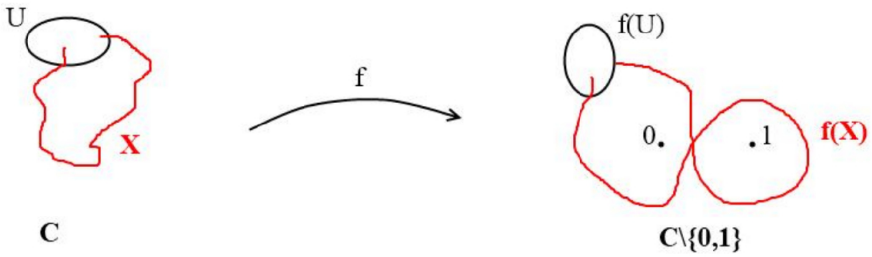
COROLLARY 4.17. *A two-dimensional Brownian motion winds clockwise and anti-clockwise arbitrarily many times about $x = 0$, but returns to the vicinity of the starting point unwound (equal number of clockwise and anti-clockwise turns) infinitely often.*

II. *Winding of Brownian about two points in the plane.*
For prove Picard's little theorem we study the winding behavior of Brownian motion in the double-punctured plane $\mathbb{C}\backslash\{0,1\}$. Recall that we want to show that a holomorphic function $f : \mathbb{C} \to \mathbb{C}\backslash\{0,1\}$ is constant.

*Idea of proof.* By Lévy's theorem on conformal invariance of Brownian motion (Theorem 4.9), the image process $f(X)$ is a time-changed Brownian motion on $\mathbb{C}\backslash\{0,1\}$, and if $f \neq$ const, the time change will be non-trivial. Recall that complex Brownian motion $X$ on $\mathbb{C}$ is recurrent on open sets.

Let $U$ be an open contractible neighbourhood of the starting point $X_0 = x$. If $U$ is sufficiently small and $x$ not a degenerate point of $f$, then $f(U)$ is also open and contractible in $\mathbb{C}\backslash\{0,1\}$.

Since every path of $X$ which returns to $U$ can be contracted relative to $U$, Little Picard's Theorem follows if we can show the following Lemma.

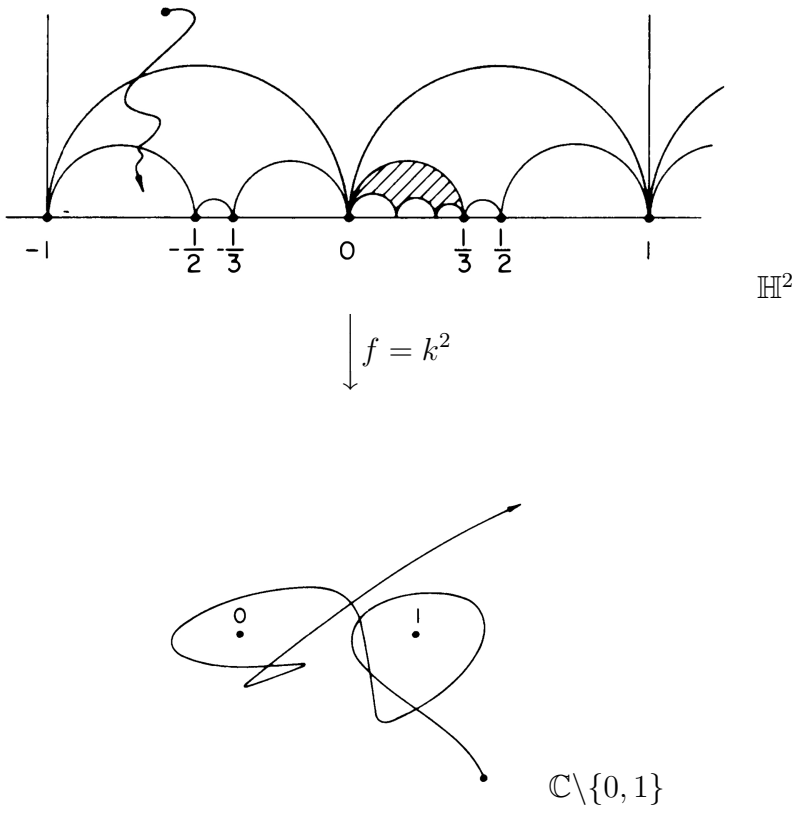



LEMMA 4.18. *There is a random time $T$ such that $\mathbb{P}\{T < \infty\} = 1$ and such that for any $t > T$, if $X_t \in U$ then the path $f(X)$ up to the time $t$ is not contractible in $\mathbb{C}\backslash\{0,1\}$ relative to $f(U)$.*

