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Calculus of Variations

Covariant Riesz transform on differential forms for $1 < p \leq 2$

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Abstract
In this paper, we study L^p-boundedness ($1 < p \leq 2$) of the covariant Riesz transform on differential forms for a class of non-compact weighted Riemannian manifolds without assuming conditions on derivatives of curvature. We present in particular a local version of L^p-boundedness of Riesz transforms under two natural conditions, namely the curvature-dimension condition, and a lower bound on the Weitzenböck curvature endomorphism. As an application, the Calderón–Zygmund inequality for $1 < p \leq 2$ on weighted manifolds is derived under the curvature-dimension condition as hypothesis.

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Contents

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1 Introduction

Let (M, g) be a complete geodesically connected m-dimensional Riemannian manifold, ∇ the Levi-Civita covariant derivative, and Δ the Laplace-Beltrami operator understood as a self-adjoint positive operator on $L^2(M)$. The Riesz transform $\nabla\Delta^{-1/2} f$, introduced by Strichartz [28] on Euclidean space, has been investigated in many subsequent papers, see e.g., [2–4, 8] and the references therein, and has been further extended to Riemannian manifolds, e.g. [1, 11–13, 23, 30]. Since the Riesz transform is bounded in $L^2(M)$, by the interpolation theorem, the weak (1, 1) property already implies $L^p(M)$-boundedness for $p \in (1, 2]$.

When it comes to Riemannian vector bundles, boundedness in L^p of the Riesz transforms $\mathrm{d}^{(k)}(\Delta^{(k)} + \sigma)^{-1/2}$ and $\delta^{(k-1)}(\Delta^{(k)} + \sigma)^{-1/2}$ has been well considered, see [4, 25], where $\Delta^{(k)}$ is the usual Hodge Laplacian acting on k-forms, $\mathrm{d}^{(k)}$ the exterior differential on k-forms and $\delta^{(k)}$ the L^2-adjoint of $\mathrm{d}^{(k)}$. Let ∇ be the Levi-Civita covariant derivative. In this paper, we aim to study covariant Riesz transforms $\nabla(\Delta^{(k)} + \sigma)^{-1/2}$ on Riemannian vector bundles for $p \in (1, 2]$ which poses comparably more difficulties to deal with. This question has already been addressed by the second and third named author in [30], where the authors adopted the method of Coulhon and Duong [11] relying on the doubling volume property, Li-Yau type heat kernel upper bounds and derivative estimates of the heat kernel. Note that the approach in [30] is of stochastic nature and derivative estimates for the heat kernel are deduced from derivative formulae for semigroups on vector bundles, by means of the methodology of Driver and the second named author [15]. In [15] estimates of certain functionals of Brownian motion with respect to the Wiener measure are required; nevertheless pointwise estimate for the heat kernel $\mathrm{e}^{-t\Delta^{(k)}}(x, y)$ and the derivative estimate of heat kernel $\nabla \mathrm{e}^{-t\Delta^{(k)}}(x, y)$ can be obtained from such general derivative formulas, where $\Delta^{(k)}$ is the unique self-adjoint realization of the Hodge-de Rham Laplacian acting on k-forms under explicit curvature condition, see [5]. By following a similar approach, Baumgarth, Devyver and Güneysu [5] studied the covariant Riesz transform on j-forms, removing the doubling volume property and using uniformly boundedness conditions of the curvature and the derivative of the curvature on differential forms. These results are further used to establish Calderón-Zygmund inequalities for $1 < p \le 2$, where for $\varphi \in C_c^\infty(M)$, the set of smooth functions of compact support, if there exist positive constants C_1 and C_2 such that

$$\| \, |\mathrm{Hess}(\varphi)| \, \|_p \le C_1 \|\varphi\|_p + C_2 \|\Delta\varphi\|_p$$

then the Calderón-Zygmund inequalities holds. Note that the argument in [5] for establishing Calderón-Zygmund inequalities is from Güneysu and Pigola's paper [20], where the Calderón-Zygmund inequalities are equivalent to the L^p boundedness of the operator $\mathrm{Hess}(\Delta + \sigma)^{-1}$ for some constant $\sigma > 0$ and this operator further can be rewritten as

$$\nabla(\Delta^{(1)} + \sigma)^{-1/2} \circ \mathrm{d}(\Delta + \sigma)^{-1/2}.$$

Thus to investigate whether these L^p-Calderón-Zygmund inequalities hold are reduced to the study of conditions for boundedness of the classical Riesz transform $\mathrm{d}(\Delta_\mu + \sigma)^{-1/2}$ on functions and boundedness of the covariant Riesz transform $\nabla(\Delta_\mu^{(1)} + \sigma)^{-1/2}$ on one-forms in L^p-sense. Therefore, in [5] combining this argument with the result in [30] yields that the

L^p-Calderón-Zygmund inequalities hold for $1 < p \le 2$ if

$$\|R\|_\infty < \infty \quad \text{and} \quad \|\nabla R\|_\infty < \infty,$$

where R is the curvature tensor.

On the other hand, very recently, Cao, Cheng and Thalmaier [7] established the L^p-Calderón-Zygmund inequality for $1 < p < 2$ by only using the natural assumption of a lower Ricci curvature bound. Since the conditions for the L^p boundedness of the classical Riesz transform $\mathrm{d}(\Delta_\mu + \sigma)^{-1/2}$ for $1 < p \le 2$ are quite weak, it arises us to wondering whether the condition for the L^p boundedness of the covariant Riesz transform on one-forms in [5] is too strong. This work is devoted to investigating this problem on the L^p boundedness of covariant Riesz transform for $1 < p \le 2$ again.

As explained in [30], it is difficult to follow the corresponding argument in [11] directly concerning derivatives of heat kernel on vector bundle, since the heat kernel $\mathrm{e}^{-t\Delta^{(k)}}(x, y)$ is a linear operator on a vector bundle $E \to M$ from E_y to E_x. In this paper, we aim to overcome this difficulty by using a different approach, that is the Weitzenböck formula and the Gaussian type estimates for some Schrödinger heat kernels on manifolds.

Before moving on, let us first introduce some basic notations. Consider a weighted Laplacian $\Delta + \nabla h$ with $h \in C_b^2(M)$. In this paper, we study the covariant Riesz transform relative to the weighted volume measure $\mu(\mathrm{d}x) = \mathrm{e}^{h(x)}\,\mathrm{vol}(\mathrm{d}x)$ where vol is the Riemannian volume measure on M. We write $\Delta_\mu := \Delta + \nabla h$ where Δ_μ is understood as a self-adjoint positive operator on $L^2(\mu)$. Let $\rho(x, y)$ be the geodesic distance of x and y and $B(x, r)$ the open ball centered at x of radius r. Given a smooth vector bundle $E \to M$ carrying a canonically given metric and a canonically given covariant derivative, we denote its fiberwise metric by $(\cdot, \cdot)_g$, the fiberwise norm by $|\cdot| = \sqrt{(\cdot, \cdot)_g}$ and the smooth sections by $\Gamma_{C^\infty}(M, E)$. We denote its covariant derivative by

$$\nabla : \Gamma_{C^\infty}(M, E) \to \Gamma_{C^\infty}\big(M, T^*M \otimes E\big).$$

The Banach space $\Gamma_{L^p}(M, E)$ consists of equivalent classes of Borel sections ψ of $E \to M$ such that

$$\|\psi\|_p \equiv \|\psi\|_{L^p} := \|\,|\psi|\,\|_{L^p} < \infty$$

where $\|\,|\psi|\,\|_{L^p}$ denotes the norm of the function $|\psi|$ with respect to $L^p(\mu)$. Then $\Gamma_{L^2}(E)$ canonically becomes a Hilbert space with the scalar product

$$\langle \psi_1, \psi_2 \rangle := \langle \psi_1, \psi_2 \rangle_{L^2} = \int (\psi_1, \psi_2)_g \, \mathrm{d}\mu.$$

Consider the spaces

$$\Omega^k = \Gamma_{C^\infty}\big(M, \Lambda^k T^*M\big) \quad \text{and} \quad \Omega_c^k = \Gamma_{C_c^\infty}\big(M, \Lambda^k T^*M\big) \quad (0 \le k \le m)$$

of smooth differential k-forms, respectively compactly supported smooth k-forms, and denote the space of smooth k-forms in L^p by

$$\Omega_{L^p}^k := \Gamma_{C^\infty \cap L^p(\mu)}\big(M, \Lambda^k T^*M\big).$$

In terms of the exterior differential $\mathrm{d}^{(k)} : \Omega^k \to \Omega^{k+1}$ on Ω^k and $\delta_\mu^{(k+1)}$ the $L^2(\mu)$-adjoint of $\mathrm{d}^{(k)}$, i.e.,

$$\big\langle \delta_\mu^{(k+1)} a, b \big\rangle := \mu\big((\delta_\mu^{(k+1)} a, b)_g\big) = \mu\big((a, \mathrm{d}^{(k)} b)_g\big) := \langle a, \mathrm{d}^{(k)} b \rangle$$

for $a \in \Omega^{k+1}$ and $b \in \Omega^k$, the weighted Hodge Laplacians acting on 0-, respectively k-forms, are given by

$$\begin{aligned}\Delta_\mu^{(0)} &:= \delta_\mu^{(1)} \mathrm{d}^{(0)} : C^\infty(M) \to C^\infty(M),\\ \Delta_\mu^{(k)} &:= \delta_\mu^{(k+1)} \mathrm{d}^{(k)} + \mathrm{d}^{(k-1)} \delta_\mu^{(k)} : \Omega^k \to \Omega^k. \end{aligned} \tag{1.1}$$

Obviously, the canonical commutation rules hold:

$$\mathrm{d}^{(k-1)} \Delta_\mu^{(k-1)} = \Delta_\mu^{(k)} \mathrm{d}^{(k-1)}.$$

To simplify the notation, we write $\Delta_\mu = \Delta_\mu^{(0)}$ and $\mathrm{d} = \mathrm{d}^{(0)}$.

To this end, we first give some assumptions where we start with notions related to the curvature. Letting ∇_μ^* be the $L^2(\mu)$-adjoint of ∇ and $\square_\mu = \nabla_\mu^* \nabla$ the weighted Bochner Laplacian, it is easy to check that

$$\square_\mu = -\operatorname{tr} \nabla^2 - \nabla_{\nabla h}, \tag{1.2}$$

where

$$\operatorname{tr} \nabla^2 \eta(\bullet) := \sum_i \nabla^2 \eta\,(\bullet, e_i, e_i)$$

with η being a differential k-form and (e_i) a local orthonormal frame. Note that by definition $\nabla^2\eta$ is a tensor of order $(0, k+2)$ and $\operatorname{tr} \nabla^2 \eta$ is independent of the choice of local orthonormal frame (e_i). The Weitzenböck formula gives the relationship between $\square_\mu$ and the Hodge Laplacian $\Delta_\mu^{(\cdot)}$: for any differential k-forms $\eta \in \Omega^k$, we have

$$\Delta_\mu^{(k)} \eta = \square_\mu \eta + \mathscr{R}^{(k)}(\eta) - (\operatorname{Hess} h)^{(k)}(\eta), \tag{1.3}$$

where in explicit terms the *Weitzenböck curvature endomorphism* $\mathscr{R}^{(k)} - (\operatorname{Hess} h)^{(k)}$ is given by

$$\big(\mathscr{R}^{(k)} - (\operatorname{Hess} h)^{(k)}\big)(\bullet) = -\sum_{i,j=1}^m \theta^j \wedge \big(e_i \lrcorner \mathrm{R}(e_j, e_i)(\bullet)\big) - \sum_{i,j=1}^m e_i\big(e_j(h)\big)\Big(\theta^j \wedge (e_i \lrcorner \bullet)\Big)$$

for any orthonormal frame $(e_i)_{1\le i\le m}$ with corresponding dual frame $(\theta^j)_{1\le j\le m}$ (see Theorem 2.2 below). When $k = 1$, $\mathscr{R}^{(1)} - (\operatorname{Hess} h)^{(1)} = \operatorname{Ric} - \operatorname{Hess} h$.

