Historical milestones of Brownian motion The martingales of Joseph Doob Itô's calculus and the Mathematics of Finance Brownian motion and curved spaces Brownian motion and global geometry Brownian motion and the diffusion of shapes

# Brownian motion: from pollen grains in water to global geometry

# Anton Thalmaier Université du Luxembourg

March 13, 2007

Anton Thalmaier Brownian motion: from pollen grains to global geometry

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# 1

#### Historical milestones of Brownian motion

- Robert Brown
- Albert Einstein
- Louis Bachelier
- Andrei Kolmogorov
- Norbert Wiener

#### The martingales of Joseph Doob

- Stochastic flows and driftless motions
- The heat equation
- The Dirichlet problem

#### Itô's calculus and the Mathematics of Finance

- Stochastic differential equations
- The pricing of options
- 4

#### Brownian motion and curved spaces

- Geodesic random walks and Brownian motion on manifolds
- Recurrence and Transience
- The asymptotic behaviour of Brownian motion

#### Brownian motion and global geometry

- Stochastic parallel transport
- Random holonomy
- Theorem of Gauß-Bonnet-Chern

### 6

#### Brownian motion and the diffusion of shapes

- Brownian motion on the diffeomorphism group of the circle
- Brownian motion and shapes

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Microscopical observations of Robert Brown

Robert Brown Albert Einstein Louis Bachelier Andrei Kolmogorov Norbert Wiener



Robert Brown 1777-1858

 Light particles suspended in water perform under the microscope a rapid oscillatory and highly irregular motion.

"Extremely minute particles of solid matter, whether obtained from organic or inorganic substances, when suspended in pure water, exhibit motions for which I am unable to account ..." (Brown 1827)

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The erratic dance of pollen grains

Robert Brown Albert Einstein

Louis Bachelier Andrei Kolmogorov Norbert Wiener

• Brown did not claim to have discovered the phenomenon.

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- He narrowed down other plausible causes, like temperature gradients, capillary actions, convection currents, etc.

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Summary of the observations

Robert Brown Albert Einstein Louis Bachelier Andrei Kolmogord Norbert Wiener

• The motion is highly irregular, composed of translations and rotations; the trajectory appears to have no tangent.

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- The motion is more active the higher the temperature.

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- The motion is more active the higher the temperature.
- The motion never ceases.

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The period before Einstein

Robert Brown

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• Already from the 1860s on, many scientists worked on the phenomenon.

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- Already from the 1860s on, many scientists worked on the phenomenon.
- By the end of the 19th century a molecular kinetic theory of gases was developed by Clausius, Maxwell and Boltzmann.
   People were battling over the issue whether atoms are real or not.

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- By the end of the 19th century a molecular kinetic theory of gases was developed by Clausius, Maxwell and Boltzmann.
   People were battling over the issue whether atoms are real or not.
- The theory that the random motion of Brownian particles is caused by collisions with the molecules of the liquid appeared in the second half of the 19th century

(Giovanni Cantoni, Joseph Delsaulx, Ignace Carbonelle, ...)

http://math.uni.lu/thalmaier/Inaugural/browianmotion/browianmotion.html

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First experimental tests

Robert Brown Albert Einstein Louis Bachelier Andrei Kolmogorov Norbert Wiener

The kinetic theory that Brownian motion of microscopic particles is caused by bombardment by the molecules of the fluid, appeared to be open to a simple test: the law of equipartition of energy in statistical mechanics implies that the kinetic energy of translation of a particle and of a molecule should be equal.

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- The latter was roughly known (by a determination of Avogadro's number by other means), the mass of a particle could be determined, so all one had to measure was the velocity of a particle in Brownian motion.

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- This was attempted by several experimenters, but the result failed to confirm the kinetic theory as the two values of kinetic energy differed by a factor of about 100,000.

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The puzzle remained

Robert Brown Albert Einstein Louis Bachelier Andrei Kolmogoro Norbert Wiener

• If the scenario that each of the apparently linear displacements of the grain is caused by a collision with a water molecule, is true, then such displacements should appear at time intervals of  $10^{-12}$  seconds.

