Available online at www.sciencedirect.com

ScienceDirect

Journal of Functional Analysis 259 (2010) 3037–3079

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Brownian measures on Jordan–Virasoro curves associated to the Weil–Petersson metric

Hélène Airault [a,b], Paul Malliavin [c], Anton Thalmaier [d,*]

[a] INSSET, Université de Picardie, 48 rue Raspail, 02100 Saint Quentin (Aisne), France
[b] Laboratoire CNRS UMR 6140, LAMFA, 33 rue Saint-Leu, 80039 Amiens, France
[c] 10 rue Saint Louis en l'Isle, 75004 Paris, France
[d] Unité de Recherche en Mathématiques, FSTC, Université du Luxembourg, 6 rue Richard Coudenhove-Kalergi, L-1359 Luxembourg

Received 26 April 2010; accepted 5 August 2010

Available online 1 September 2010

Communicated by Daniel W. Stroock

Dedicated to Len Gross for his friendship

Abstract

In this paper existence of the Brownian measure on Jordan curves with respect to the Weil–Petersson metric is established. The step from Brownian motion on the diffeomorphism group of the circle to Brownian motion on Jordan curves in $\mathbb{C}$ requires probabilistic arguments well beyond the classical theory of conformal welding, due to the lacking quasi-symmetry of canonical Brownian motion on $\mathrm{Diff}(S^1)$. A new key step in our construction is the systematic use of a Kählerian diffusion on the space of Jordan curves for which the welding functional gives rise to conformal martingales, together with a Douady–Earle type conformal extension of vector fields on the circle to the disk.

Keywords: Jordan curves; Conformal welding; Weil–Petersson metric; Douady–Earle extension; Group of diffeomorphisms; Kähler Brownian motion; Shape representation

Contents

* Corresponding author.
E-mail addresses: helene.airault@u-picardie.fr (H. Airault), anton.thalmaier@uni.lu (A. Thalmaier).

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2010.08.002

I. Kählerian Geometry on the space of C^∞ Jordan curves

A similar topic has been discussed in [5]. In August 2009 Antti Kupiainen pointed out to us that the Hölderianity stated in Section 6 of [5] is in fact not proved there. In the present paper we establish from scratch existence of the Brownian measure on Jordan curves for the Weil–Petersson metric.

New key steps of our paper are: i) the use of a Kählerian diffusion on the space of Jordan curves for which the welding functionals give rise to conformal martingales; ii) the construction of a Douady–Earle type extension of vector fields from the circle to the disk. We thank Antti Kupiainen cordially for his careful reading of [5], which has been at the origin of the present work. We also like to mention the interesting paper [8] which constructs probability measures on Jordan curves by a global approach; this method is quite different from the infinitesimal approach based on a stochastic Loewner equation which is used here.

Our work is contiguous to several branches of Mathematics: *SLE theory* (see for instance [25]); Mumford's *theory of vision* [34]; *representations of Virasoro algebra* [21,2,3,22]; *Stochastic Differential Geometry on infinite dimensional homogeneous spaces* [12,4,16]; *stochastic flows under low regularity assumptions* [28,26,6,13,14,33]; *stochastic PDE theory* as developed further in this paper has been started in [5], as resolution of the non-linear Beltrami PDE by a continuity method along a stochastic flow.

Our paper is limited to a short and self-contained proof of the result indicated in the title.

1. Structure of homogeneous Kähler manifold on C^∞ Jordan curves

A *Jordan curve* in the complex plane $\mathbb{C}$ is a closed subset $\Gamma \subset \mathbb{C}$ for which there exists a continuous injective map $\phi : S^1 \to \mathbb{C}$ of the circle S^1 satisfying $\phi(S^1) = \Gamma$. Such a parametrization ϕ is not unique: given two parametrizations ϕ_1, ϕ_2 of the same Jordan curve, there exists a homeomorphism h of S^1 such that $\phi_2 = \phi_1 \circ h$. Two parametrizations define the same orientation of Γ if h is an orientation preserving homeomorphism of the circle S^1. The inconvenience of this point of view is that indeterminacy in the parametrization depends on an element of an infinite dimensional group, namely the group of homeomorphisms of the circle.

The *holomorphic parametrization* is constructed in the following way: the complement of Γ in $\mathbb{C}$ is the union of two connected open subsets Γ^+ and Γ^- where Γ^+ is bounded and Γ^- is unbounded. The Riemann mapping theorem gives the existence of a conformal map

$$f_\Gamma \text{ of the open unit disk } D \text{ onto } \Gamma^+,$$

defined up to composition with an element of the form

$$z \mapsto \frac{az+b}{\bar{b}z+\bar{a}}, \quad a, b \in \mathbb{C}, \quad |a|^2 - |b|^2 = 1. \tag{1.1}$$

By a theorem of Caratheodory [10], see also [31], f_Γ has a continuous injective extension to the closure $\bar{D}$ of D, also denoted f_Γ, and $f_\Gamma|S^1$ gives a parametrization of Γ defined canonically up to a transformation of the form (1.1). The advantage of the holomorphic parametrization is that its indeterminacy corresponds to the finite dimensional Poincaré group H of Möbius transformations $z \mapsto (az+b)/(\bar{b}z+\bar{a})$.

Now consider Γ^- and let $D^- = \{z\colon |z| > 1\}$ be the open exterior of the unit disk. There exists a univalent function

$$h_\Gamma : D^- \to \Gamma^-, \qquad h_\Gamma(\infty) = \infty, \tag{1.2}$$

being uniquely defined up to a Möbius transformation of D^- preserving ∞, that is up to a rotation. We eliminate this ambiguity by the extra normalization that

$$\lim_{z\to\infty} \frac{h_\Gamma(z)}{z} \text{ is a positive number.} \tag{1.3}$$

As above, by the theorem of Caratheodory, h_Γ extends to the closures and $h_\Gamma|S^1$ also provides a parametrization of Γ.

Let G be the group of orientation preserving homeomorphisms of the circle $S^1 = \partial D$; further let H be the group of Möbius transformation $z \mapsto (az+b)/(\bar{b}z+\bar{a})$ of the unit disk D restricted to the boundary ∂D. Then H is a subgroup of G; we consider the homogeneous space

$$\mathscr{M} := H\backslash G. \tag{1.4}$$

Theorem 1.1. *Let $\hat{f}_\Gamma, \hat{h}_\Gamma$ be the restrictions of f_Γ, h_Γ to ∂D. The correspondence*

$$\Gamma \mapsto \hat{f}_\Gamma^{-1} \circ \hat{h}_\Gamma \text{ defines a map } \Theta : \mathcal{J} \mapsto \mathscr{M} \tag{1.5}$$

where $\mathcal{J}$ denotes the set of Jordan curves.

Proof. The indeterminacy on f_Γ through a Möbius transformation appearing on the right is equivalent to the indeterminacy on f_Γ^{-1} through a Möbius appearing on the left. □

Remark 1.2. Following Sharon and Mumford [34] we introduce the *space of shapes* $\mathscr{S}$ as the orbit space of $\mathcal{J}$ under the action of the affine group

$$z \mapsto \alpha z + \beta, \quad \alpha, \beta \in \mathbb{C},\ \alpha \neq 0. \tag{1.6}$$

We shall use the following result proved in [34]:

$$\Theta \textit{ realizes an injection of } \mathscr{S} \textit{ into } \mathscr{M}, \tag{1.7}$$

which means that $\Theta(\Gamma) = \Theta(\Gamma')$ if and only $\Gamma = \alpha\Gamma' + \beta$.

We shall say that a Jordan curve is C^∞ if its holomorphic parametrization together with its inverse are C^∞; we denote by $\mathcal{J}^\infty$ the set of C^∞ Jordan curves. We say that Γ is Hölderian if f_Γ is Hölderian together with its inverse; we denote by $\mathcal{J}^{\mathrm{h}}$ the set of Hölderian Jordan curves. In the same way we denote by G^∞, G^{h} the groups of C^∞ diffeomorphisms of S^1, respectively Hölderian homeomorphisms of S^1. Letting Θ^∞, Θ^{h} be the restrictions of Θ to $\mathcal{J}^\infty$, respectively $\mathcal{J}^{\mathrm{h}}$, then

$$\Theta^\infty : \mathcal{J}^\infty \mapsto G^\infty, \qquad \Theta^{\mathrm{h}} : \mathcal{J}^{\mathrm{h}} \mapsto G^{\mathrm{h}}. \tag{1.8}$$

Problem 1.3. The C^∞-*welding problem* is the following problem: given $g \in G^\infty$, find univalent functions f, h defined on the closures of D, resp. D^-, such that

$$\hat{f}^{-1} \circ \hat{h} = g. \tag{1.9}$$

It is a classical fact that the C^∞-welding problem has a solution (see [1] and also the examples given in [17, Sections 5 and 6]); therefore the map Θ^∞ is surjective and

$$\Theta^\infty \textit{ realizes a bijection of } \mathscr{S}^\infty \textit{ onto } \mathscr{M}^\infty := H\backslash G^\infty. \tag{1.10}$$

Denote by $\mathfrak{g} = \mathrm{diff}(S^1)$ the *right invariant* Lie algebra of G^∞ constituted by smooth vector fields on S^1. The identification of smooth vector fields and smooth functions on S^1 by the formula $u \mapsto u(\theta)\frac{d}{d\theta}$ identifies $\mathfrak{g}$ and $C^\infty(S^1)$. In terms of this identification the Lie bracket is transferred to the following expressions:

$$[u, v](\theta) = u(\theta)v'(\theta) - u'(\theta)v(\theta). \tag{1.11}$$

In the trigonometric basis the bracket has an easy expression: for instance

$$2[\cos k\theta, \cos p\theta] = (k-p)\sin(k+p)\theta + (k+p)\sin(k-p)\theta.$$

$$\textit{The Lie algebra } \mathfrak{h} \textit{ of } H \textit{ has the basis } 1, \cos\theta, \sin\theta. \tag{1.12}$$

The Weil–Petersson metric is the unique Hilbertian metric on $\mathfrak{g}$ invariant under the adjoint action of $\mathfrak{h}$. The system

$$\frac{\cos k\theta}{\sqrt{k^3 - k}}, \qquad \frac{\sin k\theta}{\sqrt{k^3 - k}}, \quad k > 1, \tag{1.13}$$

is orthonormal for the Weil–Petersson metric and induces on $\mathscr{M}^\infty$ the structure of an infinite dimensional Riemannian manifold. The Hilbert transform is defined by

$$J \cos k\theta = \sin k\theta, \qquad J \sin k\theta = -\cos k\theta \quad \text{for } k \geqslant 1. \tag{1.14}$$

As usual in harmonic analysis, we take the Hilbert transform of a constant function equal to zero. The Hilbert transform possesses the Nijenhuis property with respect to the Lie bracket (1.11): for $u, v \in \mathfrak{g}$,

$$[Ju, Jv] - [u, v] = J\big([u, Jv] + [Ju, v]\big). \tag{1.15}$$

As $J^2 = -1$, the Hilbert transform defines on $\mathfrak{g}_0$ (= the quotient of $\mathfrak{g}$ by the constant functions) a completely integrable complex structure.

It has been proved (see [4]) that

$$\mathscr{M}^\infty \textit{ has the structure of a complex Kähler manifold.} \tag{1.16}$$

Set

$$\Delta := \frac{1}{2} \sum_{k>1} \frac{1}{\sqrt{k^3 - k}} \big(\partial^2_{\cos k\theta} + \partial^2_{\sin k\theta}\big); \tag{1.17}$$

then Δ is an elliptic operator on $\mathscr{M}^\infty$. We regularize Δ by introducing

$$\Delta^r = \sum_{k>1} \frac{r^k}{k^3 - k} \big(\partial^2_{\cos k\theta} + \partial^2_{\sin k\theta}\big), \quad r \in]0, 1]. \tag{1.18}$$

Theorem 1.4. *The operators Δ^r have the following properties*:

1. *In the exponential chart, Δ^r do not involve first order derivative terms.*
2. *Any holomorphic functional Φ on $\mathcal{J}^\infty$ satisfies*

$$\Delta^r \Phi = 0. \tag{1.19}$$

Proof. The differential $d\Phi$ defines a linear form on the tangent space; the second order differential defines a bilinear form on the tangent space, or equivalently a linear form on the tensor product of the tangent space by itself. Using these notations we have:

$$\partial_{\cos k\theta}\big(\langle \cos k\theta, d\Phi\rangle\big) = -k\langle \cos k\theta \sin k\theta, d\Phi\rangle + \big\langle \cos k\theta \otimes \cos k\theta, d^2\Phi\big\rangle;$$
$$\partial_{\sin k\theta}\big(\langle \sin k\theta, d\Phi\rangle\big) = k\langle \sin k\theta \cos k\theta, d\Phi\rangle + \big\langle \sin k\theta \otimes \sin k\theta, d^2\Phi\big\rangle;$$
$$\partial_{\cos k\theta}\big(\langle \cos k\theta, d\Phi\rangle\big) + \partial_{\sin k\theta}\big(\langle \sin k\theta, d\Phi\rangle\big) = \big\langle \cos k\theta \otimes \cos k\theta, d^2\Phi\big\rangle + \big\langle \sin k\theta \otimes \sin k\theta, d^2\Phi\big\rangle.$$

For example, the first of these equations is obtained as follows: with the identification (1.8), we take $g_\varepsilon(\theta) = \theta + \varepsilon \cos(k\theta)$, then

$$\langle \cos k\theta, d\Phi\rangle = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \Phi\big(\theta + \varepsilon \cos(k\theta)\big)$$

and

$$\partial_{\cos k\theta}\big(\langle \cos k\theta, d\Phi\rangle\big) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \big\langle \cos\big(k\big(\theta+\varepsilon\cos(k\theta)\big)\big), d\Phi\big(\theta+\varepsilon\cos(k\theta)\big)\big\rangle.$$

In the same way,

$$\partial_{\sin k\theta}\big(\langle \cos k\theta, d\Phi\rangle\big) = -k\big\langle \sin^2 k\theta, d\Phi\big\rangle + \big\langle \cos k\theta \otimes \sin k\theta, d^2\Phi\big\rangle,$$
$$\partial_{\cos k\theta}\big(\langle \sin k\theta, d\Phi\rangle\big) = k\big\langle \cos^2 k\theta, d\Phi\big\rangle + \big\langle \cos k\theta \otimes \sin k\theta, d^2\Phi\big\rangle.$$

According to (1.14) the $\bar\partial$ operator on $\mathcal{J}^\infty$ (see [29]) corresponds to

$$\bar\vartheta_p := \partial_{\cos p\theta} + \sqrt{-1}\partial_{J(\cos p\theta)} = \partial_{\cos p\theta} + \sqrt{-1}\partial_{\sin p\theta}; \tag{1.20}$$
$$\Re[\vartheta_p, \bar\vartheta_p] \text{ vanishes on } \mathcal{J}^\infty, \quad p > 1, \tag{1.21}$$

where $\Re z$ denotes the real part of a complex number z; therefore

$$4\Delta^r = \sum_{p>1} \frac{r^p}{p^3-p}(\vartheta_p\bar\vartheta_p + \bar\vartheta_p\vartheta_p) = 2\sum_{p>1}\frac{r^p}{p^3-p}\Re(\vartheta_p\bar\vartheta_p). \qquad \square$$

The Brownian motion "on" $\mathcal{M}$ will be discussed in Section 4; a main feature is that it takes its values in the group G^{h}, thus getting us out of the C^∞ category where (1.9) has been established.