Let $\lambda_k(x)$ be the lowest eigenvalue of $(\mathscr{R}^{(k)} - (\operatorname{Hess} h)^{(k)})(x)$ for $x \in M$. We use the notation

$$V_k(x) := \lambda_k^-(x) = (|\lambda_k(x)| - \lambda_k(x))/2.$$

Let $P_t^{V_k}$ be the semigroup $\mathrm{e}^{-t(\Delta_\mu - V_k)}$ which has a smooth integral kernel denoted by $p_t^{V_k}(x, y)$.

Definition 1.1 We say that $\Delta_\mu - V_k + \sigma$ for some constant σ is *strongly positive* if the following condition holds: there exists $A < 1$ such that for all $f \in C_c^\infty(M)$,

$$\int_M (V_k - \sigma)|f|^2 \mathrm{d}\mu \le A \int_M |\nabla f|^2\, \mathrm{d}\mu.$$

We remark that the strong positivity condition has its origin in the Hardy inequality, see the introduction of [14].

The following theorem is our first main result.

Theorem 1.2 *Let σ_1, σ_2 and σ_2 be positive constants such that the following three conditions hold:*

(i) (local) volume doubling property: for $\alpha > 1$,

$$\mu(B(x,\alpha r)) \le C\mu(B(x,r))\,\alpha^m \exp(\sigma_1(\alpha-1)r), \quad x \in M, \tag{LD}$$

holds for all $r > 0$ and some constant $C > 0$;

(ii) local off-diagonal upper bound of the heat kernel $p_t^{V_k}$:

$$p_t^{V_k}(x,x) \le \frac{Ce^{\sigma_2 t}}{\mu(B(x,\sqrt{t}))}, \quad x \in M, \tag{UE}$$

for all $t > 0$ and some constant $C > 0$;

(iii) the operator $\Delta_\mu - V_k + \sigma_3$ is strongly positive.

Then there exists a positive constant σ depending on σ_1, σ_2 and σ_3 such that the covariant Riesz transform $\nabla(\Delta_\mu^{(k)} + \sigma)^{-1/2}$ is bounded in L^p for $p \in (1,2]$. In particular, $\sigma_1 = \sigma_2 = \sigma_3 = 0$ implies $\sigma = 0$.

The upper bound estimate of Schrödinger heat kernel with an increasing exponential factor have been well studied in [27, 29]. Often, inequality (**UE**) appears without the increasing exponential factor (i.e. $\sigma_2 = 0$), which requires stronger conditions on the curvature and the potential V_k. In the following, we use the result from [32] to consider the global covariant Riesz transform, i.e. $\sigma = 0$.

Let $P_t = e^{-\Delta_\mu t}$ be the semigroup generated by $-\Delta_\mu$ and $p_t(x,y)$ the corresponding heat kernel with respect to the measure μ. Our second main result is the following Theorem 1.3. It has been proved in [32] that the doubling volume property (**D**) and the on-diagonal estimate (**U**) in this theorem, together with condition (1.4), imply that

$$p_t^{V_k}(x,x) \le \frac{C}{\mu(B(x,\sqrt{t}))}.$$

Thus the following result is a consequence of Theorem 1.2 for $\sigma_1 = \sigma_2 = \sigma_3 = 0$.

Theorem 1.3 *Suppose the following conditions hold:*

(i) volume doubling property: for $\alpha > 1$,

$$\mu(B(x,\alpha r)) \le C\mu(B(x,r))\alpha^m \tag{D}$$

holds for some constant $C > 0$ and all $x \in M$, $r > 0$;

(ii) on-diagonal upper bound of the heat kernel: there exists constants $C, \delta > 0$ such that

$$p_t(x,x) \le \frac{C}{\mu(B(x,\sqrt{t}))} \tag{U}$$

for all $t > 0$ and $x \in M$, and

$$K(V_k) \equiv \sup_{x\in M} \int_0^\infty \int_M \frac{1}{\mu(B(x,\sqrt{s}))} e^{-\rho^2(x,y)/s} V_k(y)\,\mu(dy)\,ds < \delta; \tag{1.4}$$

iii) the operator $\Delta_\mu - V_k$ is strongly positive on Ω_c^k.

Then the Riesz transform $\nabla(\Delta_\mu^{(k)})^{-1/2}$ on $\Omega_{L^p}^k$ is bounded in L^p for $p \in (1,2]$.

Let us compare to known results. In [30] for the usual Riemannian manifold, i.e. $h = 0$, under the assumption that $\nabla \mathrm{R} + \nabla \mathscr{R}^{(k)} = 0, \mathscr{R}^{(k)} \geq 0$ and the doubling volume property, it is shown that $\nabla(\Delta^{(k)})^{-1/2}$ has the weak type $(1, 1)$ property. In the above theorem, if $\Delta_\mu - V_k$ is strongly positive on Ω_c^k, the lower bound of $\mathscr{R}^{(k)}$ can be relaxed and no condition on the derivative of curvature is needed. From this point of view, our results also improve the recent work of Baumgarth, Devyver and Güneysu [5] on the covariant Riesz transform on k-forms for $p \in (1, 2]$.

It has been observed that the curvature-dimension condition implies the local volume doubling condition (see [16]). For the localization argument towards the boundedness of Riesz transform, we need the local doubling volume property with respect to μ, which is related to the following curvature-dimension condition. Assume that

$$\Gamma_2(f, f) := -\frac{1}{2}\Delta_\mu |\nabla f|^2 + (\nabla \Delta_\mu f, \nabla f)_g \geq -K_0 |\nabla f|^2 + \frac{1}{n}(\Delta_\mu f)^2, \qquad (\mathbf{CD})$$

where $K_0 \in \mathbb{R}$ and $n \geq m$ provide a curvature lower bound and a dimension upper bound of Δ_μ, respectively. In the case $\nabla h = 0$ this condition is equivalent to $\mathrm{Ric} \geq -K_0$ and then the curvature-dimension condition holds for $n = m$. When $\nabla h \neq 0$ however, typically n is larger than m. Indeed, the curvature-dimension condition can be written as

$$\mathrm{Ric}_h^{(n-m)}(X, X) \geq -K_0 |X|^2, \quad X \in TM$$

where for $\alpha > 0$, the α-Ricci curvature of the weighted Laplacian Δ_μ is defined as

$$\mathrm{Ric}_h^\alpha := \mathrm{Ric} - \mathrm{Hess}\, h - \frac{1}{\alpha}\nabla h \otimes \nabla h.$$

This condition implies that

$$\mathrm{Ric}_h(X, X) := \mathrm{Ric}(X, X) - (\mathrm{Hess}\, h)(X, X) \geq -K_0 |X|^2. \qquad (\mathbf{Ric})$$

Assuming the curvature-dimension curvature condition (**CD**) then in particular the local doubling assumption with respect to μ holds, see [16, 26] for details, i.e., there exists a constant $L > 0$ such that

$$\mu(B(y, \alpha r)) \leq C\mu(B(y, r))\alpha^m \exp(L(\alpha - 1)r), \quad y \in M, \ r > 0, \ \alpha > 1. \qquad (1.5)$$

Assumption 1.4 For $k \in \mathbb{N}$ and $k \geq 2$, there exists $K \in \mathbb{R}$ such that

$$-K = \min\left\{\left(\left(\mathscr{R}^{(k)} - (\mathrm{Hess}\, h)^{(k)}\right)v, v\right)_g : v \in \Lambda^{(k)} T_x M, \ |v| = 1, \ x \in M\right\}.$$

Obviously, Assumption 1.4 implies that $\mathscr{R}^{(k)} - (\mathrm{Hess}\, h)^{(k)}$ is bounded below by $-K$ so that $-V_k + K^+ \geq 0$, hence the operator $\Delta_\mu - V_k + K^+$ is strongly positive. Assume that (**CD**) holds for some constant K_0. On the one hand (**CD**) implies the local volume doubling property. On the other hand, it implies the lower Ricci curvature bound (**Ric**), which is further used to derive the Gaussian type estimate of $p_t(x, y)$ (see [31, Theorem 2.4.4]). We then conclude that for any $\alpha \in (0, 1/4)$ there exist constants $C_1(\alpha), C_2(\alpha) > 0$ such that

$$|p_t^{V_k}(x, y)| \leq \mathrm{e}^{K^+ t} p_t(x, y) \leq \frac{C_1(\alpha)}{\mu(B(x, \sqrt{t}))} \exp\left(-\frac{\alpha \rho(x, y)^2}{t} + (K^+ + C_2(\alpha))t\right) \quad (1.6)$$

for $t > 0$. As a consequence, we have the following corollary from Theorem 1.2 directly.

Corollary 1.5 *Assume (**CD**) holds for some $K_0 \geq 0$. Then there exists a constant $\sigma > 0$ such that the Riesz transform $\nabla(\Delta_\mu^{(1)} + \sigma)^{-1/2}$ is bounded in $L^p(\mu)$ for $1 < p \leq 2$. If in addition Assumption 1.4 holds for some $k \geq 2$, then there exists $\sigma > 0$ such that the Riesz transform $\nabla(\Delta_\mu^{(k)} + \sigma)^{-1/2}$ on $\Omega^k_{L^p}$ is bounded in $L^p(\mu)$ for $1 < p \leq 2$.*

As explained at the beginning, the result of Theorem 1.5 implies the Calderón–Zygmund inequality for $1 < p < 2$. We say that an $L^p(\mu)$-Calderón-Zygmund inequality holds on M if there exist two constants $C_1, C_2 > 0$ such that

$$\left\| \mathrm{Hess}(\varphi) \right\|_p \leq C_1 \|\varphi\|_p + C_2 \|\Delta_\mu \varphi\|_p \qquad (\mathbf{CZ}_\mu(p))$$

for every function $\varphi \in C_c^\infty(M)$. We denote this inequality by $\mathbf{CZ}_\mu(p)$. Güneysu and Pigola [20] observed that under Calderón-Zygmund inequalities, if M is geodesically complete and admits a sequence of Laplacian cut-off functions (this is the case e.g. if M has non-negative Ricci curvature; for more general curvature conditions see [18] and [6]), then $H_0^{2,p}(M) = H^{2,p}(M)$ holds for all $1 < p < \infty$. We refer the reader to [21] for further applications of Calderón-Zygmund inequalities.

In general, $\mathbf{CZ}_\mu(p)$ inequalities may hold or fail on M, depending on the underlying Riemannian geometry, which leads to the question which geometric assumptions on M guarantee $\mathbf{CZ}_\mu(p)$ and how the $\mathbf{CZ}_\mu(p)$ -constants C_1, C_2 depend on the geometric entities. In [20] two methods appear for attacking Calderón-Zygmund inequalities: the first one depends on appropriate elliptic estimates under conditions on harmonic bounds of the injectivity radius, while the second one uses boundedness results for the covariant Riesz transform in L^p for $1 < p \leq 2$ from [30]. Whereas conditions on harmonic bounds of the injectivity radius are usually difficult to verify, the second approach relies on probabilistic covariant derivative formulae for heat semigroups and has the advantage to avoid assumptions on the injectivity radius. Along the main idea of this second method in [20], Theorem 1.3 permits to establish $\mathbf{CZ}_\mu(p)$ for $1 < p \leq 2$ on weighted manifolds along the same approach but only using the curvature-dimension condition.

