Master in Mathematics

Probabilistic Models in Finance

Sample Exam 2020

Solution

The market is arbitrage free and complete. We denote by $\mathbb{P}^*$ the unique probability measure on Ω under which discounted prices of assets are martingales. Under these conditions,

$$\mathcal{P}_n(h) = \frac{1}{(1+r)^{N-n}}\,\mathbb{E}^*[h|\mathscr{F}_n].$$

For $k \geq 0$, consider

$$h = S_N \vee k\,.$$

1. We find

$$\begin{aligned}\mathcal{P}_n(h) &= \frac{1}{(1+r)^{N-n}}\,\mathbb{E}^*[S_N \vee k|\mathscr{F}_n] \geq \frac{1}{(1+r)^{N-n}}\,\mathbb{E}^*[S_N|\mathscr{F}_n]\\ &= (1+r)^n\,\mathbb{E}^*[\tilde{S}_N|\mathscr{F}_n] = (1+r)^n\tilde{S}_n = S_n\,.\end{aligned}$$

   On the other hand,

$$\mathcal{P}_n(h) \geq \frac{1}{(1+r)^{N-n}}\,\mathbb{E}^*[k|\mathscr{F}_n] = \frac{k}{(1+r)^{N-n}}.$$

   The last expression gives the discounted guaranteed minimum value of the option.

2. We have

$$\begin{aligned}V_n(\phi) = \mathcal{P}_n(h) - \mathcal{P}_n(g) &= \frac{(1+r)^n}{(1+r)^N}\mathbb{E}^*\big[|S_N - k|\,|\mathscr{F}_n\big]\\ &\geq (1+r)^n\left|\mathbb{E}^*[\tilde{S}_N - \tilde{k}\mid\mathscr{F}_n]\right|\\ &= (1+r)^n\big|\tilde{S}_n - \tilde{k}\big| = \big|S_n - \tilde{k}(1+r)^n\big|,\end{aligned}$$

   where $\tilde{k} = k/(1+r)^N$.

3. Let $S_0 = s_0$. We show that a necessary and sufficient condition for $\mathcal{P}_n(h) = S_n\ \forall\, n$ is given by $k \leq s_0(1+a)^N$. Since $S_n = s_0\,\xi_1\xi_2\cdot\ldots\cdot\xi_N$, we first remark that $S_N \geq s_0(1+a)^N$ where independence of the random variables ξ_i under $\mathbb{P}^*$ implies that

$$\mathbb{P}^*\{S_N = s_0(1+a)^N\} = \mathbb{P}^*\{\xi_1 = 1+a,\,\xi_2 = 1+a,\ldots,\xi_N = 1+a\} = p^N > 0. \qquad (1)$$

   i) Suppose now that $k \leq s_0(1+a)^N$; then $S_N \geq s_0(1+a)^N \geq k$. Therefore we have

$$\begin{aligned}\mathcal{P}_n(h) &= \frac{1}{(1+r)^{N-n}}\,\mathbb{E}^*[S_N \vee k\,|\,\mathcal{F}_n] = \frac{1}{(1+r)^{N-n}}\,\mathbb{E}^*[S_N\,|\,\mathcal{F}_n]\\ &= (1+r)^n\,\mathbb{E}^*[\tilde{S}_N|\mathcal{F}_n] = (1+r)^n\tilde{S}_n = S_n.\end{aligned}$$

   ii) Conversely, suppose that $\mathcal{P}_n(h) = S_n\ \forall\, n$; this means in particular that $h = \mathcal{P}_N(h) = S_N$ and hence $S_N \geq k$. Since by (1), $\{\omega\,|\,S_N(\omega) = s_0(1+a)^N\} \neq \varnothing$, we have $s_0(1+a)^N \geq k$.

4. We write

$$\mathcal{P}_n(h) = \frac{1}{(1+r)^{N-n}}\,\mathbb{E}^*\big[S_N \vee k|\mathscr{F}_n\big] = \frac{1}{(1+r)^{N-n}}\,\mathbb{E}^*\big[(S_n\,\xi_{n+1}\cdot\ldots\cdot\xi_N)\vee k|\mathscr{F}_n\big] = u(n, S_n)$$

   where $u(n,s) = (1+r)^{n-N}\,\mathbb{E}^*\big[(s\,\xi_{n+1}\cdot\ldots\cdot\xi_N)\vee k\big]$. From the fact that $s \mapsto s\vee k$ is increasing, we read off immediately the monotonicity of the functions $u(n,\cdot)$.