2. Douady–Earle infinitesimal extension

Beurling and Ahlfors characterized the boundary values of quasi-conformal maps of the disk as the quasi-symmetric homeomorphisms of the circle; they gave a construction from a given quasi-symmetric boundary homeomorphism to the quasi-conformal extension. Their methodology is based on Fourier analysis. In [5] we extended the Beurling–Ahlfors construction to general infinitesimal transformations of the circle. In contrast to this, Douady and Earle constructed a canonical and conformally natural extension covariant under the action of the Möbius group. In this paper we shall use an infinitesimal version of the Douady–Earle extension which leads to a more transparent formalism than Fourier approach.

Let φ be a quasi-symmetric homeomorphism of the circle; its *Douady–Earle extension* Φ is characterized (see [11, p. 28]) by the identity

$$\int_{\partial D} \frac{\varphi(\zeta)-\Phi(z)}{1-\overline{\Phi(z)}\varphi(\zeta)}\frac{|d\zeta|}{|z-\zeta|^2} = 0. \tag{2.1}$$

In other words, if $z \in D$ then $\Phi(z)$ is the unique point in D such that (2.1) holds.

For instance, taking $\varphi_0(\zeta) = \zeta$, then we get $\Phi_0(z) = z$, which means that

$$\int_{\partial D} \frac{\zeta - z}{1-\bar z\zeta}\frac{|d\zeta|}{|z-\zeta|^2} = 0.$$

Indeed

$$\nu_z(d\zeta) := \frac{(1-|z|^2) \times |d\zeta|}{|z-\zeta|^2}$$

is the Poisson kernel of the point z; therefore the holomorphic function

$$h(\tilde{\zeta}) := \frac{\tilde{\zeta} - z}{1 - \bar{z}\tilde{\zeta}}$$

satisfies

$$\int_{\partial D} h(\tilde{\zeta})\, \nu_z(d\tilde{\zeta}) = h(z) = 0.$$

Moreover if

$$\varphi_1(\zeta) = \frac{\zeta - a}{1 - \bar{a}\zeta} \quad \text{and} \quad u := \Phi_1(z) = \frac{z-a}{1-\bar{a}z},$$

then

$$\frac{\varphi_1(\zeta) - u}{1 - \bar{u}\varphi_1(\zeta)} = \frac{1 - a\bar{z}}{1 - \bar{a}z} \times \frac{\zeta - z}{1 - \bar{z}\zeta}.$$

Thus the defining relation (2.1) is satisfied for homographic transformations.

We proceed now infinitesimally. Let φ_t be a family of diffeomorphisms of ∂D depending smoothly on the parameter t such that $\varphi_0 = \text{Identity}$; setting Φ_t the corresponding Douady–Earle extensions, we have

$$\int_{\partial D} \frac{\varphi_t(\zeta) - \Phi_t(z)}{1 - \overline{\Phi_t(z)}\varphi_t(\zeta)} \frac{|d\zeta|}{|z-\zeta|^2} = 0. \tag{2.2}$$

As in Vasil'ev [35, Section 5], we extend vector fields on the circle to vector fields inside the disk. Using

$$\frac{\zeta}{(\zeta - z)(1 - \bar{z}\zeta)} = \frac{1}{(1 - \bar{\zeta}z)(1 - \bar{z}\zeta)} = \frac{1}{|\zeta - z|^2}$$

and $d\zeta = \sqrt{-1}\,\zeta\,|d\zeta|$, we get

$$\frac{d\zeta}{(\zeta - z)(1 - \bar{z}\zeta)} = i\frac{|d\zeta|}{|\zeta - z|^2}.$$

Therefore (2.2) takes the form

$$\int_{\partial D} \frac{\varphi_t(\zeta) - \Phi_t(z)}{1 - \overline{\Phi_t(z)}\varphi_t(\zeta)} \frac{d\zeta}{(\zeta - z)(1 - \bar{z}\zeta)} = 0. \tag{2.3}$$

Differentiating (2.3) relatively to t and setting

$$v(\zeta) = \frac{d}{dt}\bigg|_{t=0} \varphi_t(\zeta), \qquad V(z) = \frac{d}{dt}\bigg|_{t=0} \Phi_t(z), \tag{2.4}$$

gives

$$\int_{S^1} \frac{v - V}{(1-\bar{z}\zeta)^2(\zeta - z)}\, d\zeta + \int_{S^1} \frac{\bar{z}v + \zeta\bar{V}}{(1-\bar{z}\zeta)^3}\, d\zeta = 0. \tag{2.5}$$

Since

$$\zeta \mapsto \frac{\zeta\bar{V}}{(1-\bar{z}\zeta)^3}$$

is holomorphic, the integral $\int_{S^1} \frac{\zeta\bar{V}}{(1-\bar{z}\zeta)^3}\, d\zeta$ is zero. With Cauchy's integral formula we obtain

$$V(z) = \frac{(1-|z|^2)^2}{2i\pi} \int_{S^1} \frac{v(\zeta)}{(1-\bar{z}\zeta)^2(\zeta - z)}\, d\zeta + \frac{\bar{z}(1-|z|^2)^2}{2i\pi} \int_{S^1} \frac{v(\zeta)}{(1-\bar{z}\zeta)^3}\, d\zeta. \tag{2.6}$$

The $\bar{\partial}$ derivative of the vector field $V(z)$ in (2.6) has been calculated in Reich and Chen [32, formulas (2.1)–(2.3)]. With the following theorem $\bar{\partial}V$ is obtained in a different manner.

Theorem 2.1.

1. *If $v(\zeta) = \zeta^p$ where $p \geqslant 0$, then*

$$V(z) = z^p. \tag{2.7}$$

2. *If $v(\zeta) = \zeta^{-p}$ where $p \geqslant 1$ then*

$$V(z) = \bar{z}^p\left(1 + p(1 - z\bar{z}) + \frac{p(p+1)}{2}(1 - z\bar{z})^2\right). \tag{2.8}$$

3. *Moreover, if $V(z)$ is given by* (2.8)*, then*

$$\frac{\partial V}{\partial \bar{z}} = \bar{z}^{p-1}\frac{p(p+1)(p+2)}{2}(1 - z\bar{z})^2, \tag{2.9}$$

and if V is given by (2.7) *then*

$$\frac{\partial V}{\partial \bar{z}} = 0. \tag{2.10}$$

For V given by (2.8) *we also have*

$$\frac{\partial V}{\partial z} = -p\bar{z}^{p+1}\big[1 + (p+1)(1 - z\bar{z})\big]. \tag{2.11}$$

Proof. We first prove (2.7); in this case the second integral of (2.6) vanishes and the first integral equals the residues at the point z.

To prove (2.8) apply elementary residues to (2.6) as follows:

$$\begin{aligned} V(z) \times \frac{2i\pi}{(1-z\bar{z})^2} &= \frac{1}{(1-z\bar{z})^2}\int_{S^1} \frac{v(\zeta)}{\zeta - z}\,d\zeta + \frac{\bar{z}}{(1-z\bar{z})^2}\int_{S^1} \frac{v(\zeta)}{1-\bar{z}\zeta}\,d\zeta \\ &\quad + \frac{\bar{z}}{1-z\bar{z}}\int_{S^1} \frac{v(\zeta)}{(1-\bar{z}\zeta)^2}\,d\zeta + \bar{z}\int_{S^1} \frac{v(\zeta)}{(1-\bar{z}\zeta)^3}\,d\zeta. \end{aligned} \tag{2.12}$$

For $v(\zeta) = \zeta^{-k}$, the first integral in (2.12) cancels since

$$\int_{S^1} \frac{d\zeta}{\zeta^k(\zeta - z)} = 0 \quad \text{if } k \geqslant 1.$$

Thus we obtain

$$\begin{aligned} \frac{V(z)}{(1-z\bar{z})^2} &= \frac{\bar{z}}{(1-z\bar{z})^2}\operatorname{Res}_{\zeta=0}\left(\frac{1}{(1-\bar{z}\zeta)\zeta^k}\right) \\ &\quad + \frac{\bar{z}}{1-z\bar{z}}\operatorname{Res}_{\zeta=0}\left(\frac{1}{(1-\bar{z}\zeta)^2\zeta^k}\right) + \bar{z}\operatorname{Res}_{\zeta=0}\left(\frac{1}{(1-\bar{z}\zeta)^3\zeta^k}\right). \end{aligned} \tag{2.13}$$

Calculating the three residues gives

$$\frac{V(z)}{(1-z\bar{z})^2} = \frac{\bar{z}}{(1-z\bar{z})^2}\bar{z}^{k-1} + \frac{\bar{z}}{(1-z\bar{z})}k\bar{z}^{k-1} + \bar{z}\frac{k(k+1)}{2}\bar{z}^{k-1}. \qquad \square$$

3. Loewner type equation of a conformal welding flow

Let $t \mapsto C_t$ be a map from [0, 1] into the space $\mathfrak{g}$, which is assumed to be continuous for the C^∞ topology; assume furthermore that

$$\text{the Fourier coefficients of } C_t \text{ on } 1, \cos\theta, \sin\theta \text{ vanish.} \tag{3.1}$$

To these data consider the flow of C^∞ diffeomorphisms of S^1 defined by

$$\frac{d}{dt}g_t(\theta) = C_t\big(g_t(\theta)\big), \quad g_0 = \text{Identity}. \tag{3.2}$$

Let $z \mapsto \tilde{C}_t(z)$ be the Douady–Earle extension of C_t to the closed disk $\bar{D}$. Because of (3.1) and (2.7)–(2.8), we have $\tilde{C}_t(0) = 0$. Now consider the flow of C^∞ diffeomorphisms $\tilde{g}_t$ of $\bar{D}$ defined by the equation

$$\frac{d}{dt}\tilde{g}_t(z) = \tilde{C}_t\big(\tilde{g}_t(z)\big), \quad \tilde{g}_0 = \text{Identity}. \tag{3.3}$$

As $g_t \in C^\infty(S^1)$, the conformal welding (1.9) for g_t exists; set

$$g_t(\theta) = \big(f_t^{-1} \circ h_t\big)\big(\exp(i\theta)\big). \tag{3.4}$$

Next define a function F_t on the whole complex plane by

$$F_t(z) := \begin{cases} h_t(z), & |z| > 1, \\ (f_t \circ \tilde{g}_t)(z), & |z| \leqslant 1. \end{cases} \tag{3.5}$$

As the restriction of $\tilde{g}_t$ to ∂D equals g_t, we observe that

$$F_t \text{ has a continuous extension to the whole complex plane.} \tag{3.6}$$

Consider the infinitesimal increment

$$\delta_t(F) := \left(\frac{d}{dt}F_t\right) \circ F_t^{-1}. \tag{3.7}$$

Given a univalent function φ_t let $F_t^\varphi := \varphi_t \circ F_t$. Then we have

$$\delta_t\big(F^\varphi\big) = \varphi_t'\big(\varphi_t^{-1}\big) \times \big(\delta_t(F) \circ \varphi_t^{-1}\big) + \delta_t(\varphi). \tag{3.8}$$

Moreover, since φ_t is holomorphic, $\bar{\partial}(\delta_t(\varphi)) = 0$ and from (3.8),

$$\bar{\partial}\big(\delta_t\big(F^\varphi\big)\big) = \bar{\partial}\big[\varphi_t'\big(\varphi_t^{-1}\big) \times \big(\delta_t(F) \circ \varphi_t^{-1}\big)\big].$$

In the special case of an affine transformation $\varphi_t(z) = \alpha_t z + \beta_t$, we find

$$\delta_t\big(F^\varphi\big) = \varphi_t \circ \delta_t(F) \circ \varphi_t^{-1} + \delta_t(\varphi) - \beta_t. \tag{3.9}$$

Theorem 3.1 *(Loewner equation along a conformal welding flow). On D we have*

$$\frac{d}{dt}f_t = \delta_t(F) \circ f_t - (\partial f_t) \times \tilde{C}_t \tag{3.10}$$

and

$$\bar{\partial}\big\{\delta_t(F) \circ f_t - (\partial f_t) \times \tilde{C}_t\big\} = \frac{d}{dt}(\bar{\partial} f_t) = 0. \tag{3.11}$$

Thus $\delta_t(F)$ satisfies on $f_t(D)$ the following identity:

$$\bar{\partial}\big[\delta_t(F)\big] = A_t, \quad A_t := \left(\frac{\partial f_t}{\overline{\partial f_t}} \times \bar{\partial}\tilde{C}_t\right) \circ f_t^{-1}. \tag{3.12}$$

We have

$$A_t = \bar{\partial} W_t \tag{3.13}$$

where W_t is the image of the vector field $\tilde{C}_t$ through the map f_t,

$$W_t(u) = f_t'\big(f_t^{-1}(u)\big)\tilde{C}_t\big(f_t^{-1}(u)\big), \tag{3.14}$$

denoting $f_t'(u) = \partial f_t(u)$. On the other hand,

$$\bar{\partial}\big[\delta_t(F)\big](z) = 0, \quad z \in \big(adherence\ f_t(D)\big)^c. \tag{3.15}$$

Proof. From

$$f_t = F_t \circ \tilde{g}_t^{-1}$$

we get formula (3.10), and by taking into account that $\bar{\partial} f_t = 0$, we arrive at (3.11). To obtain (3.12), recall the rule of change of variables for the holomorphic and antiholomorphic derivatives which can be found in [1, p. 8]:

$$\bar{\partial}(u \circ v) = \big((\bar{\partial} u) \circ v\big)\overline{\partial v} + \big((\partial u) \circ v\big)(\bar{\partial} v). \tag{3.16}$$

By means of (3.16), taking $u = \exp(\eta \delta_t(F))$ and $v = f_t$, the vector fields being considered as infinitesimal transformations, we get

$$\bar{\partial}\big(\delta_t(F) \circ f_t\big) = \bar{\partial}\delta_t(F) \times \overline{\partial f_t}. \tag{3.17}$$

On the other hand, we have

$$\bar{\partial}\big(\partial f_t \times \tilde{C}_t\big) = \partial f_t \times \bar{\partial}\tilde{C}_t. \tag{3.18}$$

Eqs. (3.17) and (3.18), along with (3.11), imply the claimed formula (3.12).

To prove (3.14)–(3.15), we calculate again exploiting formula (3.16), the expression

$$\bar{\partial}\big(\tilde{C}_t\big(f_t^{-1}(u)\big)\big) = (\bar{\partial}\tilde{C}_t)\big(f_t^{-1}(u)\big) \times \overline{\partial f_t^{-1}}(u).$$

We find

$$\bar{\partial} W_t = f_t'\big(f_t^{-1}(u)\big) \times (\bar{\partial}\tilde{C}_t)\big(f_t^{-1}(u)\big) \times \overline{\partial f_t^{-1}}(u) \tag{3.19}$$

which coincides with A_t since $f_t'(f_t^{-1}(u)) = 1/(f_t^{-1})'(u)$. □

Remark 3.2. Identity (3.10) permits to obtain Kirillov vector fields, see [21], as well as [30] where identities like (3.10) are integrated via line integrals. Identities (3.14)–(3.15) can be deduced directly from (3.8); however they are delicate since the Taylor part in the expansion of W_t is different from

$$f'\big(f^{-1}(u)\big) \times (\text{Taylor part of } \tilde{C}_t)\big(f^{-1}(u)\big).$$

We are able to integrate (3.12) as follows.