Theorem 1.6 *Let (M, g) be a complete Riemannian manifold satisfying (**CD**). Let $1 < p < 2$ be fixed. Then there exists a constant $\sigma > 0$ such that the operator $\mathrm{Hess}(\Delta_\mu + \sigma)^{-1}$ is bounded in $L^p(\mu)$, and in particular $\mathbf{CZ}_\mu(p)$ holds.*

Comparing Theorem 1.6 with existing results on $\mathbf{CZ}_\mu(p)$, it should be pointed out that the result is valid without any injectivity radius assumptions and boundedness of $\|R\|_\infty$ and $\|\nabla R\|_\infty$ as in [20]. Our result extends [7] to the weighted manifold by only requiring the curvature-dimension condition.

The paper is organized as follows. In Section 2 we present L^2 and L^1 weighted derivative estimates for the heat kernel on differential forms (see Theorems 2.6 and 2.7). These estimates are applied in Sect. 3 to study the L^p-boundedness ($1 < p \leq 2$) of Riesz transforms for differential forms on Riemannian manifolds with a metric connection (see Theorem 1.2). Moreover, Theorem 1.5 gives a local version of covariant Riesz transform on Riemannian forms. As application, Theorem 1.5 is used to obtain the Calderón-Zygmund inequalities for $p \in (1, 2]$.

2 Heat kernel estimates

2.1 Preliminaries

Let us first recall the interior product.

Definition 2.1 The interior product $X \lrcorner a \in \Omega^{k-1}$ corresponds to the contraction of $a \in \Omega^k$ with a vector field $X \in \Gamma(TM)$ and is defined as

$$X \lrcorner a\,(X_1, \ldots, X_{k-1}) := a\,(X, X_1, \ldots, X_{k-1})\,, \quad \forall X_1, \ldots, X_{k-1} \in \Gamma(TM).$$

The interior product is an anti-derivation, i.e.,

$$X \lrcorner (a \wedge b) = (X \lrcorner a) \wedge b + (-1)^k a \wedge (X \lrcorner b) \quad \forall a \in \Omega^k,\ b \in \Omega^1.$$

The Weitzenböck formula relates the weighted Hodge-de Rham Laplacian to the weighted Bochner Laplacian on (M, g).

Theorem 2.2 *(Weitzenböck formula) For all differential k-forms $\eta \in \Omega^k$, we have*

$$\Delta_\mu^{(k)} \eta = \Box\eta - \nabla_{\nabla h}\eta + \mathscr{R}^{(k)}(\eta) - (\mathrm{Hess}\, h)^{(k)}(\eta),$$

where $\mathscr{R}^{(k)} : \Omega^k \to \Omega^k$ is given by

$$\mathscr{R}^{(k)}(\eta) = -\sum_{i,j=1}^{m} \theta^j \wedge \big(e_i \lrcorner \mathrm{R}\big(e_j, e_i\big)(\eta)\big)$$

and $(\mathrm{Hess}\, h)^{(k)} : \Omega^k \to \Omega^k$ by

$$(\mathrm{Hess}\, h)^{(k)}(\eta) = \sum_{i,j=1}^{m} e_i(e_j(h))\theta^j \wedge (e_i \lrcorner \eta)\,,$$

for any orthonormal frame $(e_i)_{1\le i\le m}$ and corresponding dual frame $(\theta^j)_{1\le j\le m}$ such that $\nabla e_i = 0$, $\nabla\theta^j = 0$ and $\theta^j(e_i) = \delta_i^j$.

Proof It is well known that

$$\mathrm{d}^{(k)} = \sum_{j=1}^{m} \theta^j \wedge \nabla_{e_j} \quad \text{and} \quad \delta_\mu^{(k)}(\cdot) = -\sum_{j=1}^{m} \mathrm{e}^{-h} e_j \lrcorner \nabla_{e_j}(\mathrm{e}^h(\cdot)).$$

Let $\Delta^{(k)}$ be the usual Hodge Laplacian acting on k-form. Since orthonormal frames $(e_i)_{1\le i\le m}$ and dual frames $\big(\theta^j\big)_{1\le j\le m}$ satisfy $\nabla e_i = 0$ and $\nabla\theta^j = 0$, we obtain for $\eta \in \Omega^k$, using the summation convention,

$$\begin{aligned}
\Delta_\mu^{(k)}\eta &= -\mathrm{e}^{-h} e_j \lrcorner \nabla_{e_j}(\mathrm{e}^h(\theta^i \wedge \nabla_{e_i}\eta)) - \theta^i \wedge \nabla_{e_i}(\mathrm{e}^{-h} e_j \lrcorner \nabla_{e_j}(\mathrm{e}^h\eta)) \\
&= -e_j(h) e_j \lrcorner (\theta^i \wedge \nabla_{e_i}\eta) - e_j \lrcorner \nabla_{e_j}(\theta^i \wedge \nabla_{e_i}\eta) \\
&\quad - \theta^i \wedge \nabla_{e_i}(e_j(h) e_j \lrcorner \eta) - \theta^j \wedge \nabla_{e_j}(e_j \lrcorner \nabla_{e_j}\eta) \\
&= -e_j(h) e_j \lrcorner (\theta^i \wedge \nabla_{e_i}\eta) - e_i(e_j(h))\theta^i \wedge (e_j \lrcorner \nabla_{e_j}\eta) - e_j(h)\theta^i \wedge \nabla_{e_i}(e_j \lrcorner \eta) + \Delta^{(k)}\eta \\
&= -e_j(h)\nabla_{e_j}\eta + e_j(h)(\theta^i \wedge (e_j \lrcorner \nabla_{e_i}\eta)) - e_i(e_j(h))\theta^i \wedge (e_j \lrcorner \eta) - e_j(h)(\theta^i \wedge (e_j \lrcorner \nabla_{e_i}\eta)) \\
&\quad + \Box\eta - \mathscr{R}^{(k)}(\eta)
\end{aligned}$$

$$
\begin{aligned}
&= -e_j(h)\nabla_{e_j}\eta - e_i(e_j(h))\theta^i \wedge (e_j \lrcorner \eta) + \Box\eta - \mathscr{R}^{(k)}(\eta) \\
&= -\nabla_{\nabla h}\eta - (\operatorname{Hess} h)^{(k)}(\eta) + \Box\eta - \mathscr{R}^{(k)}(\eta),
\end{aligned}
$$

where the last equation follows from the fact that

$$\Delta^{(k)}\eta = \Box\eta - \mathscr{R}^{(k)}(\eta).$$

□

By the usual abuse of notation, the corresponding self-adjoint realizations will again be denoted by the same symbol, i.e. Δ_μ and $\Delta_\mu^{(k)}$ respectively. By local parabolic regularity, for all square-integrable k-forms $a \in \Omega^k_{L^2}$, the time-dependent k-form

$$(0,\infty) \times M \ni (t,x) \mapsto \mathrm{e}^{-\Delta_\mu^{(k)} t} a \in \Lambda^k T_x^* M$$

has a smooth representative which extends smoothly to $[0,\infty)\times M$ if a is smooth. In addition, there exists a unique smooth heat kernel of $\mathrm{e}^{-\Delta_\mu^{(k)} t}$ with respect to the measure μ, which is understood as a map

$$(0,\infty) \times M \times M \ni (t,x,y) \mapsto \mathrm{e}^{-\Delta_\mu^{(k)} t}(x,y) \in \operatorname{Hom}(\Lambda^k T_y^* M, \Lambda^k T_x^* M)$$

such that

$$\mathrm{e}^{-t\Delta_\mu^{(k)}} a(x) = \int_M \mathrm{e}^{-t\Delta_\mu^{(k)}}(x,y)a(y)\,\mu(\mathrm{d}y).$$

Let

$$-K := \min\left\{\left((\mathscr{R} - \operatorname{Hess} h)^{(k)} v, v\right)_g : v \in \Lambda^k T_x M,\ |v| = 1,\ x \in M\right\}.$$

Recall the notation $\lambda_k(x)$ defined as the smallest eigenvalue of $(\mathscr{R} - \operatorname{Hess} h)^{(k)}(x)$, $x \in M$ and let

$$V_k(x) = \lambda_k^-(x) = (|\lambda_k(x)| - \lambda_k(x))/2.$$

Then by [22],

$$\left|\exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right| \leq p_t^{V_k}(x,y).$$

We conclude that to estimate $\left|\exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right|$, it suffices to estimate the Schrödinger heat kernel $p_t^{V_k}(x,y)$. There is a lot of previous work dealing with Schrödinger heat kernels on manifolds, see for instance [14, 19, 29, 32, 33].

Theorem 2.3 *Let M be a complete non-compact Riemannian manifold satisfying (**LD**) and (**UE**). Then for any $\alpha \in (0, 1/4)$, there exists $\tilde{\sigma} > 0$ depending only on the constants in (**LD**) and (**UE**) and a constant $C > 0$ such that*

$$\left|\exp(-t\Delta_\mu^{(k)})(x,y)\right| \leq \frac{C\mathrm{e}^{\tilde{\sigma} t}}{\mu(B(y,\sqrt{t}))} \exp\left(-\alpha\rho(x,y)^2/t\right), \quad \forall x,y \in M,\ t > 0.$$

If $\sigma_1 = 0$ and $\sigma_2 = 0$, then $\tilde{\sigma} = 0$.

Proof Let $P_t^{V_k}$ be the semigroup generated by the operator $-\Delta_\mu + V_k$ and $p_t^{V_k}(x, y)$ the corresponding heat kernel. We recall that

$$\left|\exp\left(-t\Delta_\mu^{(k)}\right)(x, y)\right| \le p_t^{V_k}(x, y), \quad x, y \in M,\ t > 0.$$

From the assumptions (**LD**) and (**UE**), one can derive that $p_t^{V_k}$ satisfies the Gaussian estimate:

$$p_t^{V_k}(x, y) \le \frac{C e^{\tilde{\sigma} t}}{\mu(B(y, \sqrt{t}))} \exp\left(-\alpha \rho(x, y)^2/t\right)$$

for $x, y \in M$ and $t > 0$, by the same argument as in [17]; see [14, Theorem 3.1] for a similar argument. □

Lemma 2.4 *If the local volume doubling property (**LD**) holds, then for any $\gamma > 0$, there exist positive constants C_γ and $\tilde{c} := \sigma_1^2/2\gamma$ such that*

$$\int_{\rho(x,y)\ge\sqrt{t}} e^{-2\gamma \frac{\rho^2(x,y)}{s}} \mu(dx) \le C_\gamma\, \mu\big(B(y, \sqrt{s})\big)\, e^{-\gamma t/s} e^{\tilde{c} s} \tag{2.1}$$

for $s, t > 0$ and $x, y \in M$.

Proof By (1.5), it is easy to see that for all $\gamma > 0$, $s, t > 0$ and $y \in M$, there exist two positive constants C_γ (depending on γ and the constants in (1.5)) and $\tilde{c} = \sigma_1^2/2\gamma$ such that

$$\begin{aligned}
\int_{\rho(x,y)\ge\sqrt{t}} e^{-2\gamma \frac{\rho^2(x,y)}{s}} \mu(dx) &\le e^{-\gamma t/s} \int_M e^{-\gamma \frac{\rho^2(x,y)}{s}} \mu(dx) \\
&\le e^{-\gamma t/s} \sum_{i=0}^{\infty} \mu(B(y, (i+1)\sqrt{s})) e^{-\gamma i^2} \\
&\le C e^{-\gamma t/s} \mu(B(y, \sqrt{s})) \sum_{i=0}^{\infty} (i+1)^{m+1} e^{-\gamma i^2} e^{\sigma_1 i \sqrt{s}} \\
&\le C e^{-\gamma t/s} \mu(B(y, \sqrt{s})) \sum_{i=0}^{\infty} (i+1)^{m+1} e^{-\gamma i^2} e^{\gamma i^2/2 + \sigma_1^2 s/(2\gamma)} \\
&\le C e^{-\gamma t/s} e^{\sigma_1^2 s/(2\gamma)} \mu(B(y, \sqrt{s})) \sum_{i=0}^{\infty} (i+1)^{m+1} e^{-\gamma i^2/2} \\
&\le C_\gamma \mu(B(y, \sqrt{s}))\, e^{-\gamma t/s} e^{\tilde{c} s},
\end{aligned} \tag{2.2}$$

where the third inequality comes from condition (1.5). □

By means of this estimate, we obtain immediately the following consequence.

