Brownian motion: from pollen grains in water to global geometry

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March 13, 2007

Microscopical observations of Robert Brown

Robert Brown 1777-1858

- Light particles suspended in water perform under the microscope a rapid oscillatory and highly irregular motion.

  "*Extremely minute particles of solid matter, whether obtained from organic or inorganic substances, when suspended in pure water, exhibit motions for which I am unable to account ...*" (Brown 1827)

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- Brown ruled out vitalism: it holds for live and dead pollens.
- He narrowed down other plausible causes, like temperature gradients, capillary actions, convection currents, etc.

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- The motion never ceases.

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- By the end of the 19th century a molecular kinetic theory of gases was developed by Clausius, Maxwell and Boltzmann. People were battling over the issue whether atoms are real or not.
- The theory that the random motion of Brownian particles is caused by collisions with the molecules of the liquid appeared in the second half of the 19th century

  (Giovanni Cantoni, Joseph Delsaulx, Ignace Carbonelle, ...)

  http://math.uni.lu/thalmaier/Inaugural/browianmotion/browianmotion.html

First experimental tests

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- The latter was roughly known (by a determination of Avogadro's number by other means), the mass of a particle could be determined, so all one had to measure was the velocity of a particle in Brownian motion.
- This was attempted by several experimenters, but the result failed to confirm the kinetic theory as the two values of kinetic energy differed by a factor of about 100,000.

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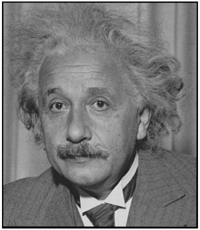
- Our eyes can resolve events that are separated in time by more than $1/30$ seconds.
- The success of Einstein's theory of Brownian motion (1905) was largely due to his circumventing this question. The puzzle was resolved later by Smoluchowski, a contemporary of Einstein.

Einstein's molecular-kinetic conception of heat



Albert Einstein 1777-1858

Einstein (1905), completely unaware of the existence of the phenomenon, and not acquainted with earlier investigations of Boltzmann and Gibbs as well, predicted it on theoretical grounds and formulated a correct quantitative theory of it.

Annalen der Physik und Chemie **1905**, 549–560

5. Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen; von A. Einstein.

In dieser Arbeit soll gezeigt werden, daß nach der molekularkinetischen Theorie der Wärme in Flüssigkeiten suspendierte Körper von mikroskopisch sichtbarer Größe infolge der Molekularbewegung der Wärme Bewegungen von solcher Größe ausführen müssen, daß diese Bewegungen leicht mit dem Mikroskop nachgewiesen werden können. Es ist möglich, daß die hier zu behandelnden Bewegungen mit der sogenannten „Brownschen Molekularbewegung" identisch sind; die mir erreichbaren Angaben über letztere sind jedoch so ungenau, daß ich mir hierüber kein Urteil bilden konnte.

Einstein's main result on particles constantly kicked by lighter water molecules

Einstein's main result can be summarized as follows:

- The mean-square displacement $\langle R^2 \rangle$ suffered by a spherical Brownian particle, of radius a, in time t is given by

$$\langle R^2 \rangle = D\,t \quad \text{where} \quad D = \frac{kT}{3\pi\, N_{av}\, a\eta},$$

T is the temperature, η the viscosity of the fluid,
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  T is the temperature, η the viscosity of the fluid,
  k the Boltzmann constant, N_{av} the Avogadro number.

- For the probability density $p(t,x)$ of the position x at time t he derived the diffusion equation

$$\frac{\partial}{\partial t} p = D \cdot \Delta p \quad \text{where} \quad \Delta = \partial_1^2 + \partial_2^2 + \partial_3^2\,.$$

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  For a spherical particle of radius 1 micron, the root-mean square displacement should be of the order of a few microns when observed over a period of one minute.
- The experimental verification of Einstein's theory silenced all skeptics who did not believe in the existence of atoms, who were quite numerous at that time (Ostwald, Mach, ...).