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- If the scenario that each of the apparently linear displacements of the grain is caused by a collision with a water molecule, is true, then such displacements should appear at time intervals of 10<sup>-12</sup> seconds.
- Our eyes can resolve events that are separated in time by more than 1/30 seconds.

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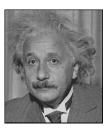
## The puzzle remained

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- Our eyes can resolve events that are separated in time by more than 1/30 seconds.
- The success of Einstein's theory of Brownian motion (1905) was largely due to his circumventing this question. The puzzle was resolved later by Smoluchowski, a contemporary of Einstein.

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Einstein's molecular-kinetic conception of heat

Robert Brown Albert Einstein Louis Bachelier Andrei Kolmogorov Norbert Wiener



Albert Einstein 1777-1858

Einstein (1905), completely unaware of the existence of the phenomenon, and not acquainted with earlier investigations of Boltzmann and Gibbs as well, predicted it on theoretical grounds and formulated a correct quantitative theory of it.

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Annalen der Physik und Chemie 1905, 549–560

Robert Brown Albert Einstein Louis Bachelier Andrei Kolmogorov Norbert Wiener

 Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen; von A. Einstein.

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In dieser Arbeit soll gezeigt werden, daß nach der molekularkinetischen Theorie der Wärme in Flüssigkeiten suspendierte Körper von mikroskopisch sichtbarer Größe infolge der Molekularbewegung der Wärme Bewegungen von solcher Größe ausführen müssen, daß diese Bewegungen leicht mit dem Mikroskop nachgewiesen werden können. Es ist möglich, daß die hier zu behandelnden Bewegungen mit der sogenannten "Brown schen Molekularbewegung" identisch sind; die mir erreichbaren Angaben über letztere sind jedoch so ungenau, daß ich mir hierüber kein Urteil bilden konnte.

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Einstein's main result on particles constantly kicked by lighter water molecules

Einstein's main result can be summarized as follows:

• The mean-square displacement  $\langle R^2 \rangle$  suffered by a spherical Brownian particle, of radius *a*, in time *t* is given by

$$\langle R^2 \rangle = D t$$
 where  $D = \frac{kT}{3\pi N_{av} a\eta}$ ,

T is the temperature,  $\eta$  the viscosity of the fluid, k the Boltzmann constant,  $N_{av}$  the Avogadro number.

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T is the temperature,  $\eta$  the viscosity of the fluid, k the Boltzmann constant,  $N_{av}$  the Avogadro number.

• For the probability density p(t, x) of the position x at time t he derived the diffusion equation

$$rac{\partial}{\partial t} p = D \cdot \Delta p \quad ext{where} \quad \Delta = \partial_1^2 + \partial_2^2 + \partial_3^2 \,.$$

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• Einstein's 1905 paper provided a testing ground for the validity of the molecular kinetic theory.

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- Einstein's 1905 paper provided a testing ground for the validity of the molecular kinetic theory.
- Einstein had a clear idea of the orders of magnitude that would make the movements visible under a microscope.
   For a spherical particle of radius 1 micron, the root-mean square displacement should be of the order of a few microns when observed over a period of one minute.

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- Einstein had a clear idea of the orders of magnitude that would make the movements visible under a microscope.
   For a spherical particle of radius 1 micron, the root-mean square displacement should be of the order of a few microns when observed over a period of one minute.
- The experimental verification of Einstein's theory silenced all skeptics who did not believe in the existence of atoms, who were quite numerous at that time (Ostwald, Mach, ...).

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Marian von Smoluchowski and Jean Perrin

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• Smoluchowski had developed the theory much before Einstein but decided to publish it only when he saw Einstein's paper which contained similar ideas (Smoluchowski 1906).

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- He not only confirmed that the mean-square displacement of the dispersed particles grow with time t, but also made a good estimate of the Avogadro number  $(N_{av} = 6.022 \cdot 10^{23}/\text{mol})$  derived from macroscopic entities.