Theorem 2.5 *Let M be a complete non-compact Riemannian manifold satisfying (**LD**) and (**UE**). Then for any $\alpha \in (0, 1/4)$ and $\gamma \in (0, \alpha)$, there exists some constant $C > 0$ such that*

$$\int_M \left|\exp\left(-t\Delta_\mu^{(k)}\right)(x, y)\right|^2 e^{\frac{2\gamma\rho^2(x,y)}{t}} \mu(dx) \le \frac{C e^{2C_0 t}}{\mu\big(B(y, \sqrt{t})\big)},$$

for all $y \in M$ and $t > 0$, where $C_0 := \tilde{\sigma} + \frac{1}{2}\tilde{c}$ and the constants $\tilde{\sigma}, \tilde{c}$ defined in Theorems 2.3 and Lemma 2.4 respectively.

Proof Letting t tend to ∞ in inequality (2.1), we obtain

$$\int_M \mathrm{e}^{-2\gamma \frac{\rho^2(x,y)}{t}} \mu(\mathrm{d}x) \leq C_\gamma\, \mu\Big(B\big(y, \sqrt{t}\big)\Big)\, \mathrm{e}^{\tilde{c}t}, \quad t > 0.$$

By Theorem 2.3 and Lemma 2.4, we conclude that there exists $\tilde{\sigma} > 0$ depending only on the constants σ_1 and σ_2 and a constant $C > 0$ such that

$$\begin{aligned}
&\int_M \left|\exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right|^2 \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}} \mu(\mathrm{d}x) \\
&\leq C \frac{\mathrm{e}^{2\tilde{\sigma}t}}{\mu\big(B(y,\sqrt{t})\big)^2} \int_M \mathrm{e}^{\frac{-(2\alpha-2\gamma)\rho^2(x,y)}{t}} \mu(\mathrm{d}x) \\
&\leq C \frac{\mathrm{e}^{(2\tilde{\sigma}+\tilde{c})t}}{\mu\big(B\big(y,\sqrt{t}\big)\big)}.
\end{aligned}$$

We then complete the proof. □

2.2 L^2-weighted derivative estimates of heat kernel

In this subsection, we start the discussion under the assumption that (**LD**) and (**UE**) hold and that the operator $\Delta_\mu - V_k + \sigma_3$ on Ω^k is strongly positive. Then we have the following result about the L^2-weighted derivative estimate of the heat kernel.

Theorem 2.6 *Let M be a complete non-compact Riemannian manifold satisfying the assumptions as in Theorem 1.2. Fix $\alpha \in (0, 1/4)$ as in Theorem 2.3. Then for any $0 < \gamma < \alpha$, there exists a constant $C > 0$ such that*

$$\int_M \left|\nabla \exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right|^2 \mathrm{e}^{2\gamma \frac{\rho^2(x,y)}{t}} \mu(\mathrm{d}x) \leq \frac{C(1+\sigma_3 t)\mathrm{e}^{2C_0 t}}{t\mu(B(y,\sqrt{t}))}$$

for all $y \in M$, $t > 0$, where the constant C_0 is defined as in Theorem 2.5.

Proof For $R > 0$, let ψ be a C^2 function on $\mathbb{R}^+$ such that $\psi(r) = 1$ for $r \in [0, R]$, $\psi(r) = 0$ for $r > 2R$ and $\|\psi\|_\infty \leq 1$, $\|\psi'\|_\infty \leq \frac{c\psi^{1/2}}{R}$ for some positive constant $c > 0$ (see [24]). An argument of Calabi, which is also used in [9], allows us to assume without loss of generality that $\mathrm{e}^{\frac{2\gamma\rho^2(\cdot,y)}{t}} \psi(\rho(\cdot,y))$ for $y \in M$ is smooth. According to the integration by parts formula, we have

$$\begin{aligned}
&\int_M \left|\nabla \exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right|^2 \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}} \psi(\rho(x,y))\mu(\mathrm{d}x) \\
&= \int_M \left(\psi(\rho(x,y))4\gamma\frac{\rho(x,y)}{t} + \psi'(\rho(x,y))\right)\left(\nabla_{\nabla\rho} \exp\left(-t\Delta_\mu^{(k)}\right)(x,y),\right. \\
&\quad \left.\exp\left(-t\ \Delta_\mu^{(k)}\right)(x,y)\right)_g \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}} \mu(\mathrm{d}x) \\
&\quad + \int_M \psi(\rho(x,y))\left(\nabla_\mu^*\nabla \exp\left(-t\Delta_\mu^{(k)}\right)(x,y), \exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right)_g \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}} \mu(\mathrm{d}x) =: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Then there exists $\alpha > \gamma' > \gamma > 0$ such that

$$
\begin{aligned}
\mathrm{I} &= \int_M 4\gamma \frac{\rho(x,y)}{t}\left(\nabla_{\nabla\rho}\exp\left(-t\Delta_\mu^{(k)}\right)(x,y), \exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right)_g \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}}\psi(\rho(x,y))\,\mu(\mathrm{d}x)\\
&\quad+\int_M \psi'(\rho(x,y))\left(\nabla_{\nabla\rho}\exp\left(-t\Delta_\mu^{(k)}\right)(x,y), \exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right)_g \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}}\mu(\mathrm{d}x)\\
&= \int_M 4\gamma \frac{\rho(x,y)}{t}\left(\nabla\exp\left(-t\Delta_\mu^{(k)}\right)(x,y), \left(\mathrm{d}\rho\otimes\exp\left(-t\Delta_\mu^{(k)}\right)\right)(x,y)\right)_g \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}}\psi(\rho(x,y))\,\mu(\mathrm{d}x)\\
&\quad+\frac{c}{R}\int_M \left|\left(\nabla\exp\left(-t\Delta_\mu^{(k)}\right)(x,y), \left(\mathrm{d}\rho\otimes\exp\left(-t\Delta_\mu^{(k)}\right)\right)(x,y)\right)_g\right| \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}}\psi(\rho(x,y))^{1/2}\,\mu(\mathrm{d}x)\\
&\le \frac{C}{\sqrt{t}}\int_M \left|\nabla\exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right|\cdot\left|\exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right| \mathrm{e}^{\frac{2\gamma'\rho^2(x,y)}{t}}\psi(\rho(x,y))\,\mu(\mathrm{d}x)\\
&\quad+\frac{c}{R}\int_M \left|\nabla\exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right|\cdot\left|\exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right| \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}}\psi(\rho(x,y))^{1/2}\,\mu(\mathrm{d}x)\\
&\le \frac{C}{\sqrt{t}}\left(\int_M \left|\nabla\exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right|^2 \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}}\psi(\rho(x,y))\,\mu(\mathrm{d}x)\right)^{1/2}\\
&\qquad\times\left(\int_M \left|\exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right|^2 \mathrm{e}^{\frac{(4\gamma'-2\gamma)\rho^2(x,y)}{t}}\psi(\rho(x,y))\,\mu(\mathrm{d}x)\right)^{1/2}\\
&\quad+\frac{c}{R}\left(\int_M \left|\nabla\exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right|^2 \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}}\psi(\rho(x,y))\,\mu(\mathrm{d}x)\right)^{1/2}\\
&\qquad\times\left(\int_M \left|\exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right|^2 \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}}\mu(\mathrm{d}x)\right)^{1/2}\\
&\le \frac{1-A}{2}\int_M \left|\nabla\exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right|^2 \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}}\psi(\rho(x,y))\,\mu(\mathrm{d}x)\\
&\quad+\frac{C^2}{(1-A)t}\int_M \left|\exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right|^2 \mathrm{e}^{(4\gamma'-2\gamma)\frac{\rho^2(x,y)}{t}}\psi(\rho(x,y))\,\mu(\mathrm{d}x)\\
&\quad+\frac{c^2}{(1-A)R^2}\int_M \left|\exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right|^2 \mathrm{e}^{2\gamma\frac{\rho^2(x,y)}{t}}\mu(\mathrm{d}x)
\end{aligned}
$$

where $(\mathrm{d}\rho)(x,y) := (\mathrm{d}\rho(\cdot,y))(x)$ and $A<1$ is the constant from the strong positivity property of $\Delta_\mu - V_k + \sigma_3$. Since $2\gamma'-\gamma<\alpha$, we can use the estimate in Theorem 2.5 to get

$$
\begin{aligned}
&\int_M 4\gamma \frac{\rho(x,y)}{t}\left(\nabla\exp\left(-t\Delta_\mu^{(k)}\right)(x,y), \left(\mathrm{d}\rho\otimes\exp\left(-t\Delta_\mu^{(k)}\right)\right)(x,y)\right)_g \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}}\psi(\rho(x,y))\,\mu(\mathrm{d}x)\\
&\le \frac{1-A}{2}\int_M \left|\nabla\exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right|^2 \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}}\psi(\rho(x,y))\,\mu(\mathrm{d}x)+\left(\frac{1}{t}+\frac{1}{R^2}\right)\frac{C\mathrm{e}^{2C_0t}}{\mu(B(y,\sqrt{t}))}
\end{aligned}
$$

for some generic constant C. As

$$\Delta_\mu^{(k)} = \square - \nabla_{\nabla h} + \mathscr{R}^{(k)} - (\mathrm{Hess}\ h)^{(k)},$$

and $(\mathscr{R}^{(k)} - (\mathrm{Hess}\ h)^{(k)})(x) \ge -V_k(x)$, we then have

$$
\begin{aligned}
\mathrm{II} &= \int_M \left(\nabla_\mu^*\nabla\exp\left(-t\Delta_\mu^{(k)}\right)(x,y), \exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right)_g \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}}\psi(\rho(x,y))\,\mu(\mathrm{d}x)\\
&= \int_M \left(\Delta_\mu^{(k)}\exp\left(-t\Delta_\mu^{(k)}\right)(x,y), \exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right)_g \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}}\psi(\rho(x,y))\,\mu(\mathrm{d}x)\\
&\quad-\int_M \left(\left(\mathscr{R}^{(k)} - (\mathrm{Hess}\,h)^{(k)}\right)\exp\left(-t\Delta_\mu^{(k)}\right)(x,y),\right.
\end{aligned}
$$

$$\begin{aligned}
&\exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\Big)_g \mathrm{e}^{2\gamma\frac{\rho^2(x,y)}{t}}\psi(\rho(x,y))\,\mu(\mathrm{d}x)\\
&\leq \int_M \left(\Delta_\mu^{(k)}\exp\left(-t\Delta_\mu^{(k)}\right)(x,y), \exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right)_g \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}}\psi(\rho(x,y))\,\mu(\mathrm{d}x)\\
&\quad+\int_M (V_k(x)-\sigma_3)\left|\exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right|^2 \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}}\psi(\rho(x,y))\,\mu(\mathrm{d}x)\\
&\quad+\sigma_3\int_M \left|\exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right|^2 \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}}\psi(\rho(x,y))\,\mu(\mathrm{d}x)\\
&\leq \int_M \left(\Delta_\mu^{(k)}\exp\left(-t\Delta_\mu^{(k)}\right)(x,y), \exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right)_g \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}}\psi(\rho(x,y))\,\mu(\mathrm{d}x)\\
&\quad+A\int_M \left|\mathrm{d}|\exp\left(-t\Delta_\mu^{(k)}\right)(x,y)|\right|^2 \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}}\psi(\rho(x,y))\,\mu(\mathrm{d}x)\\
&\quad+\sigma_3\int_M \left|\exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right|^2 \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}}\psi(\rho(x,y))\,\mu(\mathrm{d}x).
\end{aligned}$$