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- He not only confirmed that the mean-square displacement of the dispersed particles grow with time t, but also made a good estimate of the Avogadro number ($N_{av} = 6.022 \cdot 10^{23}/\mathrm{mol}$) derived from macroscopic entities.
- Einstein himself was surprised by the high level of accuracy achieved by Perrin.
  "*I did not believe that it was possible to study Brownian motion with such a precision*"

Langevin, Ornstein-Uhlenbeck

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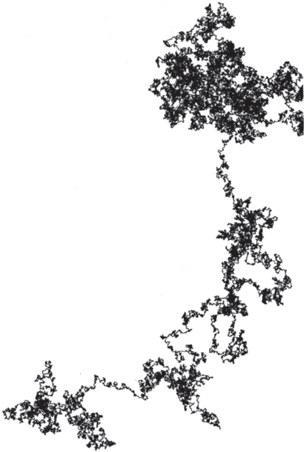
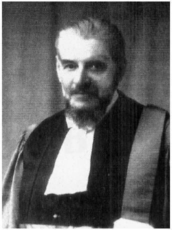
- Conceptual difficulty: The Hamiltonian dynamics is reversible and deterministic. How does the irreversible and chaotic nature fit in this picture?



Brownian path in the plan

Bachelier's Thesis

Louis Bachelier 1870 - 1946

- On March 29, 1900 Louis Bachelier defended at the Sorbonne his thesis *La théorie de la Spéculation*.

  Strongly supported by his supervisor Henri Poincaré, the thesis was published in *Annales Scientifiques de l'École normale Supérieure*.

Bachelier's Thesis

PREMIÈRE THÈSE.

THÉORIE

DE

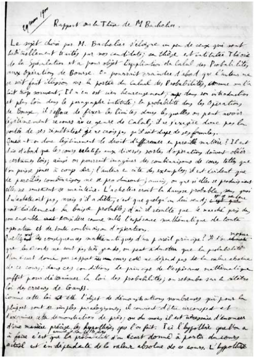
LA SPÉCULATION.

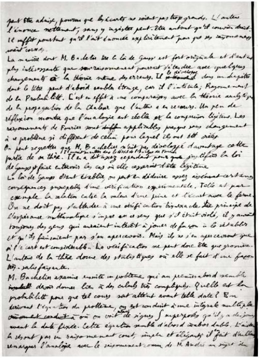
INTRODUCTION.

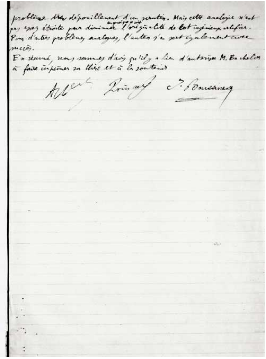
Les influences qui déterminent les mouvements de la Bourse sont innombrables, des événements passés, actuels ou même escomptables, ne présentant souvent aucun rapport apparent avec ses variations, se répercutent sur son cours.

A côté des causes en quelque sorte naturelles des variations, interviennent aussi des causes factices : la Bourse agit sur elle-même et le mouvement actuel est fonction, non seulement des mouvements antérieurs, mais aussi de la position de place.

Poincaré's Report







Bachelier's spectacular work

- Mathematical modeling of stock price movements

$$\text{log Price} = \underbrace{\text{systematic component}}_{\text{drift}} + \underbrace{\text{fluctuative component}}_{\text{Brownian part}}$$

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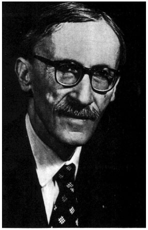
- Calculation of prices for American and path-dependant options

The mathematical career of Bachelier

- Bachelier was very active in the period from 1900–1914.
- Nevertheless his work remained in obscurity for decades.
- Blackballed in Dijon 1926



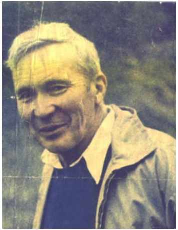
Paul Lévy 1886-1971

Dijon 1926

- "Due to a sequence of incredible circumstances ... I have found myself at the age of 56 in a situation worse than I had during the last six years; this is after twenty-six years with the doctor degree, five years of teaching as free professor at Sorbonne, and six years of official replacement of a full professor ...
- "The critique of M. Lévy is simply ridiculous: ..."
- "M. Lévy pretends not to know my other five large papers published in *Annales de l'École normale* and in *Journal de Mathématiques pures et appliquées* as well as various notes published elsewhere. He has written a work of 300 pages on probability without even opening my book on the same subject, ..."