Theorem 3.3. *The following identity is valid in the whole complex plane*:

$$\big(\delta_t(F_t)\big)(z) - \frac{1}{2\pi i}\int\limits_{f_t(D)} \frac{1}{z-z'} A_t\big(z'\big)\,dz' \wedge d\bar{z}' = \alpha_t z + \beta_t, \quad \alpha_t \in \mathbb{C},\ \beta_t \in \mathbb{C}. \tag{3.20}$$

Proof. Consider

$$H_1(z) := \frac{1}{2\pi i}\int\limits_{f_t(D)} \frac{1}{z-z'} A_t\big(z'\big)\,dz' \wedge d\bar{z}', \quad z \in \mathbb{C}. \tag{3.21}$$

Note that H_1 is continuous in $\mathbb{C}$. Using the fact that the Cauchy kernel is the elementary solution of the $\bar{\partial}$ operator we get

$$(\bar{\partial} H_1)(z) = \begin{cases} A_t(z), & z \in f_t(D), \\ 0, & z \notin \overline{f_t(D)}. \end{cases} \tag{3.22}$$

Therefore setting $H_2 := H_1 - \delta_t(F)$, we have

$$H_2 \text{ is holomorphic on } \big(f_t(\partial D)\big)^c \text{ and continuous on } \mathbb{C};$$

thus by Morera's theorem, H_2 is holomorphic on $\mathbb{C}$. As H_2 is of order $\mathrm{O}(z)$ at infinity, by Liouville's theorem, it is an affine function; we conclude by using the fact that $H_1(\infty) = 0$. □

Remark 3.4. The indeterminacy appearing in formula (3.20) through the choice of α_t and β_t relies on the fact that our construction is done for the space $\mathscr{S}$ of shapes, where objects are defined up to left multiplication by an affine transformation: indeed such a multiplication induces at the level of differentials, as shown in formula (3.9), the addition of an arbitrarily chosen infinitesimal affine transformation.

Theorem 3.5 *(Holomorphy of the welding functionals). Consider the functional Φ on $\mathscr{M}^\infty$ defined by*

$$\Phi(g) = h, \tag{3.23}$$

where h is determined by the welding relation (1.9). *Assuming the normalization*

$$\Phi(g)(z) = z + \mathrm{o}(1), \quad z \to \infty, \tag{3.24}$$

then for any fixed $z_o \in \mathbb{C}$, $|z_o| > 1$, the mapping

$$g \mapsto \Phi(g)(z_0) \tag{3.25}$$

is a holomorphic functional for the Kähler structure of $\mathscr{M}^\infty$. Consequently with Δ^r being defined as in (1.18), *we have*

$$\Delta^r(\Phi) = 0. \tag{3.26}$$

Proof. We have to compute the differential of the functional $(\Phi_{z_0})(g) := \Phi(g)(z_0)$. Given $C \in \mathfrak{g}$ satisfying (3.1), fix $g \in G^\infty$ and consider the function

$$h_\varepsilon = \Phi\big(\exp(\varepsilon C)g\big).$$

Then $h_0 = \Phi(g) = h$ and we have

$$\langle C, d\Phi_{z_0}\rangle_g = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} h_\varepsilon(z_0).$$

The derivative of the variation h_ε can be calculated by applying the results of Theorems 3.1 and 3.3. Using (3.20) and (3.24), we get according to (3.12)

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} h_\varepsilon(z_0) = \frac{1}{2\pi i}\int\limits_{f(D)} \frac{1}{h(z_0)-z'} A_C\big(z'\big)\, dz' \wedge d\bar{z}' \tag{3.27}$$

where the univalent function f on D is given via the welding of g and

$$A_C = \left(\frac{\partial f}{\overline{\partial f}} \times \bar{\partial}\tilde{C}\right) \circ f^{-1}.$$

According to (1.20), holomorphy on $\mathscr{M}^\infty$ is equivalent to

$$\big\{\big(\partial_{C_k} + \sqrt{-1}\partial_{C'_k}\big)\Phi(g)\big\}(z_0) = 0, \quad C_k = \cos k\theta,\ C'_k = \sin k\theta.$$

This vanishing is assured through (3.27) if

$$\bar{\partial}\tilde{C}_k + \sqrt{-1} \times \bar{\partial}\tilde{C}'_k = 0. \tag{3.28}$$

The associated vector fields are

$$C_k(\zeta) = \frac{1}{2}\left(\zeta^{k+1} + \frac{1}{\zeta^{k-1}}\right) \quad \text{and} \quad C'_k(\zeta) = \frac{1}{2i}\left(\zeta^{k+1} - \frac{1}{\zeta^{k-1}}\right).$$

Taking the extension of $C_k(\zeta) + iC'_k(\zeta) = \zeta^{k+1}$, we observe that (3.28) is true as a consequence of Eq. (2.10). Finally the second part of Theorem 1.4 gives (3.26). □

II. Kählerian Brownian motion

4. Canonical Brownian on the diffeomorphism group of the disk

We start by recalling the construction of the *canonical Brownian motion* "on" G^∞. The *regularized* canonical Brownian motion on the group of diffeomorphisms of the circle is the stochastic flow on the circle S^1 associated to the Stratonovich SDE

$$d\psi^r_{x,t}(\theta) = dv^r_{x,t}\big(\psi^r_{x,t}(\theta)\big) \tag{4.1}$$

where $v^r_{x,t}(\theta)$ is the regularized $\mathfrak{g}$-valued Brownian motion defined by

$$v^r_{x,t}(\theta) := \sum_{k>1} \frac{r^k}{\sqrt{k^3-k}} \big(x_{2k}(t)\cos k\theta + x_{2k+1}(t)\sin k\theta\big), \quad \theta \in S^1. \tag{4.2}$$

Here $\{x_*\}$ is a sequence of independent scalar Brownian motions and $r \in]0,1[$.

It results from Kunita's theory of stochastic flows [23] that $\theta \mapsto \psi^r_{x,t}(\theta)$ constitutes a C^∞ diffeomorphism of S^1. It can be proved that the $\lim_{r\to 1} \psi^r_{x,t} = \psi_{x,t}$ exists uniformly in θ, defining a random *homeomorphism* $\psi_{x,t}$ which is the so-called *canonical Brownian motion* "on" $\mathrm{Diff}(S^1)$; this random homeomorphism is furthermore Hölder continuous [28,13,6]. The corresponding infinitesimal generators Δ^r and Δ of these processes are given by (1.18), resp. (1.17). The fact that the construction (4.1)–(4.2) gives Brownian motion with respect to the Levi-Civita connection on $\mathscr{M}^\infty$ has been proved in [4, p. 103].

Writing $\Psi^r_{x,t}(e^{i\theta}) = e^{i\psi^r_{x,t}(\theta)}$, then $d_t\Psi^r_{x,t}(e^{i\theta}) = i\Psi^r_{x,t}(e^{i\theta})\, d_t\psi^r_{x,t}(\theta)$. Thus in the variable $\zeta = \exp(i\theta)$, we obtain from (4.2) the vector field

$$\chi^r_{x,t}(\zeta) = \sum_{k>1} \frac{r^k}{\sqrt{k^3-k}} \frac{i\zeta}{2}\left(\left(\zeta^k + \frac{1}{\zeta^k}\right)x_{2k}(t) - i\left(\zeta^k - \frac{1}{\zeta^k}\right)x_{2k+1}(t)\right). \tag{4.3}$$

When $r=1$, we denote $\chi_{x,t}(\zeta) = \chi^1_{x,t}(\zeta)$ or equivalently

$$\chi_{x,t}(\zeta) = \frac{i}{2}\sum_{k>1} \frac{1}{\sqrt{k^3-k}}\left(\zeta^{k+1}\big(x_{2k}(t) - ix_{2k+1}(t)\big) + \frac{1}{\zeta^{k-1}}\big(x_{2k}(t) + ix_{2k+1}(t)\big)\right). \tag{4.4}$$

According to Theorem 2.1 (items 1 and 2), the extension of the vector field (4.4) inside the unit disk is given by

$$V_{x,t}(z) = \frac{i}{2}\sum_{k>1} \frac{1}{\sqrt{k^3-k}} \times B^k_{x,t}(z), \quad z \in D, \tag{4.5}$$

where

$$\begin{aligned} B^k_{x,t}(z) &:= z^{k+1}\big(x_{2k}(t) - ix_{2k+1}(t)\big) \\ &\quad + \bar{z}^{k-1}\left(1 + (k-1)(1-z\bar{z}) + \frac{k(k-1)}{2}(1-z\bar{z})^2\right)\big(x_{2k}(t) + ix_{2k+1}(t)\big). \end{aligned} \tag{4.6}$$

Note that for fixed t, the function

$$z \mapsto V_{x,t}(z) \tag{4.7}$$

is C^∞ on the open disk and vanishing at 0. As a consequence of its smoothness, $V_{x,t}(*)$ induces a flow of local diffeomorphism of D in the sense of Kunita [23], via the Stratonovich SDE

$$d_t\Psi_{x,t}(z) = (d_tV_{x,t})\big(\Psi_{x,t}(z)\big). \tag{4.8}$$

Extending the vector field (4.3) to the disk, we obtain the regularization $V^r_{x,t}$ of $V_{x,t}$ defined as

$$V^r_{x,t}(z) = \frac{i}{2}\sum_{k>1}\frac{r^k}{\sqrt{k^3-k}} \times B^k_{x,t}(z). \tag{4.9}$$

The corresponding regularized Stratonovich SDE

$$d_t\Psi^r_{x,t}(z) = \left(d_t V^r_{x,t}\right)\left(\Psi^r_{x,t}(z)\right) \tag{4.10}$$

defines a local C^∞ flow on the closure $\bar{D}$ of the unit disk.

In terms of polar coordinates

$$z = e^{i\theta}e^{-y}, \quad \text{resp.} \quad \Psi_{x,t}(z) = e^{i\theta_t}e^{-y_t}, \tag{4.11}$$

the Stratonovich SDE (4.10) becomes the following Stratonovich SDE

$$\begin{cases} d\theta_t = \sum_{k>1}\frac{e^{-ky_t}r^k}{2\sqrt{k^3-k}}\left(1+R_k e^{2y_t}\right)\left(\cos(k\theta_t)\circ dx_{2k}(t) + \sin(k\theta_t)\circ dx_{2k+1}(t)\right), \\ dy_t = \sum_{k>1}\frac{e^{-ky_t}r^k}{2\sqrt{k^3-k}}\left(1-R_k e^{2y_t}\right)\left(\sin(k\theta_t)\circ dx_{2k}(t) - \cos(k\theta_t)\circ dx_{2k+1}(t)\right) \end{cases} \tag{4.12}$$

where

$$e^{2y}R_k(y) = \frac{k(k+1)}{2}e^{2y} - (k-1)(k+1) + \frac{k(k-1)}{2}e^{-2y}. \tag{4.13}$$

Remark 4.1. For $y = 0$, as it should be, we recover Eq. (4.1).

Theorem 4.2. *The Itô contraction of the Stratonovich system* (4.12) *is given by*

$$d(\theta_t + iy_t) * d(\theta_t + iy_t) = \left(\frac{1}{2}e^{-2y_t} - \frac{1}{4}e^{-4y_t}\right)dt.$$

Proof. The Itô contractions will be expressed essentially as a sum of geometric series or derivatives of geometric series. We write the Ito contractions for (4.12) when $r = 1$,

$$\begin{aligned} d\theta_t * d\theta_t &= \sum_{k>1}\frac{e^{-2ky_t}}{4(k^3-k)}\left(1+R_k e^{2y_t}\right)^2 dt, \\ d\theta_t * dy_t &= 0, \\ dy_t * dy_t &= \sum_{k>1}\frac{e^{-2ky_t}}{4(k^3-k)}\left(1-R_k e^{2y_t}\right)^2 dt. \end{aligned}$$

Thus

$$d(\theta_t + iy_t) * d(\theta_t + iy_t) = \sum_{k>1} \frac{e^{-2ky_t}}{k^3 - k} R_k e^{2y_t}\, dt$$

and we have

$$\sum_{k>1} \frac{e^{-2ky}}{k^3 - k} R_k e^{2y} = \frac{1}{2} e^{-2y} - \frac{1}{4} e^{-4y}. \qquad \square$$

Theorem 4.3. *For a given $z_0 \in D$, let $\Psi_{x,t}(z_0)$ be solution of Eq.* (4.8). *Consider the stopping time*

$$T_{z_0} = \inf\{t > 0\colon\ \Psi_{x,t}(z_0) \in \partial D\}. \tag{4.14}$$

Then $T_{z_0} = \infty$.

Proof. First remark that $t \mapsto y_t$ is a Markov process: indeed there exists an independent family of scalar Brownian motions ω_k, independent of θ, such that

$$\omega_k \simeq_{\text{law}} \cos(k\theta_t) x_{2k+1}(t) - \sin(k\theta_t) x_{2k}(t). \tag{4.15}$$

This allows to compare the Markov processes y_t with the process having as infinitesimal generator the ODE

$$\frac{1}{2} q(y) \frac{d^2}{dy^2} + w(y) \frac{d}{dy}, \tag{4.16}$$

where

$$q(y) = \sum_{k>1} \frac{e^{-2ky}}{4(k^3 - k)} \left(1 - R_k e^{2y}\right)^2, \qquad w(y) = \frac{1}{2} e^{-2y} - \frac{1}{4} e^{-4y}. \tag{4.17}$$

We have $w(y) > 0$, and by (4.13) the estimation of $q(y)$ at $y = 0$ gives the result. The comparison equation in Itô form reads as

$$d\tilde{y} = \tilde{y}\, db_t$$

where b_t is an abstract Brownian motion; therefore

$$\tilde{y}(t) = \tilde{y}(0) \exp\left(b(t) - \frac{t}{2}\right)$$

which never vanishes. $\square$

Theorem 4.4. *The process $\Psi^r_{x,t}$ takes values in the C^∞ orientation preserving diffeomorphisms of the open disk D.*

Proof. By Kunita's theory of stochastic flows [23], $\Psi^r_{x,t}(z)$ is a C^∞ diffeomorphism on the open random set $\{z\colon\ T_z > t\}$; we conclude as in Theorem 4.3. $\square$

Remark 4.5. The Ito contractions for a two points process governed by (4.12) are also expressed as sums of geometric series. Denote by $(\theta_t^{(j)}, y_t^{(j)})$, $j = 1, 2$, solutions of Eqs. (4.12) with given initial conditions at $t = 0$. It is not difficult to see that

$$d\theta_t^{(1)} * d\theta_t^{(1)} = L_{y_t^{(1)}, y_t^{(2)}}\big(\theta_t^{(1)} - \theta_t^{(2)}\big)\, dt$$

where

$$\begin{aligned} L_{y^{(1)}, y^{(2)}}(\theta) = {} & \frac{1}{2}\big(1 - \cos(\theta)\cosh\big(y^{(1)} + y^{(2)}\big)\big) \\ & \times \log\big(1 - 2e^{-(y^{(1)}+y^{(2)})}\cos(\theta) + e^{-(y^{(1)}+y^{(2)})}\big) \\ & + \text{terms bounded in } (\theta, y) \end{aligned} \tag{4.18}$$

with $\theta = \theta^{(1)} - \theta^{(2)}$ and $y = (y^{(1)}, y^{(2)})$, $y^{(1)} \geqslant 0$, $y^{(2)} \geqslant 0$. We observe that the induced flow is isotropic in θ, see [26]; moreover it is log-Lipschitzian as in the case of its restriction to the circle, see [6,13,28].