Using Kato's inequality we further obtain

$$\begin{aligned}
\mathrm{II} &\leq \int_M \left(\Delta_\mu^{(k)}\exp\left(-t\Delta_\mu^{(k)}\right)(x,y), \exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right)_g \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}}\psi(\rho(x,y))\,\mu(\mathrm{d}x)\\
&\quad+A\int_M \left|\nabla\exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right|^2 \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}}\psi(\rho(x,y))\,\mu(\mathrm{d}x)\\
&\quad+\sigma_3\int_M \left|\exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right|^2 \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}}\mu(\mathrm{d}x).
\end{aligned}$$

By Cauchy's integral formula, we get for $w \in \mathscr{B}(M)$ and $a_1, a_2 \in \Omega^k_{L^2}$,

$$\begin{aligned}
&\left\langle \Delta_\mu^{(k)}\mathrm{e}^{-t\Delta_\mu^{(k)}}a_1,\ wa_2\right\rangle\\
&= \left|\int_{z:\,|z-t|=t/2}\frac{\left\langle \mathrm{e}^{-z\Delta_\mu^{(k)}}a_1, wa_2\right\rangle}{(z-t)^2}\,\mathrm{d}z\right| \leq (2\pi)^{-1}\pi t \sup_{z:|z-t|=t/2}\left|\frac{\left\langle \mathrm{e}^{-z\Delta_\mu^{(k)}}a_1, wa_2\right\rangle}{(z-t)^2}\right|\\
&\leq \frac{t}{2}\sup_{z:\,|z-t|=t/2}\left\| |\mathrm{e}^{-z\Delta_\mu^{(k)}}a_1|\sqrt{w}\right\|_2 \left\| |a_2|\sqrt{w}\right\|_2 (t/2)^{-2}\\
&\leq \frac{2}{t}\left\| |a_1|\sqrt{w}\right\|_2 \left\| |a_2|\sqrt{w}\right\|_2,
\end{aligned}$$

which implies for $w(\cdot) = \mathrm{e}^{\frac{2\gamma\rho^2(\cdot,y)}{t}}\psi(\rho(\cdot,y))$,

$$\left|\left\langle \Delta_\mu^{(k)}\mathrm{e}^{-t\Delta_\mu^{(k)}/2}a_1, a_2\mathrm{e}^{\frac{2\gamma\rho^2(\cdot,y)}{t}}\right\rangle\right| \leq \frac{C}{t}\left\|a_1\mathrm{e}^{\frac{\gamma\rho^2(\cdot,y)}{t}}\right\|_2\left\|a_2\mathrm{e}^{\frac{\gamma\rho^2(\cdot,y)}{t}}\right\|_2.$$

Letting $a_1(x) = \exp\left(-t\Delta_\mu^{(k)}/2\right)(x,y)$ and $a_2(x) = \mathrm{e}^{-t\Delta_\mu^{(k)}}(x,y)$, we then obtain by Theorem 2.5,

$$\begin{aligned}
&\left|-\int_M \left(\Delta_\mu^{(k)}\exp\left(-t\Delta_\mu^{(k)}\right)(x,y), \exp\left(-t\Delta_\mu^{(k)}\right)(x,y)\right)_g \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}}\psi(\rho(x,y))\,\mu(\mathrm{d}x)\right|\\
&\leq \frac{C}{t}\left(\int_M \left|\exp\left(-t\Delta_\mu^{(k)}/2\right)(x,y)\right|^2 \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}}\mu(\mathrm{d}x)\right)^{1/2}
\end{aligned}$$

$$\times \left(\int_M \left| \exp\left(-t\Delta_\mu^{(k)}\right)(x,y) \right|^2 \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}} \mu(\mathrm{d}x) \right)^{1/2}$$
$$\leq \frac{C}{t} \left(\frac{C_1 \mathrm{e}^{C_0 t}}{\mu(B(y,\sqrt{t/2}))} \right)^{1/2} \left(\frac{C_2 \mathrm{e}^{2C_0 t}}{\mu(B(y,\sqrt{t}))} \right)^{1/2}$$
$$\leq \frac{C\mathrm{e}^{3/2C_0 t}}{t\mu(B(y,\sqrt{t}))}.$$

We conclude that

$$\int_{B(y,R)} \left| \nabla \exp\left(-t\Delta_\mu^{(k)}\right)(x,y) \right|^2 \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}} \mu(\mathrm{d}x)$$
$$\leq \int_M \left| \nabla \exp\left(-t\Delta_\mu^{(k)}\right)(x,y) \right|^2 \mathrm{e}^{\frac{2\gamma\rho^2(x,y)}{t}} \psi(\rho(x,y))\,\mu(\mathrm{d}x)$$
$$\leq \frac{C(1+\sigma_3 t)\mathrm{e}^{2C_0 t}}{t\mu(B(y,\sqrt{t}))} + \frac{C\mathrm{e}^{2C_0 t}}{R^2\mu(B(y,\sqrt{t}))}.$$

We then complete the proof by letting R tend to ∞. □

Combining Theorem 2.6 with Lemma 2.4, we obtain

Theorem 2.7 *Let M be a complete non-compact Riemannian manifold satisfying the same assumptions as in Theorem 1.2. Fix $\alpha \in (0, 1/4)$ as in Theorem 2.3 and let $0 < \gamma < \alpha$. There exists a constant $C > 0$ such that*

$$\int_{\rho(x,y)\geq t^{1/2}} \left| \nabla \exp\left(-s\Delta_\mu^{(k)}\right)(x,y) \right| \mu(\mathrm{d}x) \leq C(1+\sqrt{\sigma_3 s})\,\mathrm{e}^{2C_0 s - \gamma t/2s} s^{-1/2}$$

for all $y \in M$ and $s, t > 0$, where C_0 is the same as in Theorem 2.5.

Proof Let $0 < \gamma < \alpha$. By Cauchy's inequality we obtain

$$\int_{\rho(x,y)\geq t^{1/2}} \left| \nabla \exp\left\{-s\Delta_\mu^{(k)}\right\}(x,y) \right| \mu(\mathrm{d}x)$$
$$\leq \left(\int_M \left| \nabla \exp\left\{-s\Delta_\mu^{(k)}\right\}(x,y) \right|^2 \mathrm{e}^{2\gamma\rho^2(x,y)/s} \mu(\mathrm{d}x) \right)^{1/2} \left(\int_{\rho(x,y)\geq t^{1/2}} \mathrm{e}^{-2\gamma\rho^2(x,y)/s} \mu(\mathrm{d}x) \right)^{1/2}$$
$$\leq C\mathrm{e}^{C_0 s} \sqrt{\frac{1+\sigma_3 s}{s\mu(B(y,\sqrt{s}))}} \sqrt{\mu(B(y,\sqrt{s}))}\,\mathrm{e}^{-\gamma t/2s}\,\mathrm{e}^{\tilde{c}s/2}$$
$$= C\mathrm{e}^{2C_0 s} \frac{\sqrt{1+\sigma_3 s}}{\sqrt{s}}\,\mathrm{e}^{-\gamma t/2s},$$

where the second inequality follows from Theorem 2.6 and Lemma 2.4. This finishes the proof. □

3 Proof of Theorem 1.2

Let us now present the main steps of the proof of Theorem 1.2 and Theorem 1.3, following closely the approach of [11, Theorems 1.1 and 1.2]. Some of the arguments have been used already in [7] and can be taken from there. For the convenience of the reader and for the sake of completeness we give details here.

The object of our interest is for suitable $\sigma \geq 0$ the following operator on Ω^k:

$$T_\sigma^{(k)} := \nabla \left(\Delta_\mu^{(k)} + \sigma\right)^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty \nabla \exp\left(-\Delta_\mu^{(k)} s\right) \frac{\mathrm{e}^{-\sigma s}}{\sqrt{s}} \,\mathrm{d}s. \tag{3.1}$$

We ignore the normalization constant $1/\sqrt{\pi}$ in the sequel which is irrelevant for our purpose. We start with the boundedness of $T_\sigma^{(k)}$ in L^2.

Lemma 3.1 *For $k \in \mathbb{N}^+$, suppose $\Delta_\mu - V_k + \sigma$ is strongly positive on $C_c^\infty(M)$ for some constant $\sigma > 0$. Then the operator $\nabla\left(\Delta_\mu^{(k)} + \sigma\right)^{-1/2}$ on Ω_c^k is bounded in L^2-sense.*

Proof We use the Bochner formula for $a \in \Omega_c^k$. According to the Weitzenböck formula (1.3) and the strong positivity of $\Delta_\mu - V_k + \sigma$, we have

$$\begin{aligned}
\|\nabla a\|_2^2 &= \langle \Box_\mu a, a\rangle \\
&= \langle \Delta_\mu^{(k)} a, a\rangle - \langle (\mathscr{R}^{(k)} - (\mathrm{Hess}\, h)^{(k)}) a, a\rangle \\
&\leq \langle \Delta_\mu^{(k)} a, a\rangle + \int_M (V_k - \sigma)|a|^2(x)\,\mu(\mathrm{d}x) + \sigma\|a\|^2 \\
&\leq \langle \Delta_\mu^{(k)} a, a\rangle + A\int_M \big|\mathrm{d}|a|\big|^2(x)\,\mu(\mathrm{d}x) + \sigma\|a\|^2 \\
&\leq \|(\Delta_\mu^{(k)})^{1/2} a\|_2^2 + A\|\nabla a\|_2^2 + \sigma\|a\|_2^2,
\end{aligned}$$

where the last inequality comes from the Kato inequality (see e.g. [22]). This implies

$$\|\nabla a\|_2^2 \leq \frac{1}{1-A}\|(\Delta_\mu^{(k)} + \sigma)^{1/2} a\|_2^2.$$

We complete the proof by letting $a = (\Delta_\mu^{(k)} + \sigma)^{-1/2} b$ for $b \in \Omega_c^k$. □

As the local version of the Riesz transform is bounded in L^2, by the interpolation theorem, the weak (1, 1) property for $T_\sigma^{(k)}$ already implies L^p-boundedness for all $p \in (1, 2]$. Hence we aim to study the weak (1, 1) property of $T_\sigma^{(k)}$ for $\sigma \geq 0$ suitable: there exists $c > 0$ such that

$$\sup_{\lambda>0} \lambda\mu\left(\left|T_\sigma^{(k)} a\right| > \lambda\right) \leq c\,\mu(|a|), \quad a \in \Omega_c^k. \tag{3.2}$$

To this end, we use a version of the localization technique of [1, Section 4] on the finite overlap property of M, which has also been used in [7].