The lemma can be proved either directly by probability arguments, or it can be deduced from the existence of a holomorphic universal covering of $\mathbb{C}\backslash\{0,1\}$. The same method as above may then be used to study the winding of Brownian motion about two points, 0 and 1, say. The result is however quite different. The universal cover of the twice-punctured plane $\mathbb{C}\backslash\{0,1\}$ is the open upper half-plane, the covering map being the so-called *Jacobi modulus*, usually denoted by $k^2$,

$$\mathbb{H}^2 = \{z \in \mathbb{C} \colon \mathrm{Im}(z) > 0\} \xrightarrow{f=k^2} \mathbb{C}\backslash\{0,1\}.$$

Brownian motion $\hat{X}$ on $\mathbb{C}\backslash\{0,1\}$ does not hit 0 or 1 if $\hat{X}_0 \neq 0, 1$, and as the Jacobi modulus




is analytic, it lifts to a Brownian motion $X$ on the half-plane $\mathbb{H}$, running with a complicated clock $\tau_t$. The lift $X$ is confined to the half-plane, however Brownian motion will eventually try

to cross the real line. Thus $(X_{\tau_t})_{t\geq 0}$ must run with so slow a clock that it approaches a point of the bordering real axis as $t \to \infty$. To achieve this, it has ultimately to leave the fundamental domain where it started from as well as any other fundamental domain, and pass through a series of sheets of the tesselation corresponding to longer and longer words of the fundamental group with the implication that the original Brownian motion on $\mathbb{C}\backslash\{0,1\}$ gets inextricably tangled up in its winding about $0$ and $1$. Unlike the case of the once-punctured plane, the winding of the Brownian path about the two punctures $0$ and $1$ becomes progressively more complicated as $t \to \infty$ and never gets undone.

CHAPTER 5

# Semigroup derivative formulas and computation of the price sensitivities

Let $X \mathrel{\widehat{=}} (X_t)$ be a solution of an SDE on $\mathbb{R}^n$ of the type

$$dX = \underbrace{b(X)}_{n\times 1}\, dt + \underbrace{\sigma(X)}_{n\times r}\underbrace{dW_t}_{r\times 1}, \quad X_0 = x, \tag{5.1}$$

where $W$ is a Brownian motion on $\mathbb{R}^r$. We write again $X_t = X_t(x)$. We suppose that Eq. (5.1) models the time evolution of the asset prices. Consider payoffs of the type $\phi = f(X_T(x))$, where $T > 0$ and where $f\colon \mathbb{R}^n \to \mathbb{R}$ is a bounded and measurable function. Consider option prices of the type

$$P(\phi) = \mathbb{E}[f(X_T(x))]$$

where $x$ represents the current price of the underlying at time $0$.

## 5.1. Greek's Delta

We would like to calculate

$$\Delta_0 = \partial_x P(\phi) \quad (\text{"}\textit{Greek Delta}\text{"}),$$

which represents the sensitivity of the price with respect to the current price and is important for the hedging strategies.

One would like to have a formula of the type

$$\partial_x P(\phi) = \mathbb{E}[f(X_T(x))\, \pi_T]$$

where $\pi_T$ is independent of $f$ and where no derivatives of $f$ are involved in the right-hand side.

*Naive Approach:*

$$\Delta_0 = \partial_x \mathbb{E}[f(X_T(x))] = \mathbb{E}[\partial_x\big(f(X_t(x))\big)] = \mathbb{E}[(\partial_x f)(X_T(x))\, \partial_x(X_T(x))]$$

where $\partial_x X$ would represent the solution to the formally differentiated SDE, i.e.,

$$\begin{aligned} dU &= (\partial b)_X U\, dt + (\partial\sigma)_X U\, dW_t, \\ U_0 &= 1 \quad \text{(unit matrix)}. \end{aligned}$$

*Obvious problems:*

(1) In many situations $f$ is not differentiable, sometimes not even continuous, e.g. $n = 1$, and

$$\begin{aligned} &f(X_T(x)) = (X_T(x) - K)_+ \quad \textit{(European Call)}, \text{ or} \\ &f(X_T(x)) = \mathbb{1}_{[K,\infty[}(X_T(x)) \quad \textit{(digital option)}. \end{aligned}$$