Kolmogorov's theory of diffusions

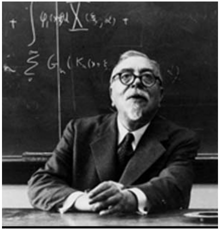
The axiomatic approach of Kolmogorov made probability probability theory to a rigorous mathematical discipline (*Grundbegriffe der Wahrscheinlichkeitstheorie*, Springer 1933).



Andrei Nikolaevich Kolmogorov 1903-1987

Über die analytischen Methoden in der Wahrscheinlichkeitstheorie. *Math. Annalen* 1931

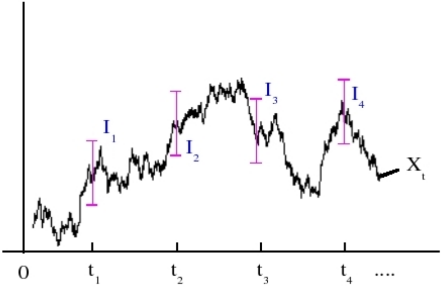
The Wiener space



Norbert Wiener 1894-1964

- Rigorous stochastic model of Brownian motion as scaling limit of random walk (Wiener 1923).

The probability of successful slaloms



Starting from the finite dimensional probabilities Wiener constructs a measure on the space $C(\mathbb{R}_+, \mathbb{R}^n)$ of continuous paths in $\mathbb{R}^n$.

Properties of Brownian motion and the Wiener measure

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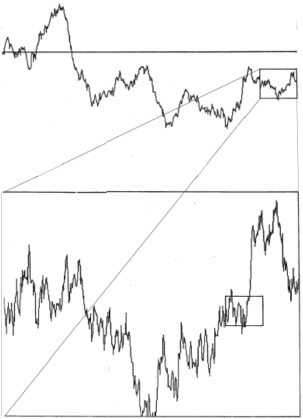
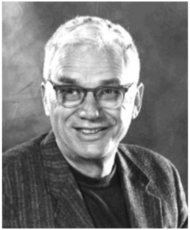
- Fractal structure of Brownian motion



Self-Similarity of Brownian motion

The modern theory of conditional expectations



Joseph Doob 1910-2004

Back to Brownian motion:

$$f(X_{t+\Delta t}) - f(X_t) = (\partial_i f)(X_t)\,\Delta X^i + \frac{1}{2}(\partial_i\partial_j f)(X_t)\,\underbrace{(\Delta X^i)(\Delta X^j)}_{=\,\delta_{ij}\,\Delta t}$$

In the language of modern probability:

The process

$$N_t = f(X_t) - f(X_0) - \int_0^t \Delta f(X_s)\,ds$$

is a martingale (driftless motion in the sense that conditional expectation of increments gives zero).

Immediate consequences

More generally:

- Let L be a second order differential operator and let X be a process such that

$$N_t = f(X_t) - f(X_0) - \int_0^t Lf(X_s)\,ds$$

is a martingale (X is called L-diffusion).

- Let $u = u(t,x)$ be a solution of the heat equation

$$\begin{cases} \frac{\partial}{\partial t} u = Lu \\ u|_{t=0} = f \end{cases}$$

The Heat equation

Then, the observation that

$$N_t = u(T-t, X_t) - u(T, X_0) - \int_0^t \underbrace{(\partial_s + L)u(T-s, X_s)}_{=0}\, ds$$

is a martingale leads to the equality $\mathbb{E}[N_T] = \mathbb{E}[N_0] = 0$.

Stochastic representation of the heat equation

$$u(T, x) = \mathbb{E}[f(X_T(x))]$$

where $X_t(x)$ is an L-diffusion starting from the point x at time 0.

The Dirichlet problem

Let u be a solution of the Dirichlet problem

$$\begin{cases} Lu = 0 \quad \text{on } D \\ u|_{\partial D} = h\,. \end{cases}$$

Then, the martingale

$$N_t = u(X_t) - u(X_0) - \int_0^t \underbrace{Lu(X_s)}_{=0}\, ds$$

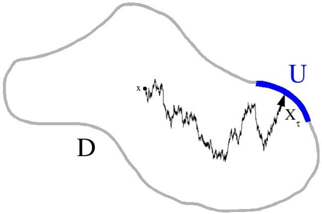
gives the equality $\mathbb{E}[N_\tau] = \mathbb{E}[N_0] = 0$ where $\tau = \inf\{t > 0 : X_t \in \partial D\}$ is the first exit time of X from D.