Lemma 3.2 *[1] Assume that condition (**LD**) holds. There exists a countable subset $\mathcal{C} = \{x_j\}_{j\in\Lambda} \subset M$ such that*

(i) $M = \cup_{j\in\Lambda} B(x_j, 1)$;
(ii) $\{B(x_j, 1/2)\}_{j\in\Lambda}$ are disjoint;
(iii) there exists $N_0 \in \mathbb{N}$ such that for any $x \in M$, at most N_0 balls $B(x_j, 4)$ contain x;
(iv) for any $c_0 \geq 1$, there exists $C > 0$ such that for any $j \in \Lambda$, $x \in B(x_j, c_0)$ and $r \in (0, \infty)$,

$$\mu\left(B(x, 2r) \cap B(x_j, c_0)\right) \leq C\mu\left(B(x, r) \cap B(x_j, c_0)\right)$$

and

$$\mu(B(x, r)) \leq C\mu\left(B(x, r) \cap B(x_j, c_0)\right)$$

for any $x \in B(x_j, c_0)$ and $r \in (0, 2c_0]$.

The following lemma provides the localization argument in order to prove (3.2).

Lemma 3.3 *Keeping the assumptions as in Theorem 1.2, let $\mathcal{C} = \{x_j\}_{j\in\Lambda}$ be a countable subset of M having the finite overlap property as in Lemma 3.2. Let $\sigma > 2C_0$ where C_0 is as in Theorem 2.5. Suppose that there exists a constant $c > 0$ such that*

$$\mu\left(\{x\colon \mathbb{1}_{B(x_j,2)}\,|T_\sigma^{(k)}a(x)| > \lambda\}\right) \le \frac{c}{\lambda}\|\,|a|\,\|_1 \tag{3.3}$$

for any $j \in \Lambda$, $\lambda \in (0,\infty)$ and $a \in \Omega_c^k$ supported in $B(x_j, 1)$. Then property (3.2) holds for any $a \in \Omega_c^k$.

Proof For $j \in \Lambda$, set $B_j := B(x_j, 1)$ and let $\{\varphi_j\}_{j\in\Lambda}$ be a C^∞-partition of the unity such that $0 \le \varphi_j \le 1$ and each φ_j is supported in B_j. Then, for $a \in \Omega_c^k$ and $x \in M$, write

$$T_\sigma^{(k)}a(x) = \sum_{j\in\Lambda}\mathbb{1}_{2B_j}T_\sigma^{(k)}(a\varphi_j)(x) + \sum_{j\in\Lambda}(1-\mathbb{1}_{2B_j})T_\sigma^{(k)}(a\varphi_j)(x),$$

which yields that for any $\lambda > 0$,

$$\begin{aligned}
&\mu(\{x\colon |T_\sigma^{(k)}a(x)| > \lambda\})\\
&\le \mu\left(\left\{x\colon \sum_{j\in\Lambda}\mathbb{1}_{2B_j}|T_\sigma^{(k)}(a\varphi_j)(x)| > \frac{\lambda}{2}\right\}\right) + \mu\left(\left\{x\colon \sum_{j\in\Lambda}(1-\mathbb{1}_{2B_j})|T_\sigma^{(k)}(a\varphi_j)(x)| > \frac{\lambda}{2}\right\}\right)\\
&=: \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}$$

For I_1, by Lemma 3.2 (iii) and condition (3.3), we have

$$\mathrm{I}_1 \le \sum_{j\in\Lambda}\mu\left(\left\{x : \mathbb{1}_{2B_j}|T_\sigma^{(k)}(a\varphi_j)(x)| > \frac{\lambda}{2N_0}\right\}\right) \lesssim \frac{1}{\lambda}\|\,|a|\,\|_1 \tag{3.4}$$

as desired, where the notation $a \lesssim b$ means $a \le Cb$ for some constant C.

To bound I_2, again by Lemma 3.2 (iii), since φ_j is supported in B_j, it is easy to see that

$$\sum_{j\in\Lambda}|(1-\mathbb{1}_{2B_j})(x)\varphi_j(y)| \le N_0\mathbb{1}_{\{\rho(x,y)\ge 1\}}.$$

Hence, according to the definition of $T_\sigma^{(k)}$ in (3.1) and Theorem 2.7, we get

$$\begin{aligned}
\mathrm{I}_2 &\le \frac{2}{\lambda}\sum_{j\in\Lambda}\left\|\,\left|\left(1-\mathbb{1}_{2B_j}\right)T_\sigma^{(k)}\left(a\varphi_j\right)\right|\,\right\|_1\\
&\lesssim \frac{1}{\lambda}\int_M\left(\int_0^\infty\frac{\mathrm{e}^{-\sigma t}}{\sqrt{t}}\int_M\left|\nabla_x\exp(-\Delta_\mu^{(k)}t)(x,y)\right|\sum_{j\in\Lambda}\left|(1-\mathbb{1}_{2B_j})(x)\varphi_j(y)\right|\,|a(y)|\,\mu(\mathrm{d}y)\,\mathrm{d}t\right)\mu(\mathrm{d}x)\\
&\lesssim \frac{1}{\lambda}\int_M\int_0^\infty\frac{\mathrm{e}^{-\sigma t}}{\sqrt{t}}\left(\int_{\rho(x,y)\ge 1}|\nabla_x\exp(-\Delta_\mu^{(k)}t)(x,y)|\,\mu(\mathrm{d}x)\right)\mathrm{d}t\,|a(y)|\,\mu(\mathrm{d}y)\\
&\le \frac{1}{\lambda}\int_M|a(y)|\,\mu(\mathrm{d}y)\int_0^\infty\mathrm{e}^{-\sigma t}(1+\sqrt{\sigma_3 t})\,\mathrm{e}^{2C_0t}\,\mathrm{e}^{-\gamma/2t}t^{-1}\,\mathrm{d}t,
\end{aligned}$$

where $\gamma \in (0,\alpha)$ and C_0 is as in Theorem 2.7. Thus, since $\sigma > 2C_0$, we obtain

$$\mathrm{I}_2 \lesssim \frac{1}{\lambda} \int_0^\infty \mathrm{e}^{(2C_0-\sigma)t-\gamma/2t} \frac{(1+\sqrt{\sigma_3 t})}{t} \,\mathrm{d}t \, \|a\|_1 \lesssim \frac{1}{\lambda} \|a\|_1$$

where as usual $\|a\|_1 = \| \, |a| \, \|_1$. This combined with the estimate of I_1 in (3.4) finishes the proof of Lemma 3.3. □

We now turn to the proof of property (3.3), where we remove the subscript j and write B for each $B(x_j, 1)$ for simplicity. Let $c_0 \ge 1$. By Lemma 3.2(iv), we have that $(c_0 B, \mu, \rho)$ is a metric measure subspace satisfying the *volume doubling property* that there exists $C_D \ge 1$ such that

$$\mu\left(B(x, 2r) \cap c_0 B\right) \le C_D \, \mu\left(B(x, r) \cap c_0 B\right) \tag{D}$$

for all $x \in c_0 B$ and $r > 0$.

We also use the following Calderón–Zygmund decomposition from [10], where $\mathcal{X}$ will replace $c_0 B$.

Lemma 3.4 *([10]) Let $(\mathcal{X}, \nu, \rho)$ be a metric measure space satisfying (**LD**). Let $f \in L^1(\mathcal{X})$ and $\lambda \in (0, \infty)$. Assume $\|f\|_{L^1} < \lambda \nu(\mathcal{X})$. Then f has a decomposition of the form*

$$f = g + h = g + \textstyle\sum_i h_i$$

such that

(a) $g(x) \le C\lambda$ for almost all $x \in M$;
(b) there exists a sequence of balls $\tilde{B}_i = B(x_i, r_i)$ so that the support of each h_i is contained in $\tilde{B}_i$:

$$\int_{\mathcal{X}} |h_i(x)| \, \nu(\mathrm{d}x) \le C\lambda \nu(\tilde{B}_i) \quad \textit{and} \quad \int_{\mathcal{X}} h_i(x) \, \nu(\mathrm{d}x) = 0;$$

(c) $\displaystyle\sum_i \nu(\tilde{B}_i) \le \frac{C}{\lambda} \int_{\mathcal{X}} |f(x)| \, \nu(\mathrm{d}x)$;
(d) there exists $k_0 \in \mathbb{N}^*$ such that each point of M is contained in at most k_0 balls $\tilde{B}_i$.

Lemma 3.5 *Let M be a complete non-compact Riemannian manifold satisfying (**LD**) and (**UE**). Let $\lambda \in (0, \infty)$ and $f = |a| \in L^1(B)$ be as in Lemma 3.4. Let furthermore $\{h_i\}$ be the sequence of bad functions of f as in Lemma 3.4 and $\left(\exp\left(-\Delta_\mu^{(k)} t - \sigma t\right)\right)_{t \ge 0}$ the heat semigroup associated to $-(\Delta_\mu^{(k)} + \sigma)$ with $\sigma > \tilde{\sigma}$, where $\tilde{\sigma}$ is as in Lemma 2.4. Then there exists a constant $C > 0$ independent of f such that*

$$\left\| \sum_i \exp\left(-t_i \Delta_\mu^{(k)} - \sigma t_i\right) \tilde{h}_i \right\|_2^2 \le C\lambda \, \|a\|_1$$

where $\tilde{h}_i = |h_i| \frac{a}{|a|}$ and $t_i = r_i^2$ with r_i denoting the radii of the balls B_i as in Lemma 3.4(b).

Proof Recall that $\operatorname{supp} h_i \subset B(x_i, \sqrt{t_i})$. Using the upper bound of the heat kernel in Lemma 2.3 and Lemma 3.4 (b), we have for $x \in M$,

$$\Big|\exp\Big(-\Delta_\mu^{(k)}t_i-\sigma t_i\Big)\tilde{h}_i(x)\Big| \le C\int_M \frac{\mathrm{e}^{-\sigma' t_i-\alpha\frac{\rho^2(x,y)}{t_i}}}{\mu(B(x,\sqrt{t_i}))}\,|\tilde{h}_i(y)|\,\mu(\mathrm{d}y)$$
$$\le \frac{C}{\mu(B(x,\sqrt{t_i}))}\,\mathrm{e}^{-\sigma' t_i-\alpha\frac{\rho^2(x,x_i)}{t_i}}\int_{\tilde{B}_i}|h_i(y)|\,\mu(\mathrm{d}y)$$
$$\le C_1\lambda\int_M \frac{\mathrm{e}^{-\sigma' t_i-\alpha\frac{\rho^2(x,y)}{t_i}}}{\mu(B(x,\sqrt{t_i}))}\,\mathbb{1}_{\tilde{B}_i}(y)\,\mu(\mathrm{d}y),$$

for suitable σ such that $\sigma'=\sigma-\tilde{\sigma}>0$. It is therefore sufficient to verify that

$$\Big\|\sum_i\int_M \frac{\mathrm{e}^{-\sigma' t_i-\alpha''\frac{\rho^2(\cdot,y)}{t_i}}}{\mu(B(\cdot,\sqrt{t_i}))}\,\mathbb{1}_{\tilde{B}_i}(y)\,\mu(\mathrm{d}y)\Big\|_2 \lesssim \Big\|\sum_i \mathbb{1}_{\tilde{B}_i}\Big\|_2, \tag{3.5}$$

since as consequence from this and Lemma 3.4 we obtain

$$\Big\|\sum_i \exp(-\Delta_\mu^{(k)}t_i-\sigma t_i)\tilde{h}_i\Big\|_2^2 \lesssim \lambda^2\Big\|\sum_i \mathbb{1}_{\tilde{B}_i}\Big\|_2^2 \lesssim \lambda^2\sum_i \mu(\tilde{B}_i) \lesssim \lambda\,\|a\|_1.$$