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- Einstein himself was surprised by the high level of accuracy achieved by Perrin.

"I did not believe that it was possible to study Brownian motion with such a precision"

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Langevin, Ornstein-Uhlenbeck

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• Paul Langevin (1908): Stochastic equation for Brownian motion in an external force field.

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- Ornstein-Uhlenbeck's theory of Brownian (1930)

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- Paul Langevin (1908): Stochastic equation for Brownian motion in an external force field.
- Ornstein-Uhlenbeck's theory of Brownian (1930)
- Conceptual difficulty: The Hamiltonian dynamics is reversible and deterministic. How does the irreversible and chaotic nature fit in this picture?

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### Brownian path in the plan

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### Bachelier's Thesis



Louis Bachelier 1870 - 1946

• On March 29, 1900 Louis Bachelier defended at the Sorbonne his thesis *La théorie de la Spéculation*.

Strongly supported by his supervisor Henri Poincaré, the thesis was published in *Annales Scientifiques de l'École normale Supérieure*.

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Bachelier's Thesis

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# PREMIÈRE THÈSE. THÉORIE LA SPÉCULATION.

#### INTRODUCTION.

Les influences qui déterminent les mouvements de la Bourse sont innombrables, des événements passés, actuels ou même escomptables, ne présentant souvent aucun rapport apparent avec ses variations, se répercutent sur son cours.

A côté des causes en quelque sorte naturelles des variations, interviennent aussi des causes factices : la Bourse agit sur elle-même et le mouvement actuel est fonction, non seulement des mouvements antérieurs, mais aussi de la position de place.

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#### Poincaré's Report

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Bachelier's spectacular work

Robert Brown Albert Einstein Louis Bachelier Andrei Kolmogorov Norbert Wiener

• Mathematical modeling of stock price movements



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• Mathematical modeling of stock price movements



Principle that "the expectation of the speculator is zero"
 "Le marché ne croit, à un instant donné, ni à la hausse, ni à la baisse du cours"

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- Mathematical treatment of Brownian motion, as well as first ideas on Markov processes, diffusions, and even weak convergence in functional spaces

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- Mathematical treatment of Brownian motion, as well as first ideas on Markov processes, diffusions, and even weak convergence in functional spaces
- Calculation of prices for American and path-dependant options

The martingales of Joseph Doob Itô's calculus and the Mathematics of Finance Brownian motion and gubal geometry Brownian motion and the diffusion of shapes

The mathematical career of Bachelier

Robert Brown Albert Einstein Louis Bachelier Andrei Kolmogorov Norbert Wiener

- Bachelier was very active in the period from 1900–1914.
- Nevertheless his work remained in obscurity for decades.
- Blackballed in Dijon 1926



Paul Lévy 1886-1971

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Dijon 1926

Robert Brown Albert Einstein Louis Bachelier Andrei Kolmogorov Norbert Wiener

- "Due to a sequence of incredible circumstances ... I have found myself at the age of 56 in a situation worse than I had during the last six years; this is after twenty-six years with the doctor degree, five years of teaching as free professor at Sorbonne, and six years of official replacement of a full professor ...
- "The critique of M. Lévy is simply ridiculous: ..."
- "M. Lévy pretends not to know my other five large papers published in Annales de l'École normale and in Journal de Mathématiques pures et appliquées as well as various notes published elsewhere. He has written a work of 300 pages on probability without even opening my book on the same subject, ..."

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Kolmogorov's theory of diffusions

Robert Brown Albert Einstein Louis Bachelier Andrei Kolmogorov Norbert Wiener

The axiomatic approach of Kolmogorov made probability probability theory to a rigorous mathematical discipline (*Grundbegriffe der Wahrscheinlichkeitstheorie*, Springer 1933).



