Master in Mathematics

Stochastic Analysis and PDE

University of Luxembourg

Homework — To be returned via email by Jan 27 at latest

Proofs given in the course need not be repeated but should be quoted properly.

1. Let X_t be a standard Brownian motion on $\mathbb{R}^2$ and let $u \in C^2(\mathbb{R}^2)$ be a harmonic function, i.e. $\Delta u = 0$ on $\mathbb{R}^2$.

   (a) Evaluate the differential of $u(X_t)$ by means of Itô's formula.

   (b) Suppose that $u \geq 0$. Explain in terms of the Brownian motion X_t why u must be constant.

2. Explain why a 2-dimensional Brownian motion X_t winds clockwise and anti-clockwise arbitrarily many times about the origin, but returns to a neighbourhood of the origin unwound infinitely often.

3. Let $D = B(x; R)$ denote the open ball of radius R in the Euclidean space $\mathbb{R}^n$ centered at x. Assume that $u \in C^2(D) \cap C(\bar{D})$ is harmonic, i.e. $\Delta u = 0$ on D.

   (a) Explain in probabilistic terms why u must be positive on D if u is positive on the boundary ∂D.

   (b) Explain in probabilistic terms why u must take its maximum on the boundary of D.

4. Let $A_0, A_1, \ldots, A_r$ be vector fields on a differentiable manifold M. Consider the partial differential operator on M given as

$$L = A_0 + \frac{1}{2}\sum_{i=1}^{r} A_i^2.$$

   (a) Describe how to construct L-diffusions on M.

   (b) Let D be an open, relatively compact domain in D and suppose that $\tau_D(x) < \infty$ almost surely for each $x \in D$ where $\tau_D(x)$ is the first hitting time of ∂D for the L-diffusion with starting point x. Let $u \in C(\bar{D}) \cap C^2(D)$ be the solution of

$$Lu - u + 1 = 0 \ \text{ on } D, \qquad u(x) = 1 \text{ if } x \in \partial D\,.$$

   Give a representation of u in terms of the L-diffusion.

   (c) Consider the following parabolic equations on M:

$$\text{(i)} \quad \frac{\partial}{\partial t}u = Lu - u + 1, \qquad \text{(ii)} \quad \frac{\partial}{\partial t}u = Lu + \sum_{i=1}^{r} A_i u$$

   on $]0,\infty[\, \times M$ subject to the boundary condition: u is continuous on $[0,\infty[\, \times M$ and $u(0,\cdot) = f$ where $f\colon M \to \mathbb{R}$ is a given bounded measurable function.

   In each of the two cases (i) and (ii) give a stochastic representation of the solution u in terms of an appropriately chosen diffusion process.