(2) Sometimes the "deterministic horizon" $T$ needs to be replaced by a stopping time:

$$\partial_x \mathbb{E}[f(X_\tau(x))]?$$

For instance, if $\tau = \tau_D(x)$ is the first exit time from some bounded domain $D \subset \mathbb{R}^n$ (*barrier option*), then

$$x \mapsto X_\tau(x)$$

can not be differentiable. Indeed, we have

$$D \ni x \mapsto X_\tau(x) \in \partial D.$$

Note that such a map even cannot be continuous, since there is no continuous retraction of $D$ to the boundary $\partial D$.

*Question* Does this mean that $x \mapsto \mathbb{E}[f(X_\tau(x))]$ is not differentiable? What can then be said about hedging strategies in this case?

Recall that

$$dX = b(X)\,dt + \sigma(X)dW_t \quad \text{(for simplicity } b \text{ and } \sigma \text{ smooth)},$$

where we adopt the *hypothesis* that $(X_t)$ is an elliptic diffusion, i.e.

$$\sigma(x)\colon \mathbb{R}^r \to \mathbb{R}^n \quad \text{surjective for all } x \in \mathbb{R}^n.$$

*Goal* We want to calculate quantities like

$$\Delta_0 = \partial_x P(\phi), \quad \text{where } P(\phi) = \mathbb{E}[\phi(x)],$$

in situations like

$$\begin{aligned}
&\phi(x) = f(X_{T\wedge\tau(x)}(x))\,, \quad \text{or}\\
&\phi(x) = f(X_T(x))\,\mathbb{1}_{\{T<\tau\}}\,, \quad \text{or}\\
&\phi(x) = f(X_\tau(x))\,, \quad \text{etc}
\end{aligned}$$

with $T > 0$ and $\tau > 0$ a first exit time.

**Step 1** The process

$$M_t(x) = \mathbb{E}^{\mathscr{F}_t}[\phi(x)], \quad t \le T,$$

is a (local) martingale expressible as a function of $(t, X_t(x))$ and differentiable in $x$ for $t < T$.

Indeed, consider for instance the case

$$P(\phi) = \mathbb{E}[f(X_{T\wedge\tau}(x))] =: u(T, x)$$

where

$$\tau = \{t > 0 : X_t(x) \in \partial D\}, \quad D \subset \mathbb{R}^n,$$

with $D$ a relatively compact domain. Then $u$ solves the PDE

$$\begin{cases}
\frac{\partial}{\partial T}u = Lu,\\
u|_{T=0} = f,\\
u(T,\cdot)|_{\partial D} = f|_{\partial D},
\end{cases}$$

where

$$L = \sum_{i=1}^n b_i\partial_i + \frac{1}{2}\sum_{i,j=1}^n (\sigma\sigma^*)_{ij}\,\partial_i\partial_j.$$

Hence

$$M_t(x) = u(T - t, X_t(x))$$

(both sides are martingales having the same value at time $T \wedge \tau$).

**Step 2** (Fact from Stochastic Analysis)
If $M_t(x)$ is a family of continuous local martingales (depending smoothly on the parameter $x$), then $\frac{\partial}{\partial x}M(x)$ is also a local martingale.

*Here:* We use this fact as follows. Write $u_t = u(t,\cdot)$, then

$$M_t(x) = u_{T-t}(X_t(x))$$

and for any $v \in \mathbb{R}^n$,

$$N_t := \left\langle \frac{\partial}{\partial x} M_t(x), v \right\rangle = (du_{T-t})_{X_t(x)} (\partial X)_t \, v, \quad t < T \wedge \tau,$$

is a local martingale as well.