Andrei Nikolaevich Kolmogorov 1903-1987

Über die analytischen Methoden in der Wahrscheinlichkeitstheorie. *Math. Annalen* 1931

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The Wiener space

Robert Brown Albert Einstein Louis Bachelier Andrei Kolmogorov Norbert Wiener



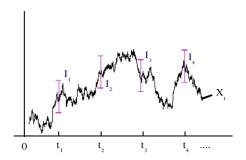
Norbert Wiener 1894-1964

 Rigorous stochastic model of Brownian motion as scaling limit of random walk (Wiener 1923).

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The probability of successful slaloms

Robert Brown Albert Einstein Louis Bachelier Andrei Kolmogorov Norbert Wiener



Starting from the finite dimensional probabilities Wiener constructs a measure on the space  $C(\mathbb{R}_+, \mathbb{R}^n)$  of continuous paths in  $\mathbb{R}^n$ .

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Properties of Brownian motion and the Wiener measure

• This famous *Wiener measure* on the space of trajectories turns out the be a natural substitute of the non-existing Lebesgue measure.

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Properties of Brownian motion and the Wiener measure

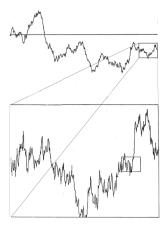
- This famous *Wiener measure* on the space of trajectories turns out the be a natural substitute of the non-existing Lebesgue measure.
- It lives on the continuous paths, and doesn't charge differentiable paths.
- Trajectories of Brownian motion are (with probability 1) nowhere differentiable.

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- It lives on the continuous paths, and doesn't charge differentiable paths.
- Trajectories of Brownian motion are (with probability 1) nowhere differentiable.
- Fractal structure of Brownian motion

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Self-Similarity of Brownian motion

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The modern theory of conditional expectations

Stochastic flows and driftless motions The heat equation The Dirichlet problem



Joseph Doob 1910-2004

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Stochastic flows and driftless motions The heat equation The Dirichlet problem

### Back to Brownian motion:

$$f(X_{t+\Delta t}) - f(X_t) = (\partial_i f)(X_t) \Delta X^i + \frac{1}{2} (\partial_i \partial_j f)(X_t) \underbrace{(\Delta X^i)(\Delta X^j)}_{= \delta_{ij} \Delta t}$$

### In the language of modern probability:

The process

$$N_t = f(X_t) - f(X_0) - \int_0^t \Delta f(X_s) \, ds$$

is a martingale (driftless motion in the sense that conditional expectation of increments gives zero).

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Immediate consequences

Stochastic flows and driftless motions The heat equation The Dirichlet problem

More generally:

• Let *L* be a second order differential operator and let *X* be a process such that

$$N_t = f(X_t) - f(X_0) - \int_0^t Lf(X_s) \, ds$$

is a martingale (X is called *L*-diffusion).

• Let u = u(t, x) be a solution of the heat equation

$$\begin{cases} \frac{\partial}{\partial t}u = Lu\\ u|_{t=0} = f \end{cases}$$



Stochastic flows and driftless motions The heat equation The Dirichlet problem

### Then, the observation that

$$N_{t} = u(T - t, X_{t}) - u(T, X_{0}) - \int_{0}^{t} \underbrace{(\partial_{s} + L)u(T - s, X_{s})}_{= 0} ds$$

is a martingale leads to the equality  $\mathbb{E}[N_{\mathcal{T}}] = \mathbb{E}[N_0] = 0.$ 

### Stochastic representation of the heat equation

 $u(T,x) = \mathbb{E}[f(X_T(x))]$ 

where  $X_t(x)$  is an *L*-diffusion starting from the point x at time 0.

Stochastic flows and driftless motions The heat equation The Dirichlet problem

The Dirichlet problem

Let u be a solution of the Dirichlet problem

 $\begin{cases} Lu = 0 \quad \text{on } D\\ u|_{\partial D} = h. \end{cases}$ 

Then, the martingale

$$N_t = u(X_t) - u(X_0) - \int_0^t \underbrace{Lu(X_s)}_{= 0} ds$$

gives the equality  $\mathbb{E}[N_{\tau}] = \mathbb{E}[N_0] = 0$  where  $\tau = \inf\{t > 0 : X_t \in \partial D\}$  is the first exit time of X from D.