5. Regularized welding process, its holomorphy

This whole section will be written for a fixed value of the regularization parameter r. We start by recalling the classical solution of the smooth welding problem. Define the complex modulus of quasi-conformality

$$\mu_{\Psi^r_{x,t}}(z) := \frac{\bar\partial \Psi^r_{x,t}}{\partial \Psi^r_{x,t}}(z)$$

and consider a solution $F^r_{x,t}$ of the following Beltrami equation:

$$\frac{\bar\partial F^r_{x,t}}{\partial F^r_{x,t}}(z) = \begin{cases} \mu_{\Psi^r_{x,t}}(z), & |z| \leqslant 1, \\ 0, & |z| > 1. \end{cases} \tag{5.1}$$

Normalizing the solution by the conditions

$$F^r_{x,t}(z) = z + \mathrm{o}(1), \quad z \to \infty, \tag{5.2}$$

then $F^r_{x,t}$ is analytically expressible by the Ahlfors–Bojarski series (see [1, Chapt. 5]; [7, Chapt. 5]). This solution is unique and therefore gives rise to a functional on the underlying probability space.

Define

$$f^r_{x,t}(z) = F^r_{x,t} \circ \big(\Psi^r_{x,t}\big)^{-1}(z), \quad z \in D; \qquad h^r_{x,t}(z) = F^r_{x,t}(z), \quad z \notin D. \tag{5.3}$$

Then

$$
\begin{aligned}
&f^r_{x,t} \text{ is holomorphic and univalent on } D, \text{ and}\\
&h^r_{x,t} \text{ is holomorphic and univalent on } \bar{D}^c.
\end{aligned}
\tag{5.4}
$$

Differential calculus along the time variable will permit to use the results of Section 3 established in the case of C^∞ welding depending smoothly on time.

Our tool for this purpose is the *transfer principle*; we proceed by smoothing the Brownian motion. To this end, we fix a *mollifier*, that is a positive C^∞ function a of compact support contained in the interval [0, 1] and integral equal to 1. To every $\varepsilon > 0$ we associate the smoothened Brownian motion defined as

$$
x_k^\varepsilon(t) = \int_0^1 x_k(t + s\varepsilon)a(s)\,ds. \tag{5.5}
$$

Note that $x_k^\varepsilon(*)$ are C^∞ functions and $\lim_{\varepsilon\to 0} x_k^\varepsilon(*) = x_k(*)$.

Replacing in Eqs. (4.5) and (4.6) the Brownian motions x_* by its smooth regularization x_*^ε, we get a C^∞ vector field depending smoothly upon time:

$$
V^r_{x^\varepsilon,t}(z), \quad z \in \bar{D}, \tag{5.6}
$$

to which we associate the following non-autonomous ODE:

$$
\frac{d}{dt}\Psi^r_{x^\varepsilon,t}(z_0) = \big(V^r_{x^\varepsilon,t}\big)\big(\Psi^r_{x^\varepsilon,t}(z_0)\big). \tag{5.7}
$$

Theorem 5.1. *We have*

$$
\lim_{\varepsilon\to 0} \Psi^r_{x^\varepsilon,t}(z) = \Psi^r_{x,t}(z), \quad \forall z \in \mathbb{C}, \ \textit{uniformly on any compact.} \tag{5.8}
$$

Proof. The transfer principle (see for instance [27, Chapt. VIII]) states that the solution of the ODE driven by the regularized Brownian x^ε converges locally uniformly towards the corresponding Stratonovich SDE driven by x. □

Theorem 5.2. *Let $F^r_{x^\varepsilon,t}$ be defined by* (5.1) *and* (5.2) *with x replaced by x^ε. The following identity is valid on the whole complex plane*:

$$
\left(\frac{d}{dt}F^r_{x^\varepsilon,t}\right) \circ \big(F^r_{x^\varepsilon,t}\big)^{-1}(z) = \frac{1}{2\pi i} \int_{f^r_{x^\varepsilon,t}(D)} \frac{1}{z - z'} A^r_{x^\varepsilon,t}\big(z'\big)\,dz' \wedge d\bar{z}' \tag{5.9}
$$

where

$$
A^r_{x^\varepsilon,t} := \left(\frac{\partial f^r_{x^\varepsilon,t}}{\overline{\partial f^r_{x^\varepsilon,t}}} \times \bar{\partial} V^r_{x^\varepsilon,t}\right) \circ \big(f^r_{x^\varepsilon,t}\big)^{-1}.
$$

Proof. Apply Loewner's equation established in Theorem 3.1 along with the normalization (5.2). □

Letting $\varepsilon \to 0$ in (5.9), we obtain a stochastic differential for $F^r_{x,t}$. Indeed, letting $z = (F^r_{x,t})^{-1}(\zeta)$, then

$$d_t F^r_{x,t}(z) = \frac{1}{2\pi i} \int\limits_{f^r_{x,t}(D)} \frac{1}{\zeta - \zeta_1} \left(\frac{\partial f^r_{x,t}}{\overline{\overline{\partial f^r_{x,t}}}} \times \left(d_t \bar{\partial} V^r_{x,t} \right) \right) \circ \left(f^r_{x,t} \right)^{-1}(\zeta_1)\, d\zeta_1 \wedge d\overline{\zeta_1}. \tag{5.10}$$

Theorem 5.3. *Fix a finite subset $\{z_j\}_{j\in\{1,2,\ldots,d\}}$ of distinct points of D^c, and define*

$$w^j_{x,t} := F^r_{x,t}(z_j).$$

There exist d complex Brownian motions b_j such that

$$w^j_{x,t} - w^j_{x,0} = \int\limits_0^t (\sqrt{A})^j_i\, db_i(t) \tag{5.11}$$

where the stochastic integrals are of Itô type and where the Hermitian matrix A is given by

$$A^j_q := \frac{1}{4\pi^2} \int\limits_{f^r_{x,t}(D)^2} \frac{1}{(w^j_{x,t} - z')(w^q_{x,t} - z'')} \mathcal{C}^r_f\left(z', z''\right) dz' \wedge d\bar{z}' \wedge dz'' \wedge d\bar{z}'' \tag{5.12}$$

where

$$\mathcal{C}^r_f\left(z', z''\right) := \mathcal{C}^r\left(\left(f^r_{x,t}\right)^{-1}(z'), \left(f^r_{x,t}\right)^{-1}(z'')\right) \frac{\partial f^r_{x,t}}{\bar{\partial} f^r_{x,t}} \left(\left(f^r_{x,t}\right)^{-1}(z')\right) \frac{\bar{\partial} f^r_{x,t}}{\partial f^r_{x,t}} \left(\left(f^r_{x,t}\right)^{-1}(z'')\right)$$

with

$$\mathcal{C}^r(u, v)\, dt := d_t\, \bar{\partial} V^r_t(u) * \overline{d_t\, \bar{\partial} V^r_t(v)}. \tag{5.13}$$

Proof. In finite dimension it is well known that the image of a Brownian motion on a Kähler manifold through a holomorphic function is a conformal martingale in $\mathbb{C}$, equal in law to a time-changed complex Brownian motion; this fact extends to finite systems of holomorphic functions.

In our case we apply the transfer principle together with the key fact of holomorphy of the conformal welding (Theorem 3.5). This implies the vanishing of Itô contractions induced by the passage from Stratonovich SDE to Itô SDE; only the martingale parts remain and (5.11) is established. Thus the computation of the martingale covariance matrix through Itô calculus involves only first order derivatives computed from (5.9) which finally establishes (5.12). □

6. Covariance for the $\bar\partial$ of Douady–Earle extension

The next step is the computation of $\mathcal{C}_f^r$ in (5.12). We start from the definition of the regularized vector fields $V_{x,t}^r(z)$ given in (4.9). According to (2.9)–(2.10) we have

$$\left(\bar\partial_z V_{x,t}^r\right)(z) = \frac{i(1-\bar z z)^2}{4}\sum_{k>1} r^k\sqrt{k^3-k}\left(x_{2k}(t)+ix_{2k+1}(t)\right)\bar z^{k-2} \tag{6.1}$$

and

$$d_t\left(\bar\partial V_{x,t}^r\right)(z) = \frac{i(1-\bar z z)^2}{4}\sum_{k>1} r^k\sqrt{k^3-k}\left(dx_{2k}(t)+i\,dx_{2k+1}(t)\right)\bar z^{k-2}. \tag{6.2}$$

The covariance associated to this random vector field is

$$\mathcal{C}^r(z_1,z_2)\,dt = d_t\,\bar\partial V_{x,t}^r(z_1) * \overline{d_t\,\bar\partial V_{x,t}^r(z_2)} \tag{6.3}$$

where $*$ denotes the Itô contraction.

Theorem 6.1. *For any $0<r\leqslant 1$, we have*

$$\mathcal{C}^r(z_1,z_2) = \frac{3r^4(1-|z_1|^2)^2(1-|z_2|^2)^2}{4(1-r^2\bar z_1 z_2)^4} \tag{6.4}$$

and

$$\left|\mathcal{C}^r(z_1,z_2)\right| \leqslant 12\exp\left(-2d_{\mathrm H}(z_1,z_2)\right),\quad z_1,z_2\in D, \tag{6.5}$$

where $d_{\mathrm H}$ is the Poincaré distance on the unit disk D.

Proof. From (6.2) we get

$$\mathcal{C}^r(z_1,z_2) = \frac{1}{8}\left(1-|z_1|^2\right)^2\left(1-|z_2|^2\right)^2\sum_{k>1}\left(k^3-k\right)r^{2k}(\bar z_1 z_2)^{k-2}. \tag{6.6}$$

Using the fact that

$$\sum_{k>1}\left(k^3-k\right)X^{k-2} = \frac{6}{(1-X)^4},$$

we obtain (6.4). Next for fixed $z_1,z_2\in D$, we verify that the function

$$u(r) = \frac{r^4}{|1-r^2\overline{z_1}z_2|^4},\quad 0\leqslant r\leqslant 1,$$

is increasing in r; thus it is enough to prove estimate (6.5) for $r=1$.

Since

$$\left(1-|z_1|^2\right)\left(1-|z_2|^2\right)=|1-\bar{z}_1z_2|^2-|z_1-z_2|^2,$$

we have

$$\mathcal{C}(z_1,z_2)=\frac{3}{4}\left(1-\frac{|z_1-z_2|^2}{|1-\bar{z}_1z_2|^2}\right)^2.$$

Note that the function

$$\phi(z_1,z_2):=\frac{|z_1-z_2|^2}{|1-\bar{z}_1z_2|^2}$$

is invariant under homographic transformations

$$\mathcal{T}(z)=\frac{z-a}{1-\bar{a}z},$$

i.e.

$$\phi(\mathcal{T}z_1,\mathcal{T}z_2)=\phi(z_1,z_2). \tag{6.7}$$

Hence it is sufficient to obtain the wanted upper bound when $z_1=0$; then

$$\mathcal{C}(0,z)=\frac{3}{4}\left(1-|z|^2\right)^2.$$

Denoting $r=|z|$, we have

$$\exp\left(-d_{\mathrm{H}}(0,z)\right)=\frac{1-r}{1+r},$$

and finally since $0<r\leqslant 1$,

$$1-r^2\leqslant 2(1-r)\leqslant 4\exp\left(-d_{\mathrm{H}}(0,z)\right);$$

thus

$$\left(1-r^2\right)^2\leqslant 16\exp\left(-2d_{\mathrm{H}}(0,z)\right). \qquad \square$$

7. A priori Hölderian estimates for the regularized welding process

Granted to the normalization (5.2) we have $h^r_{x,t}(z)=z+\sum_{k>0}c_kz^{-k}$. Setting

$$q(u):=\frac{1}{h^r_{x,t}(u^{-1})}=\frac{u}{1+\sum_{k>0}c_ku^{k+1}},$$

then q is a univalent function on the unit disk D satisfying $q(0)=0$, $q'(0)=1$. Applying the Koebe 1/4-Theorem, we get $\frac{1}{4}D\subset q(D)$.

Lemma 7.1. *Denoting for a Jordan curve $\Gamma \subset \mathbb{C}$,*

$$|\Gamma|_\infty := \sup\{|z|\colon\ z \in \Gamma\}$$

we have

$$\left|h^r_{x,t}(\partial D)\right|_\infty \leqslant 4, \quad \forall t \geqslant 0. \tag{7.1}$$

Proof. If $z \in \partial D$, then $1/z \in \partial D$. We have $q(1/z) = 1/h(z)$. The two requirements $D(0; 1/4) \subset q(D)$ and $1/z \in \partial D$ imply that $q(1/z) \geqslant 1/4$; thus $h(z) \leqslant 4$. □

Letting $\varepsilon \to 0$ in (5.9), we obtained the stochastic differential (5.10) for $F^r_{x,t}$ where as in (6.3), the stochastic differential $d_t\,\bar\partial V^r_{x,t}$ is given by (6.2). Replacing $d_t\,\bar\partial V^r_{x,t}$ by expression (6.2), we get with $z = (F^r_{x,t})^{-1}(\zeta)$,

$$d_t F^r_{x,t}(z) = \frac{1}{8\pi} \sum_{k>1} r^k \sqrt{k^3 - k}\big(dx_{2k}(t) + \sqrt{-1}\,dx_{2k+1}(t)\big) \times I_k(\zeta) \tag{7.2}$$

where we denote

$$I_k(\zeta) = \int_{f^r_{x,t}(D)} \frac{1}{\zeta - \zeta_1} \left(\frac{\partial f^r_{x,t}}{\overline{\partial f^r_{x,t}}} \times u_k\right) \circ \left(f^r_{x,t}\right)^{-1}(\zeta_1)\, d\zeta_1 \wedge d\overline{\zeta_1} \tag{7.3}$$

and $u_k(z) = (1 - z\bar z)^2 \bar z^{k-2}$.

Letting $\zeta_0, \zeta_0' \in F^r_{x,t}(\partial D)$, say $\zeta_0 = F^r_{x,t}(z_0)$, $\zeta_0' = F^r_{x,t}(z_0')$, our next objective is to evaluate the Itô contraction (see for example [24])

$$d_t\big(F^r_{x,t}(z_0) - F^r_{x,t}(z_0')\big) * d_t\big(\overline{F^r_{x,t}(z_0) - F^r_{x,t}(z_0')}\big). \tag{7.4}$$

By means of (7.3), we obtain

$$\begin{aligned} &d_t\big(F^r_{x,t}(z_0) - F^r_{x,t}(z_0')\big) * d_t\big(\overline{F^r_{x,t}(z_0) - F^r_{x,t}(z_0')}\big) \\ &\quad = \frac{2}{64\pi^2} \sum_{k>1} r^{2k}\big(k^3 - k\big)\left|I_k(\zeta_0) - I_k(\zeta_0')\right|^2 dt \end{aligned} \tag{7.5}$$

and

$$\begin{aligned} I_k(\zeta_0) - I_k(\zeta_0') &= (\zeta_0' - \zeta_0) \\ &\quad \times \int_{f^r_{x,t}(D)} \frac{1}{(\zeta_0 - \zeta_1)(\zeta_0' - \zeta_1)} \left(\frac{\partial f^r_{x,t}}{\overline{\partial f^r_{x,t}}} \times u_k\right) \circ \left(f^r_{x,t}\right)^{-1}(\zeta_1)\, d\zeta_1 \wedge d\overline{\zeta_1}. \end{aligned} \tag{7.6}$$

For the covariance of Brownian motion on the diffeomorphisms of the circle, log-Lipschitzian estimates have been established in [28,6,13]. Here, we consider $F^r_{x,t} = f^r_{x,t} \circ \Psi^r_{x,t}$ restricted to the circle.