Stochastic representation of solutions of the Dirichlet problem

$$u(x) = \mathbb{E}[h(X_\tau(x)]$$

where $X_t(x)$ is a L-diffusion starting from the point x at time 0, and τ its first hitting time of the boundary.

Indeed, $u(x) = \mathbb{E}[u(X_\tau(x)] = \mathbb{E}[h(X_\tau(x)] = \int_{\partial D} h \, d\mu_x$,
where $\mu_x(U) = \mathbb{P}\{X(x) \text{ exits } D \text{ through } U\}$.



Ito's stochastic differential equations

Kiyoshi Itô born 1915

Itô has been awarded the Gauß prize at the ICM in Madrid 2006.

Stochastic Differential Equation

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t$$

where W is a Brownian motion.

Diffusions to a given operator L

Solutions to this equation give L-diffusions for

$$L = \sum_{i=1}^{n} b_i\,\partial_i + \frac{1}{2}\sum_{i,j=1}^{n} (\sigma\sigma^*)_{i,j}\,\partial_i\partial_j$$

The evolution of stock prices

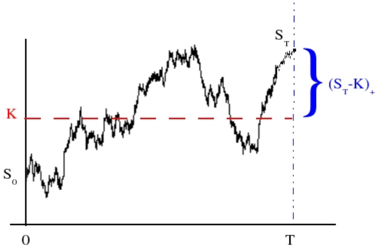
The dynamics of the prices on a logarithmic scale, is classically modeled by an SDE of the type

$$\frac{dS_t}{S_t} = b(t, S_t)\, dt + \sigma(t, S_t)\, dW_t.$$

In finance, σ^2 is called volatility and corresponds to the *agitation moléculaire* in Statistical Mechanics, while b corresponds a macroscopic velocity field.

The pricing of options

A simple example of an option is a *European Call* which gives the owner the right (but not the obligation) to buy one share of the stock at a certain future time T for the strike price K.



Applications in finance

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   - Under the simple hypothesis of **absence of arbitrage possibilities** (*"No free lunch without vanishing risk"*), both problems have a unique and numerically accessible solution.

Conclusion

- $V_t = H_t = u(t, S_t)$
  where $u(t,x)$ is the solution of the PDE

$$\begin{cases} \partial_t u + Lu = 0 \\ u(t,x)|_{t=T} = (x-K)_+ \end{cases}$$

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- The perfect hedging stragegy is given by

$$\delta_t = \frac{\partial}{\partial x} u(t,x)\Big|_{x=S_t}.$$

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  Differential calculus on Wiener space
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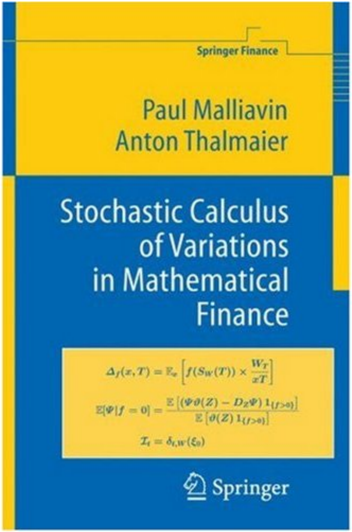
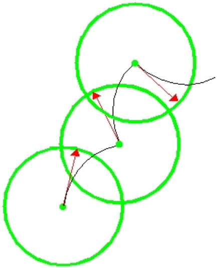
- P.-L. Lions et al., "Applications of Malliavin calculus to Monte-Carlo methods in finance", *Finance and Stochastics* (1999 et 2001)

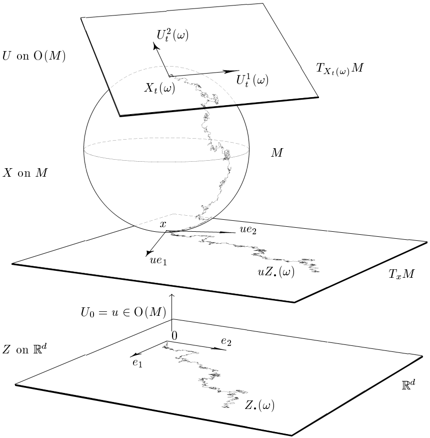

Springer Finance
Paul Malliavin
Anton Thalmaier
Stochastic Calculus of Variations in Mathematical Finance
Springer