Stochastic flows and driftless motions The heat equation The Dirichlet problem

Stochastic representation of solutions of the Dirichlet problem

 $u(x) = \mathbb{E}[h(X_{\tau}(x))]$ 

where  $X_t(x)$  is a *L*-diffusion starting from the point x at time 0, and  $\tau$  its first hitting time of the boundary.

Indeed,  $u(x) = \mathbb{E}[u(X_{\tau}(x)] = \mathbb{E}[h(X_{\tau}(x)]] = \int_{\partial D} h \, d\mu_x$ , where  $\mu_x(U) = \mathbb{P}\{X(x) \text{ exits } D \text{ through } U\}$ .



### Ito's stochastic differential equations

Stochastic differential equations The pricing of options



Kiyoshi Itô born 1915

### Itô has been awarded the Gauß prize at the ICM in Madrid 2006.

Stochastic differential equations The pricing of options

### Stochastic Differential Equation

$$dX_t = b(t, X_t) dt + \sigma(t, X_t) dW_t$$

where W is a Brownian motion.

### Diffusions to a given operator L

Solutions to this equation give L-diffusions for

$$L = \sum_{i=1}^{n} b_i \partial_i + \frac{1}{2} \sum_{i,j=1}^{n} (\sigma \sigma^*)_{i,j} \partial_i \partial_j$$



Stochastic differential equations The pricing of options

### The evolution of stock prices

The dynamics of the prices on a logarithmic scale, is classically modeled by an SDE of the type

$$\frac{dS_t}{S_t} = b(t, S_t) dt + \sigma(t, S_t) dW_t.$$

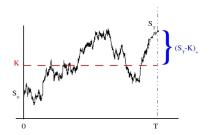
In finance,  $\sigma^2$  is called volatility and corresponds to the *agitation moléculaire* in Statistical Mechanics, while *b* corresponds a macroscopic velocity field.

Applications in finance

Stochastic differential equations The pricing of options

### The pricing of options

A simple example of an option is a *European Call* which gives the owner the right (but not the obligation) to buy one share of the stock at a certain future time T for the strike price K.



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Applications in finance

Stochastic differential equations The pricing of options

### **1** The pricing of options

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Applications in finance

Stochastic differential equations The pricing of options

### **1** The pricing of options

• Value of the option at time T:  $V_T = (S_T - K)_+$ 

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Applications in finance

Stochastic differential equations The pricing of options

### **1** The pricing of options

- Value of the option at time T:  $V_T = (S_T K)_+$
- Value at time 0:  $V_0 = ?$

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Applications in finance

Stochastic differential equations The pricing of options

## **1** The pricing of options

- Value of the option at time T:  $V_T = (S_T K)_+$
- Value at time 0:  $V_0 = ?$

# Interproblem of hedging

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Applications in finance

#### Stochastic differential equations The pricing of options

### **1** The pricing of options

- Value of the option at time T:  $V_T = (S_T K)_+$
- Value at time 0:  $V_0 = ?$

## 2 The problem of hedging

• The seller of the option wants to construct a portfolio of value  $H_t$  at time t that exactly replicates the claim  $V_T$  at time T.

Applications in finance

#### Stochastic differential equations The pricing of options

# **1** The pricing of options

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- Value at time 0:  $V_0 = ?$

# 2 The problem of hedging

- The seller of the option wants to construct a portfolio of value  $H_t$  at time t that exactly replicates the claim  $V_T$  at time T.
- Autofinancing strategy  $dH_t = \delta_t dS_t$  such that  $H_T = V_T$

Applications in finance

#### Stochastic differential equations The pricing of options

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# 8 Remarkable Fact



Stochastic differential equations The pricing of options

# **1** The pricing of options

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- Value at time 0:  $V_0 = ?$

# 2 The problem of hedging

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- Autofinancing strategy  $dH_t = \delta_t dS_t$  such that  $H_T = V_T$

# 8 Remarkable Fact

• Under the simple hypothesis of **absence of arbitrage possibilities** ( *"No free lunch without vanishing risk"*), both problems have a unique and numerically accessible solution.