One may calculate u explicitly by means of conditioning (see course): we have

$$u(n,s) = \frac{1}{(1+r)^{N-n}} \sum_{j=0}^{N-n} \binom{N-n}{j} p^j (1-p)^{N-n-j} \left[s(1+a)^j (1+b)^{N-n-j} \vee k \right],$$

and hence

$$\mathcal{P}_n(h) = \frac{1}{(1+r)^{N-n}} \sum_{j=0}^{N-n} \binom{N-n}{j} p^j (1-p)^{N-n-j} \left[S_n(1+a)^j (1+b)^{N-n-j} \vee k \right].$$

5. Let $\Phi = (\phi^0, \phi)$ denote a replicating portfolio for h.

(a) Using the definition of the price of an option, we have in terms of the replicating strategy Φ,

$$V_n(\Phi) = \mathcal{P}_n(h) = u(n, S_n) = \phi_n^0 (1+r)^n + \phi_n S_n,$$

which can also be written as

$$(1+r)^n \phi_n^0 + \phi_n S_{n-1} \xi_n = u(n, S_{n-1}\xi_n). \tag{2}$$

Since the random variables ξ_n only take the two values $1+a$ and $1+b$, Eq. (2) is equivalent to the system

$$\begin{cases} \left[(1+r)^n \phi_n^0 + \phi_n\, S_{n-1}(1+a)\right]\, 1_{\{\xi_n = 1+a\}} = u(n, (1+a)S_{n-1}) 1_{\{\xi_n=1+a\}} \\ \left[(1+r)^n \phi_n^0 + \phi_n\, S_{n-1}(1+b)\right]\, 1_{\{\xi_n = 1+b\}} = u(n, (1+b)S_{n-1}) 1_{\{\xi_n=1+b\}}. \end{cases}$$

Passing to conditional expectations with respect to $\mathscr{F}_{n-1}$ and taking into account that the sequence (Φ_n) is predictable, whereas ξ_n is independent of $\mathscr{F}_{n-1}$, we get

$$\begin{cases} (1+r)^n \phi_n^0 + \phi_n\, S_{n-1}(1+a) = u\left(n, (1+a)S_{n-1}\right) \\ (1+r)^n \phi_n^0 + \phi_n\, S_{n-1}(1+b) = u\left(n, (1+b)S_{n-1}\right). \end{cases}$$

From here we conclude that

$$\phi_n = \frac{u\left(n, (1+b)S_{n-1}\right) - u\left(n, (1+a)S_{n-1}\right)}{(b-a)S_{n-1}}. \tag{3}$$

(b) The wanted necessary and sufficient condition is the same as in part 3, i.e. $k \leq s_0(1+a)^N$. Indeed:

- If $k \leq s_0(1+a)^N$ holds, we know that $S_N = \mathcal{P}_N(h)$, and hence $\Phi = (0,1)$ is a hedging portfolio for h.
- Let $\phi^0 = 0$. On one hand, we have $\phi_{n+1} S_n = \phi_n S_n$ (Φ is self-financing), and hence

$$\phi_N = \phi_{N-1} = \ldots = \phi_0$$

where ϕ_0 is deterministic (and thus a constant). On the other hand, we have $\phi_N S_N = V_N(\Phi) = S_N \vee k$, and hence $S_N \geq k$. We have already seen in question 3 that this implies the inequality $k \leq s_0(1+a)^N$.

(c) Since the functions $u(n, \cdot)$ are increasing, we see from formula (3) that $\phi_n \geq 0\ \forall n$ as claimed.

(d) We obtain a European call $h_0(S_N)$ of strike K by taking for h_0 the function $x \mapsto (x-K)_+$. The function h_0 is increasing, hence $u(n, \cdot)$ as well for each n. As in the preceding question this shows that $\phi_n \geq 0\ \forall n$.

(e) Suppose that $N = 1$, $S_0 = 1$ and let $h_0(x) = (K-x)_+$. To replicate the option $h_0(S_1)$ the portfolio has to satisfy

$$\begin{cases} (1+r)\phi_1^0 + \phi_1\,(1+a) = h_0(1+a) \\ (1+r)\phi_1^0 + \phi_1\,(1+b) = h_0(1+b). \end{cases}$$

(note that ϕ_1 is deterministic). Thus we have

$$\phi_1 = \frac{(K-(1+b))_+ - (K-(1+a))_+}{b-a}.$$

If we suppose $K > 1+a$ then $\phi_1 < 0$. This shows that hedging of a put of strike $> 1+a$ requires short-selling of $|\phi_1|$ unities of the risky asset at time 1.