Geodesic random walks

The mechanism underlying Brownian motion easily extends to curved spaces M

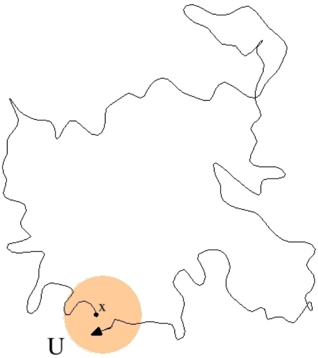


Geodesic random walk approximation of Brownian motion



Stochastic development

Transience or recurrence



The probability of coming back to the point of departure.

Transience or recurrence

- $M = \mathbb{R}^n$ for $n \leq 2$
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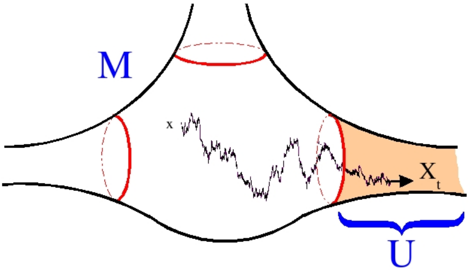
- $M = \mathbb{R}^n$ for $n \leq 2$
  With probability 1, Brownian motion comes back infinitely often.
- $M = \mathbb{R}^n$ for $n \geq 3$
  With probability 1, Brownian motion ultimately drifts off to infinity.

Exit sets

Definition

An open set $U \subset M$ is called *non-trivial exit set* for Brownian motion if, with a nontrivial probability, Brownian motion enters the set U ultimately and stays in it.



Theorem

For a Riemannian manifold M are equivalent:

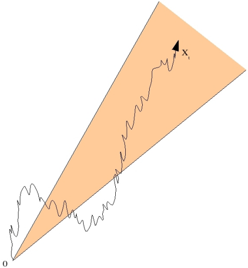
i) *There exist non-constant bounded harmonic functions on M.*

ii) BM *has non-trivial exit sets, i.e., if X is a Brownian motion on M then there exist open sets U such that*

$$\mathbb{P}\{X_t \in U \text{ eventually}\} \neq 0 \text{ or } 1.$$

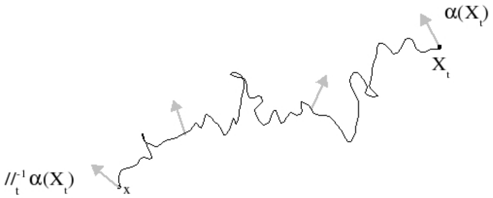
Idea The function $h(x) = \mathbb{P}\{X_t(x) \in U \text{ eventually}\}$ is harmonic, and non-constant if and only if U a non-trivial exit set.

Typical examples of exit sets are *angular sectors*.



- On the Euclidean space $\mathbb{R}^n$ the angular part of Brownian motion is metrically transitive on the sphere.
  On the hyperbolic space $\mathbb{H}^n$ Brownian motion has an asymptotic angle.

Stochastic parallel transport



There is a notion of parallel transport along Brownian paths.

Heat equation on differential forms

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- For $a \in A^{\cdot}(M)$ consider the heat flow on differential forms

$$\begin{cases} \frac{\partial}{\partial t} a_t = \Delta a_t \\ a_t\big|_{t=0} = a \end{cases}$$

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$$\begin{cases} \frac{\partial}{\partial t} a_t = \Delta a_t \\ a_t\big|_{t=0} = a \end{cases}$$

- Then we have the following stochastic representation

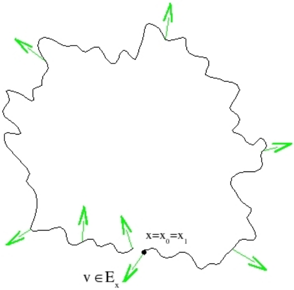
$$a_t(x) = \mathbb{E}[Q_t \, //_t^{-1} \, a(X_t(x))]$$

where Q_t is a random process taking values in the endomorphisms of $E_x = \Lambda T_x^* M$, defined in terms of the Weitzenböck curvature term $\mathcal{R}$.