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Stochastic differential equations The pricing of options

## Conclusion

•  $V_t = H_t = u(t, S_t)$ where u(t, x) is the solution of the PDE

$$\begin{cases} \partial_t u + Lu = 0\\ u(t, x)|_{t=T} = (x - K)_+ \end{cases}$$

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Stochastic differential equations The pricing of options

## Conclusion

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$$\begin{cases} \partial_t u + Lu = 0\\ u(t, x)|_{t=T} = (x - K)_+ \end{cases}$$

• L is the operator

$$Lu(t,x) = \frac{1}{2}\sigma^2(t,x) x^2 \partial_{x,x}^2 u(t,x).$$

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Stochastic differential equations The pricing of options

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$$Lu(t,x) = \frac{1}{2}\sigma^2(t,x) x^2 \partial_{x,x}^2 u(t,x).$$

• The perfect hedging stragegy is given by

$$\delta_t = \frac{\partial}{\partial x} u(t, x) \Big|_{x = S_t}$$

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Stochastic Calculus of Variations

Stochastic differential equations The pricing of options

Paul Malliavin (~ 1976)
 Differential calculus on Wiener space
 "Calcul Stochastique des Variations"

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### Stochastic Calculus of Variations

Stochastic differential equations The pricing of options

- Paul Malliavin (~ 1976) Differential calculus on Wiener space "Calcul Stochastique des Variations"
- Malliavin (1978) Probabilistic proof of Hörmander's hypoellipticity theorem Bismut (1984) Probabilistic approach to the Atiyah-Singer index theorem

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- AMS *Code de Classification* 60H07 Stochastic calculus of variations and the Malliavin calculus

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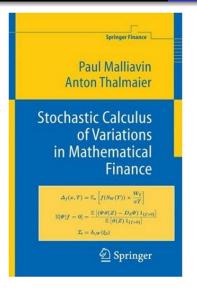
### Stochastic Calculus of Variations

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- Paul Malliavin (~ 1976)
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- AMS *Code de Classification* 60H07 Stochastic calculus of variations and the Malliavin calculus
- P.-L. Lions et al., "Applications of Malliavin calculus to Monte-Carlo methods in finance", *Finance and Stochastics* (1999 et 2001)

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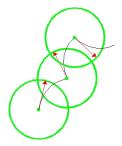


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Geodesic random walks

Geodesic random walks and Brownian motion on manifolds Recurrence and Transience The asymptotic behaviour of Brownian motion

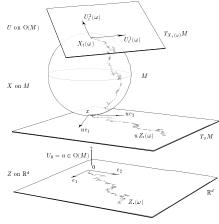
The mechanism underlying Brownian motion easily extends to curved spaces  ${\it M}$ 



Geodesic random walk approximation of Brownian motion

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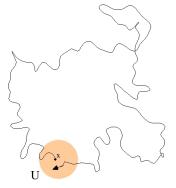


Stochastic development

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Transience or recurrence

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The probability of coming back to the point of departure.

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Transience or recurrence

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•  $M = \mathbb{R}^n$  for  $n \leq 2$ 

With probability 1, Brownian motion comes back infinitely often.

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Transience or recurrence

Geodesic random walks and Brownian motion on manifolds Recurrence and Transience The asymptotic behaviour of Brownian motion

•  $M = \mathbb{R}^n$  for  $n \leq 2$ 

With probability 1, Brownian motion comes back infinitely often.

•  $M = \mathbb{R}^n$  for  $n \ge 3$ 

With probability 1, Brownian motion ultimately drifts off to infinity.