In order to prove (3.5), we write by duality

$$\Big\|\sum_i\int_M \frac{\mathrm{e}^{-\sigma' t_i-\alpha''\frac{\rho^2(\cdot,y)}{t_i}}}{\mu(B(\cdot,\sqrt{t_i}))}\,\mathbb{1}_{\tilde{B}_i}(y)\,\mu(\mathrm{d}y)\Big\|_2$$
$$= \sup_{\|u\|_2=1}\left|\int_M\left(\sum_i\int_M \frac{\mathrm{e}^{-\sigma' t_i-\alpha''\frac{\rho^2(x,y)}{t_i}}}{\mu(B(x,\sqrt{t_i}))}\,\mathbb{1}_{\tilde{B}_i}(y)\,\mu(\mathrm{d}y)\right)u(x)\,\mu(\mathrm{d}x)\right|$$
$$\le \sup_{\|u\|_2=1}\int_M\sum_i\left(\int_M \frac{\mathrm{e}^{-\sigma' t_i-\alpha''\frac{\rho^2(x,y)}{t_i}}}{\mu(B(x,\sqrt{t_i}))}|u(x)|\,\mu(\mathrm{d}x)\right)\mathbb{1}_{\tilde{B}_i}(y)\,\mu(\mathrm{d}y). \tag{3.6}$$

By the local doubling property (**LD**), we have for any $x\in M$ and $y\in\tilde{B}_i$,

$$\mu(B(y,\sqrt{t_i}))\le C\left(1+\frac{\rho(x,y)}{\sqrt{t_i}}\right)^m \mathrm{e}^{\sigma_1\rho(x,y)/\sqrt{t_i}}\mu(B(x,\sqrt{t_i})).$$

From this, we obtain that there exist $0<\tilde{\alpha}<\alpha'<\alpha$ such that

$$\int_M \frac{\mathrm{e}^{-\sigma' t_i-\alpha\frac{\rho^2(x,y)}{t_i}}}{\mu(B(x,\sqrt{t_i}))}\,|u(x)|\,\mu(\mathrm{d}x)$$
$$\lesssim \frac{\mathrm{e}^{-\frac12\sigma' t_i}}{\mu(B(y,\sqrt{t_i}))}\int_M \mathrm{e}^{-\alpha'\frac{\rho^2(x,y)}{t_i}}|u(x)|\,\mu(\mathrm{d}x)$$
$$\lesssim \frac{1}{\mu(B(y,\sqrt{t_i}))}\left(\int_{\rho(x,y)<\sqrt{t_i}}|u(x)|\,\mu(\mathrm{d}x)+\sum_{k=0}^\infty\int_{2^k\sqrt{t_i}\le\rho(x,y)<2^{k+1}\sqrt{t_i}}\mathrm{e}^{-\alpha'\frac{\rho^2(x,y)}{t_i}}|u(x)|\,\mu(\mathrm{d}x)\right)$$
$$\le \frac{1}{\mu(B(y,\sqrt{t_i}))}\left(\int_{B(y,\sqrt{t_i})}|u(x)|\,\mu(\mathrm{d}x)+\sum_{k=0}^\infty \mathrm{e}^{-\alpha' 2^{2k}}\int_{B(y,2^{k+1}\sqrt{t_i})}|u(x)|\,\mu(\mathrm{d}x)\right)$$
$$= \left(1+\sum_{k=0}^\infty\frac{\mu(B(y,2^{k+1}\sqrt{t_i}))}{\mu(B(y,\sqrt{t_i}))}\,\mathrm{e}^{-\alpha' 2^{2k}}\right)(\mathscr{M}u)(y)$$

$$\leq \left(1 + C\sum_{k=0}^{\infty} 2^{(k+1)m} \mathrm{e}^{\sigma_1(2^{k+1}-1)\sqrt{t_i}} \mathrm{e}^{-\tilde{\alpha}2^{2k}}\right)(\mathscr{M}u)(y)$$
$$\leq \left(1 + C\sum_{k=0}^{\infty} 2^{(k+1)m} \mathrm{e}^{c2^k-\tilde{\alpha}2^{2k}}\right)(\mathscr{M}u)(y) \lesssim (\mathscr{M}u)(y),$$

where

$$(\mathscr{M}u)(y) := \sup_{r>0} \frac{1}{\mu(B(y,r))} \int_{B(y,r)} |u(x)|\, \mu(\mathrm{d}x)$$

denotes the Hardy-Littlewood maximal function of u. This together with (3.6) and the L^2-boundedness of $\mathscr{M}$ gives

$$\left\| \sum_i \int_M \frac{\mathrm{e}^{-\alpha'' \frac{\rho^2(\cdot,y)}{t_i}}}{\mu(B(\cdot,\sqrt{t_i}))} \mathbb{1}_{\tilde{B}_i}(y)\, \mu(\mathrm{d}y) \right\|_2 \lesssim \sup_{\|u\|_2=1} \int_M (\mathscr{M}u)(y) \sum_i \mathbb{1}_{\tilde{B}_i}(y)\, \mu(\mathrm{d}y) \lesssim \left\| \sum_i \mathbb{1}_{\tilde{B}_i} \right\|_2,$$

which shows that (3.5) holds true and finishes the proof of Lemma 3.5. □

With the help of the Lemmata 3.3 through 3.5, we are now in position to give a proof of Theorem 1.2. Note that Theorem 1.3 can be established along the same lines, with the slight difference that in this case σ can be taken to be 0.

Proof of Theorem 1.2 Recall that $T_\sigma^{(k)} = \nabla(\Delta_\mu^{(k)} + \sigma)^{-1/2}$. We choose σ large enough such that $\sigma > 2C_0$ where C_0 is defined as in Theorem 2.5. By Lemma 3.3, it suffices to prove

$$\mu\big(\{x \in 2B \colon |T_\sigma^{(k)} a(x)| > \lambda\}\big) \lesssim \frac{\|a\|_1}{\lambda}, \quad \lambda \in (0, \infty) \tag{3.7}$$

for all $a \in \Gamma_{C_0^\infty}(\Lambda^k T^*M)$. By means of Lemma 3.4 with $\mathcal{X} = B$, we deduce that f has a decomposition

$$|a| = g + h = g + \textstyle\sum_i h_i$$

which implies

$$\begin{aligned} &\mu\big(\{x \in 2B : |T_\sigma^{(k)} a(x)| > \lambda\}\big) \\ &\leq \mu\left(\left\{x \in 2B : |T_\sigma^{(k)} \tilde{g}(x)| > \frac{\lambda}{2}\right\}\right) + \mu\left(\left\{x \in 2B : |T_\sigma^{(k)} \tilde{h}(x)| > \frac{\lambda}{2}\right\}\right) \\ &=: \mathrm{I}_1 + \mathrm{I}_2, \end{aligned} \tag{3.8}$$

where

$$\tilde{g} = g\frac{a}{|a|} \quad \text{and} \quad \tilde{h} = h\frac{a}{|a|}.$$

Using the facts that $T_\sigma^{(k)}$ is bounded on $L^2(\mu)$ and that $|g(x)| \leq C\lambda$, we obtain as desired

$$\mathrm{I}_1 \lesssim \lambda^{-2}\|T_\sigma^{(k)}\tilde{g}\|_2^2 \lesssim \lambda^{-2}\|\,|\tilde{g}|\,\|_2^2 \lesssim \lambda^{-1}\|g\|_1 \lesssim \lambda^{-1}\|a\|_1. \tag{3.9}$$

We now turn to the estimate of I_2. Recall that $\exp(-\Delta_\mu^{(k)} t - \sigma t)$, $t \geq 0$ is the heat semigroup generated by $-(\Delta_\mu^{(k)} + \sigma)$. We write

$$T_\sigma^{(k)}\tilde{h}_i = T_\sigma^{(k)} \exp(-\Delta_\mu^{(k)} t_i - \sigma t_i)\tilde{h}_i + T_\sigma^{(k)}\Big(I - \exp(-\Delta_\mu^{(k)} t_i - \sigma t_i)\Big)\tilde{h}_i,$$

where $t_i = r_i^2$ with r_i the radius of $\tilde{B}_i$. By Lemma 3.5, we have

$$\Big\| \sum_i \exp(-\Delta_\mu^{(k)} t_i - \sigma t_i)\tilde{h}_i \Big\|_2^2 \lesssim \lambda \, \|a\|_1.$$

This combined with the L^2-boundedness of $T_\sigma^{(k)}$ yields

$$\mu\left(\left\{x \in 2B\colon \left|T_\sigma^{(k)}\left(\sum_i \exp(-\Delta_\mu^{(k)} t_i - \sigma t_i)\tilde{h}_i\right)(x)\right| > \frac{\lambda}{2}\right\}\right) \lesssim \frac{1}{\lambda}\|a\|_1 \tag{3.10}$$

as desired. Consider now the term $T_\sigma^{(k)} \sum_i (I - \exp(-\Delta_\mu^{(k)} t_i - \sigma t_i))\tilde{h}_i$. We write

$$\begin{aligned} &\mu\left(\left\{x \in 2B\colon \left|T_\sigma^{(k)}\left(\sum_i (I - \exp(-\Delta_\mu^{(k)} t_i - \sigma t_i))\tilde{h}_i\right)(x)\right| > \frac{\lambda}{2}\right\}\right) \qquad (3.11)\\ &\quad \le \sum_i \mu(2\tilde{B}_i) + \mu\left(\left\{x \in 2B \setminus \cup_i 2\tilde{B}_i\colon \left|T_\sigma^{(k)}\left(\sum_i (I - \exp(-\Delta_\mu^{(k)} t_i - \sigma t_i))\tilde{h}_i\right)\right|(x) > \frac{\lambda}{2}\right\}\right). \end{aligned}$$

From Lemma 3.4 we conclude that

$$\sum_i \mu(2\tilde{B}_i) \lesssim \frac{\|a\|_1}{\lambda}. \tag{3.12}$$

To estimate the second term, denote the integral kernel of the operator $T_\sigma^{(k)}(I - \exp(-\Delta_\mu^{(k)} t_i - \sigma t_i))$ by $k_{t_i}^{\sigma,k}(x, y)$. Note that

$$\begin{aligned} &(\Delta_\mu^{(k)} + \sigma)^{-1/2}\big(I - \exp(-\Delta_\mu^{(k)} t_i - \sigma t_i)\big)\\ &= \int_0^\infty \left(\frac{\exp(-\Delta_\mu^{(k)} s - \sigma s)}{\sqrt{s}} - \frac{\exp(-\Delta_\mu^{(k)}(t_i + s) - \sigma(t_i + s))}{\sqrt{s}}\right) \mathrm{d}s\\ &= \int_0^\infty \left(\frac{1}{\sqrt{s}} - \frac{\mathbb{1}_{\{s \ge t_i\}}}{\sqrt{s - t_i}}\right) \exp(-\Delta_\mu^{(k)} s - \sigma s)\,\mathrm{d}s \end{aligned}$$

and

$$\begin{aligned} &T_\sigma^{(k)}(I - \exp(-\Delta_\mu^{(k)} t_i - \sigma t_i))\\ &\quad = \nabla(\Delta_\mu^{(k)} + \sigma)^{-1/2}(I - \exp(-\Delta_\mu^{(k)} t_i - \sigma t_i))\\ &\quad = \int_0^\infty \left(\frac{1}{\sqrt{s}} - \frac{\mathbb{1}_{\{s \ge t_i\}}}{\sqrt{s - t_i}}\right) \mathrm{e}^{-\sigma s} \nabla \exp(-s\Delta_\mu^{(k)})\,\mathrm{d}s. \end{aligned}$$

Therefore,

$$k_{t_i}^{\sigma,k}(x, y) = \int_0^\infty \mathrm{e}^{-\sigma s}\left(\frac{1}{\sqrt{s}} - \frac{\mathbb{1}_{\{s \ge t_i\}}}{\sqrt{s - t_i}}\right) \nabla_x \exp\big(-s\Delta_\mu^{(k)}\big)(x, y)\,\mathrm{d}s. \tag{3.13}$$