**Step 3** (Allow $v$ to vary with time)
Instead of a fixed vector $v \in \mathbb{R}^n$ consider an adapted $\mathbb{R}^n$-valued process $v(t)$ with absolutely continuous paths. Then

$$h_t := (du_{T-t})_{X_t(x)} (\partial X)_t \, v(t) - \int_0^t (du_{T-r})_{X_r(x)} \, (\partial X)_r \underbrace{\dot{v}(r) dr}_{= dv(r)}$$

is a local martingale. We consider

$$M_t^1 := u_{T-t}(X_t(x)) = u(T,x) + \int_0^t (du_{T-r})_{X_r(x)} \, \sigma(X_r(x)) \, dW_r$$

and

$$M_t^2 := \int_0^t \langle \sigma^{-1}(X_t(x))(\partial X)_r \dot{v}(r), dW_r \rangle,$$

where $\sigma^{-1}$ is the right inverse to $\sigma$. Then

$$d(M_t^1 M_t^2) \stackrel{\mathrm{m}}{=} d[M_t^1, M_t^2] = (du_{T-t})_{X_t(x)} (\partial X)_t \, \dot{v}(t) \, dt$$

where $\stackrel{\mathrm{m}}{=}$ denotes equality modulo differentials of local martingales. Hence

$$m_t := (du_{T-t})_{X_t(x)} \, (\partial X)_t \, v(t) - u_{T-t}(X_t(x)) \int_0^t \left\langle \sigma^{-1}(X_r(x))(\partial X)_r \, \dot{v}(r), dW_r \right\rangle$$

is a local martingale as well.

**Step 4** Choose $v(t)$ such that

- $(m_t)$ is a true martingale,
- $v(0) = v$ and $v(T \wedge \tau) = 0$.

Then, by taking expectations, we get

$$\mathbb{E}[m_0] = \mathbb{E}[m_{T \wedge \tau}].$$

THEOREM 5.1. *Let*

$$P(\phi) = \mathbb{E}[\phi(x)] = \mathbb{E}[f(X_{T \wedge \tau}(x))],$$

*and* $v \in \mathbb{R}^n$. *Then for*

$$(dP(\phi))_x v = \nabla_v P(\phi) = \langle (\nabla P \phi)(x), v \rangle$$

*the following formula holds:*

$$dP(\phi)_x v = -\mathbb{E}\left[ \phi(x) \int_0^{T \wedge \tau} \left\langle \sigma^{-1}(X_r(x)) \, (\partial X)_r \, \dot{v}(r), dW_r \right\rangle \right]$$

*where*

- $W$ *is Brownian motion on* $\mathbb{R}^n$,
- $v(r)$ *is an adapted* $\mathbb{R}^n$*-valued process with absolutely continuous paths such that* $v(0) = v$ *and* $v(T \wedge \tau) = 0$,
- *and moreover*

$$\mathbb{E}\left[\left(\int_0^{T\wedge\tau} |\dot{v}(s)|^2\, ds\right)^{1/2}\right] < \infty.$$

EXAMPLE 5.2. Let

$$\tau = \infty, \quad v(s) = \frac{T-s}{T} v, \quad v = e_i\,.$$

Then

$$\frac{\partial}{\partial s} v(s) = -\frac{1}{T} v = -\frac{1}{T} e_i$$

and

$$\partial_i P(\phi(x)) = \frac{1}{T}\mathbb{E}\left[f(X_T(x)) \int_0^T \left\langle \sigma^{-1}(X_t(x))\,(\partial X)_t\, e_i, dW_t \right\rangle\right],$$

or

$$\Delta_0 = \nabla P(\phi)(x) = \frac{1}{T}\,\mathbb{E}\left[f(X_T(x)) \int_0^T (\partial X)_t^*\, \sigma^{-1}(X_t(x))^* dW_t\right].$$

## 5.2. The case of Black and Scholes

For instance, consider the Black-Scholes model ($n = 1$)

$$\frac{dS_t}{S_t} = r_t\, dt + \sigma_t\, dW_t, \quad S_0 = x > 0,$$

with $r_t$ and $\sigma_t$ are deterministic functions and $\inf_t \sigma_t > 0$. Thus

$$\sigma(S_t) = S_t \sigma_t$$

and

$$\frac{\partial}{\partial x} S_t(x) = \frac{1}{x} S_t(x) \mathrel{\widehat{=}} (\partial X)_t.$$

Hence

$$\Delta_0 = \mathbb{E}\left[f(X_T(x)) \frac{1}{xT} \int_0^T \frac{dW_t}{\sigma_t}\right].$$

*Special case* Consider the particular case when $r_t \equiv r$ and $\sigma_t \equiv \sigma$ are constants. Then we recover the classical formula

$$\frac{\partial}{\partial x}\mathbb{E}\left[e^{-rT} f(S_T(x))\right] = \mathbb{E}\left[e^{-rT} f(S_T(x)) \frac{W_T}{x\sigma T}\right].$$

There is no derivative of $f$ in the right-hand-side.

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# Notations

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