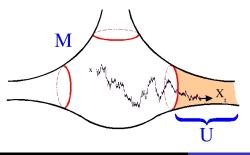
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#### Exit sets

### Definition

An open set  $U \subset M$  is called *non-trivial exit set* for Brownian motion if, with a nontrivial probability, Brownian motion enters the set U ultimately and stays in it.



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### Theorem

For a Riemannian manifold M are equivalent:

- i) There exist non-constant bounded harmonic functions on M.
- ii) BM has non-trivial exit sets, i.e., if X is a Brownian motion on M then there exist open sets U such that

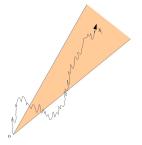
 $\mathbb{P}{X_t \in U \text{ eventually}} \neq 0 \text{ or } 1.$ 

*Idea* The function  $h(x) = \mathbb{P}\{X_t(x) \in U \text{ eventually}\}\$  is harmonic, and non-constant if and only if U a non-trivial exit set.

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## Typical examples of exit sets are angular sectors.



On the Euclidean space ℝ<sup>n</sup> the angular part of Brownian motion is metrically transitive on the sphere.
 On the hyperbolic space ℍ<sup>n</sup> Brownian motion has an asymptotic angle.

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Historical milestones of Brownian motion The martingales of Joseph Doob Itô's calculus and the Mathematics of Finance Brownian motion and curved spaces Brownian motion and global geometry

Brownian motion and the diffusion of shapes

### Stochastic parallel transport

Stochastic parallel transport Random holonomy Theorem of Gauß-Bonnet-Chern



There is a notion of parallel transport along Brownian paths.

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Heat equation on differential forms

Stochastic parallel transport Random holonomy Theorem of Gauß-Bonnet-Chern

• Let 
$$A^{\cdot}(M) := \bigoplus_{p} A^{p}(M)$$
 where  $A^{p}(M) = \Gamma^{p}(\Lambda^{p}T^{*}M)$ ,

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Heat equation on differential forms

Stochastic parallel transport Random holonomy Theorem of Gauß-Bonnet-Chern

• Let 
$$A^{\bullet}(M) := \bigoplus_{p} A^{p}(M)$$
 where  $A^{p}(M) = \Gamma^{p}(\Lambda^{p}T^{*}M)$ ,

 Δ = −(d d\* + d\* d) = □ − R the Hodge-de Rham Laplacian on A<sup>•</sup>(M), and Δ = □ − R its Weitzenböck decomposition

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Heat equation on differential forms

Stochastic parallel transport Random holonomy Theorem of Gauß-Bonnet-Chern

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- For  $a \in A^{\cdot}(M)$  consider the heat flow on differential forms

$$\left\{ egin{array}{l} rac{\partial}{\partial t} a_t = \Delta a_t \ a_t ert_{t=0} = a \end{array} 
ight.$$

Heat equation on differential forms

Stochastic parallel transport Random holonomy Theorem of Gauß-Bonnet-Chern

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- For  $a \in A^{\cdot}(M)$  consider the heat flow on differential forms

$$\begin{cases} \frac{\partial}{\partial t}a_t = \Delta a_t\\ a_t|_{t=0} = a \end{cases}$$

• Then we have the following stochastic representation

 $a_t(x) = \mathbb{E}[Q_t //_t^{-1} a(X_t(x))]$ 

where  $Q_t$  is a random process taking values in the endomorphisms of  $E_x = \Lambda T_x^* M$ , defined in terms of the Weitzenböck curvature term  $\mathcal{R}$ .

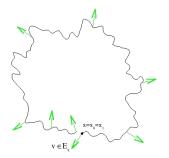
Index theorems and random holonomy

Stochastic parallel transport Random holonomy Theorem of Gauß-Bonnet-Chern

Thus

$$P_t(x,x) = \mathbb{E}[Q_t \, / / t^{-1} | X_t(x) = x] \, p_t(x,x)$$

for the corresponding heat kernel  $P_t(x, y)$  on the diagonal;  $p_t(x, y)$  is the scalar heat kernel on M.