Carrying out the change of variable $\zeta_1 = f^r_{x,t}(z_1)$ in (7.6), we get

$$\left|I_k(\zeta_0) - I_k(\zeta_0')\right|^2 = \left|(\zeta_0' - \zeta_0)\right|^2 \times L_k$$

with

$$\begin{aligned} L_k = \int_{D^2} & \frac{1}{(\zeta_0 - f^r_{x,t}(z_1))(\zeta_0' - f^r_{x,t}(z_1))\,\overline{(\zeta_0 - f^r_{x,t}(z_2))(\zeta_0' - f^r_{x,t}(z_2))}} \\ & \times \left(\frac{\partial f^r_{x,t}}{\overline{\partial f^r_{x,t}}}\right)(z_1) u_k(z_1) \left|\partial f^r_{x,t}(z_1)\right|^2 \\ & \times \left(\frac{\partial f^r_{x,t}}{\overline{\partial f^r_{x,t}}}\right)(z_2) \overline{u_k(z_2)} \left|\partial f^r_{x,t}(z_2)\right|^2 dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2}. \end{aligned} \tag{7.7}$$

From (6.6) and (6.5) we have

$$\left|\sum_{k>1} r^{2k}\left(k^3 - k\right) u_k(z_1)\overline{u_k(z_2)}\right| \leqslant 8 \times 12 \exp\left(-2 d_{\mathrm{H}}(z_1, z_2)\right).$$

Substituting in (7.5) we obtain

$$d_t\left(F^r_{x,t}(z_0) - F^r_{x,t}\left(z_0'\right)\right) * d_t\left(\overline{F^r_{x,t}(z_0) - F^r_{x,t}\left(z_0'\right)}\right) = \left|\zeta_0 - \zeta_0'\right|^2 \times J\,dt \tag{7.8}$$

where

$$\begin{aligned} J \leqslant \frac{3}{\pi^2} \int_{D^2} & \frac{\exp(-2 d_{\mathrm{H}}(z_1, z_2))}{|(\zeta_0 - f^r_{x,t}(z_1))(\zeta_0' - f^r_{x,t}(z_1))(\zeta_0 - f^r_{x,t}(z_2))(\zeta_0' - f^r_{x,t}(z_2))|} \\ & \times \left|\partial f^r_{x,t}(z_1)\right|^2 \left|\partial f^r_{x,t}(z_2)\right|^2 dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2}. \end{aligned} \tag{7.9}$$

Transforming back to the ζ-variable, $\zeta = f^r_{x,t}(z)$, this gives

$$J \leqslant \frac{3}{\pi^2} \int_{(f^r_{x,t}(D))^2} \frac{\exp(-2 d_{\mathrm{H}}((f^r_{x,t})^{-1}(\zeta_1), (f^r_{x,t})^{-1}(\zeta_2)))}{|(\zeta_0 - \zeta_1)(\zeta_0' - \zeta_1)(\zeta_0 - \zeta_2)(\zeta_0' - \zeta_2)|} d\zeta_1 \wedge d\overline{\zeta_1} \wedge d\zeta_2 \wedge d\overline{\zeta_2} \tag{7.10}$$

where as above d_{H} denotes the Poincaré distance on D. Moreover, because of (5.3), the domain of integration for the integral (7.10) is

$$F^r_{x,t}(D) = f^r_{x,t}(D). \tag{7.11}$$

The following theorem is obtained by establishing an upper bound for J.

Theorem 7.2. *Denote*

$$d_t\big(F^r_{x,t}(z_0) - F^r_{x,t}\big(z'_0\big)\big) * d_t\big(\overline{F^r_{x,t}(z_0) - F^r_{x,t}\big(z'_0\big)}\big)$$
$$=: \mathbb{E}^{\mathscr{F}_t}\big[\big|\delta_t\big(F^r_{x,t}\big)(\zeta_0) - \delta_t\big(F^r_{x,t}\big)\big(\zeta'_0\big)\big|^2\big]\,dt, \quad \zeta_0 = F^r_{x,t}(z_0),\ \zeta'_0 = F^r_{x,t}\big(z'_0\big).$$

Then there exists a numerical constant c, independent of r, such that for all $\zeta_0, \zeta'_0 \in F^r_{x,t}(\partial D)$,

$$\mathbb{E}^{\mathscr{F}_t}\big[\big|\delta_t\big(F^r_{x,t}\big)(\zeta_0) - \delta_t\big(F^r_{x,t}\big)\big(\zeta'_0\big)\big|^2\big] \leqslant c\big|\zeta_0 - \zeta'_0\big|^2 \log \frac{16}{|\zeta_0 - \zeta'_0|}. \tag{7.12}$$

Proof. We already proved that

$$\mathbb{E}^{\mathscr{F}_t}\big[\big|\delta_t\big(F^r\big)(\zeta_0) - \delta_t\big(F^r\big)\big(\zeta'_0\big)\big|^2\big] \leqslant c\big|\zeta_0 - \zeta'_0\big|^2 \times J$$

for some constant c where J satisfies estimate (7.10). Recall the definition of the approximate hyperbolic metric for an open subset $\Omega \subset D$ (see [31, p. 92], [15] and [9]):

$$d_\Omega(\eta_1, \eta_2) = \inf_\gamma \int_\gamma \frac{|d\zeta|}{\operatorname{dist}(\zeta, \partial\Omega)} \tag{7.13}$$

where γ is any rectifiable curve joining η_1 to η_2. Then by [31, formula (17), p. 9 and formula (6), p. 92] we have for the Poincaré distance d_{H} on D:

$$d_{\mathrm{H}}\big(f^{-1}(\eta_1), f^{-1}(\eta_2)\big) \geqslant \frac{1}{4} d_{\Gamma^+}(\eta_1, \eta_2) \tag{7.14}$$

where f is a univalent function mapping the disk upon Γ^+. As we used (6.5) to derive (7.10), we now have

$$\exp\big(-2d_{\mathrm{H}}\big(\big(f^r_{x,t}\big)^{-1}(\zeta_1), \big(f^r_{x,t}\big)^{-1}(\zeta_2)\big)\big) \leqslant \exp\bigg(-\frac{1}{2} d_{\Gamma^+}(\zeta_1, \zeta_2)\bigg). \tag{7.15}$$

The proof of Theorem 7.2 will be completed after some preparatory lemmas. □

Denote by Ω the complement in the complex plane of the two points $\zeta_0, \zeta'_0 \in F^r_{x,t}(\partial D)$; then $\Gamma^+ \subset \Omega$ therefore

$$d_{\Gamma^+}(\zeta_1, \zeta_2) \geqslant d_\Omega(\zeta_1, \zeta_2). \tag{7.16}$$

Set $\delta = \delta(\zeta_0, \zeta'_0) := |\zeta_0 - \zeta'_0|$, then up to a Euclidean motion of the complex plane, the distance d_Ω is characterized by δ.

Lemma 7.3. *We have*

$$J \leqslant K_\delta\big(F_{x,t}(D)\big) \leqslant K_\delta(4D), \tag{7.17}$$

where

$$K_\delta(B) := \frac{3}{\pi^2} \int\limits_{B^2} \frac{\exp(-d_\Omega(\zeta_1, \zeta_2)/2)}{|(\zeta_0 - \zeta_1)(\zeta_0' - \zeta_1)(\zeta_0 - \zeta_2)(\zeta_0' - \zeta_2)|}\, d\zeta_1 \wedge d\bar{\zeta}_1 \wedge d\zeta_2 \wedge d\bar{\zeta}_2. \tag{7.18}$$

For all $\lambda > 0$,

$$K_\delta(B) = K_{\lambda\delta}\big(H_\lambda(B)\big) \tag{7.19}$$

where H_λ *denotes the homothety of ratio* λ *and center* $(\zeta_0 + \zeta_0')/2$; *in particular*

$$J \leqslant K_1\left(\frac{8D}{\delta}\right). \tag{7.20}$$

Proof. The first inequality in (7.17) is a direct consequence of (7.13)–(7.16) and (7.10)–(7.11); the second inequality a consequence of Lemma 7.1.

On the other hand,

$$H_\lambda(\zeta) = \lambda\zeta + (1-\lambda)\frac{\zeta_0 + \zeta_0'}{2}; \tag{7.21}$$

thus

$$H_\lambda(\zeta_0) = \frac{1+\lambda}{2}\zeta_0 + \frac{1-\lambda}{2}\zeta_0', \qquad H_\lambda\big(\zeta_0'\big) = \frac{1+\lambda}{2}\zeta_0' + \frac{1-\lambda}{2}\zeta_0$$

and

$$H_\lambda\big(\zeta_0'\big) - H_\lambda(\zeta_0) = \lambda\big(\zeta_0' - \zeta_0\big), \tag{7.22}$$

i.e.,

$$H_\lambda(u_1) - H_\lambda(u_2) = \lambda(u_1 - u_2) \quad \forall u_1, u_2.$$

With the change of variables $\zeta_1 = H_\lambda(u_1)$, we have

$$\int\limits_{H_\lambda(B)} \frac{\phi(\zeta_1)}{|(H_\lambda(\zeta_0) - \zeta_1)(H_\lambda(\zeta_0') - \zeta_1)|}\, d\zeta_1 \wedge d\overline{\zeta_1} = \int\limits_B \frac{\phi(H_\lambda(u_1))}{|(\zeta_0 - u_1)(\zeta_0' - u_1)|}\, du_1 \wedge d\overline{u_1}.$$

In the same way, we carry out the change of variables in integrals of the type

$$\int\limits_{H_\lambda(B)^2} \frac{\phi(\zeta_1, \zeta_2)}{|(H_\lambda(\zeta_0) - \zeta_1)(H_\lambda(\zeta_0') - \zeta_1)(H_\lambda(\zeta_0) - \zeta_2)(H_\lambda(\zeta_0') - \zeta_2)|}\, d\zeta_1 \wedge d\overline{\zeta_1} \wedge d\zeta_2 \wedge d\overline{\zeta_2}.$$

Taking $\phi(\zeta_1, \zeta_2) = \exp(-\frac{1}{2} d_{H_\lambda(B)}(\zeta_1, \zeta_2))$ and using

$$d_\Omega(\zeta_1, \zeta_2) = d_{H_\lambda(\Omega)}\big(H_\lambda(\zeta_1), H_\lambda(\zeta_2)\big), \tag{7.23}$$

we see that integral in (7.18) is invariant under H_λ and we have (7.19).

Finally to establish (7.20) we observe that

$$K_1\big(H_{1/\delta}(4D)\big) \leqslant K_1\left(\frac{8D}{\delta}\right). \qquad \square \tag{7.24}$$

Lemma 7.4.

$$\lim_{\lambda\to\infty} d_\Omega(\lambda\zeta_1, \lambda\zeta_2) \geqslant \left|\log\frac{|\zeta_1|}{|\zeta_2|}\right|. \tag{7.25}$$

Proof. Set

$$\rho(\zeta) := |\zeta - \zeta_0|, \qquad \rho'(\zeta) := \left|\zeta - \zeta_0'\right|,$$
$$r_1 := \rho(\zeta_1), \qquad r_1' := \rho'(\zeta_1), \qquad r_2 := \rho(\zeta_2), \qquad r_2' := \rho'(\zeta_2).$$

Write $C(\eta, a)$ for the circle of center η and radius a. Since $\zeta_1 \in C(\zeta_0, r_1)$ and $\zeta_2 \in C(\zeta_0', r_2')$, we have

$$d_\Omega(\zeta_1, \zeta_2) \geqslant d_\Omega\big(C(\zeta_0, r_1), C\big(\zeta_0', r_2'\big)\big). \tag{7.26}$$

The distance $d_\Omega(C(\zeta_0, r_1), C(\zeta_0', r_2'))$ vanishes if the two circles intersect which means that

$$\left|r_1 - r_2'\right| \leqslant \delta \leqslant r_1 + r_2'. \tag{7.27}$$

Assume that $r_1 \leqslant r_2'$ and that (7.27) fails (that is the two circles do not intersect). Then either $r_2' > \delta + r_1$ or $r_1 + r_2' < \delta$.

In the first case, we have

$$d_\Omega\big(C(\zeta_0, r_1), C\big(\zeta_0', r_2'\big)\big) = \log\frac{r_2' - \delta}{r_1} \leqslant d_\Omega(\zeta_1, \zeta_2),$$

and in the second case,

$$d_\Omega\big(C(\zeta_0, r_1), \bar{D}\big(\zeta_0', r_2'\big)\big) = \log\frac{\delta - r_2'}{r_1} = \log\frac{|r_2' - \delta|}{r_1} \leqslant d_\Omega(\zeta_1, \zeta_2).$$

Thus

$$\begin{aligned} &\log\frac{|r_2' - \delta|}{r_1} \leqslant d_\Omega(\zeta_1, \zeta_2), \quad r_1 \leqslant r_2', \\ &\log\frac{|r_1' - \delta|}{r_2} \leqslant d_\Omega(\zeta_1, \zeta_2), \quad r_1 \geqslant r_2'. \end{aligned} \tag{7.28}$$

The lemma is proved by fixing δ, rewriting (7.28) for $\lambda\zeta_1, \lambda\zeta_2$, and letting $\lambda \to \infty$ in (7.28). $\square$

End of Proof of Theorem 7.2. We have

$$K_1(2\lambda D) - K_1(\lambda D) = \int\limits_{\{0<|\zeta_1|<2\lambda\}} \int\limits_{\{0<|\zeta_2|<2\lambda\}} \cdots - \int\limits_{\{0<|\zeta_1|<\lambda\}} \int\limits_{\{0<|\zeta_2|<\lambda\}} \cdots$$
$$= 2 \int\limits_{\{\lambda<|\zeta_2|<2\lambda\}} \int\limits_{\{0<|\zeta_1|<|\zeta_2|\}} \cdots . \tag{7.29}$$

The second equality in (7.29) is easily obtained by passing to polar coordinates, $|\zeta_j| = \rho_j$ and integrating over the squares

$$\{0 < \rho_j < 2\lambda,\ j = 1, 2\} \quad \text{and} \quad \{0 < \rho_j < \lambda,\ j = 1, 2\}.$$

Expressing the volume element in polar coordinates

$$|\zeta_j| = \rho_j, \quad \zeta_j = \rho_j \exp(i\psi_j),$$

$$\frac{1}{4\pi^2} \int\limits_{\{\lambda<|\zeta_2|<2\lambda\}} \int\limits_{\{0<|\zeta_1|<|\zeta_2|\}} \cdots \leqslant \int_\lambda^{2\lambda} \frac{d\rho_2}{\rho_2} \left(c + 3 \int_0^{\rho_2} \frac{d\rho_1}{\rho_1} \left(\rho_2^{-1}\rho_1\right)^{1/2} \right), \tag{7.30}$$

we obtain

$$\limsup_{\lambda\to\infty} \bigl(K_1(2\lambda D) - K_1(\lambda D)\bigr) \leqslant (c+6)\log 2 < \infty. \tag{7.31}$$

On the other hand, let $\phi(\lambda)$ be a real-valued, continuous and increasing function of the variable λ; assume that

$$\lim_{\lambda\to\infty} \bigl(\phi(2\lambda) - \phi(\lambda)\bigr) < +\infty.$$

Then $\phi(\lambda) \leqslant C \log\lambda$ for λ near ∞. This can be seen as follows: we verify that $0 < A = \lim_{n\to\infty}(\phi(2^{n+1}) - \phi(2^n)) < +\infty$ implies asymptotically

$$\phi\bigl(2^n\bigr) = \bigl[\phi\bigl(2^n\bigr) - \phi\bigl(2^{n-1}\bigr)\bigr] + \bigl[\phi\bigl(2^{n-1}\bigr) - \phi\bigl(2^{n-2}\bigr)\bigr] + \cdots \leqslant n \times A =: C\log\bigl(2^n\bigr);$$

then we extend the proof by considering $\psi(2^\lambda) = \phi(\lambda)$.