Since $\tilde{h}_i$ is supported in $\tilde{B}_i$, we have

$$\begin{aligned} &\int_{2B\setminus(2\tilde{B}_i)} \left|T_\sigma^{(k)}\left((I - \exp(-\Delta_\mu^{(k)} t_i - \sigma t_i))\tilde{h}_i\right)(x)\right| \mu(\mathrm{d}x)\\ &\quad \le \int_{2B\setminus(2\tilde{B}_i)} \left(\int_{\tilde{B}_i} |k_{t_i}^{\sigma,k}(x, y)|\,|h_i(y)|\,\mu(\mathrm{d}y)\right) \mu(\mathrm{d}x) \end{aligned}$$

$$\le \int_{\tilde{B}_i} \left(\int_{\rho(x,y)\ge t_i^{1/2}} |k_{t_i}^{\sigma,k}(x,y)|\,\mu(\mathrm{d}x) \right) |h_i(y)|\,\mu(\mathrm{d}y). \tag{3.14}$$

Now by means of (3.13) and Theorem 2.7, we get

$$\begin{aligned}
&\int_{\rho(x,y)\ge t_i^{1/2}} |k_{t_i}^{\sigma,k}(x,y)|\,\mu(\mathrm{d}x) \\
&\le \int_0^\infty \left(\int_{\rho(x,y)\ge t_i^{1/2}} |\nabla_x \exp(-\Delta_\mu^{(k)} s)(x,y)|\,\mu(\mathrm{d}x) \right) \mathrm{e}^{-\sigma s} \left| \frac{1}{\sqrt{s}} - \frac{\mathbb{1}_{\{s\ge t_i\}}}{\sqrt{s-t_i}} \right| \mathrm{d}s \\
&\le C \int_0^\infty \mathrm{e}^{-\gamma t_i/2s}\, \mathrm{e}^{2C_0 s} \frac{(1+\sqrt{\sigma_3 s})}{\sqrt{s}} \left| \frac{1}{\sqrt{s}} - \frac{\mathbb{1}_{\{s\ge t_i\}}}{\sqrt{s-t_i}} \right| \mathrm{e}^{-\sigma s}\,\mathrm{d}s \\
&\le C \int_0^\infty \mathrm{e}^{-\gamma/2u} \left| \frac{1}{u} - \frac{\mathbb{1}_{\{u\ge 1\}}}{\sqrt{u(u-1)}} \right| \mathrm{d}u \\
&= C \int_0^1 \frac{\mathrm{e}^{-\gamma/2u}}{u}\,\mathrm{d}u + C \int_1^\infty \left(\frac{1}{\sqrt{u(u-1)}} - \frac{1}{u} \right) \mathrm{d}u < \infty
\end{aligned}$$

where for the third line above we used the fact that

$$\mathrm{e}^{s(2C_0-\sigma)}(1+\sqrt{\sigma_3 s}), \quad s \in (0,\infty),$$

is bounded. The estimate above together with (3.14) and Lemma 3.4 implies that

$$\mu\left(\left\{ x \in 2B \setminus \cup_i 2\tilde{B}_i : \left| T_\sigma^{(k)} \left(\sum_i \left(I - \exp(-\Delta_\mu^{(k)} t_i - \sigma t_i)\right)\tilde{h}_i \right)(x) \right| > \frac{\lambda}{2} \right\} \right) \lesssim \frac{\|a\|_1}{\lambda}. \tag{3.15}$$

Altogether, combining (3.8) through (3.10), (3.12) and (3.15), we conclude that (3.7) holds which completes the proof of Theorem 1.3. □

4 Proof of Theorem 1.6

By the Bishop-Gromov comparison theorem and the well-known formula for the volume of balls in the hyperbolic space, the local volume doubling property (**LD**) holds if the curvature-dimension condition (**CD**) is satisfied.

Proof of Theorem 1.6 Let us sketch the main idea of the second method in [20]. Inequality $\mathbf{CZ}_\mu(p)$ is reduced to the existence of positive constants C and σ such that

$$\big\| \,|\mathrm{Hess}(\Delta_\mu+\sigma)^{-1}u|\, \big\|_p \le C\|u\|_p,$$

which is equivalent to

$$\left\| \,|\nabla(\Delta_\mu^{(1)}+\sigma)^{-1/2} \circ \mathrm{d}(\Delta_\mu+\sigma)^{-1/2}u|\, \right\|_p \le C\|u\|_p.$$

The problem is thus reduced to the study of conditions for boundedness of the classical Riesz transform $\mathrm{d}(\Delta_\mu+\sigma)^{-1/2}$ on functions and boundedness of the covariant Riesz transform $\nabla(\Delta_\mu^{(1)}+\sigma)^{-1/2}$ on one-forms.

As far as the covariant Riesz transform $\nabla(\Delta_\mu^{(1)}+\sigma)^{-1/2}$ on one-forms is concerned, this transform is bounded in $L^p(\mu)$ for $1 < p \le 2$ by Theorem 1.3. If the local volume doubling property and short time Gaussian estimate for the heat kernel hold, then boundedness of the classical Riesz transform $d(\Delta_\mu + \sigma)^{-1/2}$ holds for $1 < p \leq 2$ as well. Note that (**CD**) for $K_0 \in \mathbb{R}$ implies (**Ric**) for K_0, and the curvature condition (**Ric**) assures the short time Gaussian estimate for the heat kernel (see [31, Theorem 2.4.4]). □

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References

1. Auscher, P., Coulhon, T., Duong, X.T., Hofmann, S.: Riesz transform on manifolds and heat kernel regularity. Ann. Sci. École Norm. Sup. (4) **37**(6), 911–957 (2004)
2. Bakry, D.: Transformations de Riesz pour les semi-groupes symétriques. I. Étude de la dimension 1, Séminaire de probabilités, XIX, 1983/84, Lecture Notes in Math., vol. 1123, pp. 130–144. Springer, Berlin (1985)
3. Bakry, D.: Transformations de Riesz pour les semi-groupes symétriques. II. Étude sous la condition $\Gamma_2 \geq 0$, Séminaire de probabilités, XIX, 1983/84, Lecture Notes in Math., vol. 1123, pp. 145–174. Springer, Berlin (1985)
4. Bakry, D.: Étude des transformations de Riesz dans les variétés riemanniennes à courbure de Ricci minorée, Séminaire de Probabilités, XXI, Lecture Notes in Math., vol. 1247, pp. 137–172. Springer, Berlin (1987)
5. Baumgarth, R., Devyver, B., Güneysu, B.: Estimates for the covariant derivative of the heat semigroup on differential forms, and covariant Riesz transforms. Math. Ann. **25**, 46 (2022)
6. Bianchi, D., Setti, A.G.: Laplacian cut-offs, porous and fast diffusion on manifolds and other applications. Calc. Var. Partial Differ. Equ. **57**(1), 4 (2018)
7. Cao, J., Cheng, L.-J., Thalmaier, A.: Hessian heat kernel estimates and Calderón-Zygmund inequalities on complete Riemannian manifolds. (2021). arXiv:2108.13058
8. Chen, J.C.: Weak type (1, 1) boundedness of Riesz transform on positively curved manifolds, Chinese Ann. Math. Ser. B **13** (1992), no. 1, 1–5, A Chinese summary appears in Chinese Ann. Math. Ser. A **13**(1), 131 (1992)
9. Cheng, S.Y., Yau, S.T.: Differential equations on Riemannian manifolds and their geometric applications. Commun. Pure Appl. Math. **28**(3), 333–354 (1975)
10. Coifman, R.R., Weiss, G: Analyse harmonique non-commutative sur certains espaces homogènes, Lecture Notes in Mathematics, vol. 242. Springer, Berlin-New York (1971)
11. Coulhon, T., Duong, X.T.: Riesz transforms for $1 \leq p \leq 2$. Trans. Am. Math. Soc. **351**(3), 1151–1169 (1999)
12. Coulhon, T., Duong, X.T.: Riesz transforms for $p>2$. C. R. Acad. Sci. Paris Sér. I Math. **332**(11), 975–980 (2001)
13. Coulhon, T., Duong, X.T.: Riesz transform and related inequalities on noncompact Riemannian manifolds. Commun. Pure Appl. Math. **56**(12), 1728–1751 (2003)
14. Coulhon, T., Zhang, Q.S.: Large time behavior of heat kernels on forms. J. Differ. Geom. **77**(3), 353–384 (2007)
15. Driver, B.K., Thalmaier, A.: Heat equation derivative formulas for vector bundles. J. Funct. Anal. **183**(1), 42–108 (2001)
16. Gong, F.-Z., Wang, F.-Y.: Heat kernel estimates with application to compactness of manifolds. Q. J. Math. **52**(2), 171–180 (2001)
17. Grigor'yan, A.: Gaussian upper bounds for the heat kernel on arbitrary manifolds. J. Differ. Geom. **45**(1), 33–52 (1997)
18. Güneysu, B.: Sequences of Laplacian cut-off functions. J. Geom. Anal. **26**(1), 171–184 (2016)
19. Güneysu, B.: Covariant Schrödinger semigroups on Riemannian manifolds, Operator Theory: Advances and Applications, vol. 264. Birkhäuser/Springer, Cham (2017)
20. Güneysu, B., Pigola, S.: The Calderón-Zygmund inequality and Sobolev spaces on noncompact Riemannian manifolds. Adv. Math. **281**, 353–393 (2015)
21. Güneysu, B., Pigola, S.: L^p-interpolation inequalities and global Sobolev regularity results, With an appendix by Ognjen Milatovic, Ann. Mat. Pura Appl. (4) **198**(1), 83–96 (2019)
22. Hess, H., Schrader, R., Uhlenbrock, D.A.: Kato's inequality and the spectral distribution of Laplacians on compact Riemannian manifolds. J. Differ. Geom. **15** (1980), no. 1, 27–37 (1981)

23. Li, J.: Gradient estimate for the heat kernel of a complete Riemannian manifold and its applications. J. Funct. Anal. **97**(2), 293–310 (1991)
24. Li, P., Yau, S.-T.: On the parabolic kernel of the Schrödinger operator. Acta Math. **156**(3–4), 153–201 (1986)
25. Li, X.-D.: Riesz transforms on forms and L^p-Hodge decomposition on complete Riemannian manifolds. Rev. Mat. Iberoam. **26**(2), 481–528 (2010)
26. Qian, Z.: Estimates for weighted volumes and applications. Q. J. Math. Oxf. Ser. (2) **48**(190), 235–242 (1997)
27. Simon, B.: Schrödinger semigroups. Bull. Am. Math. Soc. (N.S.) **7**(3), 447–526 (1982)
28. Strichartz, R.S.: Analysis of the Laplacian on the complete Riemannian manifold. J. Funct. Anal. **52**(1), 48–79 (1983)
29. Sturm, K.-T.: Schrödinger semigroups on manifolds. J. Funct. Anal. **118**(2), 309–350 (1993)
30. Thalmaier, A., Wang, F.-Y.: Derivative estimates of semigroups and Riesz transforms on vector bundles. Potential Anal. **20**(2), 105–123 (2004)
31. Wang, F.-Y.: Analysis for diffusion processes on Riemannian manifolds, Advanced Series on Statistical Science & Applied Probability, vol. 18, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ (2014)
32. Zhang, Q.S.: Large time behavior of Schrödinger heat kernels and applications. Commun. Math. Phys. **210**(2), 371–398 (2000)
33. Zhang, Q.S.: Global bounds of Schrödinger heat kernels with negative potentials. J. Funct. Anal. **182**(2), 344–370 (2001)