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Local Gauß-Bonnet-Chern

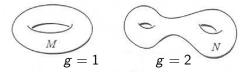
Stochastic parallel transport Random holonomy Theorem of Gauß-Bonnet-Chern

Explicit evaluation leads to the theorem of Gauß-Bonnet-Chern:

 $\lim_{t\downarrow 0} \operatorname{str} P_t(x,x) = E(x)$ 

where  $E(x) \operatorname{vol}(dx)$  is the Euler form.

**Example** n = 2:  $\chi(M) = 2 - 2g$  Euler characteristic



 $\chi(M) = \int E(x) \operatorname{vol}(dx)$  where  $E = \frac{1}{4\pi} K$  (K scalar curvature).

Brownian motion on the diffeomorphism group of the circle Brownian motion and shapes

Brownian motion on  $Diff(S^1)$  and representations of the Virasoro algebra

# **The program** (Paul Malliavin, $\sim$ 1999)

Construction of unitarizing probability measures  $\mu$  for the representation of the Virasoro algebra  $\mathcal V$  :

 $\mathcal{V} \ni u \longmapsto \left( \rho(u) : L^2(\mathcal{M}, \mu) \to L^2(\mathcal{M}, \mu) \right)$ 

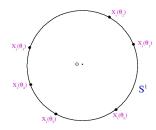
It should be  $\mathcal{M} = \text{Diff}(S^1)/\text{SU}(1,1)$ , and heuristically,

$$\mu = c_0 \exp\left(-cK\right) \, "d\lambda"$$

where  $\lambda$  is the "Lebesgue measure" on  $\mathcal{M}$ .

Brownian motion on the diffeomorphism group of the circle Brownian motion and shapes

• Approach: Construction of  $\mu$  as invariant measure to "Brownian motion on  $\mathcal{M}$  + drift". This leads to the problem of constructing Brownian motion on Diff( $S^1$ ).

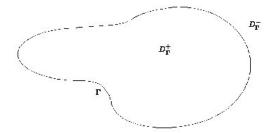


Corresponding *n*-point motion

#### Jordan curves and diffusion of shapes

Brownian motion on the diffeomorphism group of the circle Brownian motion and shapes

# $\mathcal{J} = \{ \Gamma \subset \mathbb{C} : \Gamma \text{ smooth Jordan curves} \}$



 $\Gamma \in \mathcal{J} \longleftrightarrow \exists \varphi : S^1 \to \mathbb{C}$  smooth, injective, and  $\varphi(S^1) = \Gamma$ .  $\Gamma$  splits the plane into two simply connected domains  $D_{\Gamma}^+$ ,  $D_{\Gamma}^-$ .

Brownian motion on the diffeomorphism group of the circle Brownian motion and shapes

Riemann mapping Theorem. Let  $D = \{z \in \mathbb{C} : |z| < 1\}$ .  $\exists F^+ : D \to D_{\Gamma}^+$  biholomorphic; unique mod SU(1,1)  $\exists F^- : D \to D_{\Gamma}^-$  biholomorphic; unique mod SU(1,1). Then (by Caratheodory)

$$F^+ \colon \overline{D} \to \overline{D}_{\Gamma}^+, \quad F^- \colon \overline{D} \to \overline{D}_{\Gamma}^- \text{ diffeomorphisms.}$$

In particular,  $g_{\Gamma} := (F^+)^{-1} \circ F^- | S^1 \in \operatorname{Diff}(S^1).$ 

**Theorem** (*conformal welding*; Beurling-Ahlfors-Letho) The mapping

 $\mathcal{J} \ni \Gamma \mapsto g_{\Gamma} \in \mathrm{Diff}(S^1)$ 

is surjective and induces a canonical isomorphism:

 $\mathcal{J} \cong \mathsf{SU}(1,1) \setminus \mathsf{Diff}(S^1) / \mathsf{SU}(1,1)$ .

This circle of ideas can be used to construct BM on the space  $\mathcal{J}$  of Jordan curves (H. Airault, P. Malliavin, A. Th., 2004).