The combination of formula (7.31) along with (7.20) proves (7.12). It remains to justify (7.30); to this end we need an upper bound for the integrand

$$\frac{\exp(-d_\Omega(\zeta_1, \zeta_2)/2)}{|(\zeta_0 - \zeta_1)(\zeta_0' - \zeta_1)(\zeta_0 - \zeta_2)(\zeta_0' - \zeta_2)|}. \tag{7.32}$$

Since $\zeta_0, \zeta_0' \in F^r_{x,t}(\partial D)$, by (7.1), we note that $|\zeta_0| \leqslant 4$ and $|\zeta_0'| \leqslant 4$. For $\lambda > 8$ and $\rho_2 = |\zeta_2| > \lambda$, we have

$$|\zeta_0 - \zeta_2| > \frac{\rho_2}{2} \quad \text{and} \quad \left|\zeta_0' - \zeta_2\right| > \frac{\rho_2}{2}. \tag{7.33}$$

If $\rho_1 < \rho_2$, then $\log(\rho_1/\rho_2) < 0$. By (7.25),

$$d_\Omega(\lambda\zeta_1, \lambda\zeta_2) \geqslant \log(\rho_2/\rho_1). \tag{7.34}$$

Then, for any $\lambda > 8$,

$$\exp\left(-\frac{1}{2} d_\Omega(\lambda\zeta_1, \lambda\zeta_2)\right) \leqslant \left(\frac{\rho_1}{\rho_2}\right)^{1/2}. \tag{7.35}$$

Since in polar coordinates, the volume element is $\rho_j\, d\rho_j\, d\psi_j$, $j = 1, 2$, we see that we have to estimate

$$\int_\lambda^{2\lambda} \frac{d\rho_2}{\rho_2} \int_0^{\rho_2} \frac{1}{\rho_1^2} \left(\frac{\rho_1}{\rho_2}\right)^{1/2} \rho_1\, d\rho_1. \tag{7.36}$$

This gives (7.30). □

8. Moduli of continuity of regularized welding

8.1. Local moduli of continuity of regularized welding

As $F^r_{x,t}$ is C^∞, its restriction to ∂D is Hölderian. The purpose of this subsection and the following theorem is to obtain uniform estimates in r and t.

Theorem 8.1. *Let*

$$\eta(t) \equiv \eta_x(t) := \left|F^r_{x,t}(\zeta) - F^r_{x,t}\left(\zeta'\right)\right|, \quad \zeta, \zeta' \in \partial D,$$

and

$$\gamma^+ = \sup_{s\in[0,t]} \frac{\log \eta(s)}{\log \eta(0)}, \qquad \gamma^- = \inf_{s\in[0,t]} \frac{\log \eta(s)}{\log \eta(0)}.$$

Let δ be a constant such that $0 < \delta < 1$. Then

$$\sqrt{\gamma^+} \geqslant \delta + 1 \quad \textit{if and only if} \quad \inf_{s\in[0,t]} \frac{|F^r_{x,s}(\zeta) - F^r_{x,s}(\zeta')|}{|\zeta - \zeta'|^{(1+\delta)^2}} \leqslant 1$$

and

$$\sqrt{\gamma^-} \leqslant 1 - \delta \quad \textit{if and only if} \quad \sup_{s\in[0,t]} \frac{|F^r_{x,s}(\zeta) - F^r_{x,s}(\zeta')|}{|\zeta - \zeta'|^{(1-\delta)^2}} \geqslant 1.$$

There exists a constant $\sigma(t)$ depending on t, but independent of r, such that

$$\operatorname{Prob}\left\{\sqrt{\gamma^-} \leqslant 1-\delta\right\} \leqslant \frac{2\sqrt{\sigma(t)}c}{\sqrt{2\pi}\,\delta\sqrt{\log\frac{1}{\eta(0)}}}\exp\left(-\frac{\delta^2}{2\sigma(t)c^2} \times \left|\log\eta(0)\right|\right), \tag{8.1}$$

$$\operatorname{Prob}\left\{\sqrt{\gamma^+} > \delta+1\right\} \leqslant \frac{2\sqrt{\sigma(t)}c}{\sqrt{2\pi}\,\delta\sqrt{\log\frac{1}{\eta(0)}}}\exp\left(-\frac{\delta^2}{2\sigma(t)c^2} \times \left|\log\eta(0)\right|\right). \tag{8.2}$$

The function $\sigma(t)$ is independent of r, ζ, ζ' and δ and tends to zero as $t \to 0$.

Proof. The first two assertions are straightforward. We confine ourselves to prove (8.1) and (8.2). Let

$$\gamma_s = \frac{\log\eta(s)}{\log\eta(0)} = \frac{\log\big(1/\eta(s)\big)}{\log\big(1/\eta(0)\big)}. \tag{8.3}$$

By definition, $\gamma_0 = 1$ and $F^r_{x,0}(\zeta) - F^r_{x,0}(\zeta') = \zeta - \zeta'$. If we assume that $\eta(0) = |\zeta - \zeta'| < 1$, then $\log(1/\eta(0)) > 0$.

All Itô differentials below are well defined up to the stopping time

$$T = \inf\big\{s \in [0,\infty[:\ \gamma_s < 0\big\}.$$

The subsequent computations allow to evaluate the probability of the event $\{T < t\}$ which is of small order. We may limit ourselves to the case where $\eta(0) < 1$; these facts legitimate the change of variables in (8.3).

We have

$$\sqrt{\log\frac{1}{\eta(0)}} \times \left(1 - \sqrt{\gamma_s}\,\right) = \sqrt{\log\frac{1}{\eta(0)}} - \sqrt{\log\frac{1}{\eta(s)}}, \tag{8.4}$$

from where we deduce that

$$\sqrt{\log\frac{1}{\eta(0)}} \times \big(1 - \sqrt{\gamma^-}\,\big) = \sup_{s\in[0,t]}\left(\sqrt{\log\frac{1}{\eta(0)}} - \sqrt{\log\frac{1}{\eta(s)}}\right).$$

Furthermore we notice that the condition

$$1 - \sqrt{\gamma^-} > \delta$$

is equivalent to

$$\sup_{s\in[0,t]}\left(\sqrt{\log\frac{1}{\eta(0)}} - \sqrt{\log\frac{1}{\eta(s)}}\right) > \delta\sqrt{\log\frac{1}{\eta(0)}}$$

which amounts to say

$$\inf_{s\in[0,t]}\left(\sqrt{\log\frac{1}{\eta(s)}} - \sqrt{\log\frac{1}{\eta(0)}}\right) < -\delta\sqrt{\log\frac{1}{\eta(0)}}. \tag{8.5}$$

In the same way, from (8.4) we obtain

$$\sqrt{\log \frac{1}{\eta(0)}} \times \left(\sqrt{\gamma^+} - 1\right) = \sup_{s \in [0,t]} \left(\sqrt{\log \frac{1}{\eta(s)}} - \sqrt{\log \frac{1}{\eta(0)}} \right)$$

and we conclude that

$$\sqrt{\gamma^+} - 1 > \delta$$

is equivalent to

$$\sup_{s \in [0,t]} \left(\sqrt{\log \frac{1}{\eta(s)}} - \sqrt{\log \frac{1}{\eta(0)}} \right) > \delta \sqrt{\log \frac{1}{\eta(0)}}. \tag{8.6}$$

The probabilities of the events (8.5) and (8.6) will be evaluated in Lemma F below after several intermediate results. □

The following lemma serves as a key lemma.

Lemma A. *Let*

$$v(t) \equiv v_{x,\zeta,\zeta'}(t) = F_{x,t}(\zeta) - F_{x,t}\left(\zeta'\right), \quad \zeta, \zeta' \in \partial D.$$

Then

$$dv(t) = A\left(t, v(t)\right) dz(t) + B\left(t, v(t)\right) dt \tag{8.7}$$

where $z(t)$ is a Brownian motion in the complex plane, and

$$\begin{aligned} \left|A(t)\right| &\leqslant c\left|v(t)\right|\left(\sqrt{\left|\log\left|v(t)\right|\right|} + 1\right), \\ \left|B(t)\right| &\leqslant c\left|v(t)\right|\left(1 + \left|\log\left|v(t)\right|\right|\right). \end{aligned} \tag{8.8}$$

Proof. If $v(t)$ satisfies (8.7), then the Itô contraction takes the form

$$dv(t) * d\overline{v(t)} = 2A\left(t, v(t)\right)\overline{A\left(t, v(t)\right)}\, dt.$$

By (7.12), we deduce the first inequality in (8.8).

The remaining claims result from Itô calculus applied to (7.2). For inequality (8.8), let $b(t)$ be a real Brownian motion, and compare Eq. (7.2) to the Stratonovich SDE

$$dw(t) = c_1 w(t)\left(\sqrt{\log \frac{1}{w(t)}} + 1\right) \circ db(t), \quad w(0) = \left|v(0)\right| < 1.$$

Let $\tilde{A}(w) = c_1 w(\sqrt{\log|w|} + 1)$. In this case, the drift obtained by passing to the Itô SDE is

$$\tilde{B}\,dt = \frac{1}{2}\frac{\partial\tilde{A}}{\partial w}\tilde{A}\,dt,$$

and the estimate $|\tilde{B}| \leqslant c_2|w|(\log|w| + 1)$ results. □

The next lemma is obtained from Lemma A by Itô's formula.

Lemma B. *Set $\lambda(t) := |F^r_{x,t}(\zeta) - F^r_{x,t}(\zeta')|^2 = \eta(t)^2$. Then $\lambda(t)$ is solution of the Itô equation*

$$d\lambda(t) = A_1(t)\,db(t) + B_1(t)\,dt \tag{8.9}$$

where $b(t)$ is a one-dimensional real Brownian motion and where $A_1(t)$, $B_1(t)$ satisfy

$$0 < A_1(t) \leqslant 2c\eta(t)^2\left(1 + \sqrt{\log\frac{1}{\eta(t)}}\right) \quad \textit{and}$$
$$\left|B_1(t)\right| \leqslant \left(2c + 4c^2\right)\eta(t)^2\left(1 + \log\frac{1}{\eta(t)}\right). \tag{8.10}$$

Moreover, for $\eta(t) = \sqrt{\lambda(t)}$, we obtain

$$d\eta(t) = \alpha(t)\,db(t) + \beta(t)\,dt \tag{8.11}$$

with

$$0 < \alpha(t) < 2c\eta(t)\sqrt{\log\frac{1}{\eta(t)}}, \quad \left|\beta(t)\right| < 4\left(c + 2c^2\right)\eta(t)\log\frac{1}{\eta(t)}. \tag{8.12}$$

The estimates (8.12) *are valid for small values of $\eta(t)$*; *more precisely up to the first hitting time of $\eta(t)$ at $1/e$.*

Proof. By Eq. (8.7), we have

$$dv(t) = A(t)\big(dx(t) + i\,dy(t)\big) + B(t)\,dt$$

where $x(t)$ and $y(t)$ are two independent real Brownian motions. By Itô calculus,

$$\begin{aligned}
d\lambda &= v\,d\bar{v} + \bar{v}\,dv + dv * d\bar{v} \\
&= (v\bar{A} + \bar{v}A)\,dx + i(\bar{v}A - v\bar{A})\,dy + (v\bar{B} + \bar{v}B + 2A\bar{A})\,dt \\
&= C_1\,dx + C_2\,dy + (v\bar{B} + \bar{v}B + 2A\bar{A})\,dt
\end{aligned}$$

with C_1, C_2 real-valued. Since $C_1\,dx + C_2\,dy = \sqrt{C_1^2 + C_2^2}\,db$ for some real Brownian motion b, we obtain

$$d\lambda = A_1\,db + B_1\,dt$$

with $A_1 > 0$. Moreover,

$$\begin{aligned}0 < A_1 &\leqslant 2c\big|v_{x,\zeta,\zeta'}(t)\big|^2\Big(\sqrt{\big|\log\big|v_{x,\zeta,\zeta'}(t)\big|\big|} + 1\Big),\\ |B_1| &\leqslant \big(2c + 4c^2\big)\big|v_{x,\zeta,\zeta'}(t)\big|^2\big(\big|\log\big|v_{x,\zeta,\zeta'}(t)\big|\big| + 1\big).\end{aligned}$$

This proves (8.10).

Now let $\eta = \sqrt{\lambda}$. Then again by Itô calculus

$$d\eta = \frac{1}{2\sqrt{\lambda}}\,d\lambda - \frac{1}{8\lambda\sqrt{\lambda}}\,d\lambda * d\lambda = \frac{A_1}{2\sqrt{\lambda}}\,db + \left(\frac{B_1}{2\sqrt{\lambda}} - \frac{A_1^2}{8\lambda\sqrt{\lambda}}\right)dt$$

and (8.12) is a consequence of (8.10). $\square$

Lemma C. *Introduce the function*

$$\phi(x) := \frac{1}{c}\sqrt{\log\frac{1}{x}}, \quad x > 0,$$

and consider the process $\eta(t) = |v(t)|$ as in (8.11) *and* (8.12). *Then*

$$u(t) = \phi\big(\eta(t)\big)$$

is solution of the following Itô equation:

$$du(t) = \alpha_1(t)\,db_1(t) + \beta_1(t)\,dt, \quad 0 < \alpha_1 \leqslant 1,\ |\beta_1| \leqslant c_1 u,\ c_1 := 4\big(c + c^2\big), \tag{8.13}$$

where $b_1(t)$ is a Brownian motion.