Thus

$$P_t(x,x) = \mathbb{E}[Q_t \, //_t^{-1} | X_t(x) = x] \, p_t(x,x)$$

for the corresponding heat kernel $P_t(x,y)$ on the diagonal; $p_t(x,y)$ is the scalar heat kernel on M.



Local Gauß-Bonnet-Chern

Explicit evaluation leads to the theorem of Gauß-Bonnet-Chern:

$$\lim_{t\downarrow 0} \operatorname{str} P_t(x,x) = E(x)$$

where $E(x)\,\mathrm{vol}(dx)$ is the Euler form.

Example $n=2$: $\chi(M) = 2-2g$ *Euler characteristic*

$g=1$ $g=2$

$\chi(M) = \int E(x)\,\mathrm{vol}(dx)$ where $E = \frac{1}{4\pi}K$ (K scalar curvature).

Brownian motion on Diff(S^1) and representations of the Virasoro algebra

The program (Paul Malliavin, ∼1999)

Construction of unitarizing probability measures μ for the representation of the Virasoro algebra $\mathcal{V}$:

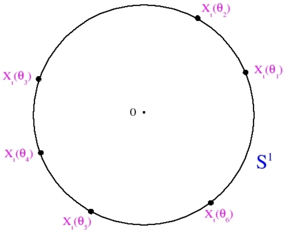
$$\mathcal{V} \ni u \longmapsto \big(\rho(u) : L^2(\mathcal{M}, \mu) \to L^2(\mathcal{M}, \mu)\big)$$

It should be $\mathcal{M} = \mathrm{Diff}(S^1)/\mathrm{SU}(1,1)$, and heuristically,

$$\mu = c_0 \exp\left(-cK\right) \text{“}d\lambda\text{”}$$

where λ is the “Lebesgue measure” on $\mathcal{M}$.

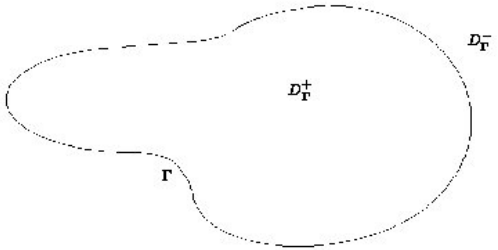
- Approach: Construction of μ as invariant measure to "Brownian motion on $\mathcal{M}$ + drift". This leads to the problem of constructing Brownian motion on Diff(S^1).



Corresponding n-point motion

Jordan curves and diffusion of shapes

$\mathcal{J} = \{\Gamma \subset \mathbb{C} : \Gamma \text{ smooth Jordan curves}\}$



$\Gamma \in \mathcal{J} \longleftrightarrow \exists\, \varphi : S^1 \to \mathbb{C}$ smooth, injective, and $\varphi(S^1) = \Gamma$.
Γ splits the plane into two simply connected domains D_Γ^+, D_Γ^-.

Riemann mapping Theorem. Let $D = \{z \in \mathbb{C} : |z| < 1\}$.
$\exists\, F^+ \colon D \to D_\Gamma^+$ *biholomorphic*; unique mod $\mathrm{SU}(1,1)$
$\exists\, F^- \colon D \to D_\Gamma^-$ *biholomorphic*; unique mod $\mathrm{SU}(1,1)$.

Then (by Caratheodory)

$$F^+ \colon \bar{D} \to \bar{D}_\Gamma^+, \quad F^- \colon \bar{D} \to \bar{D}_\Gamma^- \text{ diffeomorphisms.}$$

In particular, $\quad g_\Gamma := (F^+)^{-1} \circ F^- | S^1 \in \mathrm{Diff}(S^1)$.

Theorem (*conformal welding*; Beurling-Ahlfors-Letho)
The mapping

$$\mathcal{J} \ni \Gamma \mapsto g_\Gamma \in \mathrm{Diff}(S^1)$$

is surjective and induces a canonical isomorphism:

$$\mathcal{J} \cong \mathrm{SU}(1,1) \backslash \mathrm{Diff}(S^1) / \mathrm{SU}(1,1)\,.$$

This circle of ideas can be used to construct BM on the space $\mathcal{J}$ of Jordan curves (H. Airault, P. Malliavin, A. Th., 2004).