Proof. As

$$\begin{aligned}\phi'(x) &= -\frac{1}{2c}\left(\log\frac{1}{x}\right)^{-1/2}\frac{1}{x};\\ \phi''(x) &= -\frac{1}{4c}\left(\log\frac{1}{x}\right)^{-3/2}\frac{1}{x^2} + \frac{1}{2c}\left(\log\frac{1}{x}\right)^{-1/2}\frac{1}{x^2},\end{aligned}$$

we see that $\phi''(x) > 0$ for $x < e^{-1/2}$. Thus

$$0 < \phi''(x) < \frac{1}{2c}\left(\log\frac{1}{x}\right)^{-1/2}\frac{1}{x^2}. \tag{8.14}$$

By Itô calculus,

$$\begin{aligned} du(t) &= \phi'\big(\eta(t)\big)\,d\eta(t) + \frac{1}{2}\phi''\big(\eta(t)\big)\alpha^2(t)\,dt \\ &= \phi'\big(\eta(t)\big)\alpha(t)\,db(t) + \left[\phi'\big(\eta(t)\big)\beta(t) + \frac{1}{2}\phi''\big(\eta(t)\big)\alpha^2(t)\right]dt \\ &= \alpha_1(t)\,db_1(t) + \beta_1(t)\,dt \end{aligned}$$

where $b_1(t) = -b(t)$ is a Brownian motion and

$$\alpha_1(t) = -\phi'\big(\eta(t)\big)\alpha(t), \qquad \beta_1(t) = \phi'\big(\eta(t)\big)\beta(t) + \frac{1}{2}\phi''\big(\eta(t)\big)\alpha^2(t).$$

The function $\alpha_1(t) = -\phi'(\eta(t))\alpha(t)$ satisfies $0 < \alpha_1(t) < 1$. The upper bound for β_1 is deduced from (8.12). □

Lemma D. *Let $u(s)$ be the process given by* (8.13). *Consider the two comparison processes*

$$du^{\pm} = db_1 \pm c_1 u^{\pm}\,dt, \quad u^{\pm}(0) = u(0), \tag{8.15}$$

or equivalently

$$\begin{aligned} u^+(t) - u^+(0) &= \int_0^t \exp\big(c_1(t-s)\big)\,db_1(s), \\ u^-(t) - u^-(0) &= \int_0^t \exp\big(-c_1(t-s)\big)\,db_1(s). \end{aligned} \tag{8.16}$$

Then

$$\begin{aligned} \inf_{s\in[0,t]}\big(u^-(s) - u^-(0)\big) &\leqslant \inf_{s\in[0,t]}\big(u(s) - u(0)\big) \\ &< \sup_{s\in[0,t]}\big(u(s) - u(0)\big) \\ &\leqslant \sup_{s\in[0,t]}\big(u^+(s) - u^+(0)\big). \end{aligned}$$

Proof. We use Ikeda–Watanabe's comparison theorem [18]. □

Lemma E. *Let m be a positive integer. For the two processes $u^+(t)$ and $u^-(t)$, we have*

$$\operatorname{Prob}\Big\{\inf_{s\in[0,t]}\big(u^-(s) - u^-(0)\big) < -m\Big\} \leqslant \frac{2\sqrt{\tau^-(t)}}{m\sqrt{2\pi}}\exp\left(-\frac{m^2}{2\tau^-(t)}\right) \tag{8.17}$$

with

$$\tau^-(t) = \frac{\exp(2c_1 t) - 1}{2c_1 \exp(2c_1 t)},$$

and

$$\operatorname{Prob}\Big\{\sup_{s\in[0,t]}\big(u^+(s) - u^+(0)\big) > m\Big\} \leqslant \frac{2\sqrt{\tau^+(t)}}{m\sqrt{2\pi}} \exp\bigg(-\frac{m^2}{2\tau^+(t)}\bigg) \tag{8.18}$$

with

$$\tau^+(t) = \frac{\exp(2c_1 t) - 1}{2c_1}.$$

Proof. We may write

$$\int_0^t \exp\big(-c_1(t-s)\big)\,db_1(s) = \exp(-c_1 t) B_{\tau(t)}$$

where B_t is a Brownian motion and the rescaling $\tau(t)$ is given by

$$\tau(t) = \int_0^t \exp(2c_1 s)\,ds = \frac{e^{2c_1 t} - 1}{2c_1}.$$

Let

$$\tau^{-1}(s) = \frac{\log(2c_1 s + 1)}{2c_1}$$

be the inverse to τ. Since

$$\inf_{s\in[0,t]} e^{-c_1 s} B_{\tau(s)} = \inf_{s\in[0,\tau(t)]} e^{-c_1\tau^{-1}(s)} B_s = \inf_{s\in[0,\tau(t)]} \frac{B_s}{\sqrt{2c_1 s + 1}},$$

we obtain with the reflection principle of Brownian motion:

$$\begin{aligned}
\operatorname{Prob}\Big\{\inf_{s\in[0,t]}\big(u^-(s) - u^-(0)\big) < -m\Big\} &= \operatorname{Prob}\bigg\{\inf_{s\in[0,\tau(t)]} \frac{B_s}{2c_1 s + 1} < -m\bigg\} \\
&\leqslant \operatorname{Prob}\Big\{\inf_{s\in[0,\tau(t)]} B_s < -m\sqrt{2c_1\tau(t) + 1}\Big\} \\
&= \operatorname{Prob}\big\{|B_{\tau(t)}| \geqslant m_*\big\} \quad \big[\text{where } m_* := m\sqrt{2c_1\tau(t) + 1}\,\big] \\
&= 2\int_{m_*}^{+\infty} \exp\bigg(-\frac{x^2}{2\tau(t)}\bigg) \frac{dx}{\sqrt{2\pi\tau(t)}}
\end{aligned}$$

$$\leqslant \frac{2\sqrt{\tau(t)}}{m_*\sqrt{2\pi}}\exp\left(-\frac{m_*^2}{2\tau(t)}\right).$$

This proves (8.17) with

$$\tau^-(t)=\frac{\int_0^t \exp(2c_1s)\,ds}{2c_1\int_0^t \exp(2c_1s)\,ds+1}=\frac{1}{2c_1}\frac{\exp(2c_1t)-1}{\exp(2c_1t)}.$$

For inequality (8.18) we proceed in the same way; in this case we find

$$\tau^+(t)=\frac{\int_0^t \exp(-2c_1s)\,ds}{1-2c_1\int_0^t \exp(-2c_1s)\,ds}=\frac{1}{2c_1}\frac{1-\exp(-2c_1t)}{\exp(-2c_1t)}=\frac{\exp(2c_1t)-1}{2c_1}. \qquad \square$$

Lemma F. *The two following estimates hold*:

$$\operatorname{Prob}\left\{\inf_{s\in[0,t]}\left(\sqrt{\log\frac{1}{\eta(s)}}-\sqrt{\log\frac{1}{\eta(0)}}\right)<-\delta\sqrt{\log\frac{1}{\eta(0)}}\right\}$$
$$\leqslant \frac{2\sqrt{\tau^-(t)}c}{\sqrt{2\pi}\delta\sqrt{\log\frac{1}{\eta(0)}}}\exp\left(-\frac{\delta^2\log\frac{1}{\eta(0)}}{2\tau^-(t)c^2}\right) \tag{8.19}$$

and

$$\operatorname{Prob}\left\{\sup_{s\in[0,t]}\left(\sqrt{\log\frac{1}{\eta(s)}}-\sqrt{\log\frac{1}{\eta(0)}}\right)>\delta\sqrt{\log\frac{1}{\eta(0)}}\right\}$$
$$\leqslant \frac{2\sqrt{\tau^+(t)}c}{\sqrt{2\pi}\delta\sqrt{\log\frac{1}{\eta(0)}}}\exp\left(-\frac{\delta^2\log\frac{1}{\eta(0)}}{2\tau^+(t)c^2}\right). \tag{8.20}$$

Proof. The inequality

$$\inf_{s\in[0,t]}\left(\sqrt{\log\frac{1}{\eta(s)}}-\sqrt{\log\frac{1}{\eta(0)}}\right)<-\delta\sqrt{\log\frac{1}{\eta(0)}}$$

is equivalent to

$$\inf_{s\in[0,t]}\big(u(s)-u(0)\big)<-\frac{\delta}{c}\sqrt{\log\frac{1}{\eta(0)}}$$

which implies that

$$\inf_{s\in[0,t]}\big(u^-(s)-u^-(0)\big)<-\frac{\delta}{c}\sqrt{\log\frac{1}{\eta(0)}}.$$

We apply (8.17) of Lemma E to obtain (8.19).

Eq. (8.20) is proved in a similar way,

$$\sup_{s\in[0,t]}\left(\sqrt{\log\frac{1}{\eta(s)}}-\sqrt{\log\frac{1}{\eta(0)}}\right)>\delta\sqrt{\log\frac{1}{\eta(0)}}$$

is equivalent to

$$\sup_{s\in[0,t]}\big(u(s)-u(0)\big)>\frac{\delta}{c}\sqrt{\log\frac{1}{\eta(0)}}.$$

This implies

$$\sup_{s\in[0,t]}\big(u^+(s)-u^+(0)\big)>\frac{\delta}{c}\sqrt{\log\frac{1}{\eta(0)}}$$

and we may use (8.18) of Lemma E to conclude. □

End of Proof of Theorem 8.1. Taking into account that $\tau^-(t)<\tau^+(t)$ and that the function

$$\varphi(\tau)=\frac{2\sqrt{\tau}c}{\sqrt{2\pi}\delta\sqrt{\log\frac{1}{\eta(0)}}}\exp\left(-\frac{\delta^2\log\frac{1}{\eta(0)}}{2\tau c^2}\right)$$

is increasing in τ, we obtain (8.1)–(8.2) with $\sigma(t)=\tau^+(t)$. □

8.2. Moduli of continuity of regularized welding

In the previous section, we derived local estimates of modulus of continuity; these are estimates of $|F^r_{x,t}(\zeta_0)-F^r_{x,t}(\zeta_1)|$ for ζ_0,ζ_1 fixed. Moduli of continuity are obtained from local estimates through estimates for

$$\omega^r_{x,t}(\varepsilon)=\sup_{|\zeta_0-\zeta_1|<\varepsilon}\big|F^r_{x,t}(\zeta_0)-F^r_{x,t}(\zeta_1)\big|. \tag{8.21}$$

We shall implement in this section the classical Kolmogorov methodology of deriving estimates of moduli of continuity of a stochastic process in terms of estimates of its local moduli. Given a continuous function $u(\theta)$ defined for $\theta\in[0,2\pi]$, we consider small intervals where the function has a small variation and we prove Hölderianity on these intervals by means of the triangular inequality, then considering bigger intervals we obtain an estimate in probability of the Hölderian norm.

Introduce the dyadic numbers:

$$A=\bigcup_{n\geqslant 1}A_n$$

where

$$A_n = \{\theta\colon\ \theta = \pi \times p \times 2^{-n},\ p = 1, \ldots, 2^{n+1}\}, \quad n \in \mathbb{N}.$$

The *reduced Hölder exponent* is defined as

$$\gamma_q^{++} := \inf_{n \geqslant q} \inf_{\theta \in A_n} \frac{\log |u(\theta) - u(\theta + 2^{-n}\pi)|}{\log(2^{-n}\pi)}. \tag{8.22}$$

Obviously,

$$\gamma_q^{++} \leqslant \gamma_{q+1}^{++},$$

and for any $\theta \in A_n$, $n \geqslant q$,

$$\bigl|u(\theta) - u(\theta')\bigr| \leqslant \bigl|\theta - \theta'\bigr|^{\gamma_q^{++}} \quad \text{with } \theta' = \theta + \frac{\pi}{2^n}. \tag{8.23}$$

Lemma 8.2. *For all θ and θ' such that $2^{-q}\pi < |\theta - \theta'| \leqslant 2^{-q+1}\pi$, we have*

$$\bigl|u(\theta) - u(\theta')\bigr| \leqslant 2\frac{2^{-\gamma_q^{++}}}{1 - 2^{-\gamma_q^{++}}}\bigl|\theta - \theta'\bigr|^{\gamma_q^{++}}. \tag{8.24}$$

Proof. Given $0 < \theta < \theta' \leqslant 2\pi$, let $q \in \mathbb{N}$ be such that $2^{-q}\pi < \theta' - \theta < 2^{-q+1}\pi$ and $k \in \mathbb{N}$ such that $(k-1)\pi 2^{-q} \leqslant \theta < k\pi 2^{-q}$. The integers q and k then fulfill

$$\theta < 2^{-q}k\pi \leqslant \theta', \quad 2^{-q}\pi < \theta' - \theta < 2^{-q+1}\pi.$$

Consider the dyadic development

$$\frac{\theta}{\pi} = 2^{-q}k - \sum_{\ell > q} 2^{-l}\varepsilon_\ell, \qquad \frac{\theta'}{\pi} = 2^{-q}k + \sum_{\ell > q} 2^{-l}\varepsilon'_\ell, \quad \varepsilon_*, \varepsilon'_* = 0, 1,$$

and write

$$\begin{aligned} u\bigl(2^{-q}k\pi\bigr) - u(\theta) = {} & u\bigl(2^{-q}k\pi\bigr) - u\bigl(2^{-q}k\pi - \varepsilon_{q+1}2^{-(q+1)}k\pi\bigr) \\ & + u\bigl(2^{-q}k\pi - \varepsilon_{q+1}2^{-(q+1)}k\pi\bigr) \\ & - u\bigl(2^{-q}k\pi - \varepsilon_{q+1}2^{-(q+1)}k\pi - \varepsilon_{q+2}2^{-(q+2)}k\pi\bigr) + \cdots. \end{aligned}$$

Since $\gamma_\ell^{++} \geqslant \gamma_q^{++}$ for $\ell \geqslant q$, we get by (8.23)

$$\bigl|u(\theta) - u\bigl(2^{-q}k\pi\bigr)\bigr| \leqslant \pi^{\gamma_q^{++}} \sum_{\ell > q} 2^{-\ell\gamma_q^{++}} \leqslant 2^{-q\gamma_q^{++}} \times \frac{2^{-\gamma_q^{++}}}{1 - 2^{-\gamma_q^{++}}} \times \pi^{\gamma_q^{++}}.$$

We proceed similarly for θ'. Since $2^{-q}\pi < \theta' - \theta$, we obtain estimate (8.24). $\square$

On the other hand,

$$\frac{|u(\theta)-u(\theta')|}{|\theta-\theta'|^\alpha} \leqslant 1 \quad \forall \theta, \theta' \in A_n,\ n \geqslant q, \tag{8.25}$$

implies that

$$\frac{|u(\theta)-u(\theta')|}{|\theta-\theta'|^\alpha} \leqslant 1 \quad \text{for all } \theta' = \theta + \frac{\pi}{2^n},\ \theta \in A_n,\ n \geqslant q. \tag{8.26}$$

Taking inequalities (8.23) and (8.24) into account, we see that (8.26) implies

$$\sup_{2^{-q}\pi < |\theta-\theta'| \leqslant 2^{-q+1}\pi} \frac{|u(\theta)-u(\theta')|}{|\theta-\theta'|^\alpha} \leqslant \frac{2 \times 2^{-\alpha}}{1-2^{-\alpha}}. \tag{8.27}$$

Let $c \geqslant 2^{1-\alpha}/(1-2^{-\alpha})$ be a constant, then by (8.25)–(8.27), the condition

$$\sup_{2^{-q}\pi < |\theta-\theta'| \leqslant 2^{-q+1}\pi} \frac{|u(\theta)-u(\theta')|}{|\theta-\theta'|^\alpha} \geqslant c \tag{8.28}$$

assures existence of $\theta, \theta' \in A_n$ such that

$$\frac{|u(\theta)-u(\theta')|}{|\theta-\theta'|^\alpha} \geqslant 1. \tag{8.29}$$

Let

$$B_n = \left\{ u\colon \exists \theta, \theta' \in A_n \text{ such that } \frac{|u(\theta)-u(\theta')|}{|\theta-\theta'|^\alpha} \geqslant 1 \right\},$$

then

$$B_n \subset B_{n+1} \quad \text{and} \quad \operatorname{Prob}\left(\bigcup_{n \geqslant q} B_n \right) \leqslant \sup_{n \geqslant q} \operatorname{Prob}(B_n), \tag{8.30}$$

for any probability measure on the set of considered functions u.

Consider the following Hölderian norms:

$$\|u\|_{\mathscr{H}^\alpha} = \sup_{\theta,\theta'} \frac{|u(\theta)-u(\theta')|}{|\theta-\theta'|^\alpha},$$

respectively

$$\left\| F^r_{x,t} \right\|_{\mathscr{H}^\alpha} = \sup_{\zeta,\zeta' \in \partial D} \frac{|F^r_{x,t}(\zeta) - F^r_{x,t}(\zeta')|}{|\zeta-\zeta'|^\alpha}. \tag{8.31}$$

With Theorem 8.1, we obtain uniform estimates in s, r for $0 < s \leqslant t$ and $0 < r < 1$.

Theorem 8.3. *We have for any* $0 < s \leqslant t$ *and* $0 < r < 1$,

$$\operatorname{Prob}\bigl\{\bigl\|F^r_{x,s}\bigr\|_{\mathscr{H}^{(1-\delta)^2}} > 8 \times 2^{q_0(1-\delta)^2}\bigr\} \leqslant \frac{2\sqrt{\sigma(t)}c}{\sqrt{2\pi}\delta} \times \sum_{q>q_0} \frac{1}{\sqrt{q\log 2}} \times \exp\biggl(-\frac{\delta^2}{2\sigma(t)c^2} \times q\log 2\biggr) \tag{8.32}$$

which is a converging series.

Proof. By (7.1) it holds that $|F^r_{x,t}(\partial D)|_\infty \leqslant 4$ for $t \geqslant 0$ and $0 < r < 1$. Thus

$$\sup_{|\zeta-\zeta'|\geqslant 2^{-q_0}} \frac{|F^r_{x,t}(\zeta) - F^r_{x,t}(\zeta')|}{|\zeta-\zeta'|^\alpha} \leqslant 8 \times 2^{q_0\alpha}, \tag{8.33}$$

hence for $|\zeta - \zeta'| \geqslant 2^{-q_0}$ we have Hölderianity.

Next we consider ζ, ζ' such that $|\zeta - \zeta'| < 2^{-q_0}$. We have

$$\bigl\{\bigl|\zeta-\zeta'\bigr| < 2^{-q_0}\bigr\} = \bigcup_{q\geqslant q_0} \bigl\{2^{-(q+1)} \leqslant \bigl|\zeta-\zeta'\bigr| < 2^{-q}\bigr\}. \tag{8.34}$$

For a positive constant K, we may estimate by means of (8.34) as follows:

$$\operatorname{Prob}\bigl\{\bigl\|F^r_{x,t}\bigr\|_{\mathscr{H}^\alpha} > K\bigr\} \leqslant \operatorname{Prob}\biggl\{\sup_{|\zeta-\zeta'|\geqslant 2^{-q_0}} \frac{|F^r_{x,t}(\zeta) - F^r_{x,t}(\zeta')|}{|\zeta-\zeta'|^\alpha} > K\biggr\} + \sum_{q\geqslant q_0} \operatorname{Prob}\biggl\{\sup_{2^{-(q+1)}\leqslant|\zeta-\zeta'|<2^{-q}} \frac{|F^r_{x,t}(\zeta) - F^r_{x,t}(\zeta')|}{|\zeta-\zeta'|^\alpha} > K\biggr\}.$$

Because of (8.33), the first term on the right-hand side is zero for $K = 8 \times 2^{q_0\alpha}$. On the other hand, let

$$A = \sup_{2^{-(q+1)}\leqslant|\zeta-\zeta'|<2^{-q}} \frac{|F^r_{x,t}(\zeta) - F^r_{x,t}(\zeta')|}{|\zeta-\zeta'|^\alpha}$$

and consider the event $\{A > 8 \times 2^{q_0\alpha}\}$.

Taking $\alpha = (1-\delta)^2$, then by (8.1) and (8.28)–(8.30), we get an upper bound for $\operatorname{Prob}\{A > 8 \times 2^{q_0\alpha}\}$. In estimate (8.1) we have $\eta(0) = |\zeta - \zeta'|$. For $2^{-(q+1)} \leqslant |\zeta - \zeta'| < 2^{-q}$, observe that $\eta(0) \sim 2^{-q}$ and $-\log\eta(0) \sim q\log 2$. We conclude by using estimate (8.1) since $\sigma(t)$ does not depend upon ζ and ζ'. $\square$

8.3. Moduli of continuity of the inverse of regularized welding

Theorem 8.4. *There exist a positive constant α and a function $\varphi(M)$ independent of r, such that $\varphi(M) \to 0$ as $M \to \infty$, and such that*

$$\operatorname{Prob}\left\{\left\|\left(\hat{F}_{x,t}^r\right)^{-1}\right\|_{\mathscr{H}^\alpha} > M\right\} < \varphi(M) \tag{8.35}$$

where $\hat{F}_{x,t}^r$ denotes the restriction of $F_{x,t}^r$ to ∂D.

Proof. We use Theorem 8.1 and Lemma 8.2. □

9. Welding Brownian measures to Hölderian Jordan curves

9.1. Welding of random homeomorphisms

Theorem 9.1. *Fix δ such that $0 < \delta < 1$. Then*

$$A := \left\{x\colon \limsup_{r \to 1} \left\| F_{x,t}^r \right\|_{\mathscr{H}^{(1-\delta)^2}} = \infty\right\} \quad \textit{satisfies } \operatorname{Prob}(A) = 0. \tag{9.1}$$

Proof. Assume that

$$\operatorname{Prob}(A) = \varepsilon > 0. \tag{9.2}$$

Fix q_0 such that the right-hand side of (8.32) is smaller than $\varepsilon/3$. Define

$$B_m := \left\{x\colon \inf_{r \geqslant 1-m^{-1}} \left\| F_{x,t}^r \right\|_{\mathscr{H}^{(1-\delta)^2}} > 8 \times 2^{q_0(1-\delta)^2}\right\};$$

then B_m is an increasing sequence of measurable sets and we have

$$A \subset \bigcup_m B_m; \quad \text{therefore } \lim_m \operatorname{Prob}(B_m) \geqslant \varepsilon. \tag{9.3}$$

Fix m_0 such that

$$\operatorname{Prob}(B_{m_0}) > 2\varepsilon/3. \tag{9.4}$$

As

$$B_{m_0} \subset \left\{x\colon \left\| F_{x,t}^{1-m_0^{-1}} \right\|_{\mathscr{H}^{\delta^{-1}}} \geqslant 8 \times 2^{q_0\delta^{-1}}\right\},$$

we deduce by means of (8.32) that

$$\operatorname{Prob}(B_{m_0}) \leqslant \varepsilon/3.$$

By (9.4) however this would imply that $2/3 \leqslant 1/3$. □

Theorem 9.2 *(Stochastic welding theorem). Almost surely there exist univalent functions* $h_{x,t}$, $f_{x,t}$ *such that for some* $\alpha \in]0, (1-\delta)^2]$,

$$\|h_{x,t}\|_{\mathscr{H}^\alpha} < \infty, \qquad \|f_{x,t}\|_{\mathscr{H}^\alpha} < \infty, \tag{9.5}$$

and

$$f_{x,t}(\zeta) = \bigl(h_{x,t} \circ \psi_{x,t}^{-1}\bigr)(\zeta), \quad \zeta \in \partial D. \tag{9.6}$$

Proof. According to (9.1), for almost all x, we find a sequence depending upon x, say $r_k(x)$, such that $r_k(x) \to 1$, together with

$$\sup_k \bigl\| F_{x,t}^{r_k(x)} \bigr\|_{\mathscr{H}^\alpha} < \infty. \tag{9.7}$$

We extract a subsequence $r_{k_q}(x)$ such that $F_{x,t}^{r_k(x)}$ converges for $|z| > 8$. Since the limit satisfies $\simeq z$ nearby $z = \infty$, the limit will not be constant but a univalent function $h_{x,t}$ belonging to the space $\mathscr{H}^\alpha$. We have

$$f_{x,t}^{r_{k_q}}(\zeta) = h_{x,t}^{r_{k_q}} \circ \bigl(\psi_{x,t}^{r_{k_q}}\bigr)^{-1}(\zeta).$$

We conclude using [6] which assures that $(\psi_{x,t}^{r_{k_q}})^{-1}$ converges towards $\psi_{x,t}^{-1}$ in some Hölderian norm. □

9.2. Hölderianity of $h_{x,t}^{-1}$

Estimate (8.35) has to be used.

9.3. Uniqueness of the welding

We take the point of view of [20, p. 304]. The circle S^1 is the boundary of the two closed hemispheres of the Riemann sphere. Let S_+^1 be the North hemisphere and S_-^1 the South hemisphere.

Given $h \in \mathrm{Homeo}(S^1)$, we define on $S_+^1 \oplus S_-^1$ an equivalence relation where the equivalence classes are composed of single points with the exception of the boundaries $\partial S_\pm^1$ which are identified using h. The set of equivalence classes has the structure of a topological manifold Δ_h. A continuous function Φ on Δ_h is given by the data of a couple of continuous function $\Phi_\pm$ defined on the closed hemispheres such that $\Phi_+(s) = \Phi_-(h(s))$ on the equator. This family of functions forms an algebra $\mathscr{A}_\mathrm{h}$; another equivalent definition is to define Δ_h as the Gelfand spectrum of $\mathscr{A}_\mathrm{h}$.

The welding problem is equivalent to the following question.

Question. Does there exist a conformal structure on Δ_h which restricted to each of the open hemispheres coincides with the given conformal structure on the hemisphere? For such a conformal structure $\mathscr{C}$, we denote $\Delta_\mathrm{h}^{\mathscr{C}}$ the corresponding Riemann surface.

By Poincaré's uniformization theorem, it is known that up to a homeomorphism, there is a unique conformal structure on the sphere; this means that there is a homeomorphism Θ carrying Δ_{h} onto $\Delta_{\mathrm{Identity}}$. The image of the equator $\Theta(\partial S^1_+)$ is a Jordan curve $\Gamma_{\mathrm{h}}^{\mathscr{C}}$. A *Hölderian Jordan curve* is by definition a curve which is parametrizable by a univalent function φ such that φ is Hölderian together with its inverse.

Theorem 9.3. *Assume that there exists a welding conformal structure $\mathscr{C}_0$ such that $\Gamma_{\mathrm{h}}^{\mathscr{C}_0}$ is a Hölderian Jordan curve. Then every welding structure $\mathscr{C}$ coincides with $\mathscr{C}_0$.*

Proof. Let Θ_0, Θ be the corresponding homeomorphisms of Δ_{h}; then $v := \Theta \circ \Theta_0^{-1}$ defines a new conformal structure on the complement of $\Gamma_{\mathrm{h}}^{\mathscr{C}_0}$. By [19, Cor. 2 and Cor. 4, pp. 267–268], there is a unique conformal structure which coincides with the trivial one on the complement of $\Gamma_{\mathrm{h}}^{\mathscr{C}_0}$; therefore $\mathscr{C} = \mathscr{C}_0$. $\square$

Acknowledgments

We thank I. Markina, Yu. Neretin, A. Vasiliev for helpful discussions on related topics.

References

[1] L.V. Ahlfors, Lectures on Quasiconformal Mappings, Van Nostrand Math. Stud., vol. 10, D. Van Nostrand Co., Inc., Princeton, Toronto, New York, London, 1966.

[2] H. Airault, V. Bogachev, Realization of Virasoro unitarizing measures on the set of Jordan curves, C. R. Math. Acad. Sci. Paris 336 (2003) 429–434.

[3] H. Airault, P. Malliavin, Unitarizing probability measures for representations of Virasoro algebra, J. Math. Pures Appl. (9) 80 (2001) 627–667.

[4] H. Airault, P. Malliavin, Quasi-invariance of Brownian measures on the group of circle homeomorphisms and infinite-dimensional Riemannian geometry, J. Funct. Anal. 241 (2006) 99–142.

[5] H. Airault, P. Malliavin, A. Thalmaier, Canonical Brownian motion on the space of univalent functions and resolution of Beltrami equations by a continuity method along stochastic flows, J. Math. Pures Appl. (9) 83 (2004) 955–1018.

[6] H. Airault, J. Ren, Modulus of continuity of the canonic Brownian motion "on the group of diffeomorphisms of the circle", J. Funct. Anal. 196 (2002) 395–426.

[7] K. Astala, T. Iwaniec, G. Martin, Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane, Princeton Math. Ser., vol. 48, Princeton University Press, Princeton, NJ, 2009.

[8] K. Astala, P. Jones, A. Kupiainen, E. Saksman, Random curves by conformal welding (Courbes aléatoires par soudure conforme), C. R. Math. Acad. Sci. Paris 348 (2010) 257–262.

[9] A.F. Beardon, C. Pommerenke, The Poincaré metric of plane domains, J. London Math. Soc. (2) 18 (1978) 475–483.

[10] C. Carathéodory, Über die gegenseitige Beziehung der Ränder bei der konformen Abbildung des Inneren einer Jordanschen Kurve auf einen Kreis, Math. Ann. 73 (1913) 305–320.

[11] A. Douady, C.J. Earle, Conformally natural extension of homeomorphisms of the circle, Acta Math. 157 (1986) 23–48.

[12] B.K. Driver, M. Gordina, Heat kernel analysis on infinite-dimensional Heisenberg groups, J. Funct. Anal. 255 (2008) 2395–2461.

[13] S. Fang, Canonical Brownian motion on the diffeomorphism group of the circle, J. Funct. Anal. 196 (2002) 162–179.

[14] S. Fang, T. Zhang, A class of stochastic differential equations with non-Lipschitzian coefficients: pathwise uniqueness and no explosion, C. R. Math. Acad. Sci. Paris 337 (2003) 737–740.

[15] F.W. Gehring, B.P. Palka, Quasiconformally homogeneous domains, J. Anal. Math. 30 (1976) 172–199.

[16] M. Gordina, M. Wu, Diffeomorphisms of the circle and Brownian motions on an infinite-dimensional symplectic group, Commun. Stoch. Anal. 2 (2008) 71–95.

[17] E. Grong, P. Gumenyuk, A. Vasil'ev, Matching univalent functions and conformal welding, Ann. Acad. Sci. Fenn. Math. 34 (2009) 303–314.

[18] N. Ikeda, S. Watanabe, A comparison theorem for solutions of stochastic differential equations and its applications, Osaka J. Math. 14 (1977) 619–633.
[19] P.W. Jones, S.K. Smirnov, Removability theorems for Sobolev functions and quasiconformal maps, Ark. Mat. 38 (2000) 263–279.
[20] Y. Katznelson, S. Nag, D.P. Sullivan, On conformal welding homeomorphisms associated to Jordan curves, Ann. Acad. Sci. Fenn. Math. 15 (1990) 293–306.
[21] A.A. Kirillov, Geometric approach to discrete series of unireps for Virasoro, J. Math. Pures Appl. (9) 77 (1998) 735–746.
[22] M. Kontsevich, Y. Suhov, On Malliavin measures, SLE, and CFT, Proc. Steklov Inst. Math. 258 (2007) 100–146.
[23] H. Kunita, Stochastic Flows and Stochastic Differential Equations, Cambridge Stud. Adv. Math., vol. 24, Cambridge University Press, Cambridge, 1990.
[24] H. Kunita, S. Watanabe, On square integrable martingales, Nagoya Math. J. 30 (1967) 209–245.
[25] G.F. Lawler, O. Schramm, W. Werner, Analyticity of intersection exponents for planar Brownian motion, Acta Math. 189 (2002) 179–201.
[26] Y. Le Jan, O. Raimond, Integration of Brownian vector fields, Ann. Probab. 30 (2002) 826–873.
[27] P. Malliavin, Stochastic Analysis, Grundlehren Math. Wiss. (Fundamental Principles of Mathematical Sciences), vol. 313, Springer-Verlag, Berlin, 1997.
[28] P. Malliavin, The canonic diffusion above the diffeomorphism group of the circle, C. R. Math. Acad. Sci. Paris 329 (1999) 325–329.
[29] J. Morrow, K. Kodaira, Complex Manifolds, Holt, Rinehart and Winston, Inc., New York, 1971.
[30] S. Nag, Singular Cauchy integrals and conformal welding on Jordan curves, Ann. Acad. Sci. Fenn. Math. 21 (1996) 81–88.
[31] C. Pommerenke, Boundary Behaviour of Conformal Maps, Grundlehren Math. Wiss. (Fundamental Principles of Mathematical Sciences), vol. 299, Springer-Verlag, Berlin, 1992.
[32] E. Reich, J. Chen, Extensions with bounded $\bar{\partial}$-derivative, Ann. Acad. Sci. Fenn. Math. 16 (1991) 377–389.
[33] J. Ren, X. Zhang, Stochastic flow for SDEs with non-Lipschitz coefficients, Bull. Sci. Math. 127 (2003) 739–754.
[34] E. Sharon, D. Mumford, 2D-shape analysis using conformal mapping, Internat. J. Comput. Vision 70 (2006) 55–75.
[35] A. Vasil'ev, Evolution of conformal maps with quasiconformal extensions, Bull. Sci. Math. 129 (2005) 831–859.